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Efficiency and Generalized Convex Duality for Multiobjective Programs

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The concept of efficiency (Pareto optimum) is used to formulate duality for multiobjective non-linear programs. The results are obtained for convex functions, ρ-convex (i.e., weakly convex and strongly convex) functions, and pseudoconvex functions. For the convex and ρ-convex functions a Wolfe type of dual is formulated, while for the pseudoconvex and ρ-convex functions, a Mond–Weir type dual is proposed.

1. INTRODUCTION AND PRELIMINARIES

The aim of this paper is to use the concept of efficiency (Pareto optimum) to formulate duality relationships between the multiobjective non-linear program

$$\text{(VOP)} \qquad \text{Minimize } (f_1(x), f_2(x), ..., f_p(x)) \quad \text{Subject to } g(x) \leqq 0; \tag{1}$$

and the two dual multiobjective programs

Wolfe Vector Dual [11]

$$\text{(WVD)} \qquad \text{Maximize } (f_1(u) + y^t g(u), ..., f_p(u) + y^t g(u))$$

$$\text{Subject to } \sum_{i=1}^{p} \tau_i \nabla f_i(u) + \nabla y^t g(u) = 0, \tag{2}$$

$$y \geqq 0, \tag{3}$$

$$\tau_i \geqq 0,\ i = 1, 2, ..., p,\ \sum_{i=1}^{p} \tau_i = 1; \tag{4}$$

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Mond–Weir Vector Dual [6]

(DVOP) Maximize $(f_1(u), f_2(u), ..., f_p(u))$

Subject to (2) through to (4) and

$$y'g(u) \geqq 0. \tag{5}$$

The functions f_i: $\mathbb{R}^n \to \mathbb{R}$, $i=1, 2, ..., p$ and g: $\mathbb{R} \to \mathbb{R}^n$ are assumed to be differentiable.

Optimization in (VOP), (WVD), and (DVOP) means obtaining efficient solutions for the corresponding programs.

DEFINITION 1. A feasible solution x^0 for (VOP) is efficient for (VOP) if and only if there is no other feasible x for (VOP) such that for some $i \in P = \{1, 2, ..., p\}$

$$f_i(x) < f_i(x^0) \tag{6}$$

and

$$f_j(x) \leqq f_j(x^0) \qquad \text{for all} \quad j \in P. \tag{7}$$

In the case of maximization, the signs of the inequalities (6) and (7) are reversed (i.e., they become $>$ and $\geqq$, respectively).

For the case where τ_i, $i=1, 2, ..., p$, are all strictly positive, f_i, $i=1, 2, ..., p$, and g are convex; Weir [9] has used proper efficiency [3] to establish some duality results between (VOP) and (WVD). Also, Egudo [2] and Weir [9] have used proper efficiency to obtain duality relationships between (VOP) and (DVOP) where the multipliers τ_i, $i=1, 2, ..., p$, are all strictly positive and a positive linear combination of the objective function components is assumed to be pseudoconvex. Recently Weir [10] obtained a duality result between (VOP) and (DVOP) whereby the objective function components are pseudoconvex but their positive linear combination need not be pseudoconvex.

In [2, 9, 10], the proofs of strong duality results use the following Geoffrion [3] characterization of proper efficiency.

LEMMA 1. *If for some fixed $\lambda \in 0 \in \mathbb{R}^p$, x^0 solves the single objective program*

(P_λ) *Minimize* $\lambda' f(x)$

Subject to $g(x) \leqq 0$;

then x^0 is properly efficient for (VOP).

In this paper proofs for strong duality results will invoke the following.

LEMMA 2 (Theorem 4.1 of Chankong and Haimes [1]). *x^0 is an efficient solution for (VOP) if and only if x^0 solves*

$$P_k(\varepsilon^0) \qquad \text{Minimize } f_k(x)$$

$$\text{Subject to } f_j(x) \leqq f_j(x^0) \quad \text{for all} \quad j \neq k,$$

$$g(x) \leqq 0;$$

for each $k = 1, 2, ..., p$.

2. WOLFE VECTOR DUALITY

Here we prove weak and strong duality relations between (VOP) and (WVD). First we consider a weak duality result when the functions are convex.

THEOREM 1. *Assume that for all feasible x for (VOP) and all feasible (u, τ, y) for (WVD), f_i, $i = 1, 2, ..., p$, and g are convex functions. If also either*

(a) $\tau_i > 0$ *for all* $i = 1, 2, ..., p$ *or*

(b) $\sum_{i=1}^{p} \tau_i f_i(\cdot) + \sum_{j=1}^{m} y_j g_j(\cdot)$ *is strictly convex at* u,

then the following cannot hold:

$$f_j(x) \leqq f_j(u) + y^t g(u) \qquad \text{for all} \quad j \in P = \{1, 2, ..., p\} \tag{8}$$

and

$$f_i(x) < f_i(u) + y^t g(u) \qquad \text{for some} \quad i \in P. \tag{9}$$

Proof. Suppose contrary to the result that (8) and (9) hold. Then since x is feasible for (VOP) and $y \geqq 0$, (8) and (9) imply

$$f_j(x) + y^t g(x) \leqq f_j(u) + y^t g(u) \qquad \text{for all} \quad j \in P \tag{10}$$

and

$$f_i(x) + y^t g(x) < f_i(u) + y^t g(u) \qquad \text{for some} \quad i \in P, \tag{11}$$

respectively. Now hypothesis (a) and $\sum_{i=1}^{p} \tau_i = 1$ imply

$$\sum_{i=1}^{p} \tau_i f_i(x) + y^t g(x) < \sum_{i=1}^{p} \tau_i f_i(u) + y^t g(u) \tag{12}$$

and since f_i's and g are convex and $\tau_i > 0$, $i = 1, 2, ..., p$, $y \geqq 0$, it now follows from (12) that

$$(x-u)^t \left(\sum_{i=1}^{p} \tau_i \nabla f_i(u) + \nabla y^t g(u) \right) < 0, \tag{13}$$

which contradicts (2).

Also, since $\tau_i \geqq 0$, $i = 1, 2, ..., p$, and $\sum_{i=1}^{p} \tau_i = 1$, (10) and (11) imply

$$\sum_{i=1}^{p} \tau_i f_i(x) + y^t g(x) \leqq \sum_{i=1}^{p} \tau_i f_i(u) + y^t g(u). \tag{14}$$

Now (14) and hypothesis (b) imply (13), again contradicting (2). ∎

Next we state and prove a weak duality result between (VOP) and (WVD) under ρ-convexity. But first we define ρ-convex functions [7, 8].

DEFINITION 2. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be ρ-convex [7, 8] if there exists a real number ρ such that for each x, $u \in \mathbb{R}^n$ and $0 \leqq \lambda \leqq 1$,

$$f(\lambda x + (1-\lambda) u) \leqq \lambda f(x) + (1-\lambda) f(u) - \rho\lambda(1-\lambda) \|x-u\|^2.$$

For a differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$, f is ρ-convex if and only if for all x, $u \in \mathbb{R}^n$

$$f(x) - f(u) \geqq (x-u)^t \nabla f(u) + \rho \|x-u\|^2.$$

If ρ is positive then f is said to be strongly convex [7, 8]; and if ρ is negative then f is said to be weakly convex [8].

THEOREM 2 (Weak Duality). *Assume that for all feasible x for (VOP) and all feasible (u, τ, y) for (WVD), f_i, $i = 1, 2, ..., p$, are ρ_i-convex and g_j, $j = 1, 2, ..., m$, are σ_j-convex. If also either*

(a) $\tau_i > 0$ *for all* $i = 1, 2, ..., p$ *and* $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j \geqq 0$ *or*

(b) $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j > 0$,

then the following cannot hold:

$$f_j(x) \leqq f_j(u) + y^t g(u) \qquad \textit{for all} \quad j \in P = \{1, 2, ..., p\} \tag{15}$$

and

$$f_i(x) < f_i(u) + y^t g(u) \qquad \textit{for some} \quad i \in P. \tag{16}$$

Remark 1. Hypothesis (a) can be interpreted as follows: when all the objective function multipliers are strictly positive then the linear combination of the objective function components plus the non-negative linear combination of the constraint functions should be either convex or strongly convex. Hypothesis (b) can be interpreted as follows: the non-negative linear combination of the objective function components and the constraint functions should be strongly convex. These conditions are weaker than those in Theorem 1 because they allow some of the objective function components and constraint functions to be weakly convex [8] provided that their non-negative linear combination is either convex or strongly convex.

Proof. Suppose contrary to the result that (15) and (16) hold. Then since x is feasible for (VOP) and $y \geqq 0$, (15) and (16) imply

$$f_j(x) + y'g(x) \leqq f_j(u) + y'g(u) \qquad \text{for all} \quad j \in P \tag{17}$$

and

$$f_i(x) + y'(x) < f_i(u) + y'g(u) \qquad \text{for some} \quad i \in P. \tag{18}$$

Now if hypothesis (a) holds, then from $\tau_i > 0$ for all $i \in P$, (17), and (18) we obtain

$$\sum_{i=1}^{p} \tau_i f_i(x) + y'g(x) \sum_{i=1}^{p} \tau_i < \sum_{i=1}^{p} \tau_i f_i(u) + y'g(u) \sum_{i=1}^{p} \tau_i$$

and since $\sum_{i=1}^{p} \tau_i = 1$, this inequality reduces to

$$\sum_{i=1}^{p} \tau_i (f_i(x) - f_i(u)) + y'g(x) - y'g(u) < 0. \tag{19}$$

Now from (19), ρ_i-convexity of f_i's, and σ_j-convexity of g_j's we obtain

$$(x-u)' \left(\sum \tau_i \nabla f_i(u) + \nabla y'g(u) \right) + \left(\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j \right) \|x-u\|^2 < 0$$

and since by hypothesis (a), $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j \geqq 0$, this implies

$$(x-u)' \left(\sum_{i=1}^{p} \tau_i \nabla f_i(u) + y' \nabla g(u) \right) < 0, \tag{20}$$

which contradicts (2).

Also from (17), (18), and $\tau_i \geqq 0$, $i = 1, 2, ..., p$, $\sum_{i=1}^{p} \tau_i = 1$, we obtain

$$\sum_{i=1}^{p} \tau_i (f_i(x) - f_i(u)) + y'g(x) - y'g(u) \leqq 0; \tag{21}$$

and since f_i's are ρ_i-convex and g_j's are σ_j-convex, (21) implies

$$(x-u)'\left(\sum_{i=1}^{p} \tau_i \nabla f_i(u) + \nabla y' g(u)\right) + \left(\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j\right) \|x-u\|^2 \leqq 0. \tag{22}$$

Now by hypothesis (b), $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j > 0$; hence (22) implies (20), again contradicting (2). ∎

COROLLARY 1. *Assume that weak duality (Theorem 1 or 2) holds between (VOP) and (WVD). If (u^0, τ^0, y^0) is feasible for (WVD) with $y^{0\prime} g(u^0) = 0$ and u^0 is feasible for (VOP), then u^0 is efficient for (VOP) and (u^0, τ^0, y^0) is efficient for (WVD).*

Proof. Suppose that u^0 is not efficient for (VOP); then there exists a feasible x for (VOP) such that for some $i \in P = \{1, 2, ..., p\}$,

$$f_i(x) < f_i(u^0) \tag{23}$$

and

$$f_j(x) \leqq f_j(u^0) \qquad \text{for all} \quad j \in P. \tag{24}$$

By hypothesis $y^{0\prime} g(u^0) = 0$, so (23) and (24) can be written as

$$f_u(x) < f_i(u^0) + y^{0\prime} g(u^0) \qquad \text{for some} \quad i \in P$$

$$\text{and } f_j(x) \leqq f_j(u^0) + y^{0\prime} g(u^0) \qquad \text{for all} \quad j \in P,$$

respectively; and since (u^0, τ^0, y^0) is feasible in (WVD) and x is feasible for (VOP), these inequalities contradict weak duality (Theorem 1 or 2).

Also suppose that (u^0, τ^0, y^0) is not efficient ofr (WVD). Then there exists a feasible (u, τ, y) for (WVD) such that for some $i \in P = \{1, 2, ..., p\}$

$$f_i(u) + y' g(u) > f_i(u^0) + y^{0\prime} g(u^0) \tag{25}$$

and

$$f_j(u) + y' g(u) \geqq f_j(u^0) + y^{0\prime} g(u^0) \qquad \text{for all} \quad j \in P; \tag{26}$$

and since $y^{0\prime} g(u^0) = 0$, (25) and (26) reduce to

$$f_i(u) + y' g(u) > f_i(u^0) \qquad \text{for some} \quad i \in P$$

and

$$f_j(u) + y' g(u) \geqq f_j(u^0) \qquad \text{for all} \quad j \in P,$$

respectively. Since u^0 is feasible for (VOP), these inequalities contradict weak duality (Theorem 1 or 2). Therefore u^0 and (u^0, τ^0, y^0) are efficient for their respective programs. ▮

THEOREM 3 (Strong Duality). *Let x^0 be an efficient solution for (VOP) and assume that x^0 satisfies a constraint qualification [4, 5] for $P_k(\varepsilon^0)$ for at least one $k = 1, 2, ..., p$; then there exist $\tau^0 \in \mathbb{R}^p$ and $y^0 \in \mathbb{R}^m$ such (x^0, τ^0, y^0) is feasible for (WVD) and $y^{0t}g(x^0) = 0$. If also weak duality (Theorem 1 or 2) holds between (VOP) and (WVD) then (x^0, τ^0, y^0) is efficient for (WVD).*

Proof. Since x^0 is efficient for (VOP), from Lemma 2, x^0 solves $P_k(\varepsilon^0)$ for all $k = 1, 2, ..., p$. By hypothesis there exists a $k \in P = \{1, 2, ..., p\}$ for which x^0 satisfies a constraint qualification of $P_k(\varepsilon^0)$. Now from Kuhn–Tucker necessary conditions [4, 5] there exist $\tau_i \geqq 0$ for all $i \neq k$ and $y \geqq 0 \in R^m$ such that

$$f_k(x^0) + \sum_{i \neq k} \tau_i \nabla f_i(x^0) + \sum_{j=1}^{m} y_j \nabla g_j(x^0) = 0, \tag{27}$$

$$y^t g(x^0) = 0. \tag{28}$$

Now dividing all terms in (27) and (28) by $1 + \sum_{i \neq k} \tau_i$ and setting

$$\tau_k^0 = \frac{1}{1 + \sum_{i \neq k} \tau_i} > 0, \qquad \tau_j^0 = \frac{\tau_j}{j + \sum_{1 \neq k} \tau_i} \geqq 0,$$

and

$$y^0 = \frac{y}{1 + \sum_{i \neq k} \tau_i} \geqq 0,$$

we conclude that (x^0, τ^0, y^0) is feasible for (WVD). Efficiency of (x^0, τ^0, y^0) for (WVD) now follows from Corollary 1. ▮

3. MOND–WEIR VECTOR DUALITY

In this section we present weak and strong duality relations between programs (VOP) and (DVOP). The weak duality results are given under two conditions: one when objective function components are pseudoconvex and the constraint functions are quasiconvex and the other when the functions are ρ-convex.

THEOREM 4 (Weak Duality). *Assume that for all feasible X for (VOP) and all feasible (u, τ, y) for (DVOP), $y^t g(\cdot)$ is quasiconvex at u. If also any of the following holds.*

(a) $\tau_i > 0$, $\forall i \in P = \{1, 2, ..., p\}$, *and* f_i, $i = 1, 2, ..., p$, *are pseudoconvex at* u;

(b) $\tau_i > 0$, *for all* $i \in P$ *and* $\sum_{i=1}^{p} \tau_i f_i(\cdot)$ *is pseudoconvex at* u;

(c) $\sum_{i=1}^{p} \tau_i f_u(\cdot)$ *is strictly pseudoconvex at* u,

then the following cannot hold:

$$f_j(x) \leqq f_j(u) \qquad \text{for all} \quad j \in P = \{1, 2, ..., p\} \tag{29}$$

and

$$f_j(x) < f_i(u) \qquad \text{for some} \quad i \in P. \tag{30}$$

Proof. For each feasible x for (VOP) and each feasible (u, τ, y) for (DVOP) we have $y^t g(x) - y^t g(u) \leqq 0$; and since $y^t g(\cdot)$ is quasiconvex at u this implies

$$(x - y)^t \nabla y^t g(u) \leqq 0. \tag{31}$$

Applying (31) to $\sum_{i=1}^{p} \tau_i \nabla f_i(u) + \nabla y^t g(u) = 0$ yields

$$(x - u)^t \sum_{i=1}^{p} \tau_i \nabla f_i(u) \geqq 0. \tag{32}$$

Now suppose contrary to the result of the theorem that (29) and (30) hold. If $\tau_i > 0$ for all $i = 1, 2, ..., p$ then (29) and (30) imply

$$\tau_j f_j(x) \leqq \tau_j f_j(u) \qquad \text{for all} \quad j \in P = \{1, 2, ..., p\} \tag{33}$$

and

$$\tau_i f_i(x) < \tau_i f_i(u) \qquad \text{for some} \quad i \in P, \tag{34}$$

respectively. Equations (33) and (34) also imply

$$\sum_{i=1}^{p} \tau_i f_i(x) < \sum_{i=1}^{p} \tau_i f_i(u). \tag{35}$$

By hypothesis (a), i.e., f_i's are pseudoconvex, (33) and (34) imply

$$(x - u)^t \left(\sum_{i=1}^{p} \tau_i \nabla f_i(u) \right) < 0, \tag{36}$$

contradicting (32).

By hypothesis (b), i.e., $\sum_{i=1}^{p} \tau_i f_i(\cdot)$ is pseudoconvex at u, (35) implies (36), again contradicting (32).

Now from $\tau_i \geqq 0$, $i = 1, 2, ..., p$, (29), and (30) we obtain

$$\sum_{i=1}^{p} \tau_i f_i(x) \leqq \sum_{i=1}^{p} \tau_i f_i(u) \tag{37}$$

and by hypothesis (c), i.e., $\sum_{i=1}^{p} \tau_i f_i(\cdot)$ is strictly pseudoconvex, (37) implies (36), again contradicting (32). ▮

THEOREM 5 (Weak Duality). *Assume that for all feasible x for (VOP) and all feasible (u, τ, y) for (DVOP), f_i, $i = 1, 2, ..., p$, are ρ_i-convex and g_j, $j = 1, 2, ..., m$, are σ_j-convex. If either*

(a) $\tau_i > 0$ *for all* $i \in P = \{1, 2, ..., p\}$ *and* $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j \geqq 0$, *or*

(b) $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j > 0$,

then the following cannot hold:

$$f_i(x) < f_i(u) \qquad \text{for some} \quad i \in P = \{1, 2, ..., p\} \tag{38}$$

and

$$f_j(x) \leqq f_j(u) \qquad \text{for all} \quad j \in P. \tag{39}$$

Proof. Suppose contrary to the result that (38) and (39) hold; then for $\tau_i > 0$ for each $i = 1, 2, ..., p$, (38) and (39) imply

$$\sum_{i=1}^{p} \tau_i f_i(x) < \sum_{i=1}^{p} \tau_i f_i(u); \tag{40}$$

and for $\tau_i \geqq 0$, $i = 1, 2, ..., p$, (38) and (39) imply

$$\sum_{i=1}^{p} \tau_i f_i(x) \leqq \sum_{i=1}^{p} \tau_i f_i(u). \tag{41}$$

Now since f_i's are ρ_i-convex (40) and (41) imply

$$(x-u)^t \left(\sum_{i=1}^{p} \tau_i \nabla f_i(u) \right) + \left(\sum_{i=1}^{p} \tau_i \rho_i \right) \|x-u\|^2 < 0, \tag{42}$$

$$(x-u)^t \left(\sum_{i=1}^{p} \tau_i \nabla f_i(u) \right) + \left(\sum_{i=1}^{p} \tau_i \rho_i \right) \|x-u\|^2 \leqq 0, \tag{43}$$

respectively. Also, since (u, τ, y) is feasible for (DVOP) and x is feasible for (VOP) we have

$$y^t g(x) - y^t g(u) \leqq 0; \tag{44}$$

and because g_j's are σ_j-convex (44) implies

$$(x-u)^t \nabla y^t g(u) + \|x-u\|^2 \sum_{j=1}^{m} y_j \sigma_j \leqq 0. \tag{45}$$

Now adding (42) and (45) and then applying hypothesis (a), i.e.,

$$\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j \geqq 0,$$

yield

$$(x-u)^t \left(\sum_{i=1}^{p} \tau_i \nabla f_i(u) + \sum_{j=1}^{m} y_j \nabla g_j(u) \right) < 0, \tag{46}$$

which contradicts (2). Also, adding (43) and (45) and then applying hypothesis (b) yield (46), again contradicting (2). ∎

COROLLARY 2. *Assume weak duality (Theorem 4 or 5) holds between (VOP) and (DVOP). If (u^0, τ^0, y^0) is feasible for (DVOP) such that u^0 is feasible for (VOP), then u^0 is efficient for (VOP) and (u^0, τ^0, y^0) is efficient for (DVOP).*

Proof. First we show that u^0 is efficient for (VOP). Suppose that u^0 is not efficient for (VOP); then there exists a feasible x for (VOP) such that (29) and (30) or (38) and (39) hold. But (u^0, τ^0, y^0) is feasible for (DVOP); hence the result of weak duality (Theorem 4 or 5) is contradicted. Therefore u^0 must be efficient for (VOP). Similarly assuming that (u^0, τ^0, y^0) is not efficient for (DVOP) leads to a contradiction, and hence (u^0, τ^0, y^0) is efficient for (DVOP). ∎

THEOREM 6 (Strong Duality). *Let x^0 be efficient for (VOP) and assume that x^0 satisfies a constraint qualification [4, 5] for $P_k(\varepsilon^0)$ for at least one $k = 1, 2, ..., p$. Then there exist $\tau^0 \in \mathbb{R}^p$ and $y^0 \in \mathbb{R}^m$ such that (x^0, τ^0, y^0) is feasible for (DVOP). If also weak duality (Theorem 4 or 5) holds between (VOP) and (DVOP) then (x^0, τ^0, y^0) is efficient for (DVOP).*

Proof. Since x^0 is an efficient solution of (VOP), then from Lemma 2, x^0 solves $P_k(\varepsilon^0)$ for each $k = 1, 2, ..., p$. By hypothesis there exists at least one $k = 1, 2, ..., p$ such that x^0 satisfies a constraint qualification [4, 5] for $P_k(\varepsilon^0)$. From Kuhn–Tucker necessary conditions [4, 5] we obtain $\tau_i \geqq 0$ for all $i \neq k$, and $y \geqq 0 \in \mathbb{R}^m$ such that

$$\nabla f_k(x^0) + \sum_{i \neq k} \tau_i \nabla f_i(x^0) + \nabla y^t g(x^0) = 0 \tag{47}$$

$$y^t g(x^0) = 0. \tag{48}$$

Now dividing (47) and (48) by $1+\sum_{i\neq k}\tau_i$ and defining

$$\tau_k^0=\frac{1}{1+\sum_{i\neq k}\tau_i}>0,\quad \tau_j^0=\frac{\tau_j}{1+\sum_{i\neq k}\tau_i}\geqq 0\qquad \text{for all}\quad j\neq k$$

and

$$y^0=\frac{y}{1+\sum_{i\neq k}\tau_i}\geqq 0,$$

we conclude that (x^0, τ^0, y^0) is feasible for (DVOP). The efficiency of (x^0, τ^0, y^0) for (DVOP) now follows from Corollary 2. ∎

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