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A Supporting Hyperplane Derivation of the Hamilton–Jacobi–Bellman Equation of Dynamic Programming*

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A wide class of nonlinear relaxed optimal control problems are shown to be equivalent to convex optimization problems over a linear space of boundaries. This space provides a framework in which dynamic programming optimality conditions can be interpreted as hyperplane sypport results, in line with the majority of necessary conditions in optimization.

1. Introduction

In a previous paper [1], a new necessary condition for optimality applicable to a wide class of optimal control problems was given. This was based upon the equivalence of the control problem with a convex problem over a set of measures [2] and resulted from a study of the Fenchel dual of the new program. The statement of the new necessary condition is a weakened version of that of a well-known verification or sufficient condition, the continuously differentiable verification function being replaced by a sequence of such functions. Not surprisingly we cannot in general demonstrate the convergence of the sequence, except along the optimal trajectory.

Here, using a different approach, we obtain convergence of the sequence for a restricted class of problems. A parametric version of the equivalent convex program is rewritten as a convex optimization problem over a linear space of boundaries and optimality is characterized by a supporting hyperplane result involving continuous linear functionals on the space of boundaries. A representation theory for the linear functionals leads us to the main result. It should be emphasized that the existence of the hyperplane support depends upon a restrictive sensitivity hypothesis being imposed upon the original control problem and that therefore the results are not much different from those given in [3].

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The contribution of this paper is the novel derivation and interpretation of the necessary conditions. It can reasonably be asserted that all first-order necessary conditions in optimization have their origins in the hyperplane support of a convex set. Here we offer the first such interpretation for dynamic programming like necessary conditions in optimal control problems.

2. Problem Formulation and Existing Results

We consider optimal control problems slightly less general than those in [4], in that variable terminal points are admitted but the initial point is fixed. Otherwise notation and terminology are as in [4].

The original, or "strong," optimal control problem is

$$(\mathrm{S})\begin{cases} \text{Minimize } \int_{t_0}^{t_1}\int_\Omega l(x(t), t, u)\, d\mu_t(u)\, dt & (2.1)\\ \text{subject to} & \\ \dot{x}(t) = \int_\Omega f(x(t), t, u)\, d\mu_t(u) \quad \text{a.e.} \quad t \in [t_0, t_1] & (2.2)\\ \text{where} \quad \mu_{(\cdot)}: [t_0, t_1] \to P^n(\Omega) \quad \text{is a relaxed control} & (2.3)\\ \text{and} \quad x(t_0) = x_0, \quad (x(t_1), t_1) \in \Gamma_1. & (2.4) \end{cases}$$

Here $\Omega \subset \mathbb{R}^m$, the control constraint set, and $\Gamma_1 \subset \mathbb{R}^{n+1}$, the target (or terminal) set, are assumed compact and $(x_0, t_0) \notin \Gamma_1$. $l: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ and $f: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^n$ are continuous functions. The hypothesis H of [4] permits us to restrict our attention to trajectories $(x(\cdot), \cdot)$ such that $(x(t), t) \in A \subset \mathbb{R}^{n+1}$ for all $t \in [t_0, t_1]$ where A is some $n + 1$ cube (determination of t_1 is implicitly part of the problem of solving (S)) and to assert that (S) has a solution.

The central result of [4] (see also [2]) is that (S) is equivalent to a "weak" control problem (W), formulated over a set of Radon measures, in the sense that their minima, or values, are the same, $\xi(W) = \xi(S)$ [4, Theorem (2.1)].

$$(\mathrm{W})\begin{cases} \text{Minimize } \int_{A\times\Omega} l(x, t, u)\, d\mu & (2.5)\\ \text{subject to} \quad \mu \in P^{\oplus}(A \times \Omega) \quad \text{and the existence of} & (2.6)\\ \left.\begin{array}{l} \beta_1 \in P^n(\Gamma_1) \quad \text{such that for all } \phi \in C^1(A) \\ \int_{A\times\Omega} \phi_t + \phi_x f\, d\mu = \int_{\Gamma_1} \phi\, d\beta_1 - \phi(t_0, t_0). \end{array}\right\} & (2.7) \end{cases}$$

$P^{\oplus}(A \times \Omega)$ denotes the positive polar cone in $C^*(A \times \Omega)$, the dual space of $C(A \times \Omega)$, and $P^n(\Gamma_1)$ denotes the set of probability measures on Γ_1. In (W), the dynamic constraints (2.2) and (2.3) and the boundary constraints (2.4) are imposed weakly by (2.6) and (2.7). Note that (W) is convex, since the set of feasible elements $\mathscr{W} = \{\mu \in P^{\oplus}(A \times \Omega): (2.7)$ holds for some $\beta_1 \in P^n(\Gamma_1)\}$ is convex and (2.5) is linear in μ.

As a corollary to equivalence we obtain [4, Corollary (2.2)].

PROPOSITION 2.8. *Every element $\mu \in \mathscr{W}$ is a unit positive mixture of elements feasible for* (S), $\mathscr{S}$, *that is, to every μ there corresponds a $\Lambda \in P^n(\mathscr{S})$ such that $\mu = \int_{\mathscr{S}} s \, d\Lambda(s)$.*

$\mathscr{S}$ is of course the set of control–trajectory pairs $\{\mu_t, x(t): t_0 \leqslant t \leqslant t_1\}$ satisfying (2.2)–(2.4).

Equivalence and its corollary are proved indirectly by introducing parametric versions of (S) and (W). Observe that in these problems the variables x and t do not have the same status, for in (S) x is parametrized by t which is itself nonparametric. Changing the parameter to σ, i.e., writing $x(\sigma)$ and $t(\sigma)$ we must replace d/dt by $(1/\dot{t}(\sigma))d/d\sigma$ and $\int_{\dots} dt$ by $\int_{\dots} | \dot{t}(\sigma)| \, d\sigma$ where $\dot{t}(\sigma) = dt(\sigma)/d\sigma$. The motivation for this is that in the parametric framework the set of boundaries is a linear space, [5, 6]; viz., if (x_0, t_0) and (x_1, t_1) are the initial and terminal points of a parametric curve $\{(x(\sigma), t(\sigma)): 0 \leqslant \sigma \leqslant 1\}$, then (x_1, t_1) and (x_0, t_0) are the initial and terminal points of the reversed curve with parameter $\rho = 1 - \sigma$: in the nonparametric framework this reversal is not possible and we do not have a linear space of boundaries. This linear space structure is vital to the proof in [6] and necessitates the indirectness of the equivalence proofs in [2, 4]. Here we turn the indirectness to our advantage and use the space of boundaries to derive necessary conditions for optimality in the "strong" control problem.

The parametric version of the weak problem is

$$
\text{(P)} \begin{cases}
\text{Minimize} \int_{A \times B} L(y, \dot{y}) \, d\mu & (2.9) \\
\text{subject to} \quad \mu \in P^{\oplus}(A \times B) & (2.10) \\
\int_{A \times B} M \, d\mu = \int_{A \times B} D \, d\mu = 0 \quad \text{and the existence of} & (2.11) \\
\left.\begin{array}{l} \beta_1 \in P^n(\Gamma_1) \quad \text{such that for all } \phi \in C^1(A) \\ \int_{A \times B} \phi_y \, \dot{y} \, d\mu = \int_{\Gamma_1} \phi \, d\beta_1 - \phi(x_0, t_0). \end{array}\right\} & (2.12)
\end{cases}
$$

A is as before, B is the unit sphere (surface of the unit ball) in $\mathbb{R}^{n+1}$, and $y = (x, t)$ and $\dot{y} = (\dot{x}, \dot{t})$ denote points in $\mathbb{R}^{n+1}$. The functions L, M, and D are defined by (see [2, 4])

$$
\begin{aligned}
L(y, \dot{y}) &= \underline{l}(x, t, \dot{x}/\dot{t}) \, | \dot{t} \, |, && \dot{t} \neq 0 \\
&= 0, && \dot{t} = 0, \\
M(y, \dot{y}) &= \max[-\dot{t}, 0],
\end{aligned}
$$

$$D(y, y) = \operatorname{dist}(\dot{x}/\dot{t}, f(x, t, \Omega)) \mid \dot{t} \mid, \qquad \dot{t} \neq 0$$
$$= \| \dot{x} \|, \qquad \dot{t} = 0,$$
$$\underline{l}(x, t, z) = \min\{l(x, t, u) \colon z = f(x, t, u), u \in \Omega\}.$$

The method of proof of equivalence in [4] is to use the fact established in [6], that elements $\mu \in P^{\oplus}(A \times B)$ feasible for (P), known as generalized flows, are unit positive mixtures of generalized curves feasible for (P). This is the parametric equivalent of Proposition 2.8. Therefore, (P) has a solution which is a generalized curve and the proof is completed by showing that there is a one-to-one mapping of these generalized curves onto the set $\mathscr{S}$ of control-trajectory pairs feasible for (S). Denoting the set of elements feasible for (P) by $\mathscr{P}$ and the subset comprised of generalized curves by $\mathscr{G}$ we have ($\simeq$ denotes isomorphism)

$$\begin{array}{ccc} \mathscr{P} & = \overline{\text{co}} & \mathscr{G} \\ \simeq & & \simeq \\ \mathscr{W} & = \overline{\text{co}} & \mathscr{S}. \end{array} \tag{2.13}$$

An extension of (2.13) is used in the following section.

3. A Convex Problem over a Space of Boundaries

In the terminology of [4–6], elements $\mu \in P^{\oplus}(A \times B)$ are known as *generalized flows* and the functions $h \in C(A \times B)$ which they operate on are called *integrands*. The restriction of a generalized flow μ to exact integrands $\phi_y \dot{y}$, where $\phi \in C^1(A)$, is called the *boundary* of μ and is denoted by $\partial\mu$. The *norm* of a boundary ρ is given by $| \rho | = \inf\{\| \bar{\mu} \| \colon \partial\bar{\mu} = \rho\}$ where $\| \cdot \|$ is the strong norm in $C^*(A \times B)$. A generalized flow μ has a finite boundary if there exist $\beta_0, \beta_1 \in P^{\oplus}(A)$ with $\| \beta_0 \| = \| \beta_1 \|$ such that for all $\phi \in C^1(A)$

$$\partial\mu(\phi_y \dot{y}) = \int_{A \times B} \phi_y \dot{y} \, d\mu = \int_A \phi \, d\beta_1 - \int_A \phi \, d\beta_0 . \tag{3.1}$$

We denote a finite boundary by $\{\beta_1 - \beta_0\}$ and agree that β_0 and β_1 should always be the unique pair of minimum norm, guaranteed by the Hahn–Jordan decomposition theorem, such that (3.1) is satisfied. Automatically then $\beta_0(\operatorname{supp}\{\beta_1\}) = \beta_1(\operatorname{supp}\{\beta_0\}) = 0$. The *modulus* of a finite boundary $\rho = \{\beta_1 - \beta_0\}$ is defined by

$$\operatorname{mod}(\rho) = \| \beta_1 \| = \| \beta_0 \| . \tag{3.2}$$

This is unrelated to the norm, $| \rho |$.

A *simplicial* boundary is a finite boundary for which both the measures β_0 and

β_1 have supports in a finite number of points in A. The set of boundaries of interest is the set Σ, the completion of the set of simplicial boundaries in the boundary norm. It is shown in [5] that with the boundary norm, Σ is a complete normed linear vector space. Σ contains all finite boundaries (see [6]) and other nonfinite boundaries [5].

Remarks. The generalized flows admissible for (P) have finite boundaries $\{\beta_1 - \delta(x_0, t_0)\}$ where $\beta_1 \in P^n(A)$ and $\delta(x_0, t_0)$ is the unit measure concentrated at (x_0, t_0). The boundaries all have unit moduli.

For any $0 \leqslant a < \infty$ that subset of Σ of finite boundaries of moduli less than or equal to a will be denoted by Σ_a. The following lemma is proved in Appendix I.

LEMMA 3.3. *For each fixed* $0 \leqslant a < \infty$, Σ_a *is sequentially compact.*

Now take L, M, and D as defined for the parametric problem and let $\mathscr{C}$ be the set of generalized flows given by

$$\mathscr{C} \triangleq \left\{\mu \in P^{\oplus}(A \times B): \int D\, d\mu = \int M\, d\mu = 0 \text{ and } \partial\xi \in \Sigma_2\right\}. \tag{3.4}$$

Define

$$\Sigma_{\mathscr{C}} \triangleq \{\rho \in \Sigma: \rho = \partial\mu \text{ for some } \mu \in \mathscr{C}\}. \tag{3.5}$$

Clearly $\Sigma_{\mathscr{C}}$ is convex and is the intersection of Σ_2 with the cone $\{\rho \in \Sigma: \rho = \partial\mu$ for some μ such that $\int D\, d\mu = \int M\, d\mu = 0\}$. Further, for each $\rho \in \Sigma_{\mathscr{C}}$, the set of $\mu \in \mathscr{C}$ such that $\partial\mu = \rho$ is weak-star compact (the proof is the same as that of proposition (3.11) in [4]) and L is lower-semicontinuous on $A \times B$ ([4, Lemma (3.2) and definition (3.4)]). On $\Sigma_{\mathscr{C}}$ therefore the real-valued function q given below is well defined and finite.

$$q(\rho) \triangleq \min \left\{\int L\, d\mu: \mu \in \mathscr{C}, \partial\mu = \rho\right\}. \tag{3.6}$$

LEMMA 3.7. *q is convex on $\Sigma_{\mathscr{C}}$.*

Proof. For $\rho_1, \rho_2 \in \Sigma_{\mathscr{C}}$ let $\mu_1, \mu_2 \in \mathscr{C}$ give the values of q at ρ_1 and ρ_2 respectively, as in (3.6). For any $0 < \alpha < 1$ $\mu \triangleq \alpha\mu_1 + (1 - \alpha)\mu_2 \in \mathscr{C}$ and $\partial\mu = \alpha\rho_1 + (1 - \alpha)\rho_2$ so that

$$\begin{aligned} q(\alpha\rho_1 + (1-\alpha)\rho_2) &\leqslant \int L\, d\mu \\ &= \alpha \int L\, d\mu_1 + (1-\alpha)\int L\, d\mu_2 \\ &= \alpha q(\rho_1) + (1-\alpha)\, q(\rho_2). \qquad \blacksquare \end{aligned}$$

q is homogeneous on $\Sigma_{\mathscr{C}}$ in the following sense. If ρ and $\alpha\rho$ both belong to $\Sigma_{\mathscr{C}}$ for any $\alpha \geqslant 0$, then $\partial(\alpha\rho) = \alpha q(\rho)$, since by definition if μ, $\bar{\mu}$ give q at ρ and $\alpha\rho$, respectively then $\partial(\alpha\mu) = \alpha\rho$ and $\partial([1/\alpha]\,\bar{\mu}) = \rho$, assuming $\alpha > 0$, hence $q(\alpha\rho) \leqslant \alpha q(\rho)$ and $q(\rho) \leqslant [1/\alpha]\, q(\alpha\rho)$. When $\alpha = 0$, $q(\theta) = 0 = \alpha q(\rho)$ for any $\rho \in \Sigma_{\mathscr{C}}$.

Note. θ is the zero or closed boundary. That $q(\theta) = 0$ follows from the tact that $\mu \in \mathscr{C}$ and $\partial\mu = \theta$ implies $\mu = 0$, the trivial generalized flow. ([4, Theorem (3.14) *et seq.*]).

To generate the new necessary condition we need to extend q to a function convex and continuous on Σ. For this we make the following sensitivity hypothesis.

HYPOTHESIS 3.8. *For f and l as in the strong control problem* (S), *if* $\eta(t_0, x_0, t_1, x_1) \triangleq \min \int_{t_0}^{t_1} \int_\Omega l(x(t), t, u)\, d\mu_t(u)\, dt$, *over relaxed control–trajectory pairs* $\{\mu_t, x(t): t_0 \leqslant t \leqslant t_1\}$ *satisfying* (2.2) *and* (2.3) *and* $x(t_0) = x_0$, $x(t_1) = x_1$, *then we assume that η is Lipschitz continuous in* $(t_0, x_0, t_1, x_1) \in \mathbb{R}^{2n+2}$ *on its domain of definition.*

Remarks. (i) The assumption of Lipschitz continuity of the value function η is slightly weaker than the assumption of the existence of directional derivatives of η, made by Janin in [3].

(ii) Examples involving discontinuous η are known [7]. Sufficiency criteria for such problems are developed in [8].

Now recall diagram (2.13). Between control–trajectory pairs $\{\mu_t, x(t): t_0 \leqslant t \leqslant t_1\}$ satisfying (2.2) and (2.3) and $x(t_0) = x_0$, $x(t_1) = x_1$ and generalized curves $\mu \in \mathscr{C}$ (i.e., satisfying $\int M\, d\mu = \int D\, d\mu = 0$) with endpoints (x_0, t_0) and (x_1, t_1), there is exactly the same one-to-one relationship as between $\mathscr{S}$ and $\mathscr{G}$ in (2.13). (Here we allow the endpoints to be outside the boundary constraints of (S) but the dynamic constraints still hold.)

Similarly, any generalized flow $\bar{\mu} \in \mathscr{C}$ is a mixture of curves in $\mathscr{C}$, of weight $\operatorname{mod}(\partial\bar{\mu})$. (The flows feasible for (P) are unit mixtures, i.e., of weight unity, precisely because their boundaries have unit moduli.)

These considerations mean that Lipshitz continuity of η with respect to the endpoints of control–trajectory pairs is lifted to Lipschitz continuity of $\min \int L\, d\bar{\mu}$ with respect to $\partial\bar{\mu}$, for $\bar{\mu} \in \mathscr{C}$. Thus Hypothesis (3.8) ensures that the convex function q is Lipschitz continuous on $\Sigma_{\mathscr{C}}$, with respect to the boundary norm topology.

DEFINITION 3.9. For any boundary $\rho \in \Sigma$ set

$$\hat{q}(\rho) = \min[k \mid \rho - \bar{\rho} \mid + q(\bar{\rho}): \bar{\rho} \in \Sigma_{\mathscr{C}}],$$

where k is the Lipschitz constant of q on $\Sigma_{\mathscr{C}}$.

THEOREM 3.10. *$\hat{q}$ agrees with q on $\Sigma_{\mathscr{C}}$ and is continuous and convex on Σ.*

Proof. First note that as $\Sigma_{\mathscr{C}}$ is a sequentially closed subset of Σ_2, by Lemma (3.3), for each $\rho \in \Sigma$ there exists a $\bar{\rho} \in \Sigma_{\mathscr{C}}$ at which the minimum in (3.9) is attained.

Suppose $\rho \in \Sigma_{\mathscr{C}}$ but that $\hat{q}(\rho) \neq q(\rho)$. Then

$$\begin{aligned} \hat{q}(\rho) &= k \mid \rho - \bar{\rho} \mid + q(\bar{\rho}) \qquad \text{for some } \bar{\rho} \in \Sigma_{\mathscr{C}} \\ &< k \mid \rho - \rho \mid + q(\rho). \end{aligned}$$

However, $q(\rho) - q(\bar{\rho}) > k \mid \rho - \bar{\rho} \mid$ contradicts the Lipshitz property of q. The restriction of $\hat{q}$ to $\Sigma_{\mathscr{C}}$ is therefore q.

If $\hat{q}$ is discontinuous at ρ, there is a fixed $c > 0$ such that for all $\epsilon > 0$ there is a ρ_1 with $\mid \rho - \rho_1 \mid < \epsilon$ and $\mid \hat{q}(\rho) - \hat{q}(\rho_1) \mid \geqslant c$. Take $\bar{\rho}$ and $\bar{\rho}_1$ corresponding to ρ and ρ_1 as in (3.9) so that

$$\begin{aligned} \hat{q}(\rho_1) &= k \mid \rho_1 - \bar{\rho}_1 \mid + q(\bar{\rho}_1) \\ &\leqslant k \mid \rho_1 - \bar{\rho}_1 \mid + q(\bar{\rho}) \\ &\leqslant k \mid \rho_1 - \rho \mid + k \mid \rho - \bar{\rho} \mid + q(\bar{\rho}) \\ &< k\epsilon + \hat{q}(\rho).: \end{aligned}$$

Similarly $\hat{q}(\rho) \leqslant k\epsilon + \hat{q}(\bar{\rho})$, a contradiction if $\epsilon < c/k$.

Convexity of $\hat{q}$ is an immediate consequence of its definition and convexity of $\mid \cdot \mid$ and $q(\cdot)$ on $\Sigma_{\mathscr{C}}$.

$\hat{q}$ is the desired extension of q to Σ. ▮

4. THE NEW NECESSARY CONDITION

The following theorem is proved in Appendix II.

THEOREM 4.1. *$F: \Sigma \to \mathbb{R}$ is a continuous linear functional on Σ if and only if there exists a sequence $\{\psi^i\} \subset C^1(A)$ converging to a Lipshitz continuous function ψ, such that for $\rho \in \Sigma$.*

$$F(\rho) = \lim_{i \to \infty} \int \psi_y{}^i \dot{y} \, d\mu \tag{4.2}$$

for any generalized flow μ with $\partial\mu = \rho$. Further, if k is the Lipshitz constant of ψ, k_i that of ψ^i, then we require $\{k_i\}$ to be bounded and $k^i \to k$.

Consider now the epigraph of $\hat{q}$ in $\Sigma \times \mathbb{R}$,

$$\text{Epi}[\hat{q}] = \{(\rho, r) \in \Sigma \times \mathbb{R}: r \geqslant \hat{q}(\rho)\}.$$

On $\Sigma \times \mathbb{R}$ take the topology generated by the norm $\|(\rho, r) - (\sigma, s)\| = |\rho - \sigma| + |r - s|$. From (3.10) it follows that Epi$[\hat{q}]$ is convex and has nonempty interior in $\Sigma \times \mathbb{R}$. There is therefore a nonzero tangent hyperplane supporting Epi$[\hat{q}]$ at each of its boundary points $(\rho, \hat{q}(\rho))$ [9]. More precisely, for every $\rho \in \Sigma$ there is a continuous linear functional $F \in \Sigma^*$ and real numbers a and b such that

$$\begin{aligned} F(\rho) + a\hat{q}(\rho) &= b, \\ F(\sigma) + as &\leqslant b \qquad \text{for all } (\sigma, s) \in \text{Epi}[\hat{q}]. \end{aligned} \tag{4.3}$$

Suppose $a = 0$, then $F(\sigma) \leqslant b$ and $F(-\sigma) \leqslant b$ imply that $F(\sigma) = b$ for all $\sigma \in \Sigma$ and since $F(\theta) = 0$, that $b = 0$. However $F = 0$ contradicts the non-tradicts the nontriviality of the tangent hyperplane, so $a \neq 0$. If $a > 0$, choosing $s > q(\sigma)$, s sufficiently large, gives $F(\sigma) + as > b$; hence $a < 0$ and can be normalized to take the value -1.

Since $F(\theta) = \hat{q}(\theta) = 0$, $b \geqslant F(\theta) - \hat{q}(\theta) = 0$. Suppose ρ in (4.3) has modulus less than 2 and is in $\Sigma_{\mathscr{C}}$, as will be the case when we look at boundaries related to the control problem. In this case there is an $\alpha \in \mathbb{R}$, $\alpha > 1$, with $\alpha\rho \in \Sigma_{\mathscr{C}}$ and we have $\hat{q}(\alpha\rho) = q(\alpha\rho) = \alpha q(\rho) = \alpha\hat{q}(\rho)$ (recall the homogeneity property of q). So if $b > 0$,

$$\begin{aligned} F(\alpha\rho) - \hat{q}(\alpha\rho) &= \alpha F(\rho) - \alpha\hat{q}(\rho) \\ &= \alpha b > b, \end{aligned}$$

hence $b = 0$. For such ρ, (4.3) becomes

$$\begin{aligned} F(\rho) - q(\rho) &= 0, \\ F(\sigma) - s &\leqslant 0 \qquad \text{for all } (\sigma, s) \in \text{Epi}[\hat{q}]. \end{aligned} \tag{4.4}$$

We now choose such a ρ related to the optimal control problem. The target set for the control problem, Γ_1, is a compact subset of A. Let $y_1 = (x_1, t_1) \in \Gamma_1$, be the endpoint of the optimal trajectory and for any $\epsilon > 0$ choose an integer $J < \infty$ and $y_j \in \Gamma_1$, $j = 2,..., J$, such that $\Gamma_1 \subset \bigcup_{j=1}^{J} N(y_j, \epsilon)$ where $N(y, \epsilon)$ is the ϵ neighborhood of y. The initial point of every admissible trajectory for the control problem is $y_0 = (x_0, t_0)$; we define ρ by

$$\rho = \left\{ \sum_{j=1}^{J} (1/J)\, \delta(y_j) - \delta(y_0) \right\}. \tag{4.5}$$

As in [1] we assume that Γ_1 is reachable in the sense that every point $y \in \Gamma_1$ is the endpoint of an admissible trajectory. By the one-to-one relationship of these trajectories (and their associated relaxed controls) with generalized curves in $\mathscr{C}$ having the same endpoints, we see that ρ defined by (4.5) is in $\Sigma_{\mathscr{C}}$. As it has modulus unity, (4.4) is applicable at ρ.

Invoking the representation theorem (4.1), there is a sequence of functions $\{\psi^i\} \subset C^1(A)$ converging to a Lipschitz continuous ψ, with $F(\sigma) = \lim_i \int \psi_y{}^i \dot{y}\, d\mu$ for any μ with $\partial\mu = \sigma$. By definition of q, (4.4) becomes

$$\lim_i \int \psi_y{}^i \dot{y}\, d\mu - \int L\, d\mu \leqslant 0 \qquad \text{for all } \mu \in \mathscr{C},$$
$$\lim_i \int \psi_y{}^i \dot{y}\, d\bar{\mu} - \int L\, d\bar{\mu} = 0, \tag{4.6}$$

where $\bar{\mu} \in \mathscr{P} \subset \mathscr{C}$ satisfies $\partial\bar{\mu} = \rho$ and $q(\rho) = \int L\, d\bar{\mu}$. ($\mathscr{P}$ is the subset of $\mathscr{C}$ of elements feasible for the parametric problem (P.)

In particular, the first inequality in (4.6) holds for any generalized curve in $\mathscr{C}$, i.e., any curve satisfying (2.11). By virtue of the one-to-one relationship between these curves and control-trajectory pairs satisfying (2.2) and (2.3) we have for the arbitrary pair $\{\mu_t, x(t): \tau_0 \leqslant t \leqslant \tau_1\}$ with endpoints $x(\tau_0) = \eta_0$ and $x(\tau_1) = \eta_1$

$$\lim_{i\to\infty} \int_{\tau_0}^{\tau_1} \left\{\psi_t{}^i(x(t), t) + \int_\Omega [\psi_x{}^i(x(t)\, t)\, f(x(t), t, u) - l(x(t), t, u)]\, d\mu_t(u)\right\} dt.$$
$$\leqslant 0. \tag{4.7a}$$

Note. The only restriction on the endpoints is that (η_0, τ_0), $(\eta_1, \tau_1) \in A$ and that (η_1, τ_1) is reachable from (η_0, τ_0).

By definition $q(\rho) = \min\{\int L\, d\mu: \mu \in \mathscr{C}, \partial\mu = \rho\}$. By Theorem (4.2) of [6] minimum can be attained by a generalized flow $\mu = \sum_{j=1}^J (1/J)\, \mu_j$ where μ_j are generalized curves in $\mathscr{C}$ with initial and final points y_0 and $y,$, respectively, $j = 1,\ldots, J$. Therefore letting $\{\mu_t{}^j, x^j(t): t_0 \leqslant t \leqslant t_j\}$ be the pairs minimizing (2.1) subject to (2.2) and (2.3) and $x^j(t_0) = x_0$, $x^j(t_j) = x_j$ where $(x_j, t_j) = y_j$ are as in the definition of the boundary ρ, these pairs are in one-to-one correspondence with the generalized curves μ_j.

Note. The pairs exist because we have assumed that all the points y_j are reachable. The pair $\{\mu_t{}^1, x^1(t): t_0 \leqslant t \leqslant t_1\}$ is a solution to the "strong" optimal control problem.

The second inequality in (4.6) therefore yields

$$\lim_{i\to\infty}(1/J) \sum_{j=1}^J \left[\int_{t_0}^{t_j} \left\{\psi_t{}^i(x^j(t), t) + \int_\Omega [\psi_x{}^i(x^j(t)\, t)\, f(x^j(t), t, u)\right.\right.$$
$$\left.\left. - l(x^j(t), t, u)]\, d\mu_t{}^j(u)\right\} dt\right] = 0 \tag{4.7b}$$

By replacing ψ^i by $\psi^i - t\alpha^i$ for a suitable sequence $\{\alpha^i\}$ converging to zero, we can ensure that the inequality (4.7a) holds for all i and that the limit in (4.7b) is unaltered.

LEMMA 4.8. (a) *$\psi_t^i + \psi_x^i f - l \leqslant 0$ on $A \times \Omega$ for all i.*
(b) *$\psi^i \geqslant 0$ on Γ_1 for all i.*

Proof. (a) Assuming the contrary, at some point $(\bar{x}, \bar{t}, \bar{u}) \in A \times \Omega$, $\psi_t^i + \psi_x^i f - l = c > 0$. By continuity of the left-hand side, there exists $\epsilon > 0$ and an ϵ neighborhood of $(\bar{x}, \bar{t}, \bar{u})$, N say, in which $\psi_t^i + \psi_x^i f - l \geqslant c/2 > 0$. Since f is continuous, the trajectory starting from $\bar{x}$ at time $\bar{t}$ with control $\mu_t = \delta(\bar{u})$ (the ordinary control with constant value $\bar{u} \in \Omega$) remains in a neighborhood of $\bar{x}$ on $\bar{t} \leqslant t \leqslant \bar{t} + \eta$ for sufficiently small $\eta > 0$. That is, we can arrange that $(x(t), t, \bar{u}) \in N$ for $\bar{t} \leqslant t \leqslant \bar{t} + \eta$. The integral of $\psi_t^i + \psi_x^i f - l$ along this trajectory-control pair from $\bar{t}$ to $\bar{t} + \eta$ is greater than $c\eta/2$. The contradiction of (4.7a) establishes (a).

(b) By part (a) and (4.7b), for each $1 \leqslant j \leqslant J$

$$0 = \lim_i \int_{t_0}^{t_j} \left\{ \psi_t^i(x^j(t), t) + \int_\Omega [\psi_x^i(x^j(t), t) f(x^j(t), t, u) - l(x^j(t), t, u)] \, d\mu_t^j(u) \right\} dt$$

$$= \lim_i [\psi^i(x_j, t_j) - \psi^i(x_0, t_0)] - \int_{t_0}^{t_j} \int_\Omega l(x^j(t), t, u) \, d\mu_t^j(u) \, dt$$

$$= \psi(x_j, t_j) - \psi(x_0, t_0) - \int_{t_0}^{t_j} \int_\Omega l(x^j(t), t, u) \, d\mu_t^j(u) \, dt.$$

The trajectory-control pair given by $j = 1$ is the solution to the "strong" control problem, while those given by $j = 2, \ldots, J$ are admissible. So for $j = 2, \ldots, J$,

$$\psi(x_j, t_j) = \psi(x_0, t_0) + \int_{t_0}^{t_j} \int_\Omega l(x^j(t), t, n) \, d\mu_t^j(u) \, dt$$

$$\geqslant \psi(x_0, t_0) + \int_{t_0}^{t_1} \int_\Omega l(x^1(t), t, n) \, d\mu_t^1(u) \, dt$$

$$= \psi(x_1, t_1).$$

As the ϵ neighborhoods of the points (x_j, t_j) cover Γ_1,

$$\psi(x, t) \geqslant \psi(x_1, t_1) \qquad \text{for all } (x, t) \in \Gamma_1 .$$

The proof is completed by replacing ψ by $\psi - \psi(x_1, t_1)$ and ψ^i by $\psi^i - \psi^i(x_1, t_1) + \beta^i$ for some suitable sequence $\{\beta^i\}$ converging to zero, noting that the addition of constants does not affect (4.7a) and (4.7b) or part (a) above. ∎

Let us define the set Φ of differentiable functions by

$$\Phi = \{\phi \in C^1(A): \phi_t + \phi_x f - l \leqslant 0 \text{ on } A \times \Omega \text{ and } \phi \geqslant 0 \text{ on } \Gamma_1\}.$$

Lemma 4.8 leads to the following characterizations of the value of the "strong" control problem (S) and of its solutions.

THEOREM 4.9. $\eta(S) = \sup\{-\phi(x_0, t_0): \phi \in \Phi\}$ *and the supremum is given by a sequence converging to a Lipshitz function* ψ.

Proof. From Lemma 4.8

$$\begin{aligned}\eta(S) &= \int_{t_0}^{t_1}\int_\Omega l(x^1(t), t, u)\, d\mu_t^1(u)\, dt \\ &= \psi(x_1, t_1) - \psi(x_0, t_0) \\ &= -\psi(x_0, t_0) \\ &= \lim_{i\to\infty}[-\psi^i(x_0, t_0)]\end{aligned}$$

and each ψ^i belongs to Φ. For any other $\phi \in \Phi$,

$$\begin{aligned}\eta(S) &= \int_{t_0}^{t_1}\int_\Omega l(x^1(t), t, u)\, d\mu_t(u)\, dt \\ &\geqslant \int_{t_0}^{t_1}\left[\phi_t(x^1(t), t) + \int_\Omega \phi_x(x^1(t), t) f(x^1(t), t, u)\, d\mu_t(u)\right] dt \\ &= \phi(x_1, t_1) - \phi(x_0, t_0) \\ &\geqslant -\phi(x_0, t_0).\end{aligned}$$

THEOREM 4.10. *A necessary and sufficient condition for the admissible control-trajectory pair* $\{\mu_t, x(t): t_0 \leqslant t \leqslant \tau\}$ *to be optimal in the "strong" control problem* (S) *is the existence of a sequence* $\{\psi^i\} \subset \Phi$, *converging to a Lipshitz continuous* ψ, *with*

(i) $\lim_{i\to\infty}\{\psi_t^i(x(t), t) + \int_\Omega [\psi_x^i(x(t), t) f(x(t), t, u) - l(x(t), t, u)]d\mu_t(u)\} = 0$,

(ii) $\lim_{i\to\infty} \psi^i(x(\tau), \tau) = \psi(x(\tau), \tau) = 0$.

The limit in (i) *is the strong limit in* $L_1[t_0, \tau]$.

Proof. Necessity: take the sequence $\{\psi^i\}$ as in Theorem (4.9). Let $m^i(t)$ denote the nonpositive function in the braces on the left-hand side of (i). If the control-trajectory pair is optimal,

$$\begin{aligned} 0 &\geqslant \lim_i \int_{t_0}^{\tau} m^i(t)\, dt \\ &= \lim_i \psi^i(x(\tau), \tau) - \lim_i \psi^i(x_0, t_0) - \int_{t_0}^{\tau}\int_{\Omega} l(x(t), t, u)\, d\mu_t(u)\, dt \\ &= \lim_i \psi^i(x(\tau), \tau) + \eta(S) - \eta(S) \\ &\geqslant 0. \end{aligned}$$

So $0 = \lim_i \int_{t_0}^{\tau} m^i(t)\, dt = \int_{t_0}^{\tau} \lim_i m^i(t)\, dt$ as $m^i(t)$ is uniformly bounded and the nonpositivity of $m^i(t)$ gives (i). (ii) is clear from the above.

Sufficiency: this is an elementary extension of the well-known verification theorem [10]. ▮

We refer to Theorem (4.10) as a dynamic-programming-like necessary and sufficient condition for optimality because of the similarity of the statement

$$\begin{aligned} &\psi_t^i + \psi_x^i f - l \leqslant 0 \quad \text{on} \quad A \times \Omega, \qquad \psi^i \geqslant 0 \quad \text{on} \quad \Gamma_1 \\ &\text{with equality in the limit along the optimal solution} \\ &\text{and at the optimal endpoint, respectively,} \end{aligned} \tag{4.11}$$

with that of the Hamilton–Jacobi–Bellman equation of dynamic programming [10]. Sufficiency of such criteria is well known but necessity has previously been demonstrated only for problems where the optimal control satisfies certain restrictive assumptions [10] or where more restrictive sensitivity assumptions than Hypothesis (3.8) are used [3]. The fact that the above statements involve sequences instead of a single differentiable function is the price we pay for added generality.

Generality is not however the most important feature of these results. Their importance lies in the fact that Theorem (4.10) is derived from the existence of a hyperplane supporting the epigraph of a convex function related to the control problem. This interpretation, common to necessary conditions in optimization problems, was previously lacking in dynamic programming, earlier derivations being based upon the structure of the value function.

5. Extensions

As pointed out in the Introduction, Theorems (4.9) and (4.10) are stronger versions of theorems first stated in [1]. In [1] the sensitivity restriction (3.8) was not made, so control problems with discontinuous value functions were admitted, and in this case the statements of Theorems (4.9) and (4.10) involve sequences $\{\psi^i\} \subset \Phi$ which do not necessarily have Lipshitz continuous limits.

In the approach used here (which differs from that in [1]), Hypothesis (3.8) is necessary for the construction of the continuous convex functional $\hat{q}$. However, in the absence of Hypothesis (3.8) one could still construct a homogeneous, convex (but not continuous) functional $\bar{q}$ on Σ, similar to q. The general Hahn–Banach theorem ([9], p. 62]) then yields a linear functional supporting the epigraph of q at the desired point. In this case the support functional is not necessarily continuous—it is most likely that such linear functionals on Σ are characterized by sequences $\{\psi^i\} \subset C^1(A)$ which converge to a limit in a more general class of functions than that of Lipshitz continuous ones. If this is so we recover immediately the results of [1].

The foundations upon which Theorems (4.9) and (4.10) are built are the equivalences expressed by (2.13). These are valid for more general control problems than those studied here, for example problems with explicit state constraints: the limits of validity of (2.13) are discussed in [4]. Theorems (4.9) and (4.10) can be generalized to these problems.

Appendix I: Proof of Lemma (3.3)

That Σ_a is closed is evident. Denote the ball of radius a in $P^{\oplus}(A)$ by $P_a(A)$, then if $\{\rho_i\}$ is any sequence in Σ_a, $\rho_i = \{\beta_{1i} - \beta_{0i}\}$ where β_{0i}, $\beta_{1i} \in P_a(A)$. $P_a(A)$ is weak-star compact hence there is a w-$*$ convergent subsequence of $\{\beta_{0i}\}$, $\{\beta_{0i_j}\}$ and a w-$*$ convergent subsequence of $\{\beta_{1i_j}\}$; $\{\beta_{1i_k}\}$. Both $\{\beta_{0i_k}\}$ and $\{\beta_{1i_k}\}$ are w-$*$ convergent so (relabeling with index i) there are β_0, $\beta_1 \in P_a(A)$ such that $\beta_{0i} \to^{w^*} \beta_0$ and $\beta_{1i} \to^{w^*} \beta_1$ and $\rho \triangleq \{\beta_1, \beta_0\}$ is in Σ_a (see note after proof). We shall prove that $\rho_i \to \rho$ in boundary norm in Σ.

First, note that changing ρ_i to $\bar{\rho}_i$ below does not affect the limit. Let $\bar{\beta}_{0i} = (\|\beta_0\|/\|\beta_{0i}\|)\beta_{0i}$ and $\bar{\beta}_{1i} = (\|\beta_1\|/\|\beta_{1i}\|)\beta_{1i}$, then $\bar{\beta}_{0i} \to^{w^*} \beta_0$, $\bar{\beta}_{1i} \to^{w^*} \beta_1$ and if $\bar{\rho}_i \triangleq \{\bar{\beta}_{1i} - \bar{\beta}_{0i}\}$

$$\begin{aligned} |\rho_i - \bar{\rho}_i| &= |1 - \|\beta_0\|/\|\beta_{0i}\|| \, |\{\beta_{1i} - \beta_{0i}\}| \\ &\leqslant |1 - \|\beta_0\|/\|\beta_{0i}\|| \cdot K, \end{aligned}$$

where K is any upper bound on the norm of elements in Σ_a and can be taken to be $a \times \operatorname{diam}(A) < \infty$ as A is compact. So for $\epsilon > 0$, $|\rho_i - \bar{\rho}_i| < \epsilon$ for sufficiently large i.

$$\begin{aligned} |\bar{\rho}_i - \rho| &= |\{\bar{\beta}_{1i} - \bar{\beta}_{0i}\} - \{\beta_1 - \beta_0\}| \\ &= |\{\bar{\beta}_{1i} - \beta_1\} - \{\bar{\beta}_{0i} - \beta_0\}| \\ &\leqslant |\{\bar{\beta}_{1i} - \beta_1\}| + |\{\bar{\beta}_{0i} - \beta_0\}| \,. \end{aligned}$$

The proof is completed by showing that these two terms can be made arbitrarily small.

Consider $\{\bar{\beta}_{1i} - \beta_1\}$. First suppose that $\bar{\beta}_{1i}$ and β_1 have finite supports, i.e., for some $M < \infty$, $\beta_{1i} = \sum_{j=1}^{M} a_i^j \delta(y_i^j)$ and $\beta_1 = \sum_{j=1}^{M} b^j \delta(y^j)$ where $y_i^j, y^j \in A$, $a_i^j, b^j \geqslant 0$, $\sum_{j=1}^{M} a_i^j = \sum_{j=1}^{M} b^j = \|\beta_{1i}\| = \operatorname{mod}(\bar{\rho}_i)$ (by definition of $\bar{\beta}_{1i}$) and δ denotes the unit atomic measure. Then $\bar{\beta}_{1i} \to^{w^*} \beta_1$ implies that for each $j =$

$1,\ldots, M$ either $a_i^j \to 0$ as $i \to \infty$ or $y_i^j \to y^k$ for some $1 \leqslant k \leqslant M$. In the latter case let J_k denote the set of indices j such that $y_i^j \to y^k$ then $\lim_i \sum_{j\in J_k} a_i^j = b^k$ (see proof of Lemma 6.1 in [6]). Let s_i^j denote the segment from y^k to y_i^j for $j \in J_k$, $k = 1,\ldots, M$, then the generalized flow $\mu_i = \sum_{j=1}^M a_i^j s_i^j$ has boundary $\partial\mu_i = \{\bar{\beta}_{1i} - \beta_1\}$ and

$$\| \mu_i \| = \sum_{j=1}^{M} a_i^j \| S_i^j \| = \sum_{k=1}^{M} \sum_{j\in J_k} a_i^j \| y^k - y_i^j \| < \epsilon \qquad \text{for } i$$

sufficiently large. Thus when β_{1i} and β_1 have finite supports, $[\{\bar{\beta}_{1i} - \beta_1\}| \to 0$ as $i \to \infty$.

The general case of nonfinite support is tackled by approximation. In [6, Sect. 5] we showed how for any boundary, $\{\bar{\beta}_{1j} - \beta_1\}$ say, it is possible to construct sequences of measures $\{\bar{\beta}_{1i}^j\}$ and $\{\beta_1^j\}$ in $P_a(A)$ with finite supports, such that $\bar{\beta}_{1i}^j \to^{w^*} \beta_{1i}$, $\beta_1^j \to^{w^*} \beta_1$, $|\{\beta_1^j - \beta_1\}| \to 0$, and $|\{\bar{\beta}_{1i}^j - \bar{\beta}_{1i}\}| \to 0$ as $j \to \infty$. Further, since $\bar{\beta}_{1i} \to^{w^*} \beta_1$ it is evident from the constructions that for each j, $\bar{\beta}_{1i}^j \to^{w^*} \beta_1^j$ as $i \to \infty$. With ϵ as before let i_j denote the index corresponding to j such that $|\{\bar{\beta}_{1i_j}^j - \beta_1^j\}| < \epsilon$ which exists because the measures concerned have finite support. Finally, for large enough j, setting $i = i_j$,

$$\begin{aligned} |\{\bar{\beta}_{1i} - \beta_1\}| &\leqslant |\{\bar{\beta}_{1i} - \bar{\beta}_{1i}^j\}| + |\{\bar{\beta}_{1i}^j - \beta_1^j\}| + |\{\beta_1^j - \beta_1\}| \\ &< 3\epsilon. \end{aligned}$$

The same holds for $|\{\beta_0 - \bar{\beta}_{0i}\}|$ so that $|\bar{\rho} - \rho| < 6\epsilon$. This completes the proof.

Note. As written, $\rho = \{\beta_1 - \beta_0\}$ may not be in reduced Hahn–Jordan form. Neverless $\operatorname{mod}(\rho) \leqslant \lim_i \operatorname{mod}(\rho_i) \leqslant a$, hence $\rho \in \Sigma_a$.

Appendix II: Proof of Theorem (4.1)

(a) Suppose given a sequence $\{\psi^i\}$ as in the statement of (4.1), then F defined by (4.2) is certainly linear. For any $\rho_1, \rho_2 \in \Sigma$ with $|\rho_1 - \rho_2| < \epsilon$,

$$\begin{aligned} |F(\rho_1) - F(\rho_2)| &= |F(\rho_1 - \rho_2)| \\ &= \left| \lim_{i\to\infty} \int \psi_v^i \dot{y}\, d\mu \right|, \qquad \text{where} \qquad \partial\mu = \rho_1 - \rho_2 \\ &\leqslant \lim_{i\to\infty} \int \max_y \| \psi_y^i \| \, \| \dot{y} \| \, d\mu \\ &\leqslant \lim_{i\to\infty} \int K^i \, d\mu \qquad \text{as} \qquad \dot{y} \in B, \qquad \text{the unit sphere} \\ &= K \int d\mu = K \| \mu \| . \end{aligned}$$

By definition of the boundary norm $|\cdot|$, there exists μ with $\partial\mu = \rho_1 - \rho_2$ and $\|\mu\| < 2\epsilon$, hence $|F(\rho_1) - F(\rho_2)| < 2K\epsilon$, i.e., F is continuous.

(b) Now let F be a continuous linear functional on Σ and define

$$S(y) \triangleq F(\partial s)$$

where s is the segment from 0 to $y \in A$. Thus

$$S(y_1) - S(y_2) = F(\partial s_{1,2}) \leqslant \|F\| \, |\partial s_{1,2}| = \|F\| \, \|y_1 - y_2\|,$$

$s_{1,2}$ being the segment from y_2 to y_1. The reverse segment yields $S(y_2) - S(y_1) \leqslant \|F\| \, \|y_1 - y_2\|$ hence S is Lipshitz continuous on A. If ξ denotes Lebesgue measure on $\mathbb{R}^{n+1}$, then $\bar{S}$ defined by

$$\bar{S}(y) = \int_A S(y + \epsilon\xi)\, d\xi \tag{A1}$$

belongs to $C^1(A)$ for all $\epsilon > 0$ (cf. [5, Sect. 84]).

For any $\rho \in \Sigma$ there exists a sigma-polynomial flow g with $\partial g = \rho$ ([5, 89.2]). Let $g = \sum_{i=1}^{\infty} a_i s_i$ with s_i segments from $y_0{}^i$ to $y_1{}^i$ (recall $a_i \geqslant 0$ and $\sum_{i=1}^{\infty} a_i \|s_i\| = \|g\| < \infty$). Let $\delta > 0$ be given, chose N sufficiently large that $\sum_{i=N+1}^{\infty} a_i \|s_i\| < \delta$, and select ϵ in the definition of $\bar{S}$ such that $\epsilon < \delta / \sum_{i=1}^{N} a_i$. Then

$$\left| F(\rho) - \int \bar{S}_y \dot{y}\, d\mu \right| \qquad \mu \text{ any flow with } \partial\mu = \rho$$

$$= \left| F(\rho) - \int \bar{S}_y \dot{y}\, dg \right| \qquad \text{as } \bar{S} \in C^1(a)$$

$$= \left| \sum_{i=1}^{\infty} a_i [S(y_1{}^i) - S(y_0{}^i)] - \sum_{i=1}^{\infty} [\bar{S}(y_1{}^i) - \bar{S}(y_0{}^i)] \right|$$

$$\geqslant \sum_{i=1}^{N} a_i \int_A |S(y_1{}^i) - S(y_1{}^i + \epsilon\xi)| + |S(y_0{}^i) - S(y_0{}^i + \epsilon\xi)|\, d\xi$$

$$+ \sum_{i=N+1}^{\infty} a_i \int_A , S(y_1{}^i) - S(y_0{}^i)| + |S(y_1 + \epsilon\xi) - S(y_0 + \epsilon\xi)|\, d\xi$$

$$\leqslant \sum_{i=1}^{N} a_i 2 \|F\| \int_A \|\epsilon\xi\|\, d\xi + \sum_{i=N+1}^{\infty} 2a_i \|y_1{}^i - y_0{}^i\| \, \|F\| \tag{A2}$$

$$\leqslant 2\|F\| \left[\epsilon \sum_{i=1}^{N} a_i + \sum_{i=N+1}^{\infty} a_i \|s_i\| \right]$$

$$< 4\|F\| \delta. \tag{A.3}$$

Choose a sequence $\delta_j \to 0$, the corresponding ϵ_j, and define $\psi^j = \bar{S}$ (with ϵ_j in (A1)), then $\psi^i \to \psi \triangleq S$, the convergence being as stated. From (A3)

$$F(\rho) = \lim_{j\to\infty} \int \psi_y{}^j \dot{y}\, d\mu. \qquad \blacksquare$$

Notes. (i) Convergence of $\int \psi_y{}^i \dot{y}\, d\mu$ to $F(\rho)$ is not uniform over ρ (obviously since Σ is not norm bounded). However, the same sequence $\{\psi^j\}$ suffices for all ρ. To see this, take $\rho_0 \neq \rho$ (ρ as used in the above proof). Equation (A2) remains valid, with a_i, $y_0{}^i$, $y_1{}^i$ corresponding to ρ_0, for any N and ϵ. With δ fixed we can find N_0 such that $\sum_{i=N_0+1}^{\infty} a_i \| s_i \| < \delta$ and then, since the ϵ^j associated with ψ^j tend to zero, a $J < \infty$ such that for all $j \geqslant J$, $\epsilon^j < \delta / \sum_{i=1}^{N_0} a_i$. Equation (A3) follows immediately.

(ii) The reader will note that the above proof of Theorem 4.1 is very similar to [5, Sect. 84].

(iii) To ensure that F given by (4.2) is properly defined, we require:

LEMMA (A4). *Take F, $\{\psi^i\}$, and $\{\psi\}$ as above, then if $\{\phi^i\} \subset C^1(A)$ is any other sequence converging, as required in Theorem 4.1, to ψ, $F(\rho) = \lim_{i\to\infty} \int \phi_y{}^i \dot{y}\, d\mu$ for all $\rho \in \Sigma$, $\partial\mu = \rho$.*

Proof. A slight modification of the previous argument suffices. For any $\rho \in \Sigma$, $\epsilon > 0$, find N such that $| F(\rho) - \int \psi_y{}^i y\, d\mu \, | < \epsilon$ for all $i \geqslant N$. Using a sigma-polygonal μ, choose M sufficiently large that $\sum_{m=M+1}^{\infty} a_m \| s_m \| < \epsilon$ and then i and j sufficiently large that $| \psi^i(y) - \phi^j(y) | < \epsilon / \sum_{i=1}^{M} a_m$ for all $y \in A$. (This is possible because of the uniformity of congergence of $\{\psi^i\}$ and $\{\phi^j\}$ to the same function ψ.) Thus

$$\left| \int \psi_y{}^i \dot{y}\, d\mu - \int \phi_y{}^j \dot{y}\, d\mu \right|$$

$$\leqslant \sum_{m=1}^{M} a_m [| \psi^i(y_1{}^m) - \phi^j(y_1{}^m)| + | \psi^i(y_0{}^m) - \phi^j(y_0{}^m)|]$$

$$+ \sum_{m+M+1}^{\infty} a_m [| \psi^i(y_1{}^m) - \psi^i(y_0{}^m)| + | \phi^j(y_1{}^m) - \phi^j(y_0{}^m)|]$$

$$< 2\epsilon \sum_{m=M+1}^{\infty} a_m \, 2K \| s_m \|$$

$$< \epsilon(2 + 2K), \qquad \text{where } K \text{ is the Lipshitz constant of } \psi.$$

Therefore

$$\left| F(_\nu) - \int \phi_y{}^j \dot{y}\, d\mu \right| \leqslant \epsilon(3 + 2K). \qquad \blacksquare$$

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