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Separation of a diagonal

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ABSTRACT

We investigate different separation properties of the diagonal of a space X. Namely, we study spaces X in which the diagonal of X^2 and every closed subset of X^2 off the diagonal can be separated from each other by means of open sets, or continuous functions, or some other tools.

1. Introduction

There are numerous results in the literature that show that the diagonal of X^2 carries enormous amount of information about X. In particular, a space X satisfies axiom T_2 iff the diagonal $\Delta_X = \{\langle x, x\rangle\colon\ x \in X\}$ is closed in X^2. Another classical example is the Sneider theorem stating that a compactum with a G_δ-diagonal is metrizable [10]. A good account of results of such a kind can be found in any survey on generalized metric spaces. Diagonal properties often appear in formulas that estimate certain cardinal invariants of a space. Thus, no justification is needed for our curiosity in diagonals. However, to motivate our interest in this particular study let us consider the following quite obvious statement: *If X has a G_δ-diagonal and X^2 is normal then X has a zeroset diagonal.* Having a zeroset diagonal is considerably a stronger property than having just a G_δ-diagonal. Indeed, a separable space with a zeroset diagonal admits a continuous injection into a second-countable metrizable space [8]. A separable space with a G_δ-diagonal need not have metrizable subtopology as witnessed by the classical Mrowka space [9]. A quick analysis of a mental proof of the above statement reveals that not much is needed from the normality property of X^2. We only need to know that for any closed set A of X^2 that misses the diagonal there exists a continuous map from X^2 to $[0,1]$ that maps A to $\{0\}$ and the diagonal to $\{1\}$. This example and many other results by different mathematicians motivate the following definitions:

Definition 1.1. A space X is Δ-normal if for every $A \subset X^2 \setminus \Delta_X$ closed in X^2 there exist disjoint open U and V in X^2 such that $A \subset U$ and $\Delta_X \subset V$.

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Definition 1.2. A space X is functionally Δ-normal if for every $A \subset X^2 \setminus \Delta_X$ closed in X^2 there exists a continuous function $f: X^2 \to [0,1]$ such that $f(A) = \{1\}$ and $f(\Delta_X) = \{0\}$.

Definition 1.3. A space X is Δ-paracompact if for every $A \subset X^2 \setminus \Delta_X$ closed in X^2 there exists a locally finite open cover $\mathcal{U}$ of X such that $\bigcup\{U \times U\colon\ U \in \mathcal{U}\}$ does not meet A.

Definition 1.4. A space X is regular Δ-paracompact if for every $A \subset X^2 \setminus \Delta_X$ closed in X^2 there exists a locally finite open cover $\mathcal{U}$ of X such that $\bigcup\{\overline{U} \times \overline{U}\colon\ U \in \mathcal{U}\}$ does not meet A.

Definition 1.5. A space X is functionally Δ-paracompact if for every $A \subset X^2 \setminus \Delta_X$ closed in X^2 there exists a locally finite cover $\mathcal{U}$ of X by functionally open sets such that $\bigcup\{U \times U\colon\ U \in \mathcal{U}\}$ does not meet A.

The notion of Δ-normality was already studied in [3] and the introduction of the concept is attributed to Eva Lowen-Colebunders. After reading this manuscript, K.P. Hart informed the author that he can prove the equivalence of functional Δ-paracompactness and divisibility. Recall that a space X is *divisible* [1] if for every open set U containing ΔX there exists an open set V containing ΔX such that $V \circ V \subset U$, where $V \circ V = \{\langle x, z\rangle\colon\ \langle x, y\rangle \in V \text{ and } \langle y, z\rangle \in V,\ \text{ for some } y\}$. In this paper we will nevertheless use the name "functional Δ-paracompactness" to stress a unified approach to different separation properties of the diagonal of a space. Another reason for using this name instead of divisibility is because in our proofs we will use the property described in the definition of functional Δ-paracompactness, which some may find more intuitively clear than that in the definition of divisibility. We believe that almost everyone who worked with properties of a diagonal or the square of a space implicitly used some of the five properties we just listed or their natural modifications. Since these properties have already justified their right for existence we think it is quite natural to give them names and devote a study. We will concentrate our study around the following three general problems.

Problem 1.6. Which of these properties are distinguishable in the class of Tychonoff spaces? In the class of normal spaces?

Problem 1.7. Find classes of spaces in which two or more of these properties are equivalent or one property is stronger than another.

Problem 1.8. Let X have one of the properties. What additional conditions on X guarantee that X is normal?

It is not hard to show that in the class of Tychonoff spaces, the property of being functionally Δ-paracompact implies the other four properties as well as normality. In the class of countably compact normal spaces, functional Δ-paracompactness is equivalent to Δ-paracompactness. Majority of our results are proved for countably compact spaces. We show, in particular, that a Δ-paracompact countably compact space need not be normal, while any functionally Δ-paracompact space is necessarily normal. We also show that a countably compact normal space need not be Δ-paracompact. This follows from an interesting, yet not hard to prove, connection between Δ-paracompactness and the theory of fixed-point free maps. We find a wide class of spaces in which Δ-normality is stronger than normality. This class consists of all spaces whose closed subsets are star-Lindelöf. Recall that a space X is *star-Lindelöf* if for every open cover $\mathcal{U}$ of X one can find a countable subset $C \subset X$ such that $X = \bigcup\{St(c, \mathcal{U})\colon\ c \in C\}$. While the description of the class is rather technical it contains such nice objects as countably compact spaces, and more generally, spaces of countable extent. The class also contains hereditarily separable spaces.

In notation and terminology we will follow [2]. All spaces are assumed to be Tychonoff. To distinguish ordered pairs from open intervals, we will use angular parenthesis $\langle a, b\rangle$ for ordered pairs and regular parenthesis (a, b) for open intervals.

2. Study

In our proofs we will often use the fact that every locally finite cover $\{U_s\}_{s\in S}$ of X by functionally open sets admits a functionally closed shrinking $\{F_s\}_{s\in S}$ and a functionally open shrinking $\{W_s\}_{s\in S}$ such that $F_s \subset W_s \subset \overline{W_s} \subset U_s$ [2, Exercise 7.1.B]. The following needs no proof.

Fact 2.1. *The following hold*:

1. *If X is functionally Δ-normal then X is Δ-normal*;
2. *If X is functionally Δ-paracompact then X is regular Δ-paracompact*;
3. *If X is regular Δ-paracompact then X is Δ-paracompact*;
4. *If X is regular Δ-paracompact then X is Δ-normal.*

The next two statements are folklore-type results.

Lemma 2.2. *(Folklore) If X is paracompact then X is functionally Δ-paracompact.*

Proof. Fix $A \subset X^2 \setminus \Delta_X$ closed in X^2. For each $x \in X$ fix an open neighborhood U_x of x such that $U_x \times U_x$ does not meet A. Since X is paracompact there exists a locally finite open cover $\mathcal{U}$ of X inscribed in $\{U_x\colon x \in X\}$ such that U is functionally open for any $U \in \mathcal{U}$ [2, Theorem 5.1.9]. It is clear that $\bigcup\{U \times U\colon U \in \mathcal{U}\}$ does not meet A. □

Theorem 2.3. *(Folklore) If X is functionally Δ-paracompact then X is functionally Δ-normal.*

Proof. Fix $A \subset X^2 \setminus \Delta_X$ closed in X^2. Fix a locally finite cover $\mathcal{U}$ of X by functionally open sets such that $\bigcup\{U \times U\colon U \in \mathcal{U}\}$ misses A. Since $\mathcal{U}$ is locally finite and consists of functionally open sets there exists a shrinking $\mathcal{F} = \{F_U\colon F_U \subset U,\ U \in \mathcal{U}\}$ by functionally closed sets.

Now for every $U \in \mathcal{U}$ fix a family $\mathcal{W}_U = \{W_U^q\colon q \in (0,1) \text{ is rational}\}$ of open sets such that

a. $F_U \subset W_U^q \subset \overline{W}_U^q \subset U$;
b. $\overline{W}_U^p \subset W_U^q$ if $p < q$.

Such families exist because F_U and $X \setminus U$ are disjoint functionally closed subsets. For each rational number $q \in (0,1)$ put $W_q = \bigcup_{U \in \mathcal{U}} W_U^q \times W_U^q$. The family $\{W_q\colon q \in (0,1) \text{ is rational}\}$ has the following properties:

1. $\Delta_X \subset W_q$ for every q. This is because $\{W_U^q\colon U \in \mathcal{U}\}$ is a cover of X.
2. $\overline{W}_p \subset W_q$ if $p < q$. This is because of (b) and the fact that $\{W_U^q\colon U \in \mathcal{U}\}$ is a shrinking of $\mathcal{U}$, which is locally finite.
3. $W_q \subset \bigcup_{U \in \mathcal{U}} U \times U$.

By 1–3, the family $\{W_q\colon q \in (0,1) \text{ is rational}\}$ is Urysohn-type and, therefore, defines a continuous function $f : X^2 \to [0,1]$ such that $f(\Delta_X) = \{0\}$ and $f(A) = \{1\}$. □

Fact 2.1 and statements 1–3 collectively give the following information about paracompact spaces.

Theorem 2.4. *Every paracompact space is Δ-normal, functionally Δ-normal, Δ-paracompact, regular Δ-paracompact, and functionally Δ-paracompact.*

In our next result we will show that every generalized ordered space is functionally Δ-paracompact and therefore has all the properties under consideration. As was mentioned in the introduction, K.P. Hart communicated to the author that he proved the equivalence of functional Δ-paracompactness and divisibility. In the same communication, K.P. Hart mentioned that his proof in conjunction with an alteration of one of Mansfield's results give an alternative proof of our next theorem. Namely, Mansfield [7] proved that every linearly ordered space is $\aleph_0$-normal which implies divisibility. Mansfield's proof also can be re-written for generalized ordered spaces. Thus Hart's equivalence result and a modified version of Mansfield theorem already imply what we are about to prove. However, the author has decided to keep the proof since it uses the property described in the definition of functional Δ-paracompactness and gives another strategy for establishing divisibility in certain spaces. For our proof we need the following classical theorem.

Theorem. *(R. Engelking and D. Lutzer [6]) A generalized ordered space X is paracompact iff no closed subspace of X is homeomorphic to a stationary subset of a regular uncountable cardinal.*

Theorem 2.5. *Every generalized ordered space is Δ-paracompact, regular Δ-paracompact, functionally Δ-paracompact, Δ-normal, and functionally Δ-normal.*

Proof. Fix a generalized ordered space X. Let bX be a linearly ordered compactification of X such that every point $x \in bX \setminus X$ is a limit point for X from one side only. That is, either $[-\infty, x)_{bX}$ or $(x, \infty]_{bX}$ is clopen in bX, where $-\infty = \min bX$ and $\infty = \max bX$. Such a compactification can be obtained from the Dedekind compactification by replacing every point x in the remainder of the Dedekind compactification, which is a limit point for X on both sides, by a pair $\{x', x''\}$. That is, perform the double arrow trick on all such points.

Fix closed $A \subset X^2$ that misses Δ_X. For every $x \in bX$ select, if possible, a convex open neighborhood J_x of x in bX that satisfies the following property:

$$(I_x \times I_x) \cap A = \emptyset, \quad \text{where } I_x = J_x \cap X.$$

Denote the set of all such x's by S. Clearly, $X \subset S$.

Claim. *S is paracompact.*

To prove the claim, assume the contrary. By the Engelking–Lutzer theorem stated before this theorem, there exist $z \in bX \setminus S$ and a set $T_z \subset S$ homeomorphic to a stationary subset of an uncountable regular cardinal τ_z such that z is the only complete accumulation point for T_z in bX. We may assume that all of T_z are to the left of z. We may also assume that if $y \in T_z$ is a limit point for T_z then y is a limit point for T_z on the left side. For every $y \in T_z$, which is a limit point for T_z, fix $a_y \in J_y \cap T_z \cap [-\infty, y)_{bX}$. Such an a_y exists because J_y is an open neighborhood of y in bX and y is a limit point for T_z on the left side. The correspondence $y \to a_y$ defines a regressive function from a stationary subset of T_z into T_z. By the Pressing Down Lemma (see, for example, [5, 6.15]), there exist $a \in T_z$ and a τ_z-sized $P_z \subset T_z$ such that $a_y = a$ for every $y \in P_z$.

Now consider $I_z = (a, z]_{bX} \cap X$. Since $X \subset S$, we have $z \in bX \setminus X$. Since z is a limit point for X from one side only, $(a, z]_{bX}$ is a convex open neighborhood of z in bX. To reach a contradiction we need to show that $I_z \times I_z$ does not meet A. For this it is enough to show that $I_z \times I_z \subset \bigcup\{I_y \times I_y\colon\ y \in P_z\}$. Fix $\langle x_0, y_0\rangle \in I_z \times I_z$. Since P_z is not bounded on the right in T_z, there exists $y \in P_z$ such that $x_0, y_0 < y$. By the choice of P_z, we have $(a, y]_{bX} \cap X \subset I_y$. Applying the fact that $x_0, y_0 \in (a, y)_X$ we conclude that $\langle x_0, y_0\rangle$ is in $I_y \times I_y$. The claim is proved.

Thus, we have $Cl_{S\times S}(A)$ does not meet Δ_S. Since S is paracompact, by Theorem 2.4, there exists a continuous function from $S \times S$ that sends Δ_S to $\{0\}$ and $Cl_{S\times S}(A)$ to $\{1\}$. The restriction of the function to $X \times X$ proves that X is functionally Δ-normal. The other three properties are proved similarly applying Theorem 2.4 to $Cl_{S\times S}(A)$ and Δ_S. □

Next we study when one or more of our properties implies normality.

Proposition 2.6. *If X is regular Δ-paracompact then X is normal.*

Proof. Let A and B be closed and disjoint subsets of X. Since A and B are disjoint closed sets the set $A \times B$ is closed and misses the diagonal. Fix a locally finite open cover $\mathcal{V}$ such that $\bigcup\{\overline{V} \times \overline{V}\colon\ V \in \mathcal{V}\}$ misses $A \times B$. Put $\mathcal{V}_A = \{V \in \mathcal{V}\colon\ V \cap A \neq \emptyset\}$. Clearly, $\mathcal{V}_A$ is a cover of A. Hence $\bigcup \mathcal{V}_A$ is an open neighborhood of A in X. Since $\bigcup\{\overline{V} \times \overline{V}\colon\ V \in \mathcal{V}\}$ misses $A \times B$ the set $\overline{V}$ does not meet B for every $V \in \mathcal{V}_A$. Since $\mathcal{V}_A$ is locally finite we have $B \subset X \setminus \overline{\bigcup \mathcal{V}_A}$. Normality is proved. □

Corollary 2.7. *If X is Δ-paracompact and Δ-normal then X is normal.*

Theorem 2.8. *Let X^2 be pseudocompact. If X is functionally Δ-normal then X is normal.*

Proof. Since X^2 is pseudocompact $\beta(X \times X) = \beta X \times \beta X$. Assume X is not normal. Then there exists $x \in \beta X \setminus X$ and disjoint closed $A, B \subset X$ such that $x \in Cl_{\beta X}(A) \cap Cl_{\beta X}(B)$. Consider $C = A \times B$. Since A and B are closed, C is closed as well. Since A and B are disjoint C does not meet the diagonal. Since $x \in Cl_{\beta X}(A) \cap Cl_{\beta X}(B)$ we have $\langle x, x\rangle \in Cl_{\beta X\times\beta X}(A \times B)$. Therefore, C and Δ_X are not functionally separated. Hence X is not functionally Δ-normal. □

Our next result gives a wide class of spaces in which Δ-normality implies normality. To prove it we will use Lemma 1.5.14 in [2] that states the following: *Let X be a T_1-space. Suppose for any closed $F \subset X$ and any open W containing F there exists a countable family $\mathcal{U}$ of open sets in X such that $\overline{U} \subset W$ for every $U \in \mathcal{U}$ and $F \subset \bigcup \mathcal{U}$. Then X is normal.*

Theorem 2.9. *Suppose every closed subset of X is star-Lindelöf. If X is Δ-normal then X is normal.*

Proof. Let A and B be disjoint closed subsets of X. The set $S = A \times B$ is closed and misses the diagonal. Therefore there exists an open W that contains Δ_X and whose closure misses S. Let $\mathcal{U}$ be the collection of all open subsets $U \subset X$ with $U \times U \subset W$.

Claim. *$\overline{St(a, \mathcal{U})} \cap B = \emptyset$ for all $a \in A$.*

To prove the claim assume the contrary and pick any b in the intersection. Since $\langle a, b\rangle \in S$ there exist O_a and O_b open neighborhoods of a and b, respectively, such that $(O_a \times O_b) \cap W = \emptyset$. Since $b \in \overline{St(a, \mathcal{U})}$ there exists $U_a \in \mathcal{U}$ such that $a \in U_a$ and $U_a \cap O_b \neq \emptyset$. Pick $b_1 \in U_a \cap O_b$. Then $\langle a, b_1\rangle \in U_a \times U_a \subset W$. On the other hand, since $a \in O_a$ we have $\langle a, b_1\rangle \in O_a \times O_b \subset X \times X \setminus W$. This contradiction proves the claim.

Since A is closed it is star-Lindelöf. Therefore there exists a countable $C \subset A$ such that $A \subset \bigcup\{St(c, \mathcal{U}),\ c \in C\}$. By Claim and Lemma 1.5.14 of [2], X is normal. □

Recall that a space X has *countable extent* if every closed discrete subset of X is countable. In particular, every countably compact space has countable extent. It is a known folklore fact that every closed subset of a space of countable extent or a hereditarily separable space is star-Lindelöf. Therefore, we have the following corollaries.

Corollary 2.10. *Let X have countable extent. If X is Δ-normal then X is normal.*

Corollary 2.11. *Let X be hereditarily separable. If X is Δ-normal then X is normal.*

A particular case of the above theorem, namely, when X is first-countable and countably compact, was proved in [3]. Our theorem prompts to ask the following: *Let X be pseudocompact and Δ-normal. Is X countably compact*? Observe that the answer is "yes" if the space is star-Lindelöf due to Theorem 2.9.

Next example shows that Δ-paracompactness does not imply normality.

Example 2.12. The space $X = \{\langle \alpha, \beta\rangle \in (\omega_1 + 1) \times (\omega_1 + 1) \setminus \{\langle \omega_1, \omega_1\rangle\}\colon\ \alpha \leqslant \beta\}$ is Δ-paracompact.

Proof. First observe that X^2 is countably compact. Therefore, $\beta X^2 = \beta X \times \beta X$ and is equal to the closure of X^2 in $(\omega_1 + 1)^4$. Let $F \subset X^2$ be closed and $F \cap \Delta_X = \emptyset$. If $Cl_{\beta X^2}(F)$ does not contain $\langle\langle \omega_1, \omega_1\rangle, \langle \omega_1, \omega_1\rangle\rangle$ then $Cl_{\beta X^2}(F)$ and $Cl_{\beta X^2}(\Delta_X)$ do not meet. Since βX is compact, the conclusion follows.

We now assume that $Cl_{\beta X^2}(F)$ contains $\langle\langle \omega_1, \omega_1\rangle, \langle \omega_1, \omega_1\rangle\rangle$. Our strategy is completely contained in the statement of Claim 5 and a short proof after Claim 5. Before we dive into the details of our argument let us agree that by π_1 and π_2 we denote the projections of X^2 onto the first and second coordinate axes, respectively. Also if $p \in X^2$, by $p(1), \dots, p(4)$ we denote the coordinates of p in $(\omega_1 + 1)^4$.

Claim 1. *There exists $\lambda < \omega_1$ such that $\pi_1(F \cap [\lambda, \omega_1]^4)$ is first-countable.*

To prove the claim, assume the contrary. Then we have the following

Property: *For every $\beta < \omega_1$ there exists $\alpha > \beta$ such that a closed ω_1-sized subset of $\{\alpha\} \times (\omega_1 + 1)$ is in $\pi_1(F)$.*

We have two cases to consider.

Case 1. Assume that for any $\alpha < \omega_1$ there exists $p_\alpha \in F \cap [\alpha, \omega_1]^4$ such that $\pi_1(p_\alpha)$ and $\pi_2(p_\alpha)$ have countable characters in X.

We can select a strictly increasing sequence $\{\alpha_n\}_n$ of countable ordinals and $p_{\alpha_n} \in F \cap [\alpha, \omega_1]^4$ with the following properties:
1. $\pi_i(p_{\alpha_n})$ has countable character in X for $i = 1, 2$;
2. $p_{\alpha_n}(j) < p_{\alpha_{n+1}}(i)$ for $i, j = 1, \dots, 4$.

Put $\lim_{n\to\infty} p_{\alpha_n}(1) = x$. By 2, $\lim_{n\to\infty} p_{\alpha_n}(i) = x$ for every $i = 1, \dots, 4$. Since F is closed, we have $\langle\langle x, x\rangle, \langle x, x\rangle\rangle \in F$, contradicting the fact that $F \cap \Delta_X = \emptyset$.

Case 2. Assume a failure of Case 1.

By *the Property* and the failure of Case 1, we conclude that for every $\beta < \omega_1$ there exist countable $\alpha > \beta$ and subset $F_\alpha \subset F$ such that $\pi_1(F_\alpha)$ is an ω_1-sized closed subset of $\{\alpha\} \times (\omega_1 + 1)$ and $\pi_2(F_\alpha) \subset \{\langle \gamma, \omega_1\rangle\colon\ \gamma < \omega_1\}$.
Therefore, we can select a strictly increasing sequence $\{\alpha_n\}_n$ of countable ordinals and $p_{\alpha_n} \in F \cap [\alpha_n, \omega_1]^4$ with the following properties:
1. $p_{\alpha_n}(2) = p_{\alpha_n}(4) = \omega_1$;
2. $p_{\alpha_{n+1}}(1), p_{\alpha_{n+1}}(3) > \max\{p_{\alpha_n}(1), p_{\alpha_n}(3)\}$.

Put $\lim_{n\to\infty} p_{\alpha_n}(1) = x$. By 2, $\lim_{n\to\infty} p_{\alpha_n}(3) = x$. Since F is closed, we have $\langle\langle x, \omega_1\rangle, \langle x, \omega_1\rangle\rangle \in F$, contradicting the fact that $F \cap \Delta_X = \emptyset$.

The claim is proved.

Claim 2. *If S is a first-countable closed subspace of X, then $S \setminus \{\langle \alpha, \omega_1\rangle\colon\ \alpha < \omega_1\}$ is closed in S.*

Claim 3. *If S is closed in X^2, $S \cap \Delta_X = \emptyset$, and $\pi_1(S) \subset \{\langle \alpha, \omega_1\rangle\colon\ \alpha < \omega_1\}$, then there exists $\lambda < \omega_1$ such that $\pi_2(S \cap [\lambda, \omega_1]^4)$ does not meet $\{\langle \alpha, \omega_1\rangle\colon\ \alpha < \omega_1\}$.*

Assume the conclusion is not true. Then we can select a strictly increasing sequence $\{\alpha_n\}_n$ of countable ordinals and $p_{\alpha_n} \in S \cap [\alpha_n, \omega_1]^4$ with the following properties:
1. $\pi_i(p_{\alpha_n}) \in \{\langle \alpha, \omega_1\rangle\colon\ \alpha < \omega_1\}$ for $i = 1, 2$;
2. $p_{\alpha_{n+1}}(1), p_{\alpha_{n+1}}(3) > \max\{p_{\alpha_n}(1), p_{\alpha_n}(3)\}$.

Put $\lim_{n\to\infty} p_{\alpha_n}(1) = x$. By 2, $\lim_{n\to\infty} p_{\alpha_n}(3) = x$. Since S is closed, we have $\langle\langle x,\omega_1\rangle, \langle x,\omega_1\rangle\rangle \in S$, contradicting the fact that $S \cap \Delta_X = \emptyset$. The claim is proved.

Claim 4. *Suppose S is closed in X^2, $\Delta_X \cap S = \emptyset$, and $\pi_1(S) \subset X \setminus \{\langle\alpha,\omega_1\rangle\colon \alpha < \omega_1\}$. Then there exists $\lambda < \omega_1$ such that $\pi_2(S \cap [\lambda,\omega_1]^4) \subset \{\langle\alpha,\omega_1\rangle\colon \alpha < \omega_1\}$.*

The proof is similar to that of Claim 3 but the final point that leads to a contradiction is $\langle\langle x,x\rangle, \langle x,x\rangle\rangle$.

Claim 5. *There exist disjoint closed subsets A and B of X and $\lambda < \omega_1$ such that $F \cap [\lambda,\omega_1]^4$ is a subset of $(A\times B)\cup(B\times A)$.*

To prove the claim let λ satisfy the conclusion of Claim 1 for both π_1 and π_2. By Claim 2, $\pi_i(F\cap[\lambda,\omega_1]^4) \subset A_i \oplus B_i$, where A_i is closed and does not meet $\{\langle\alpha,\omega_1\rangle\colon \alpha<\omega_1\}$ and B_i is a closed subset of $\{\langle\alpha,\omega_1\rangle\colon \alpha<\omega_1\}$. Since Claims 3 and 4 hold if we interchange π_1 with π_2, we may assume that λ satisfies the conclusions of Claims 3 and 4 for every $S \in \{\pi_1^{-1}(A_1)\cap F, \pi_2^{-1}(A_2)\cap F, \pi_1^{-1}(B_1)\cap F, \pi_2^{-1}(B_2)\cap F\}$.
We have, $F\cap[\lambda,\omega_1]^4 \subset (A_1\times A_2)\cup(A_1\times B_2)\cup(B_1\times A_2)\cup(B_1\times B_2)$. By Claims 3 and 4, F meets neither $A_1\times A_2$ nor $B_1\times B_2$. Thus, $F\cap[\lambda,\omega_1]^4$ is a subset of $(A_1\times B_2)\cup(B_1\times A_2)$. Put $A = A_1\cup A_2$ and $B = B_1\cup B_2$. The claim is proved.

Let A, B, and λ be as in Claim 5. We may assume that λ is isolated. To finish the proof, put $U_A = (X\cap[\lambda,\omega_1]^2)\setminus A$ and $U_B = (X\cap[\lambda,\omega_1]^2)\setminus B$. We have $F\cap[\lambda,\omega_1]^4$ is a subset of $(A\times B)\cup(B\times A)$. Since $A\cap B=\emptyset$ we have $U_A\times U_A\cup U_B\times U_B$ covers $\Delta_X\cap[\lambda,\omega_1]^4$ and does not meet $(A\times B)\cup(B\times A)$, which, in its turn, contains $F\cap[\lambda,\omega_1]^4$.
If $\pi_1(x,x)$ misses $[\lambda,\omega_1]^2$, then $x\in[0,\lambda)\times[0,\omega_1]$. Since $X\cap([0,\lambda)\times[0,\omega_1])$ is compact $\Delta_X\cap([0,\lambda)\times[0,\omega_1])^2$ is compact as well. Therefore there exists a finite open cover $\mathcal{U}$ of $\Delta_X\cap([0,\lambda)\times[0,\omega_1])^2$ such that $\bigcup\{U\times U\colon U\in\mathcal{U}\}$ does not meet F. Clearly, the cover $\mathcal{U}\cup\{U_A,U_B\}$ proves that X is Δ-paracompact. □

The space in the example is not normal. The example together with Fact 2.1, Proposition 2.6, and Corollary 2.7 lead to

Corollary 2.13. *Δ-paracompactness implies neither normality, nor Δ-normality, nor functional Δ-normality, nor regular Δ-paracompactness, nor functional Δ-paracompactness.*

Next we will show that Δ-paracompactness cannot be distinguished from functional Δ-paracompactness in the class of countably compact normal spaces. For this we need the following folklore statement.

Lemma 2.14 *(Folklore). Let X be normal and let $\{U_1,\dots,U_n\}$ be a finite open cover of X. Let U_i' be the largest open set in βX such that $U_i = X\cap U_i'$. Then $\beta X = U_1'\cup\dots\cup U_n'$.*

Proof. Since X is normal, by Lemma 1.5.H of [2], there exists an open cover $\{V_1,\dots,V_n\}$ of X such that $\overline{V}_i\subset U_i$. Then $Cl_{\beta X}(V_1)\cup\dots\cup Cl_{\beta X}(V_n) = U_1'\cup\dots\cup U_n'$. Since the left side covers βX we are done. □

Theorem 2.15. *Let X be countably compact and normal. If X is Δ-paracompact, then X is functionally Δ-paracompact.*

Proof. Fix $A\subset X^2\setminus\Delta_X$ closed in X^2 and fix an open locally finite cover $\mathcal{U}$ of X such that $\bigcup\{U\times U\colon U\in\mathcal{U}\}$ misses A. Since X is countably compact $\mathcal{U}$ is finite. By Lemma 2.14, $\mathcal{U}' = \{U'\colon U\in\mathcal{U}\}$ is a cover of βX, where U' is the largest open set in βX such that $U = U'\cap X$. Since βX is compact, we can inscribe a finite open cover $\mathcal{V}$ of βX in $\mathcal{U}'$ that consists of functionally open sets. Then $\bigcup\{V\times V\colon V\in\mathcal{V}\}$ covers $\Delta_{\beta X}$ in $\beta X\times\beta X$ and misses $Cl_{\beta X\times\beta X}(A)$. This proves that X is functionally Δ-paracompact. □

The above theorem and Fact 2.1 imply the following.

Corollary 2.16. *Δ-paracompactness, regular Δ-paracompactness, and functional Δ-paracompactness are equivalent in the class of countably compact normal spaces.*

Thus, if a countably compact normal space X is Δ-paracompact then it has the other four properties as well. Our next goal is to show that normality does not imply Δ-paracompactness.

Lemma 2.17. *Let X be normal and countably compact. Suppose there exists a continuous fixed-point free map $f: X\to X$ such that $\tilde{f}:\beta X\to\beta X$ fixes a point. Then X is not Δ-paracompact.*

Table 1

	n	Δn	fΔn	Δp	rΔp	fΔp
n	yes	no [3, 3.1]	no [3, 3.1] + 2.1	no 2.18	no 2.18 + 2.1	no 2.18 + 2.1
Δn	?	yes	?	?	?	?
fΔn	?	yes 2.1	yes	?	?	?
Δp	no 2.13	no 2.13	no 2.13	yes	no 2.13	no 2.13
rΔp	yes 2.6	yes 2.1	?	yes 2.1	yes	?
fΔp	yes 2.6 + 2.1	yes 2.1	yes 2.3	yes 2.1	yes 2.1	yes

Table 2

	Δn	fΔn	Δp	rΔp	fΔp
Δn	yes	?	?	?	?
fΔn	yes 2.1	yes	?	?	?
Δp	yes 2.16 + 2.3 + 2.1	yes 2.16 + 2.3	yes	2.16	yes 2.16
rΔp	yes 2.1	yes 2.16 + 2.3	yes 2.1	yes	yes 2.16
fΔp	yes 2.1	yes 2.3	yes 2.1	yes 2.1	yes

Proof. By the hypothesis there exists $p \in \beta X \setminus X$ such that $\tilde{f}(p) = p$. Put $A = \{\langle x, f(x)\rangle\colon\ x \in X\}$. Since f is continuous, the set A is closed in X^2. Since f is fixed-point free, A does not meet the diagonal.

Let $\mathcal{U}$ be a locally finite open cover of X. We need to show that there exists $U \in \mathcal{U}$ such that $U \times U$ meets A.

Since X is countably compact, $\mathcal{U}$ is finite. By Lemma 2.14, $\{U'\colon\ U \in \mathcal{U}\}$ is a cover of βX, where U' is the largest open set of βX such that $X \cap U' = U$. Therefore, there exists $U_p \in \mathcal{U}$ such that $p \in U'_p$. Since $\tilde{f}(p) = p$ and f is continuous there exists $x \in U_p$ such that $f(x) \in U_p$. Then $\langle x, f(x)\rangle \in U_p \times U_p$. □

Example 2.18. Assume Continuum Hypothesis. Then there exists a countably compact normal space which is not Δ-paracompact.

Proof. In [4, 3.1], the authors show that under Continuum Hypothesis there exists an example of a space that meets the hypothesis of Lemma 2.17. □

Examples 2.12 and 2.18 show that normality and Δ-paracompactness are incomparable in the class of countably compact spaces.

We finish this work with two tables that present and justify questions that naturally arise from the presented results. In the tables, “n”, “Δn”, “fΔn”, “Δp”, “rΔp”, and “fΔp” stand for normality, Δ-normality, and so on. In Table 1, an entry that appears in row “R” and column “C” gives an answer to the question “Is it true that every Tychonoff space with property R has property C?”.

In Table 2, an entry that appears in row “R” and column “C” gives an answer to the question “Is it true that every countably compact normal space with property R has property C?”.

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