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Generating self-complementary uniform hypergraphs

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ABSTRACT

In 2007, Szymański and Wojda proved that for positive integers n, k with $k \leq n$, a self-complementary k-uniform hypergraph of order n exists if and only if $\binom{n}{k}$ is even. In this paper, we characterize the cycle type of a k-complementing permutation in $\mathrm{Sym}(n)$ which has order equal to a power of 2. This yields a test for determining whether a finite permutation is a k-complementing permutation, and an algorithm for generating all self-complementary k-hypergraphs of order n, up to isomorphism, for feasible n. We also obtain an alternative description of the necessary and sufficient conditions on the order of a self-complementary k-uniform hypergraph, in terms of the binary representation of k. This extends previous results for the cases $k = 2, 3, 4$ due to Ringel, Sachs, Suprunenko, Kocay and Szymański.

1. Introduction

For a finite set V and a positive integer k, let $V^{(k)}$ denote the set of all k-subsets of V. A *hypergraph* with vertex set V and edge set E is a pair (V, E) in which V is a finite set and E is a collection of subsets of V. A hypergraph (V, E) is called *k-uniform* (or a *k-hypergraph*) if E is a subset of $V^{(k)}$. The parameters k and $|V|$ are called the *rank* and the *order* of the k-hypergraph, respectively. The vertex set and the edge set of a hypergraph X will often be denoted by $V(X)$ and $E(X)$, respectively. Note that a 2-hypergraph is a *graph*.

An *isomorphism* between k-hypergraphs X and X' is a bijection $\phi : V(X) \to V(X')$ which induces a bijection from $E(X)$ to $E(X')$. If such an isomorphism exists, the hypergraphs X and X' are said to be *isomorphic*. The *complement* X^C of a k-hypergraph $X = (V, E)$ is the hypergraph with vertex set V and edge set $V^{(k)} \setminus E$. A k-hypergraph X is called *self-complementary* if it is isomorphic to its complement. An isomorphism between a self-complementary k-hypergraph $X = (V, E)$ and its complement X^C is called an *antimorphism* of X. The set of all antimorphisms of X will be denoted by $\mathrm{Ant}(X)$.

An antimorphism of a self-complementary k-hypergraph is also called a *k-complementing permutation*, and we have the following natural characterization.

Proposition 1.1 ([7]). *Let V be a finite set, let k be a positive integer, and let $\theta \in \mathrm{Sym}(V)$. Then the following three statements are equivalent:*

1. *θ is a k-complementing permutation.*
2. *$A^{\theta^j} \neq A$, for all $A \in V^{(k)}$, for all j odd.*
3. *The sequence $A, A^{\theta}, A^{\theta^2}, A^{\theta^3}, \ldots$ has even length, for all $A \in V^{(k)}$.*

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Conditions (2) and (3) of Proposition 1.1 depend entirely on the cycle lengths in the disjoint cycle decomposition of θ. Wojda [7] gave the following characterization of the k-complementing permutations in $\mathrm{Sym}(n)$.

Theorem 1.2 (*Wojda [7]*). *Let k, m and n be positive integers, let V be a finite set, $|V| = n$, and let us have $\sigma \in \mathrm{Sym}(V)$ with orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_m$. Let $2^{q_i}(2s_i + 1)$ denote the cardinality of the orbit $\mathcal{O}_i$, for $i = 1, 2, \ldots, m$. The permutation σ is a k-complementing permutation if and only if, for every $\ell \in \{1, 2, \ldots, k\}$ and for every decomposition*

$$k = k_1 + k_2 + \cdots + k_\ell$$

of k, where $k_j = 2^{p_j}(2r_j + 1)$ for nonnegative integers p_j, r_j, for $j = 1, 2, \ldots, \ell$, and for every subsequence of orbits

$$\mathcal{O}_{i_1}, \mathcal{O}_{i_2}, \ldots, \mathcal{O}_{i_\ell}$$

such that $k_j \le |\mathcal{O}_{i_j}|$ for $j = 1, 2, \ldots, \ell$, there is a subscript $j_0 \in \{1, 2, \ldots, \ell\}$ such that $p_{j_0} < q_{i_{j_0}}$. □

In Theorem 2.2 we give a more transparent characterization of the cycle types of k-complementing permutations which have order equal to a power of 2, and Corollary 2.4 shows how we can use it to test whether a finite permutation is a k-complementing permutation. In Section 4, we use the characterization of Theorem 2.2 to obtain an algorithm for generating all of the self-complementary k-hypergraphs of order n, up to isomorphism, for feasible n.

In 2007, Szymański and Wojda [6] solved the problem of the existence of a self-complementary k-hypergraph of order n.

Theorem 1.3 (*[6]*). *Let k and n be positive integers such that $k \le n$. A self-complementary k-uniform hypergraph of order n exists if and only if $\binom{n}{k}$ is even.*

In Section 3, we give an alternative description of the condition that $\binom{n}{k}$ is even, in terms of the binary representation of k (see Corollary 3.2). This yields more transparent conditions on the order of a self-complementary k-hypergraph when the rank k is a sum of consecutive powers of 2.

2. Cycle lengths of antimorphisms

Theorem 2.2 gives a characterization of the cycle types of antimorphisms of k-hypergraphs which have order equal to a power of 2, in terms of the binary representation of k. This yields an alternative description of the necessary and sufficient conditions of Theorem 1.3. We will make use of the following technical lemma to prove Theorem 2.2.

Lemma 2.1. *Let ℓ and x be positive integers, where $x \ge 2$. Let $a_0, a_1, \ldots, a_{\ell-1}$ be nonnegative integers such that $\sum_{i=0}^{\ell-1} a_i x^i \ge x^\ell$. Then there exist integers $c_0, c_1, \ldots, c_{\ell-1}$, where $0 \le c_i \le a_i$, such that $\sum_{i=0}^{\ell-1} c_i x^i = x^\ell$.*

Proof. The proof is by induction on ℓ.

Base step: The statement is certainly true if $\ell = 1$, for if there is a nonnegative integer a_0 such that $a_0 x^0 \ge x^1 = x$, then $a_0 \ge x$, and so the result holds with $c_0 = x$.

Induction step: Let $\ell \ge 2$ and assume that the statement is true for $\ell - 1$. That is, assume that if there is a sequence of nonnegative integers $\hat{a}_0, \ldots, \hat{a}_{\ell-2}$ such that $\sum_{i=0}^{\ell-2} \hat{a}_i x^i \ge x^{\ell-1}$, then there exists a sequence of integers $\hat{c}_0, \ldots, \hat{c}_{\ell-2}$ with $0 \le \hat{c}_i \le \hat{a}_i$, for $0 \le i \le \ell - 2$, such that $\sum_{i=0}^{\ell-2} \hat{c}_i x^i = x^{\ell-1}$.

Now suppose that $a_0, \ldots, a_{\ell-1}$ is a sequence of nonnegative integers such that $\sum_{i=0}^{\ell-1} a_i x^i \ge x^\ell$. If $a_{\ell-1} \ge x$, then to obtain the desired sequence, set $c_i = 0$ for $0 \le i \le \ell - 2$, and set $c_{\ell-1} = x$. Then $0 \le c_i \le a_i$ for all i, and $\sum_{i=0}^{\ell-1} c_i x^i = x^\ell$, as required.

Hence we may assume that $a_{\ell-1} \le x - 1$. Suppose that $a_{\ell-1} = x - k$ for an integer k such that $1 \le k \le x$. In this case $a_0, a_1, \ldots, a_{\ell-2}$ is a sequence such that

$$\sum_{i=0}^{\ell-2} a_i x^i \ge x^\ell - (x-k)x^{\ell-1} = kx^{\ell-1},$$

and so we may apply the induction hypothesis k times to obtain k sequences of integers $\{c_i^1\}, \{c_i^2\}, \ldots, \{c_i^k\}$ such that $0 \le \sum_{j=0}^{k} c_i^j \le a_i$, for $0 \le i \le \ell - 2$, and $\sum_{i=0}^{\ell-2} c_i^j x^i = x^{\ell-1}$, for $1 \le j \le k$. Now to obtain the desired sequence, set $c_i = \sum_{j=1}^{k} c_i^j$ for $0 \le i \le \ell - 2$, and set $c_{\ell-1} = a_{\ell-1} = x - k$. Then certainly $0 \le c_i \le a_i$ for $0 \le i \le \ell - 1$. Moreover,

$$\begin{aligned}
\sum_{i=0}^{\ell-1} c_i x^i &= \sum_{i=0}^{\ell-2} c_i x^i + c_{\ell-1} x^{\ell-1} \\
&= \sum_{i=0}^{\ell-2} \left[\sum_{j=1}^{k} c_i^j \right] x^i + (x-k)x^{\ell-1}
\end{aligned}$$

$$= \sum_{j=1}^{k}\left[\sum_{i=0}^{\ell-2} c_i^j x^i\right] + (x-k)x^{\ell-1}$$
$$= \sum_{j=1}^{k} x^{\ell-1} + (x-k)x^{\ell-1}$$
$$= kx^{\ell-1} + (x-k)x^{\ell-1} = x^{\ell},$$

as required. □

To state and prove Theorem 2.2, we require some terminology and notation. We will denote the *binary representation* of an integer k by a vector $b = (b_m, b_{m-1}, \ldots, b_1, b_0)_2$. That is, b is the vector such that $k = \sum_{i=0}^{m} b_i 2^i$, $b_m = 1$, and $b_i \in \{0, 1\}$ for $0 \le i \le m$. The *support* of the binary representation b is the set $\{i \in \{0, 1, 2, \ldots, m\} : b_i = 1\}$, and is denoted by $\mathrm{supp}(b)$. For positive integers m and n, let $n_{[m]}$ denote the unique integer in $\{0, 1, \ldots, m-1\}$ such that $n \equiv n_{[m]} (\mathrm{mod}\ m)$.

Theorem 2.2. *Let V be a finite set, let k be a positive integer such that $k \le |V|$, and let $b = (b_m, b_{m-1}, \ldots, b_2, b_1, b_0)_2$ be the binary representation of k. Let $\theta \in \mathrm{Sym}(V)$ be a permutation whose order is a power of 2. Given $\ell \in \mathrm{supp}(b)$, let A_ℓ denote those points of V contained in cycles of θ of length $< 2^\ell$, and let B_ℓ denote those points of V contained in cycles of θ of length $> 2^\ell$. Then θ is a k-complementing permutation if and only if, for some $\ell \in \mathrm{supp}(b)$, $V = A_\ell \cup B_\ell$ and $|A_\ell| < k_{[2^{\ell+1}]}$.*

Proof. ($\Rightarrow$) Suppose that θ is a k-complementing permutation of order a power of 2. Then every cycle of θ has length a power of 2. If θ contained a cycle of length 2^i for every $i \in \mathrm{supp}(b)$, then there would be an invariant set of θ of cardinality $\sum_{i\in\mathrm{supp}(b)} 2^i = k$, a contradiction. Hence, for some $\ell \in \mathrm{supp}(b)$, θ does not contain a cycle of length 2^ℓ.

Let

$$L = \{\ell \in \mathrm{supp}(b) : \theta \text{ does not contain a cycle of length } 2^\ell\}. \tag{1}$$

Then $V = A_\ell \cup B_\ell$ for all $\ell \in L$. It remains to show that $|A_\ell| < k_{[2^{\ell+1}]}$ for some $\ell \in L$.

Suppose to the contrary that $|A_\ell| \ge k_{[2^{\ell+1}]}$ for all $\ell \in L$. Write $|A_\ell| = \sum_{i=0}^{\ell-1} a_i 2^i$, where a_i is the number of cycles of θ of length 2^i. Note that $k_{[2^{\ell+1}]} = \sum_{i=0}^{\ell} b_i 2^i$. Thus, by assumption, $|A_\ell| \ge \sum_{i=0}^{\ell} b_i 2^i$ for all $\ell \in L$. Suppose $L = \{\ell_1, \ell_2, \ldots, \ell_t\}$ where $\ell_1 < \ell_2 < \cdots < \ell_t$.

- **Claim A.** *Let $x \in \{1, 2, \ldots, t\}$. If $|A_{\ell_j}| \ge \sum_{i=0}^{\ell_j} b_i 2^i$ for all $j \in \{1, 2, \ldots, x\}$, then $\theta|_{A_{\ell_x}}$ has an invariant set of size $\sum_{i=0}^{\ell_x} b_i 2^i$.*

Proof of Claim A. The proof is by induction on x. First note that $\sum_{i=0}^{\ell_j-1} a_i 2^i = |A_{\ell_j}|$, for $j = 1, 2, \ldots, t$. Also, for any sequence of integers $\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{\ell_j-1}$ such that $0 \le \hat{a}_i \le a_i$ for $0 \le i \le \ell_j - 1$, the sum $\sum_{i=0}^{\ell_j-1} \hat{a}_i 2^i$ is the sum of the lengths of a collection of cycles of $\theta|_{A_{\ell_j}}$, and hence it is the size of an invariant set of $\theta|_{A_{\ell_j}}$. Conversely, any invariant set S of $\theta|_{A_{\ell_j}}$ corresponds to a collection of cycles of $\theta|_{A_{\ell_j}}$ whose lengths sum to $|S|$, and hence there exists a sequence of integers $\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{\ell_j-1}$ such that $0 \le \hat{a}_i \le a_i$ for $0 \le i \le \ell_j - 1$, and $|S| = \sum_{i=0}^{\ell_j-1} \hat{a}_i 2^i$.

Base step: If $x = 1$ and $|A_{\ell_1}| \ge \sum_{i=0}^{\ell_1} b_i 2^i$, then

$$|A_{\ell_1}| = \sum_{i=0}^{\ell_1-1} a_i 2^i \ge \sum_{i=0}^{\ell_1} b_i 2^i. \tag{2}$$

By the definition of L in (1), it follows that $a_i \ge b_i$ for $0 \le i \le \ell_1 - 1$. Hence (2) implies that

$$\sum_{i=0}^{\ell_1-1} (a_i - b_i) 2^i \ge 2^{\ell_1}$$

holds with $a_i - b_i \ge 0$ for all $i = 1, 2, \ldots, \ell_1 - 1$. Thus by Lemma 2.1, there is a sequence $c_0, c_1, \ldots, c_{\ell_1-1}$ such that $0 \le c_i \le (a_i - b_i)$ for $0 \le i \le \ell_1 - 1$, and

$$\sum_{i=0}^{\ell_1-1} c_i 2^i = 2^{\ell_1}.$$

Now let $\hat{a}_i = b_i + c_i$. Then

$$0 \le \hat{a}_i = b_i + c_i \le b_i + (a_i - b_i) = a_i$$

and hence

$$0 \le \hat{a}_i \le a_i$$

for $0 \leq i \leq \ell_1 - 1$, and

$$\sum_{i=0}^{\ell_1-1} \hat{a}_i 2^i = \sum_{i=0}^{\ell_1-1} b_i 2^i + \sum_{i=0}^{\ell_1-1} c_i 2^i = \sum_{i=0}^{\ell_1-1} b_i 2^i + 2^{\ell_1} = \sum_{i=0}^{\ell_1} b_i 2^i.$$

Thus $\theta|_{A_{\ell_1}}$ has an invariant set of size $\sum_{i=0}^{\ell_1} b_i 2^i$, as required.

Induction step: Let $2 \leq x \leq t$ and assume that if $|A_{\ell_j}| \geq \sum_{i=0}^{\ell_j} b_i 2^i$ for all $j \in \{1, 2, \ldots, x-1\}$, then $\theta|_{A_{\ell_{x-1}}}$ has an invariant set of size $\sum_{i=0}^{\ell_{x-1}} b_i 2^i$. Now suppose that $|A_{\ell_j}| \geq \sum_{i=0}^{\ell_j} b_i 2^i$ for all $j \in \{1, 2, \ldots, x\}$. Then certainly $|A_{\ell_j}| \geq \sum_{i=0}^{\ell_j} b_i 2^i$ for all $j \in \{1, 2, \ldots, x-1\}$, and so by the induction hypothesis, $\theta|_{A_{\ell_{x-1}}}$ has an invariant set of size $\sum_{i=0}^{\ell_{x-1}} b_i 2^i$. This implies that there is a sequence of integers $c_0, c_1, \ldots, c_{\ell_{x-1}-1}$ such that $0 \leq c_i \leq a_i$ for $0 \leq i \leq \ell_{x-1} - 1$, and

$$\sum_{i=0}^{\ell_{x-1}-1} c_i 2^i = \sum_{i=0}^{\ell_{x-1}} b_i 2^i. \tag{3}$$

Since $|A_{\ell_x}| \geq \sum_{i=0}^{\ell_x} b_i 2^j$, we have

$$\sum_{i=0}^{\ell_x-1} a_i 2^i \geq \sum_{i=0}^{\ell_x} b_i 2^i. \tag{4}$$

Since $\ell_{x-1} \in L$, $a_{\ell_{x-1}} = 0$, so (4) implies that

$$|A_{\ell_x}| = \sum_{i=0}^{\ell_x-1} a_i 2^i = \sum_{i=0}^{\ell_{x-1}-1} a_i 2^i + \sum_{i=\ell_{x-1}+1}^{\ell_x-1} a_i 2^i \geq \sum_{i=0}^{\ell_x} b_i 2^i.$$

Hence by (3), we have

$$\sum_{i=0}^{\ell_{x-1}-1} (a_i - c_i) 2^i + \sum_{i=\ell_{x-1}+1}^{\ell_x-1} a_i 2^i \geq \sum_{i=\ell_{x-1}+1}^{\ell_x} b_i 2^i.$$

This implies that

$$\sum_{i=0}^{\ell_{x-1}-1} (a_i - c_i) 2^i + \sum_{i=\ell_{x-1}+1}^{\ell_x-1} (a_i - b_i) 2^i \geq 2^{\ell_x}. \tag{5}$$

By the definition of L in (1), we have $a_i - b_i \geq 0$ for $\ell_{x-1} + 1 \leq i \leq \ell_x - 1$. Also, $a_i - c_i \geq 0$ for $0 \leq i \leq \ell_{x-1} - 1$. Thus (5) and Lemma 2.1 guarantee that there exists a sequence of integers $d_0, d_1, \ldots, d_{\ell_x-1}$ such that $0 \leq d_i \leq (a_i - c_i)$ for $0 \leq i \leq \ell_{x-1} - 1$, $d_{\ell_{x-1}} = 0$, $0 \leq d_i \leq (a_i - b_i)$ for $\ell_{x-1} + 1 \leq i \leq \ell_x - 1$, and

$$\sum_{i=0}^{\ell_x-1} d_i 2^i = 2^{\ell_x}.$$

Now define a sequence of integers $\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{\ell_x-1}$ by

$$\hat{a}_i = \begin{cases} c_i + d_i, & \text{if } 0 \leq i \leq \ell_{x-1} - 1 \\ 0, & \text{if } i = \ell_{x-1} \\ b_i + d_i, & \text{if } \ell_{x-1} + 1 \leq i \leq \ell_x - 1. \end{cases}$$

Then one can check that $0 \leq \hat{a}_i \leq a_i$ for $0 \leq i \leq \ell_x - 1$, and

$$\sum_{i=0}^{\ell_x-1} \hat{a}_i 2^i = \sum_{i=0}^{\ell_x} b_i 2^i.$$

Thus $\theta|_{A_{\ell_x}}$ has an invariant set of size $\sum_{i=0}^{\ell_x} b_i 2^i$, as required.

Hence by mathematical induction, Claim A holds for all $x \in \{1, 2, \ldots, t\}$.

Now applying Claim A with $x = t$, we observe that $|A_{\ell_j}| \geq \sum_{i=0}^{\ell_j} b_i 2^i$ for all $j \in \{1, 2, \ldots, t\}$. Hence $\theta|_{A_{\ell_t}}$ has an invariant set of size $\sum_{i=0}^{\ell_t} b_i 2^i$. But since ℓ_t is the largest element of L, $\theta|_{B_{\ell_t}}$ (and hence θ) contains a cycle of length 2^ℓ for all $\ell \in \mathrm{supp}(b)$

with $\ell_t < \ell \le m$, and hence θ contains an invariant set of size $\sum_{i=0}^{m} b_i 2^i = k$. This contradicts the fact that θ is a k-complementing permutation.

We conclude that for some $j \in \{1, 2, \ldots, t\}$, $|A_{\ell_j}| < \sum_{i=0}^{\ell_j} b_i 2^i$. For this j, set $\ell = \ell_j$. Then $\ell \in \text{supp}(b)$ and $|A_\ell| < k_{[2^{\ell+1}]}$, as required.

($\Leftarrow$) Let $\theta \in \text{Sym}(V)$ with order a power of 2 and suppose that, for some $\ell \in \text{supp}(b)$, $V = A_\ell \cup B_\ell$ and $|A_\ell| < k_{[2^{\ell+1}]}$. This implies that θ does not have an invariant set of size k. Moreover, since the order of θ is a power of 2, for each odd integer j, θ^j has the same cycle type as θ, and hence θ^j also has no invariant set of size k. Hence $A^{\theta^j} \ne A$ for all odd integers j and all $A \in V^{(k)}$, and so Proposition 1.1 implies that θ is a k-complementing permutation. □

Lemma 2.3. *Let V be a finite set, and let s be an integer. A permutation $\theta \in \text{Sym}(V)$ is a k-complementing permutation if and only if θ^{2s+1} is a k-complementing permutation.*

Proof. If $\theta \in \text{Sym}(V)$ is a k-complementing permutation, then $\theta \in \text{Ant}(X)$ for some self-complementary k-hypergraph $X = (V, E)$, and so θ is a bijection from E to E^C and a bijection from E^C to E. It follows that $\theta^{2s+1} \in \text{Ant}(X)$.

Conversely, suppose that θ^{2s+1} is a k-complementing permutation. Then Proposition 1.1 guarantees that each orbit of θ^{2s+1} on $V^{(k)}$ has even cardinality. Observe that each orbit of θ^{2s+1} on $V^{(k)}$ is contained in an orbit of θ on $V^{(k)}$. Also, every k-subset in an orbit of θ on $V^{(k)}$ must certainly lie in an orbit of θ^{2s+1} on $V^{(k)}$. Since the orbits of θ^{2s+1} on $V^{(k)}$ are pairwise disjoint, it follows that every orbit of θ on $V^{(k)}$ is a union of pairwise disjoint orbits of θ^{2s+1} on $V^{(k)}$, each of which has even cardinality. Hence every orbit of θ on $V^{(k)}$ has even cardinality, and so by Proposition 1.1, θ is a k-complementing permutation. □

For a permutation θ on a set V, the symbol $|\theta|$ denotes the order of θ in $\text{Sym}(V)$. Lemma 2.3 and Theorem 2.2 together yield the following characterization of k-complementing permutations.

Corollary 2.4. *Let k be a positive integer, let b be the binary representation of k, and let V be a finite set. A permutation $\sigma \in \text{Sym}(V)$ is a k-complementing permutation if and only if $|\sigma| = 2^i(2t+1)$ for some integers t, i such that $i \ge 1$ and $t \ge 0$, and $\theta = \sigma^{2t+1}$ satisfies the conditions of Theorem 2.2 for some $\ell \in \text{supp}(b)$.* □

Corollary 2.4 and the conditions of Theorem 2.2 can be used to test a permutation algorithmically to determine whether it is a k-complementing permutation.

3. Necessary and sufficient conditions on order

In this section, we present an alternative description of the necessary and sufficient condition on the order n of a self-complementary k-hypergraph of Theorem 1.3.

Lemma 3.1. *A self-complementary k-hypergraph has an antimorphism whose order is equal to a power of 2.*

Proof. Let X be a self-complementary k-hypergraph, and let $\theta \in \text{Ant}(X)$. Proposition 1.1 guarantees that θ has even order, so $|\theta| = 2^z s$ for some positive integer z and some odd integer s. Since s is odd, $\theta^s \in \text{Ant}(X)$, and θ^s has order equal to a power of 2. □

Lemma 3.1 and Theorem 2.2 immediately imply the following necessary and sufficient conditions on the order of a self-complementary uniform hypergraph of rank k.

Corollary 3.2. *Let k and n be positive integers, $k \le n$, and let b be the binary representation of k. There exists a self-complementary k-hypergraph of order n if and only if*

$$n_{[2^{\ell+1}]} < k_{[2^{\ell+1}]} \quad \text{for some } \ell \in \text{supp}(b). \quad \square \tag{6}$$

In Lemma A.1 it is shown directly that condition (6) is equivalent to the condition that $\binom{n}{k}$ is even.

When $k = 2^\ell$ or $k = 2^\ell + 1$, Corollary 3.2 yields the following result.

Corollary 3.3. *Let ℓ be a positive integer.*

1. *If $k = 2^\ell$, then there exists a self-complementary k-hypergraph of order n if and only if $n_{[2^{\ell+1}]} < k$.*
2. *If $k = 2^\ell + 1$, then there exists a self-complementary k-hypergraph of order n if and only if n is even or $n_{[2^{\ell+1}]} < k$.* □

For example, Corollary 3.3 states that there exists a self-complementary graph of order n if and only if $n \equiv 0$ or 1 (mod 4), there exists a self-complementary 3-hypergraph of order n if and only if $n \equiv 0$, 1, or 2 (mod 4), and there exists a self-complementary 4-hypergraph of order n if and only if $n \equiv 0$, 1, 2 or 3 (mod 8).

In the case where k is a sum of consecutive powers of 2, the condition of Corollary 3.2 holds for the largest integer in the support of the binary representation of k, as the next result shows.

Corollary 3.4. *Let r and ℓ be nonnegative integers, and suppose that $k = \sum_{i=0}^{r} 2^{\ell+i}$. Then there exists a self-complementary k-hypergraph of order n if and only if $n_{[2^{\ell+r+1}]} < k$.*

Proof. Suppose that there exists a self-complementary k-hypergraph of order n, and let b be the binary representation of k. Then

$$\text{supp}(b) = \{\ell, \ell+1, \dots, \ell+r\},$$

and so Corollary 3.2 guarantees that

$$n_{[2^{\ell+j+1}]} < k_{[2^{\ell+j+1}]}, \tag{7}$$

for some $j \in \{0, 1, 2, \dots, r\}$. If (7) holds for some $j < r$, then the fact that

$$n_{[2^{\ell+(j+1)+1}]} \leq 2^{\ell+j+1} + n_{[2^{\ell+j+1}]}$$

implies that

$$n_{[2^{\ell+(j+1)+1}]} < 2^{\ell+j+1} + k_{[2^{\ell+j+1}]}. \tag{8}$$

Now since $2^{\ell+j+1} + k_{[2^{\ell+j+1}]} = 2^{\ell+j+1} + \sum_{i=0}^{j} 2^{\ell+i} = k_{[2^{\ell+(j+1)+1}]}$, (8) implies that

$$n_{[2^{\ell+(j+1)+1}]} < k_{[2^{\ell+(j+1)+1}]},$$

and hence (7) also holds for $j+1$. Thus, by induction on j, the fact that (7) holds for some $j \in \{0, 1, \dots, r\}$ implies that (7) holds for $j = r$. Hence $n_{[2^{\ell+r+1}]} < k_{[2^{\ell+r+1}]} = k$.

Conversely, Corollary 3.2 guarantees that there exists a self-complementary k-hypergraph of order n for every integer n such that $n_{[2^{\ell+r+1}]} < k_{[2^{\ell+r+1}]} = k$. □

Corollary 3.5. *Let ℓ be a positive integer and suppose that $k = 2^{\ell} - 1$. There exists a self-complementary k-hypergraph of order n if and only if $n_{[2^{\ell}]} < k$.*

Proof. Since $k = 2^{\ell} - 1 = \sum_{i=0}^{\ell-1} 2^{i}$, this follows directly from Corollary 3.4. □

4. Generating self-complementary hypergraphs

We will describe a simple algorithm which takes a k-complementing permutation in $\text{Sym}(V)$ as input, and returns the set $\mathcal{H}_\theta$ of all self-complementary k-hypergraphs X on V that have θ as an antimorphism. This algorithm was previously described by Sachs [3] and Ringel [2] for $k = 2$, by Suprunenko [4] and Kocay [1] for $k = 2, 3$, and by Szymański [5] for $k = 3, 4$. From each orbit $A, A^{\theta}, A^{\theta^2}, \dots$ of θ on $V^{(k)}$, we either take the alternating k-sets $A, A^{\theta^2}, A^{\theta^4}, \dots$ for X, or the alternating k-sets $A^{\theta}, A^{\theta^3}, A^{\theta^5}, \dots$ for X. Then within each orbit, θ maps edges of X onto non-edges of X, and vice versa. If there are m orbits of θ on $V^{(k)}$, we can use this method to generate the set $\mathcal{H}_\theta$ of all 2^m self-complementary k-hypergraphs on V for which θ is an antimorphism. Lemma 3.1 guarantees that every self-complementary k-hypergraph has an antimorphism which has order a power of 2, and so we can generate all of the self-complementary k-hypergraphs of order n, up to isomorphism, by applying this simple algorithm to find $\mathcal{H}_\theta$ for every permutation θ in $\text{Sym}(n)$ satisfying the conditions of Theorem 2.2. Moreover, if we just wish to generate at least one representative of each *isomorphism class* of self-complementary k-hypergraphs of order n, it suffices to apply this algorithm to one permutation θ from each conjugacy class of permutations in $\text{Sym}(n)$ satisfying the conditions of Theorem 2.2.

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Appendix

In Lemma A.1, we will show directly that the necessary and sufficient condition (6) of Corollary 3.2 on the order n of a self-complementary k-hypergraph is equivalent to Szymański and Wojda's condition that $\binom{n}{k}$ is even. First we will need some notation.

For positive integers m and n, recall that $n_{[m]}$ denotes the unique integer in $\{0, 1, \dots, m-1\}$ such that $n \equiv n_{[m]} (\text{mod } m)$. Let $\left[\frac{n}{m}\right]$ denote the quotient upon division of n by m. Finally, for any prime number p, let $n_{(p)}$ denote the largest integer i such that p^i divides n.

It is well known that for any positive integer m and prime number p, we have

$$m!_{(p)} = \sum_{r \geq 1} \left[\frac{m}{p^r}\right].$$

It follows that

$$\binom{m}{n}_{(p)} = \left(\frac{m!}{n!(m-n)!}\right)_{(p)} = m!_{(p)} - n!_{(p)} - (m-n)!_{(p)} = \sum_{r\geq 1}\left\{\left[\frac{m}{p^r}\right] - \left[\frac{n}{p^r}\right] - \left[\frac{m-n}{p^r}\right]\right\}. \quad (9)$$

We can evaluate each term in the sum above using the well-known fact that

$$\left[\frac{m}{p^r}\right] - \left[\frac{n}{p^r}\right] - \left[\frac{m-n}{p^r}\right] = \begin{cases} 1 & \text{if } m_{[p^r]} < n_{[p^r]} \\ 0 & \text{otherwise.}\end{cases} \quad (10)$$

Lemma A.1. *Let k and n be positive integers, $k \leq n$, and let b be the binary representation of k. Then $\binom{n}{k}$ is even if and only if*

$$n_{[2^{\ell+1}]} < k_{[2^{\ell+1}]} \quad \textit{for some } \ell \in \operatorname{supp}(b). \quad (11)$$

Proof. Observe that $\binom{n}{k}$ is even if and only if $\binom{n}{k}_{(2)} \geq 1$. By (9) we have

$$\binom{n}{k}_{(2)} = \sum_{r\geq 1}\left\{\left[\frac{n}{2^r}\right] - \left[\frac{k}{2^r}\right] - \left[\frac{n-k}{2^r}\right]\right\}. \quad (12)$$

By (10), for each $r \geq 1$ we have

$$\left[\frac{n}{2^r}\right] - \left[\frac{k}{2^r}\right] - \left[\frac{n-k}{2^r}\right] = \begin{cases} 1 & \text{if } n_{[2^r]} < k_{[2^r]} \\ 0 & \text{otherwise.}\end{cases}$$

Hence (12) implies that $\binom{n}{k}$ is even if and only if

$$n_{[2^r]} < k_{[2^r]} \quad \text{for some } r \geq 1,$$

that is, if and only if

$$n_{[2^{\ell+1}]} < k_{[2^{\ell+1}]} \quad \text{for some } \ell \geq 0. \quad (13)$$

Now we will show that the condition in (13) holds for some $\ell \geq 0$ if and only if it holds for some $\ell \in \operatorname{supp}(b)$. If (13) holds for some $\ell \in \operatorname{supp}(b)$, then (13) certainly holds for some $\ell \geq 0$. Conversely, assume for the sake of contradiction that the condition in (13) does not hold for any $\ell \in \operatorname{supp}(b)$, but it holds for some $\ell \notin \operatorname{supp}(b)$. Now if $i \notin \operatorname{supp}(b)$ for all i such that $0 \leq i \leq \ell$, then $k_{[2^{\ell+1}]} = \sum_{i=0}^{\ell} b_i 2^i = 0$, and so (13) implies that $n_{[2^{\ell+1}]} < 0$, giving a contradiction. Hence there must exist a nonnegative integer $i < \ell$ such that $i \in \operatorname{supp}(b)$. Let ℓ_* denote the largest such integer i. Then $k_{[2^{\ell+1}]} = \sum_{i=0}^{\ell_*} b_i 2^i = k_{[2^{\ell_*+1}]}$, and so (13) implies that

$$n_{[2^{\ell+1}]} < k_{[2^{\ell_*+1}]}. \quad (14)$$

Since $\ell_* < \ell$, we have $n_{[2^{\ell_*+1}]} \leq n_{[2^{\ell+1}]}$, and so (14) implies that

$$n_{[2^{\ell_*+1}]} < k_{[2^{\ell_*+1}]}.$$

Hence $\ell_* \in \operatorname{supp}(b)$ and (13) holds for ℓ_*, contradicting our assumption. We conclude that (13) holds if and only if $n_{[2^{\ell+1}]} < k_{[2^{\ell+1}]}$ for some $\ell \in \operatorname{supp}(b)$, and thus $\binom{n}{k}$ is even if and only if (11) holds. □

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