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# ON FINDING A MINIMUM VERTEX COVER OF A SERIES-PARALLEL GRAPH

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**Abstract:** We present a simple linear time algorithm for finding a minimum vertex cover for series-parallel graphs.

## 1. Introduction

The problem of finding a minimum vertex cover of a graph has been extensively studied in the literature, [1], [2], [3], [7]. In this paper we present a simple algorithm that finds a minimum vertex cover of a series-parallel graph in linear time. Our algorithm is based on an algorithm developed in [5] for solving the resource location problem on series-parallel graphs.

Let $G=(V,E)$ be a finite, simple, undirected graph with vertex set $V$ and edge set $E$. We say that a vertex $v$ covers an edge $e$ if $e$ is incident upon $v$. A *vertex cover* $C$ is a set of vertices of $G$ such that for every edge $(v,w)$ in $E$ at least one of $v$ and $w$ belongs to $C$. An *independent set* $I$ is a set of vertices of $G$ such that no two vertices in $I$ are joined by an edge in $E$. Vertex covers and independent sets are quite closely related. Namely, the set $C$ is a vertex cover if and only if the set $V-C$ is an independent set. Typically, one would like to find a minimum vertex cover $C$. This is a very practical problem because it arises in many applications. Notice that $C$ is a minimum vertex cover if and only if $I=V-C$ is a maximum independent set.

It is very unlikely that there exists a polynomial-time algorithm to solve the minimum vertex cover problem (mVC) (or equivalently, the maximum independent set problem (MIS)), since the corresponding decision problems are NP-complete, [1]. However, MIS can be solved by polynomial-time algorithms for bipartite graphs, edge graphs, graphs with no vertex degree exceeding 2 [1], chordal graphs [2], circle graphs [3], comparability graphs [4], and claw-free graphs [1].

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Takamizawa, Nishizeki, and Saito developed a general technique for series-parallel graphs and proved that several combinatorial problems, including the minimum vertex cover problem, are linear-time computable. Although all their algorithms run in time linear on the size of the input graph, they take time exponential on the size of the collection of forbidden graphs [7]. This means that the application of their general technique does not always yield a practical algorithm.

The main result of this paper is a linear-time algorithm that finds a minimum vertex cover for any series-parallel graph; this algorithm is based on a simple transformation to the resource location problem.

## 2. The main result

The *Resource Location Problem* is defined as follows: Consider a graph $G=(V,E)$ with source $s$ and sink $t$. Each edge has a positive integer length and may be directed or undirected. Let $R$ (range) be a given positive integer. We wish to locate the minimum number of *gas-stations* (or any facility center) on the vertices or edges of $G$ such that a car with range $R$ may start at $s$ with no gas and drive to $t$ along any simple path in $G$. Note that we do not mind arriving at $t$ with an empty tank.

The decision problem associated with Resource Location Problem is NP-complete, but there exists a polynomial-time algorithm for the case when $G$ is a series-parallel graph [5]. The time complexity of the algorithm of [5] is $O(|E| \min\{R, |V|\})$.

A graph is called *essentially one-way* if, for each edge $e$, all the simple paths from $s$ to $t$ using $e$, traverse $e$ in the same direction. An *integer point* is a point (i.e., a position) on an edge whose distance from either end of the edge is an integer. Given a graph $G$ with positive edge-lengths, a range $R$, and a configuration of gas-stations, a simple path $P$ from $s$ to $t$ is *reliable* if there is a gas-station at $s$, one on $P$ at most distance $R$ away from $t$, and if any two consecutive gas-stations on $P$ are at most $R$ apart. All distances here are measured along $P$. The configuration itself is *reliable* if every simple path from $s$ to $t$ is reliable. An *optimum reliable configuration* is one for which the number of gas-stations is minimum. The following result was proved in [5]:

**Theorem 1.** Suppose that an essentially one-way graph has only integer edge-lengths and $R$ is an integer. Then there is an optimum reliable configuration for which all gas-stations lie on integer points. $\square$

The class of *series-parallel* graphs is recursively defined as follows:

(i) A graph consisting of two vertices $s$ and $t$ joined by a single edge is a series-parallel graph with *source* $s$ and *sink* $t$.

(ii) If $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are two series-parallel graphs, with sources $s_1,s_2$ and sinks $t_1,t_2$ respectively, so are the graphs constructed by each of the following operations:

(a) *Two-terminal series connection:* The graph $H=ser(G_1,G_2)$ is formed from $G_1 \cup G_2$ by identifying $t_1$ and $s_2$, then taking $s_1$ as the source and $t_2$ as the sink of $H$.

(b) *Two-terminal parallel connection:* The graph $H=par(G_1,G_2)$ is formed from $G_1 \cup G_2$ by identifying $s_1$ and $s_2$ (with the resulting single vertex becoming the source of $H$) and identifying $t_1$ and $t_2$ (making the sink of $H$).

(iii) Only graphs constructed by a finite number of applications of connections described above are series-parallel graphs.

These operations are depicted in Figure 1. Note that series-parallel graphs are essentially one-way graphs, [6].

Consider now a series-parallel graph $G=(V,E)$ with source $s$ and sink $t$. Let all edge-lengths be equal to 1 and $R=2$. Let $G'=(V',E')$ be the series-parallel graph constructed from $G$ as follows, see Figure 2:

$$V'=V \cup \{s',t'\},\ E'=E \cup \{(s',s),(t,t')\}.$$

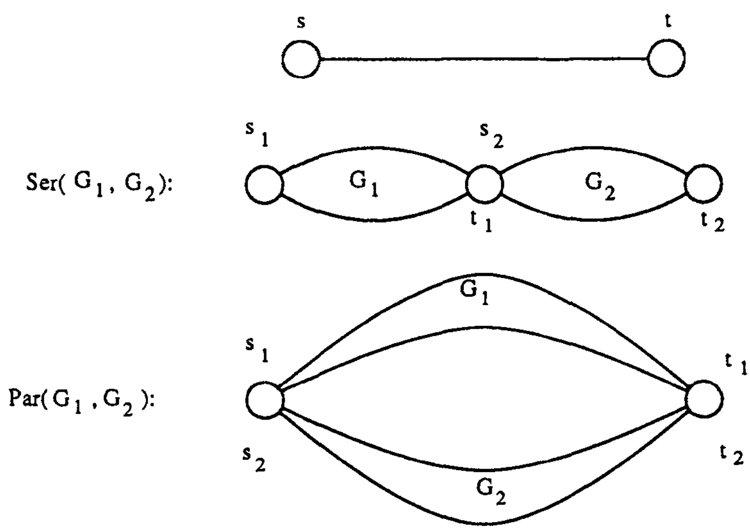


Figure 1. Series-Parallel Graphs.

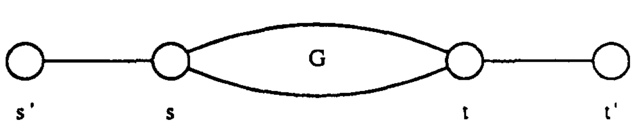


Figure 2. Construction of G' from G.

An optimum reliable configuration for $G'$ can be found by the algorithm of [5] in $O(|E'|)$ time, since $R=2$. Notice that $|E'|=O(|V'|)$ since $G'$ is a planar graph. Furthermore, by Theorem 1, all gas-stations lie on vertices of $G'$. Let $S$ be the set of all vertices of $G'$ that contain gas-stations in an optimum reliable configuration for $G'$. Then we have the following:

**Lemma 1.** The set $C = V \cap S = S - \{s'\}$ is a vertex cover for $G$.

**Proof.** Suppose that $C$ is not a vertex cover. Then there exists an edge $(u, v) \in E$ such that neither of $u$ and $v$ is in $C$. Since $G'$ is a series-parallel graph, there is a path $P$ in $G'$ from $s$ to $t$ that contains the subpath $wuvy$, for some $w, y \in V'$ such that $(w,u) \in E'$ and $(v,y) \in E'$. However, the length of the subpath is 3 and since $u$ and $v$ do not contain any gas-station the path $P$ is unreliable (not reliable), a contradiction. Finally, notice that $s' \in S$. Therefore, $V \cap S = S - \{s'\}$. □

Let $S \subseteq V'$. The configuration in which all the vertices of $S$ contain a gas-station and all the other vertices in $V'-S$ contain no gas-stations is the configuration *induced* by $S$.

**Lemma 2.** If $C$ is a vertex cover for $G$, then the set $S$ of vertices of $G'$, where $S = C \cup \{s'\}$, induces a reliable configuration of gas-stations on $G'$.

**Proof.** Suppose that the configuration induced by $S$ is unreliable. Then there is a path $P$ in $G'$ from $s'$ to $t'$ which is unreliable. This means that there exist four consecutive vertices on $P$ $w,u,v,y$ such that $u$ and $v$ are not in $S$. Hence, $(u,v) \in E$ and neither of $u$ and $v$ belongs to $C$. Therefore, $C$ is not a vertex cover, a contradiction. □

Given an optimum reliable configuration on $G'$, let $S$ be the set of all vertices of $G'$ that contain a gas-station. We can prove the following theorem:

**Theorem 2.** The set $C = V \cap S$ is a minimum vertex cover for $G$.

**Proof.** From Lemma 1 we know that $C$ is a vertex cover for $G$. We will prove that it is a minimum one. Let $C_{\min}$ be a minimum vertex cover with $|C_{\min}| < |C|$. Then by Lemma 2 the set $S_{\min} = C_{\min} \cup \{s'\}$ induces a reliable configuration on $G'$, and $|S_{\min}| < |S|$. But this is a contradiction since we assumed that the set $S$ induces a minimum reliable configuration on $G'$. □

It is now a simple matter to see that the following algorithm computes a minimum vertex cover for a series-parallel graph $G$:

Step 1. Construct $G'$ from $G$ as discussed above. All edge-lengths are equal to 1 and $R=2$.

Step 2. Find an optimum reliable configuration for $G'$ in $O(|V'|)$ time, [5].

Step 3. The set of vertices of $G$ that contain a gas-station in the optimum reliable configuration found in Step 2 for $G'$ is a minimum vertex cover for $G$.

**Theorem 3.** The above algorithm computes a minimum vertex cover in time $O(|V|)$.
□

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