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On the coloring of the annihilating-ideal graph of a commutative ring

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ABSTRACT

Suppose that R is a commutative ring with identity. Let $\mathbb{A}(R)$ be the set of all ideals of R with non-zero annihilators. The annihilating-ideal graph of R is defined as the graph $\mathbb{AG}(R)$ with the vertex set $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{(0)\}$ and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. In Behboodi and Rakeei (2011) [8], it was conjectured that for a reduced ring R with more than two minimal prime ideals, $girth(\mathbb{AG}(R)) = 3$. Here, we prove that for every (not necessarily reduced) ring R, $\omega(\mathbb{AG}(R)) \geq |\mathrm{Min}(R)|$, which shows that the conjecture is true. Also in this paper, we present some results on the clique number and the chromatic number of the annihilating-ideal graph of a commutative ring. Among other results, it is shown that if the chromatic number of the zero-divisor graph is finite, then the chromatic number of the annihilating-ideal graph is finite too. We investigate commutative rings whose annihilating-ideal graphs are bipartite. It is proved that $\mathbb{AG}(R)$ is bipartite if and only if $\mathbb{AG}(R)$ is triangle-free.

1. Introduction

In recent years, assigning graphs to rings has played an important role in the study of structures of rings, for instance see [3,4,6,7,11,14]. Throughout this paper, all rings are assumed to be commutative with unity. We denote by $Z(R)$, $\mathrm{Min}(R)$ and $\mathrm{Max}(R)$, the set of zero-divisors, the set of minimal ideals and the set of maximal ideals of R, respectively. The *Jacobson radical* and the *nilradical* of R are denoted by $J(R)$ and $Nil(R)$, respectively. The ring R is said to be *reduced* if it has no non-zero nilpotent element. The set of all non-zero ideals of R is denoted by $\mathbb{I}(R)$. We call an ideal I of R, an *annihilating-ideal* if there exists a non-zero ideal J of R such that $IJ = (0)$. We use the notation $\mathbb{A}(R)$ for the set of all annihilating-ideals of R. A subset S of a commutative ring R is called a *multiplicative closed subset* (*m.c.s*) of R if $1 \in S$ and $x, y \in S$ implies that $xy \in S$. If S is an *m.c.s* of R and M is an R-module, then we denote by R_S and M_S, the ring of fractions of R and the module of fractions of M with respect to S, respectively. If $\mathfrak{p}$ is a prime ideal of R and $S = R \setminus \mathfrak{p}$, we use the notation $M_{\mathfrak{p}}$, for the localization of M with respect to S. By $T(R)$, we mean the *total ring* of R that is the ring of fractions, where $S = R \setminus Z(R)$. A non-zero ideal I of a ring R is said to be *minimal* if there is no non-trivial ideal of R contained in I. We denote by $Soc(R)$, the sum of all minimal ideals of R (If there is no minimal ideal, then we define $Soc(R) = (0)$).

Let G be a graph with the vertex set $V(G)$. For every positive integer n, we denote the path of order n, by P_n. The *distance* between two vertices in a graph is the number of edges in a shortest path connecting them. The *diameter* of a connected graph G, denoted by $diam(G)$, is the maximum distance between any pair of the vertices of G. The *girth* of G, denoted by $girth(G)$, is the order of a shortest cycle contained in G. If G does not contain a cycle, then $girth(G)$ is defined to be infinity.

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The *complete graph* is a graph in which any two distinct vertices are adjacent. A complete graph of order n is denoted by K_n. A *bipartite graph* is a graph whose vertices can be partitioned into two disjoint sets U and V such that every edge connects a vertex in U to one in V. A *complete bipartite graph* is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. If the size of one of the parts is 1, then the complete bipartite graph is said to be a *star graph*. The *center* of a star graph is a vertex that is adjacent to all other vertices. A *clique* of a graph is a complete subgraph and the supremum of the sizes of cliques in G, denoted by $\omega(G)$, is called the *clique number* of G. If the graph has no vertex, then its clique number is defined to be 0. By $\chi(G)$, we denote the *chromatic number* of G, i.e., the minimum number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. For some $U \subseteq V(G)$, we denote by $N(U)$, the set of all vertices of G adjacent to at least one vertex of U. For every vertex $v \in V(G)$, the size of $N(v)$ is denoted by $d(v)$. For any subset U of vertices, we use the notation U^c for the complement of U. Let G and G' be two graphs. A *graph homomorphism* from G to G' is a mapping $\phi : V(G) \longrightarrow V(G')$ such that for every edge $\{u, v\}$ of G, $\{\phi(u), \phi(v)\}$ is an edge of G'. A *retract* of G is a subgraph H of G such that there exists a homomorphism $\phi : G \longrightarrow H$ such that $\phi(x) = x$, for every vertex x of H. The homomorphism ϕ is called the *retract* (*graph*) *homomorphism*.

Let R be a ring. The *zero-divisor graph* of R, $\Gamma(R)$, is a graph with the vertex set $Z(R) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $xy = 0$. The concept of the zero-divisor graph was first introduced by Beck (see [6]). Since most properties of a ring are closely tied to the behavior of its ideals, it is worthy to replace the vertices of the zero-divisor graph by the non-zero annihilating-ideals. By the *annihilating-ideal graph* $\mathbb{AG}(R)$ of R, we mean the graph with the vertex set $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{(0)\}$. Two distinct vertices I and J are adjacent if and only if $IJ = (0)$. Thus $\mathbb{AG}(R)$ has no vertex if and only if R is an integral domain. The concept of the annihilating-ideal graph of a commutative ring was first introduced in [7].

The following theorems reflect some fundamental properties of the annihilating-ideal graph of a ring which will be used in this paper.

Theorem A ([7, Theorem 1.4]). *Let R be a ring which is not an integral domain. Then the following statements are equivalent:*

(1) *$\mathbb{AG}(R)$ is a finite graph.*
(2) *R has finitely many ideals.*
(3) *Every vertex of $\mathbb{AG}(R)$ has finite degree.*

Moreover, $\mathbb{AG}(R)$ has n vertices, $n \geq 1$, if and only if R has exactly n non-trivial ideals.

Theorem B ([7,9]). *For every ring R, the annihilating-ideal graph $\mathbb{AG}(R)$ is connected and diam$(\mathbb{AG}(R)) \leq 3$. Moreover, if $\mathbb{AG}(R)$ contains a cycle, then girth$(\mathbb{AG}(R)) \leq 4$.*

2. The clique number and the chromatic number of the annihilating-ideal graphs

In this section, we study the vertex coloring of the annihilating-ideal graphs of some rings of fractions. Next, we provide some formulas for the clique and the chromatic numbers of the annihilating-ideal graph of a direct product of rings.

It is easy to see that if S is an *m.c.s* of R containing no zero-divisors, then for every ideal I of R, the localization $I_S = 0$ if and only if $I = 0$.

Theorem 1. *Let R be a ring and S be an m.c.s of R containing no zero-divisors. Then $\omega(\mathbb{AG}(R_S)) \leq \omega(\mathbb{AG}(R))$. Moreover, $\mathbb{AG}(R_S)$ is a retract of $\mathbb{AG}(R)$ if R is reduced. In particular, $\omega(\mathbb{AG}(R_S)) = \omega(\mathbb{AG}(R))$, whenever R is reduced.*

Proof. Consider a vertex map $\phi : V(\mathbb{AG}(R)) \longrightarrow V(\mathbb{AG}(R_S))$, $I \longrightarrow I_S$. Clearly, $I_S \neq J_S$ implies $I \neq J$ and $IJ = (0)$ if and only if $I_S J_S = (0)$. Thus ϕ is surjective, and hence $\omega(\mathbb{AG}(R_S)) \leq \omega(\mathbb{AG}(R))$.

In what follows, assume that R is reduced. If $I \neq J$ and $IJ = (0)$, then we show that $I_S \neq J_S$. On the contrary suppose that $I_S = J_S$. Then $I_S^2 = I_S J_S = (IJ)_S = (0)$ and so $I^2 = (0)$, a contradiction. This shows that the map ϕ is a graph homomorphism. Now, for any vertex I_S of $\mathbb{AG}(R_S)$, by the Axiom of Choice, fix an I. Then ϕ is a retract (graph) homomorphism. It clearly follows that $\omega(\mathbb{AG}(R_S)) = \omega(\mathbb{AG}(R))$ under the assumption. □

Corollary 2. *If R is a reduced ring, then $\omega(\mathbb{AG}(T(R))) = \omega(\mathbb{AG}(R))$.*

Since the chromatic number $\chi(G)$ of a graph G is the least positive integer r such that there exists a retract homomorphism $\psi : G \longrightarrow K_r$, the following corollaries follow directly from the proof of Theorem 1.

Corollary 3. *Let R be a ring and S be an m.c.s of R containing no zero-divisor. Then $\chi(\mathbb{AG}(R_S)) \leq \chi(\mathbb{AG}(R))$. Moreover, if R is reduced, then $\chi(\mathbb{AG}(R_S)) = \chi(\mathbb{AG}(R))$.*

Corollary 4. *If R is a reduced ring, then $\chi(\mathbb{AG}(T(R))) = \chi(\mathbb{AG}(R))$.*

Before stating the next theorem, we need the following interesting result, due to Eben Matlis.

Theorem 5 ([13, Proposition 1.5]). *Let R be a ring and $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ be a finite set of distinct minimal prime ideals of R. Let $S = R \setminus \bigcup_{i=1}^{n} \mathfrak{p}_i$. Then $R_S \cong R_{\mathfrak{p}_1} \times \cdots \times R_{\mathfrak{p}_n}$.*

Conjecture 1.11 of [8] states that for every reduced ring R with more than two minimal prime ideals, $girth(\mathbb{AG}(R)) = 3$. The following theorem settles this conjecture.

Theorem 6. *Let R be a ring and $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ be a finite set of distinct minimal prime ideals of R. Then there exists a clique of $\mathbb{AG}(R)$ of size n.*

Proof. Let $S = R \setminus \bigcup_{i=1}^{n} \mathfrak{p}_i$. By Theorem 5, there exists a ring isomorphism $\phi : R_{\mathfrak{p}_1} \times \cdots \times R_{\mathfrak{p}_n} \longrightarrow R_S$. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ and $\phi(e_i) = \frac{a_i}{s_i}$, where $1 \le i \le n$ and 1 is in the i-th position of e_i. Consider the principal ideals $I_i = \left(\frac{a_i}{s_i}\right) = \left(\frac{a_i}{1}\right)$ in the ring R_S. Since ϕ is an isomorphism, there exists $t_{ij} \in S$ such that $t_{ij}a_ia_j = 0$, for every i, j, $1 \le i < j \le n$. Let $t = \prod_{1 \le i < j \le n} t_{ij}$. We show that $\{(ta_1), \ldots, (ta_n)\}$ is a clique of size n in $\mathbb{AG}(R)$. Since $(ta_i)_S = \left(\frac{a_i}{1}\right) = I_i$, we deduce that (ta_i) are distinct non-trivial ideals of R. Clearly, $(ta_i)(ta_j) = (0)$ and so $\mathbb{AG}(R)$ has a clique of size n, as desired. □

Corollary 7. *For every ring R, $\omega(\mathbb{AG}(R)) \ge |\mathrm{Min}(R)|$.*

The next theorem improves [8, Corollary 2.11].

Theorem 8. *Let R be a reduced ring. Then $\chi(\mathbb{AG}(R)) = \omega(\mathbb{AG}(R)) = |\mathrm{Min}(R)|$.*

Proof. If $|\mathrm{Min}(R)| = \infty$, then by Corollary 7 there is nothing to prove. Thus, suppose that $\mathrm{Min}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, for some positive integer n. Let $S = R \setminus \bigcup_{i=1}^{n} \mathfrak{p}_i$. By Theorem 5, we have $R_S \cong R_{\mathfrak{p}_1} \times \cdots \times R_{\mathfrak{p}_n}$. Clearly, $\omega(\mathbb{AG}(R_S)) \ge n$. Now, we show that $\chi(\mathbb{AG}(R_S)) \le n$. Since R is reduced, by [13, Proposition 1.1], Part (1), every $R_{\mathfrak{p}_i}$ is a field. Now, we define the map $c : V(\mathbb{AG}(R_S)) \longrightarrow \{1, \ldots, n\}$ by $c(I_1 \times \cdots \times I_n) = min\{i \mid I_i \ne (0)\}$. Since each $R_{\mathfrak{p}_i}$ is a field, c is a proper vertex coloring of $\mathbb{AG}(R_S)$. Thus $\chi(\mathbb{AG}(R_S)) \le n$ and so $\chi(\mathbb{AG}(R_S)) = \omega(\mathbb{AG}(R_S)) = n$. By [13, Proposition 1.1], Part (3), $S \cap Z(R) = \varnothing$. Therefore, Theorem 1 and Corollary 3 complete the proof. □

Remark 9. Using Theorems 3.2, 3.11 and 4.3 in [12], one can see that for every reduced semigroup S with 0, the equality $\omega(\Gamma(S)) = \chi(\Gamma(S))$ always holds, where $\Gamma(S)$ denotes the zero-divisor graph of S, and thus the first equality in Theorem 8 is obtained, since every annihilating-ideal graph of a ring is always the zero-divisor graph of a semigroup with 0.

Lemma 10. *Let R be a ring, I and J be two non-trivial ideals of R. If for every $\mathfrak{m} \in \mathrm{Max}(R)$, $I_{\mathfrak{m}} = J_{\mathfrak{m}}$, then $I = J$.*

Proof. Since $I_{\mathfrak{m}} = J_{\mathfrak{m}}$, we conclude that $((I + J)/J)_{\mathfrak{m}} = (I_{\mathfrak{m}} + J_{\mathfrak{m}})/J_{\mathfrak{m}} = (0)$, for every $\mathfrak{m} \in \mathrm{Max}(R)$. So, by [5, Proposition 3.8], we deduce that $I \subseteq J$. Similarly, one can see that $J \subseteq I$ and the proof is complete. □

Theorem 11. *Let R be a ring and $|\mathrm{Max}(R)| < \infty$. If for every $\mathfrak{m} \in \mathrm{Max}(R)$, $\omega(\mathbb{AG}(R_{\mathfrak{m}}))$ is finite, then $\omega(\mathbb{AG}(R))$ is finite.*

Proof. Suppose that $\mathrm{Max}(R) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$. On the contrary assume that $C = \{J_i\}_{i=1}^{\infty}$ is a clique of $\mathbb{AG}(R)$. Since $\omega(\mathbb{AG}(R_{\mathfrak{m}_1})) < \infty$, we deduce that there exists an infinite subset $A_1 \subseteq \mathbb{N}$ such that for every $i, j \in A_1$, $(J_i)_{\mathfrak{m}_1} = (J_j)_{\mathfrak{m}_1}$. Now, using $\omega(\mathbb{AG}(R_{\mathfrak{m}_2})) < \infty$, we conclude that there exists an infinite subset $A_2 \subseteq A_1$ such that for every $i, j \in A_2$, $(J_i)_{\mathfrak{m}_2} = (J_j)_{\mathfrak{m}_2}$. By continuing this procedure one can see that there exists an infinite subset $A_n \subseteq A_{n-1}$ such that for every $i, j \in A_n$, $(J_i)_{\mathfrak{m}_l} = (J_j)_{\mathfrak{m}_l}$, for every l, $l = 1, \ldots, n$. Therefore, by Lemma 10, we get a contradiction. □

Theorem 12. *Let R be a ring and $|\mathrm{Max}(R)| < \infty$. If for every $\mathfrak{m} \in \mathrm{Max}(R)$, $\chi(\mathbb{AG}(R_{\mathfrak{m}}))$ is finite, then $\chi(\mathbb{AG}(R))$ is finite.*

Proof. Let $\mathrm{Max}(R) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ and $f_i : V(\mathbb{AG}(R_{\mathfrak{m}_i})) \longrightarrow \{1, \ldots, \chi(\mathbb{AG}(R_{\mathfrak{m}_i}))\}$ be a proper vertex coloring of $\mathbb{AG}(R_{\mathfrak{m}_i})$, for every i, $1 \le i \le n$. Now, by defining $f_i(L) = 1$, for every $L \in \mathbb{I}(R_{\mathfrak{m}_i}) \setminus V\big(\mathbb{AG}(R_{\mathfrak{m}_i})\big)$ (we recall that $\mathbb{I}(R_{\mathfrak{m}_i})$ denotes the set of non-zero ideals of $R_{\mathfrak{m}_i}$), we extend the map f_i to $\mathbb{I}(R_{\mathfrak{m}_i})$. We define a function f on $\mathbb{A}(R)^*$ by $f(I) = (g_1(I_{\mathfrak{m}_1}), \ldots, g_n(I_{\mathfrak{m}_n}))$, where

$$g_i(I_{\mathfrak{m}_i}) = \begin{cases} 0; & I_{\mathfrak{m}_i} = (0) \\ -1; & I_{\mathfrak{m}_i} = R_{\mathfrak{m}_i} \\ f_i(I_{\mathfrak{m}_i}); & \text{otherwise.} \end{cases}$$

Now, by Lemma 10, it is not hard to check that f is a proper vertex coloring of $\mathbb{AG}(R)$. □

In the next theorem, we provide some formulas for the clique number and the chromatic number of the annihilating-ideal graph of a direct product of rings.

Theorem 13. *Let $A = R_1 \times \cdots \times R_k$, $T = T_1 \times \cdots \times T_l$, R_i be a non-domain and T_j be a reduced ring, for $i = 1, \ldots, k$ and $j = 1, \ldots, l$. If $R = A \times T$, then the following statements hold:*

(i) *If T is not an integral domain, then $\omega(\mathbb{AG}(R)) = \omega(\mathbb{AG}(A)) + \omega(\mathbb{AG}(T))$.*
(ii) *If every T_i is an integral domain, then $\omega(\mathbb{AG}(R)) = \omega(\mathbb{AG}(A)) + l$.*
(iii) $\sum_{i=1}^{k} \omega(\mathbb{AG}(R_i)) + max\left\{l, \sum_{i=1}^{l} \omega(\mathbb{AG}(T_i))\right\} \le \omega(\mathbb{AG}(R))$.
(iv) $\omega(\mathbb{AG}(A)) \le \prod_{i=1}^{k}(\omega(\mathbb{AG}(R_i)) + 1) - 1$. *Moreover, the equality holds if and only if every R_i has a maximum clique C_i such that for every I, $I \in C_i$, $I^2 = (0)$.*

Proof. (i) If one of the sides of the equality is infinite, then obviously the assertion holds. So, suppose that both sides of the equality are finite. Now, we show that $\mathbb{AG}(R)$ has a maximum clique of the form $\{I \times (0) \mid I \in C_1\} \cup \{(0) \times J \mid J \in C_2\}$, where C_1 and C_2 are two maximum cliques of $\mathbb{AG}(A)$ and $\mathbb{AG}(T)$, respectively. Let $C = \{I_i \times J_i\}_{i\in\Lambda}$ be a maximum clique of $\mathbb{AG}(R)$. Assume that $I_k \times J_k \in C$. If $I_k \neq (0)$ and $J_k \neq (0)$, then let $C_0 = (C \setminus \{I_k \times J_k\}) \cup \{I_k \times (0), (0) \times J_k\}$. Since T is a reduced ring, we conclude that $|C_0| \geq |C|$. Since $|C_0| < \infty$, by repeating this procedure, we obtain a maximum clique C_0 such that every element of C_0 is of the form $I_r \times (0)$ or $(0) \times J_s$. Let $C_1 = \{I \mid I \times (0) \in C_0\}$ and $C_2 = \{J \mid (0) \times J \in C_0\}$. Then C_1 and C_2 are maximum cliques of $\mathbb{AG}(A)$ and $\mathbb{AG}(T)$, respectively. Thus $C_0 = \{I_1 \times (0) \mid I_1 \in C_1\} \cup \{(0) \times I_2 \mid I_2 \in C_2\}$ and the proof of (i) is complete.

(ii) If $l = 1$, then it is not hard to check that the equality holds. Suppose that $l \geq 2$. By (i), $\omega(\mathbb{AG}(R)) = \omega(\mathbb{AG}(A)) + \omega(\mathbb{AG}(T))$. Since $|\mathrm{Min}(T)| = l$, Theorem 8 completes the proof.

(iii) It is obvious.

(iv) It is clear that every clique of $\mathbb{AG}(A)$ is a subset of $\prod_{i=1}^{k}(C_i \cup \{(0)\}) \setminus \{(0)\}$, where C_i is a clique of $\mathbb{AG}(R_i)$. Thus $\omega(\mathbb{AG}(A)) \leq \prod_{i=1}^{k}(\omega(\mathbb{AG}(R_i)) + 1) - 1$. Moreover, the equality holds if and only if every R_i has a maximum clique C_i with $I^2 = (0)$, for every $I, I \in C_i$, $1 \leq i \leq k$. □

Now, using Theorem 13, we provide a simple proof for [2, Lemma 1].

Corollary 14. *Let R_1 and R_2 be two rings and $R = R_1 \times R_2$. If $\mathbb{AG}(R)$ is a triangle-free graph, then exactly one of the following statements holds.*

(i) *Both R_1 and R_2 are integral domains.*

(ii) *One R_i is an integral domain and the other one is a ring with a unique non-trivial ideal.*

Moreover, $\mathbb{AG}(R)$ has no cycle if and only if either $R \cong F \times S$ or $R \cong F \times D$, where F is a field, S is a ring with a unique non-trivial ideal and D is an integral domain.

Proof. Since $\mathbb{AG}(R)$ is triangle-free and $R_1 \times (0)$ and $(0) \times R_2$ are adjacent, we conclude that $\omega(\mathbb{AG}(R)) = 2$. If at least one of the $\omega(\mathbb{AG}(R_1))$ and $\omega(\mathbb{AG}(R_2))$, say $\omega(\mathbb{AG}(R_1))$, is 2, then by Theorem 13, Part (iii), we deduce that $\omega(\mathbb{AG}(R_2)) = 0$ and so R_2 is an integral domain. Hence Theorem 13, Part (ii) implies that $\omega(\mathbb{AG}(R)) = 3$, a contradiction. Therefore; one may assume that both $\omega(\mathbb{AG}(R_1))$ and $\omega(\mathbb{AG}(R_2))$ are at most 1. If $\omega(\mathbb{AG}(R_1)) = \omega(\mathbb{AG}(R_2)) = 1$, then Theorems A and B imply that every R_i is a ring with the unique non-trivial ideal $\mathfrak{m}_i$ and so by Nakayama's Lemma $\mathfrak{m}_i^2 = (0)$. Thus by Theorem 13, Part (iv), we conclude that $\omega(\mathbb{AG}(R)) = 3$, a contradiction. Hence $\omega(\mathbb{AG}(R_1)) + \omega(\mathbb{AG}(R_2)) \leq 1$ and so by Theorem A, (i) and (ii) are proved. The last part of the corollary is clear. □

Now, we would like to state some similar results for the chromatic number of the annihilating-ideal graph of direct product of rings.

Theorem 15. *Let $A = R_1 \times \cdots \times R_k$, $T = T_1 \times \cdots \times T_l$, R_i be a non-domain and T_j be a reduced ring, for $i = 1, \ldots, k$ and $j = 1, \ldots, l$. If $R = A \times T$, then the following statements hold:*

(i) *If T is not an integral domain, then $\chi(\mathbb{AG}(R)) = \chi(\mathbb{AG}(A)) + \chi(\mathbb{AG}(T))$.*

(ii) *If every T_i is an integral domain, then $\chi(\mathbb{AG}(R)) = \chi(\mathbb{AG}(A)) + l$.*

(iii) $\sum_{i=1}^{k}\chi(\mathbb{AG}(R_i)) + max\left\{l, \sum_{i=1}^{l}\chi(\mathbb{AG}(T_i))\right\} \leq \chi(\mathbb{AG}(R))$.

(iv) $\chi(\mathbb{AG}(A)) \leq \prod_{i=1}^{k}\big(\chi(\mathbb{AG}(R_i)) + 1\big) - 1$.

Proof. (iii) Let f be a proper vertex coloring of $\mathbb{AG}(R)$ with $\chi(\mathbb{AG}(R))$ colors. Then the restriction of f to $\{(0) \times \cdots \times I \times \cdots \times (0) \mid I \in V(\mathbb{AG}(S_t))\}$ yields a proper vertex coloring of $\mathbb{AG}(S_t)$, where $S_t = R_t$, for every t, $1 \leq t \leq k$ and $S_t = T_j$, if $t = k + j$ for some j, $1 \leq j \leq l$. Since every vertex of $\{(0) \times \cdots \times I \times \cdots \times (0) \mid I \in \mathbb{I}(S_u)\}$ is adjacent to every vertex of $\{(0) \times \cdots \times J \times \cdots \times (0) \mid J \in \mathbb{I}(S_v)\}$, for each $u \neq v$ and $\mathbb{AG}(T)$ contains a clique of size l, we conclude that $\sum_{i=1}^{k}\chi(\mathbb{AG}(R_i)) + max\left\{l, \sum_{i=1}^{l}\chi(\mathbb{AG}(T_i))\right\} \leq \chi(\mathbb{AG}(R))$, as desired.

(i) By Part (iii), $\chi(\mathbb{AG}(R)) \geq \chi(\mathbb{AG}(A)) + \chi(\mathbb{AG}(T))$. Now, we define a proper vertex coloring of $\mathbb{AG}(R)$ with $\chi(\mathbb{AG}(A)) + \chi(\mathbb{AG}(T))$ colors. Let $g : V(\mathbb{AG}(A)) \longrightarrow \{1, \ldots, \chi(\mathbb{AG}(A))\}$ and $h : V(\mathbb{AG}(T)) \longrightarrow \{1, \ldots, \chi(\mathbb{AG}(T))\}$ be proper vertex colorings of $\mathbb{AG}(A)$ and $\mathbb{AG}(T)$, respectively. By defining $g(I) = 1$ and $h(J) = 1$, for every $I \notin \mathbb{A}(A)$ and every $J \notin \mathbb{A}(T)$, we extend the maps g and h to $\mathbb{I}(A)$ and $\mathbb{I}(T)$, respectively. Define a map ϕ on $V(\mathbb{AG}(R))$ as follows

$$\phi(I \times J) = \begin{cases} (g(I), 0); & J = (0) \\ (0, h(J)); & \text{otherwise.} \end{cases}$$

Since T is a reduced ring, we deduce that ϕ is a proper vertex coloring of $\mathbb{AG}(R)$ with $\chi(\mathbb{AG}(A)) + \chi(\mathbb{AG}(T))$ colors and so (i) is proved.

(ii) First suppose that $l = 1$ and $g : V(\mathbb{AG}(A)) \longrightarrow \{1, \ldots, \chi(\mathbb{AG}(A))\}$ is a proper vertex coloring of $\mathbb{AG}(A)$. By defining $g(I) = 1$, for every $I \notin \mathbb{A}(A)$, we extend the map g to $\mathbb{I}(A)$. Define a map ϕ on $V(\mathbb{AG}(R))$ as follows

$$\phi(I \times J) = \begin{cases} g(I); & J = (0) \\ \chi(\mathbb{AG}(A)) + 1; & \text{otherwise.} \end{cases}$$

It is not hard to check that ϕ is a proper vertex coloring of $\mathbb{AG}(R)$ with $\chi(\mathbb{AG}(R)) + 1$ colors and so by Part (iii) the equality holds for $l = 1$. Now, suppose that $l \geq 2$. By (i), $\chi(\mathbb{AG}(R)) = \chi(\mathbb{AG}(A)) + \chi(\mathbb{AG}(T))$. Since $|\mathrm{Min}(T)| = l$, Theorem 8 completes the proof.

(iv) First we claim that

$$\chi(\mathbb{AG}(R_1 \times R_2)) \leq (\chi(\mathbb{AG}(R_1)) + 1)(\chi(\mathbb{AG}(R_2)) + 1) - 1.$$

Let $c_i : V(\mathbb{AG}(R_i)) \longrightarrow \{1, \ldots, \chi(\mathbb{AG}(R_i))\}$ be a proper vertex coloring of $\mathbb{AG}(R_i)$, $i = 1, 2$. We extend the map c_i to $\mathbb{I}(R_i)$ by defining $c_i(J) = 1$, for every $J \notin \mathbb{A}(R_i)$. Now, define a vertex coloring c for $\mathbb{AG}(R)$ as follows:

$$c(I \times J) = \begin{cases} (0, c_2(J)); & I = (0) \\ (c_1(I), 0); & J = (0) \\ (c_1(I), c_2(J)); & I \neq (0) \text{ and } J \neq (0). \end{cases}$$

It is not hard to check that c is a proper vertex coloring of $\mathbb{AG}(R_1 \times R_2)$. So, the claim is proved. Now, by induction on k, we find that $\chi(\mathbb{AG}(A)) \leq \prod_{i=1}^{k}(\chi(\mathbb{AG}(R_i)) + 1) - 1$. □

Now, we have the following immediate corollary.

Corollary 16. *Let $R_1, \ldots, R_k$ be rings and $R = R_1 \times \cdots \times R_k$. Then*
(i) *$\omega(\mathbb{AG}(R)) < \infty$ if and only if $\omega(\mathbb{AG}(R_i)) < \infty$, for every i, $1 \leq i \leq k$.*
(ii) *$\chi(\mathbb{AG}(R)) < \infty$ if and only if $\chi(\mathbb{AG}(R_i)) < \infty$, for every i, $1 \leq i \leq k$.*

In the sequel, the chromatic numbers of two graphs $\mathbb{AG}(R)$ and $\Gamma(R)$ are compared. The following example shows that the chromatic numbers of the zero-divisor graph and the annihilating-ideal graph of a ring are not comparable in general.

Example 17. (a) Let k be a finite field with at least three elements, x be an indeterminate and $R = \frac{k[x]}{(x^2)}$. Then $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 1$ but $\chi(\Gamma(R)) \geq \omega(\Gamma(R)) \geq |k| - 1$.
(b) If $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = \omega(\Gamma(R)) = \chi(\Gamma(R)) = 2$.
(c) If $R = \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$, then $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 4$ but $\omega(\Gamma(R)) = \chi(\Gamma(R)) = 3$. Indeed, $\mathbb{AG}(R) \cong K_4$ and $\Gamma(R) \cong K_3$.

Theorem 18. *Let R be a ring and $\chi(\Gamma(R)) < \infty$. Then $\chi(\mathbb{AG}(R)) \leq 2^{\chi(\Gamma(R))} - 1$.*

Proof. Suppose that $\chi(\Gamma(R)) = k$ and $c : V(\Gamma(R)) \longrightarrow \{1, \ldots, k\}$ is a proper vertex coloring of $\Gamma(R)$. Let $A_i = c^{-1}(\{i\})$, for $i = 1, \ldots, k$. We define a map $f : V(\mathbb{AG}(R)) \longrightarrow P(\{1, \ldots, k\}) \setminus \{\varnothing\}$ by $f(I) = \{i \mid 1 \leq i \leq k,\ I \cap A_i \neq \varnothing\}$, where $P(X)$ denotes the power set of X. Now, we claim that f is a proper vertex coloring of $\mathbb{AG}(R)$. Suppose that I and J are two distinct vertices of $\mathbb{AG}(R)$ such that $IJ = (0)$ and $f(I) = f(J)$. We show that $I \cap A_t = J \cap A_t$, for every $t \in f(I)$. Let $x \in I \cap A_t$. Then $t \in f(I) = f(J)$ and so there exists $y \in J \cap A_t$. Since every A_i is an independent set of $\Gamma(R)$ and $IJ = (0)$, we deduce that $x = y \in J \cap A_t$. Thus $I \cap A_t \subseteq J \cap A_t$. Similarly, $J \cap A_t \subseteq I \cap A_t$ and hence $I \cap A_t = J \cap A_t$, for every $t \in f(I)$. Since for every $u \in \{1, \ldots, k\} \setminus f(I)$, $I \cap A_u = J \cap A_u = \varnothing$ and $\{A_1, \ldots, A_k\}$ is a partition of $Z(R) \setminus \{0\}$, we deduce that $I = J$, a contradiction. So, the claim is proved and the proof is complete. □

Theorem 19. *Let R be a ring, $\chi(\Gamma(R)) < \infty$ and every non-zero annihilating-ideal of R is infinite. Then $\chi(\mathbb{AG}(R)) \leq \chi(\Gamma(R))$.*

Proof. Suppose that $\chi(\Gamma(R)) = k$ and $c : V(\Gamma(R)) \longrightarrow \{1, \ldots, k\}$ is a proper vertex coloring of $\Gamma(R)$. Let $A_i = c^{-1}(\{i\})$. We define a map $f : V(\mathbb{AG}(R)) \longrightarrow \{1, \ldots, k\}$, where $f(I)$ is the maximum value of i for which $|I \cap A_i| = \infty$. Since every A_i is an independent set of $\Gamma(R)$, it is not hard to check that f is a proper coloring of $\mathbb{AG}(R)$. □

3. Bipartite annihilating-ideal graphs

This section is devoted to the study of rings whose the chromatic numbers of their annihilating-ideal graphs is two. For this aim, we start with the following theorem characterizing non-reduced rings with bipartite annihilating-ideal graphs.

Theorem 20. *Let R be a non-reduced ring and $\mathbb{AG}(R)$ be a triangle-free graph. Then $\mathbb{AG}(R)$ is bipartite and R has at most two minimal ideals. Moreover, one of the following statements holds.*

(i) *R has exactly two minimal ideals and $R \cong F \times S$, where F is a field and S is a ring with exactly one non-trivial ideal.*
(ii) *R has exactly one minimal ideal, say (x), $(x)^2 = (0)$ and $V_1 = N((x))$ and V_1^c are the parts of $\mathbb{AG}(R)$. Furthermore, for $A = \{L \in V_1 \mid (x) \subseteq L\}$, the induced subgraph on $V_1^c \cup (V_1 \setminus A)$ is a complete bipartite graph and $d(L) = 1$, for every $L \in A$.*

Proof. Since R is non-reduced, there exists $0 \neq z \in R$ such that $z^2 = 0$. By hypothesis, (z) contains at most one non-trivial ideal of R. Thus we can assume that R has at least one minimal ideal, say (x), with $x^2 = 0$. Also, R has at most two minimal ideals. Two cases can be considered:

Case 1. R has exactly two minimal ideals (x) and J. Then we show that $J^2 \neq (0)$. On the contrary suppose that $J^2 = (0)$. Then J, (x) and $J + (x)$ form a triangle, a contradiction. Thus $J^2 \neq (0)$ and hence by Brauer's Lemma (see [10, 10.22]), there exists an idempotent element $e \in R$ such that $J = Re$ and $R \cong Re \times R(1 - e)$. Since R is non-reduced, Corollary 14 implies that $R \cong D \times S$, where D is an integral domain and S is a ring with exactly one non-trivial ideal. Since R has two minimal ideals (x) and J, D is a field (Note that every integral domain which is not a field has no minimal ideal). Thus Part (i) is proved.

Case 2. Assume that (x) is the unique minimal ideal of R. Let $V_1 = N((x))$, $V_2 = V_1^c$, $A = \{L \in V_1 \mid (x) \subseteq L\}$, $B = V_1 \setminus A$ and $C = V_2 \setminus \{(x)\}$. We show that $\mathbb{AG}(R)$ is a bipartite graph with parts V_1 and V_2 and the induced subgraph on $B \cup V_2$ is a complete bipartite graph. Since $\mathbb{AG}(R)$ is triangle-free, we deduce that V_1 is an independent set. We claim that one end of every edge of $\mathbb{AG}(R)$ is adjacent to (x) and another end contains (x). To see this, suppose that $\{I, J\}$ is an edge of $\mathbb{AG}(R)$ and $(x) \neq I$, $(x) \neq J$. Since $I(x) \subseteq (x)$ and (x) is a minimal ideal of R, either $I(x) = (0)$ or $(x) \subseteq I$. The latter case implies that $J(x) = (0)$. If $I(x) = (0)$, then $J(x) \neq (0)$ and hence $(x) \subseteq J$. Therefore, the claim is proved.

This implies that V_2 is an independent set and $\mathbb{AG}(R)$ is a bipartite graph with the parts V_1 and V_2. Since every vertex of A contains (x) and $\mathbb{AG}(R)$ is triangle-free, we deduce that $d(L) = 1$, for every $L \in A$. So, $N(C) \subseteq B$. Since V_2 is an independent set and (x) is a minimal ideal of R, we deduce that every vertex of C contains (x) and so every vertex of $A \cup V_2$ contains (x). Now, we show that the induced subgraph of $\mathbb{AG}(R)$ on $B \cup V_2$ is a complete bipartite graph. Let $I \in V_2$ and $J \in B$. If $IJ \neq (0)$, then IJ is a vertex of $\mathbb{AG}(R)$. Since every vertex of $A \cup V_2$ contains (x) and $(x) \nsubseteq J$, we conclude that $IJ \in B$. Since every vertex of A is just adjacent to (x) and by Theorem B, $\mathbb{AG}(R)$ is connected, I has a neighbor in B, say K. If $K \neq IJ$, then K and IJ are two adjacent vertices in B, a contradiction. So suppose that $K = IJ$. Since (x) is the unique minimal ideal of R, there exists an ideal L properly contained in $K = IJ$ and $L \in B$. It is clear that L is adjacent to IJ, a contradiction. Therefore, $IJ = (0)$. This proves (ii). □

Example 21. Let $R = D \times S$, where D is an integral domain and S is a ring with exactly one non-trivial ideal. Then $\mathbb{AG}(R)$ is a bipartite graph, satisfying Part (ii) of the previous theorem.

Remark 22. For a reduced ring R, if $\mathbb{AG}(R)$ is a triangle-free graph, then it is a complete bipartite graph. To see this, by Theorem 6, $|\mathrm{Min}(R)| \leq 2$. If $|\mathrm{Min}(R)| = 1$, then R is an integral domain. Thus one can assume that $\mathrm{Min}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. In this case, $\mathbb{AG}(R)$ is a complete bipartite graph with the parts $X = \{I \mid I \in \mathbb{A}(R)^*,\ I \subseteq \mathfrak{p}_1\}$ and $Y = \{I \mid I \in \mathbb{A}(R)^*,\ I \subseteq \mathfrak{p}_2\}$.

Now, using Theorem 20 and Corollary 14, we provide a simple proof for [2, Theorem 2].

Corollary 23. *Let R be a ring. Then $\mathbb{AG}(R)$ is a tree if and only if $\mathbb{AG}(R)$ is isomorphic to either a star graph or P_4.*

Proof. First suppose that $\mathbb{AG}(R)$ is a tree. If R is a reduced ring, then Remark 22 implies that $\mathbb{AG}(R)$ is a star graph. Hence suppose that R is a non-reduced ring. If the Part (i) of Theorem 20 occurs, then $\mathbb{AG}(R) \cong P_4$. Thus by Theorem 20, Part (ii) one may assume that $B = V_1 \setminus A \neq \varnothing$ and $C = V_1^c \setminus \{(x)\} \neq \varnothing$. Since $\mathbb{AG}(R)$ is tree, by Part (ii) of Theorem 20, we deduce that $B = \{J\}$. One can easily check that J is a minimal ideal. If $J^2 = (0)$, then J, (x) and $J + (x)$ form a triangle, a contradiction. Thus, by Brauer's Lemma and Corollary 14, $\mathbb{AG}(R)$ is isomorphic to P_4. The converse is trivial. □

Corollary 24. *Let R be a ring. Then $\mathbb{AG}(R)$ is a complete bipartite graph if and only if $\mathbb{AG}(R)$ is a star graph or R is a reduced ring and* $|\mathrm{Min}(R)| = 2$.

We note that empty graphs and the isolated vertex graphs are bipartite graphs. If R is a ring and $|V(\mathbb{A}G(R))| \geq 2$, then we obtain the following corollary immediately, which is a special case of [11, Theorem 2.1]. This result was also proved in [1] (see Theorem 13) by a different proof.

Corollary 25. *Let R be a ring. Then $\mathbb{AG}(R)$ is bipartite if and only if $\mathbb{AG}(R)$ is triangle-free.*

The next corollary in some sense is a generalization of Theorem 3 in [2].

Corollary 26. *Let R be a ring.*

(i) *If R is reduced, then $\mathbb{AG}(R)$ is a star graph if and only if $R \cong F \times D$, where F is a field and D is an integral domain.*
(ii) *If R is non-reduced, then $\mathbb{AG}(R)$ is a star graph if and only if $\mathbb{AG}(R)$ is a bipartite graph and $Z(R) = Ann(x)$, for some $x \in R$.*

Proof. (i) Suppose that $\mathbb{AG}(R)$ is a star graph with center I. Since R is reduced, I is a minimal ideal of R. Thus by Brauer's Lemma, R is decomposable and so by Corollary 14, $R \cong F \times D$, where F is a field and D is an integral domain. The converse is trivial.

(ii) Suppose that $\mathbb{AG}(R)$ is a star graph. By Theorem 20, Part (ii), $V_1^c = \{(x)\}$ and so (x) is the center of $\mathbb{AG}(R)$. Therefore, $Z(R) = Ann(x)$. The converse is clear. □

In the next theorem, we obtain some properties of rings whose annihilating-ideal graphs are bipartite.

Theorem 27. *Let $\mathbb{AG}(R)$ be a bipartite graph. Then one of the following holds.*

(i) $\mathbb{AG}(R)$ *is a star graph.*
(ii) $\mathbb{AG}(R) \cong P_4$.
(iii) $Nil(R) = Soc(R)$.

Proof. First suppose that R is reduced. If R has no minimal ideal, then $Soc(R) = (0) = Nil(R)$. Thus one may assume that R has a minimal ideal, say I, and $I^2 \neq (0)$. Since $\mathbb{AG}(R)$ is bipartite, Brauer's Lemma and Corollary 14 imply that $R \cong D_1 \times D_2$, where D_1 and D_2 are integral domains. But R has at least one minimal ideal. So, at least one of the integral domains D_1 and D_2, say D_1, is a field. Thus $\mathbb{AG}(R)$ is a star graph. Now, suppose that R is a non-reduced ring. By Theorem 20, we have two cases. If (i) happens, then $R \cong F \times S$, where F is a field and S is a ring with exactly one non-trivial ideal and $\mathbb{AG}(R) \cong P_4$. So, let (ii) happen. If $Nil(R) = (x)$, then $Nil(R) = Soc(R)$. Therefore, one can assume that $Nil(R) \neq (x)$. Since $Nil(R) \subseteq J(R) \subseteq Ann(x)$, we deduce that $(x)Nil(R) = (0)$ and so $Nil(R) \in V_1$. Let $K \in V_1 \setminus A$. Then $K \cap Nil(R)$ is a vertex of $\mathbb{AG}(R)$ not containing (x). Since $\mathbb{AG}(R)$ is triangle-free, we can find an element $z \in K \cap Nil(R)$ such that (z) is a minimal ideal of R and $(z)^2 = (0)$. Since (x) is the unique minimal ideal of R, we conclude that $(x) = (z) \subseteq K \cap Nil(R)$, a contradiction. Therefore, every vertex of V_1 contains (x). So by Theorem 20, Part (ii), $\mathbb{AG}(R)$ is a star graph with the center (x) and the proof is complete. □

We close this paper with the following theorem.

Theorem 28. *Let R be a ring and $\mathbb{AG}(R)$ be a bipartite graph. If $|\mathrm{Min}(R)| = 1$, then $\mathbb{AG}(R)$ is a star graph.*

Proof. Let $\mathbb{AG}(R)$ be a bipartite graph and $\mathfrak{p}$ be the unique minimal prime ideal of R. On the contrary suppose that $\mathbb{AG}(R)$ is not a star. By Theorem 27, $Soc(R) = Nil(R)$ and so Theorem 20, Part (ii) implies that $(x) = \mathfrak{p}$. Hence $\mathbb{AG}(R)$ is a star graph with the center $\mathfrak{p} = (x)$. □

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