A Robinson–Schensted-Type Algorithm for \( SO(2n, \mathbb{C}) \)

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INTRODUCTION

G. de B. Robinson [8] and C. Schensted [9] constructed an algorithm, called insertion algorithm, which transforms a pair of a semi-standard tableau \( T \) and a positive integer \( r \) into another semi-standard tableau denoted by \( T \leftarrow r \). Here a semi-standard tableau of shape \( \lambda \) (\( \lambda \) is a partition) is a filling of the squares of the Young diagram of \( \lambda \) with positive integers such that the entries strictly increase down each column and weakly increase across each row. We denote by \( \text{SSTab}(\lambda; \mathbb{N}) \) the set of all semi-standard tableaux of shape \( \lambda \) with entries in \( \mathbb{N} = \{1, 2, \ldots, n\} \). Then this insertion algorithm gives a bijection

\[
\text{SSTab}(\lambda; \mathbb{N}) \times \mathbb{N} \rightarrow \bigsqcup \text{SSTab}(\mu; \mathbb{N})
\]

\[
(T, r) \mapsto T \leftarrow r
\]

where \( \mu \) runs over all partitions such that the Young diagram of \( \mu \) is obtained from that of \( \lambda \) by adjoining one square. If \( \nu \) is a partition of \( k \), let \( \text{STab}(\nu) \) be the set of all semi-standard tableaux of shape \( \nu \) such that each letter \( i \in \mathbb{N} \) appears just once. Then, by iterating the above algorithm, we obtain a bijection

\[
\mathbb{N} \rightarrow \bigsqcup \text{STab}(\nu) \times \text{SSTab}(\nu; \mathbb{N}),
\]

where \( \nu \) runs over all partitions of \( k \) into at most \( n \) parts. If we consider the generating functions, this algorithm gives a bijective proof of the equations

\[
s_{\lambda}(x_1, \ldots, x_n) \times (x_1 + \cdots + x_n) = \sum_{\mu} s_{\mu}(x_1, \ldots, x_n)
\]

\[
(x_1 + \cdots + x_n)^k = \sum_{\nu} f^{\nu}s_{\nu}(x_1, \ldots, x_n),
\]

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where the summations are taken over the same ranges as in (0.1) and (0.2), respectively, and \( f^* = \# \text{STab}(v) \), \( s_\nu(x_1, \ldots, x_n) \) is the generating function of \( \text{SSTab}(v; [n]) \) called the Schur function corresponding to \( v \).

These identities (0.3) and (0.4) can be interpreted from the viewpoint of the representation theory. The irreducible representations of \( \text{GL}(n, \mathbb{C}) \) are indexed by partitions \( \lambda \) into at most \( n \) parts. Let \( (\rho_\lambda, V_\lambda) \) be the irreducible representation corresponding to \( \lambda \). Let \( V \) be the \( n \)-dimensional vector space \( \mathbb{C}^n \) on which the general linear group \( \text{GL}(n, \mathbb{C}) \) acts naturally. Then \( \text{GL}(n, \mathbb{C}) \) acts on the tensor spaces \( V_\lambda \otimes V \) and \( V \otimes V \). If we decompose these \( \text{GL}(n, \mathbb{C}) \)-modules into irreducible components, then

\[
V_\lambda \otimes V = \bigoplus_\mu V_\mu,
\]

\[
V \otimes V = \bigoplus_\nu V_\nu \otimes \nu^*.
\]

If we note that the Schur function \( s_\lambda(x_1, \ldots, x_n) \) is the character of the irreducible representation \( (\rho_\lambda, V_\lambda) \), the identities (0.3) and (0.4) can be considered as describing the irreducible decomposition of the above tensor spaces.

The object of this paper is to construct an analogous bijection for the special orthogonal group \( \text{SO}(2n, \mathbb{C}) \). For a partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \) into at most \( n \) parts, we denote by \( \lambda_{D(n)} \) the character of the representation which is obtained by restricting to \( \text{SO}(2n, \mathbb{C}) \) the irreducible representation of \( \text{O}(2n, \mathbb{C}) \) corresponding to \( \lambda \). It is well known that \( \lambda_{D(n)} \) is irreducible if \( \lambda_n = 0 \). But, if \( \lambda_n \neq 0 \), then \( \lambda_{D(n)} \) is the sum of the two irreducible characters

\[
\lambda_{D(n)} = \lambda_{D(n)}^+ + \lambda_{D(n)}^-,
\]

where \( \lambda_{D(n)}^+ \) are irreducible and \( \lambda_{D(n)}^+ \neq \lambda_{D(n)}^- \). (See Section 1 for the precise definition of \( \lambda_{D(n)}^+ \).) Then our main result is to give a bijective proof of the following theorem. (See Theorem 5.1 and Corollary 5.2.)

**Theorem.** Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) be a partition into at most \( n \) parts.

1. If \( \lambda_n = 0 \), then
   \[
   \chi_{D(n)}(x_1 + x_1^{-1} + \cdots + x_n + x_n^{-1}) = \sum_\mu \mu_{D(n)},
   \]
   where \( \mu \) runs over all partitions into at most \( n \) parts such that the Young diagram of \( \mu \) is obtained by that of \( \lambda \) by adjoining or removing one square.

2. If \( \lambda_n \neq 0 \), then
   \[
   \chi_{D(n)}^+(x_1 + x_1^{-1} + \cdots + x_n + x_n^{-1}) = \begin{cases} \sum_\mu \mu_{D(n)}^+ & (\lambda_n \geq 2) \\ \sum_\mu \mu_{D(n)}^+ + \tilde{\lambda}_{D(n)} & (\lambda_n = 1) \end{cases}
   \]
where \( \mu \) runs over all partitions into \( n \) parts such that the Young diagram of \( \mu \) is obtained from that of \( \lambda \) by adjoining or removing one square and \( \hat{\lambda} = (\lambda_1, \ldots, \lambda_{n-1}) \).

We can also give a bijection which describes the irreducible decomposition of the tensor product \( W \otimes \lambda^k \), where \( W = C^{2n} \) is the natural representation space of \( SO(2n, C) \). (See Theorem 6.6 and Corollary 6.7.)

As for the other classical groups, Berele [1] has given an analogous algorithm for the symplectic groups \( Sp(2n, C) \) and Sundaram [10] for the special orthogonal groups \( SO(2n + 1, C) \). Proctor [7] constructs an algorithm for orthogonal groups \( O(N, C) \). However, he does not deal with \( SO(2n, C) \), in particular the characters \( \lambda^\frac{1}{2} \).

This paper is composed as follows. In Section 2, we define the notion of "signed \( SO(2n) \)-tableau," which is the correspondent to that of semi-standard tableau for the case of \( GL(n, C) \). In Sections 3 and 4, we study some properties of the ordinary insertion algorithm and the ordinary sliding algorithm applied to \( SO(2n) \)-tableaux. The Robinson–Schensted-type algorithm (insertion algorithm) for \( SO(2n, C) \) is given in Section 5.

1. Preliminaries

In this paper we denote the set of positive integers by \( \mathbb{P} \) and the set of non-negative integers by \( \mathbb{N} \).

A partition is a (finite or infinite) weakly decreasing sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of non-negative integers \( \lambda_i \) with finite sum \( |\lambda| := \sum_{i \geq 1} \lambda_i \). The number of nonzero terms \( \lambda_i \) is called the length of \( \lambda \) and denoted by \( l(\lambda) \).

We usually identify a partition \( \lambda \) with its Young diagram defined by \( Y(\lambda) = \{(i, j) \in \mathbb{P} \times \mathbb{P} : 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\} \). The Young diagram is visualized by placing squares at the points in \( Y(\lambda) \). For example, the Young diagram of \( (4, 3, 1) \) is as in Fig. 1.

The conjugate partition of a partition \( \lambda \) is the partition \( \lambda' \) whose Young diagram is obtained from \( Y(\lambda) \) by reflection in the main diagonal. For example, \( (4, 3, 1)' = (3, 2, 2, 1) \).

For two partitions \( \lambda \) and \( \mu \), we write \( \lambda \succeq \mu \) if \( Y(\lambda) \supset Y(\mu) \). And we use

![Fig. 1. Young diagram of (4, 3, 1).](image-url)
the notation $\lambda \triangleright= \mu$ if $\lambda \triangleright= \mu$ and $|\lambda| = |\mu| + 1$; i.e., the Young diagram of $\lambda$ is obtained from that of $\mu$ by adding one square.

Let $\Gamma$ be a partially ordered set of symbols. A tableau of shape $\lambda$ on $\Gamma$ is a map $T: Y(\lambda) \to \Gamma$. For a symbol $\gamma \in \Gamma$, we put $m_\gamma(T) = \# \{(i, j) \in Y(\lambda): T(i, j) = \gamma\}$. A tableau $T$ of shape $\lambda$ is visualized by filling the symbol $T(i, j)$ in the $(i, j)$-cell of the Young diagram of $\lambda$. A tableau $T$ is called semi-standard if it satisfies the following conditions:

1. $T(i, 1) \leq T(i, 2) \leq \cdots \leq T(i, \lambda_i) \ (1 \leq i \leq l(\lambda))$.
2. $T(1, j) < T(2, j) < \cdots < T(\lambda_j, j) \ (1 \leq j \leq \lambda_1)$.

For example, $T$ in Fig. 2 is a semi-standard tableau of shape $(4, 3, 1)$ on \{1, 2, ..., 5\}.

Now we recall the representation theory of the special orthogonal groups $SO(2n, \mathbb{C})$. Let $SO(2n, \mathbb{C})$ be the special orthogonal group defined by

$$SO(2n, \mathbb{C}) = \{X \in SL(2n, \mathbb{C}): XJ'X = J\},$$

where $J$ is the $2n \times 2n$ anti-diagonal matrix given by

$$J = \begin{pmatrix}
1 & & & \\
& 1 & & \\
& & \ddots & \\
& & & 1
\end{pmatrix}.$$

We can take as a maximal torus $T$ of $SO(2n, \mathbb{C})$ the set of all diagonal matrices in $SO(2n, \mathbb{C})$

$$T = \{\text{diag}(t_1, ..., t_n, t_n^{-1}, ..., t_1^{-1}): t_i \in \mathbb{C}^\times\}.$$ Let $e_i: \text{Lie}(T) \to \mathbb{C}$ be the linear map defined by

$$e_i(\text{diag}(H_1, ..., H_n, -H_n, ..., -H_1)) = H_i.$$

Then $\Delta = \{\pm e_i \pm e_j: i < j\}$ is the root system of $SO(2n, \mathbb{C})$ and $\Pi = \{e_1 - e_2, ..., e_{n-1} - e_n, e_{n-1} + e_n\}$ is a fundamental system of $\Delta$.

$$T = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & 4 & 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & 4 & 5 \\ 4 \end{pmatrix}$$

Fig. 2. A tableau of shape $(4, 3, 1)$. 
THEOREM 1.1. There exists a bijection between the set of equivalence classes of the irreducible representations of $SO(2n, \mathbb{C})$ and the set of dominant integral weights

$$P^+_{D(n)} = \{ \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n : \lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n| \}.$$ 

And this bijection is afforded by taking the highest weight of the irreducible representation of $SO(2n, \mathbb{C})$.

DEFINITION. Let $\lambda$ be a partition with length $\leq n$. If $l(\lambda) \leq n - 1$, we denote by $\lambda^+_{D(n)}$ the character of the irreducible representation of $SO(2n, \mathbb{C})$ with the highest weight $\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_{n-1} e_{n-1}$. If $l(\lambda) = n$, we denote by $\lambda^+_{D(n)}$ (resp. $\lambda^-_{D(n)}$) the character of the irreducible representation of $SO(2n, \mathbb{C})$ with the highest weight $\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_{n-1} e_{n-1} + \lambda_n e_n$ (resp. $\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_{n-1} e_{n-1} - \lambda_n e_n$). It is convenient to use the following notations for a partition $\lambda$ with $l(\lambda) = n$:

$$\lambda_{D(n)} = \lambda^+_{D(n)} + \lambda^-_{D(n)}$$

and

$$\tilde{\lambda}_{D(n)} = \lambda^+_{D(n)} - \lambda^-_{D(n)}.$$ 

The character $\lambda_{D(n)}$ or $\tilde{\lambda}_{D(n)}$ is uniquely determined by the values on a maximal torus $T$. Hence it can be considered as a symmetric Laurent polynomial in $n$ variables $x_1, \ldots, x_n$.

2. $SO(2n)$-TABLEAUX

In this section we describe the weight structure of the irreducible representations of $SO(2n, \mathbb{C})$ in terms of signed $SO(2n)$-tableaux. Let $\Gamma_n$ be the totally ordered set of $2n$ symbols $\{1, \bar{1}, 2, \bar{2}, \ldots, n, \bar{n}\}$ with ordering given by

$$1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n}.$$ 

The elements 1, 2, ..., $n$ (resp. $\bar{1}$, $\bar{2}$, ..., $\bar{n}$) are referred to as unbarred (resp. barred) elements.

DEFINITION. Let $\lambda$ be a partition with $l(\lambda) \leq n$. An $SO(2n)$-tableau of shape $\lambda$ is a map $T : \Gamma_n \rightarrow \Gamma_n$ satisfying the following conditions:

\begin{itemize}
  \item[(D1)] $T(i, 1) \leq T(i, 2) \leq \cdots \leq T(i, \lambda_i)$ (1 $\leq i \leq l(\lambda)$).
  \item[(D2)] $T(1, j) < T(2, j) < \cdots < T(\lambda', j)$ (1 $\leq j \leq \lambda_1$).
  \item[(D3)] $T(i, j) \geq i$ ((i, j) $\in Y(\lambda)$).
  \item[(D4)] If $T(i, 1) = i$, then $T(i, j) = i$ implies $T(i - 1, j) = i$.
\end{itemize}
It should be noted that an $SO(2n)$-tableau has no rows containing both 1 and $\bar{1}$. For an $SO(2n)$-tableau $T$, we put $x^T = x_1^{d_1(T)} \cdots x_n^{d_n(T)}$ where $d_i(T) = m_i(T) - m_{\bar{i}}(T)$. For example,

$$
T = \begin{pmatrix}
\bar{1} & \bar{1} & 2 & 2 \\
2 & 2 & 2 & 3 \\
4 & & & \\
\bar{4}
\end{pmatrix}
$$

is an $SO(8)$-tableau with $x^T = x_1^{-2}x_2^3x_3x_4^{-1}$.

**Definition.** Let $T$ be an $SO(2n)$-tableau. A signature vector $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{\pm 1\}^n$ is compatible with $T$ if it satisfies the following conditions:

(S1) If $C(T) \cap \{i, \bar{i}\} = \emptyset$, then $\varepsilon_i = 1$;
(S2) If $C(T) \cap \{i, \bar{i}\} = \{i\}$, then $\varepsilon_i = 1$;
(S3) If $C(T) \cap \{i, \bar{i}\} = \{\bar{i}\}$, then $\varepsilon_i = -1$;
(S4) If $C(T) \cap \{i, \bar{i}\}$ and if $\bar{i}$ is not in the $i$th row, then $\varepsilon_i = 1$,

where $C(T)$ is the set of symbols appearing in the first column of $T$.

A pair $(T, \varepsilon)$ of an $SO(2n)$-tableau $T$ of shape $\lambda$ and a compatible signature vector $\varepsilon$ is called a signed $SO(2n)$-tableau of shape $\lambda$. For example, the possible signature vectors compatible with the above $T$ are $\varepsilon = (-1, 1, 1, 1)$ and $(-1, 1, 1, -1)$.

**Definition.** For a partition $\lambda$ with $l(\lambda) \leq n$, let $\text{Tab}(\lambda_{D(\varepsilon)})$ denote the set of all signed $SO(2n)$-tableaux of shape $\lambda$. If $l(\lambda) = n$, we put

$$\text{Tab}(\lambda^+_D(n)) = \left\{ (T, \varepsilon) \in \text{Tab}(\lambda_{D(\varepsilon)}) : \prod_{i=1}^{n} \varepsilon_i = 1 \right\}$$

$$\text{Tab}(\lambda^-_D(n)) = \left\{ (T, \varepsilon) \in \text{Tab}(\lambda_{D(\varepsilon)}) : \prod_{i=1}^{n} \varepsilon_i = -1 \right\}.$$ 

**Theorem 2.1.** For a partition $\lambda$ with $l(\lambda) \leq n$, we have

$$\lambda_D(n) = \sum_{(T, \varepsilon) \in \text{Tab}(\lambda_{D(\varepsilon)})} x^T.$$ 

Moreover, if $l(\lambda) = n$, we have

$$\lambda^+_D(n) = \sum_{(T, \varepsilon) \in \text{Tab}(\lambda^+_D(n))} x^T.$$
Remark. King and El-Sharkaway [2] have given a weight parametrization in terms of tableau similar to ours. However, their formulations and proofs seem to leave something to be desired. Koike and Terada [3] have also defined the notion of \("SO(2n)\)-tableau." But they did not describe the character $\lambda_{\tilde{D}(n)}$ as a generating function of $SO(2n)$-tableaux.

In order to prove Theorem 2.1, we use the following determinantal formulas.

**Proposition 2.2** ([11, p. 228], see also [6]). For a partition $\lambda$ with $l(\lambda) \leq n$,

$$\lambda_{\tilde{D}(n)}(x_1, ..., x_n) = \det((\lambda_i - i + j)_{D(n - j + 1)}(x_j, ..., x_n))_{1 \leq i, j \leq n}.$$  

**Proposition 2.3** ([4, p. 247], see also [6]). For a partition $\lambda$ with $l(\lambda) = n$,

$$\tilde{\lambda}_{\tilde{D}(n)}(x_1, ..., x_n) = \det((x_j - x_j^{-1})
\times (\lambda_i - 1 - i + j)_{C(n - j + 1)}(x_j, ..., x_n))_{1 \leq i, j \leq n},$$

where $(k)_{C(n)}$ is a Laurent polynomial in $x_1, ..., x_n$ defined by using the generating function

$$\sum_{k=0}^{\infty} (k)_{C(n)} t^k = \prod_{i=1}^{n} (1 - x_i t)^{-1}(1 - x_i^{-1} t)^{-1}.$$  

Let $G = SO(2n, \mathbb{C})$ and $H$ be the subgroup of $G$ defined by

$$H = \left\{ \begin{pmatrix} z & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^\times, Y \in SO(2n - 2, \mathbb{C}) \right\}$$

$$\cong GL(1, \mathbb{C}) \times SO(2n - 2, \mathbb{C}).$$

We consider the restriction of the irreducible characters of $G$ to $H$. For an irreducible character $\chi$ of $SO(2n, \mathbb{C})$ and $\xi$ of $SO(2n - 2, \mathbb{C})$, we define the character $[\chi : \xi]$ of $GL(1, \mathbb{C})$ by the relation

$$\chi_{\tilde{G}_H} = \sum_{\xi} [\chi : \xi] \times \xi,$$

where $\xi$ runs over all irreducible characters of $SO(2n - 2, \mathbb{C})$. For a partition $\lambda$ with $l(\lambda) \leq n$, we can write
\[
\hat{\lambda}_{D(n)}(x_1, \ldots, x_n) = \sum_{l(\mu) < n-1} \left[ \hat{\lambda}_{D(n)} \cdot \mu_{D(n-1)} \right](x_n) \cdot \mu_{D(n-1)}(x_1, \ldots, x_{n-1}) \\
+ \sum_{l(\mu) = n-1} \left[ \hat{\lambda}_{D(n)} \cdot \mu_{D(n-1)}^+ \right](x_n) \cdot \mu_{D(n-1)}^+(x_1, \ldots, x_{n-1}) \\
+ \sum_{l(\mu) = n-1} \left[ \hat{\lambda}_{D(n)} \cdot \mu_{D(n-1)}^- \right](x_n) \cdot \mu_{D(n-1)}^-(x_1, \ldots, x_{n-1})
\]
if \( l(\lambda) < n \) \hfill (2.1)

\[
\hat{\lambda}^\mu_{D(n)}(x_1, \ldots, x_n) = \sum_{l(\mu) < n-1} \left[ \hat{\lambda}^\mu_{D(n)} \cdot \mu_{D(n-1)} \right](x_n) \cdot \mu_{D(n-1)}(x_1, \ldots, x_{n-1}) \\
+ \sum_{l(\mu) = n-1} \left[ \hat{\lambda}^\mu_{D(n)} \cdot \mu_{D(n-1)}^+ \right](x_n) \cdot \mu_{D(n-1)}^+(x_1, \ldots, x_{n-1}) \\
+ \sum_{l(\mu) = n-1} \left[ \hat{\lambda}^\mu_{D(n)} \cdot \mu_{D(n-1)}^- \right](x_n) \cdot \mu_{D(n-1)}^-(x_1, \ldots, x_{n-1})
\]
if \( l(\lambda) = n \). \hfill (2.2)

Let \( \lambda \) and \( \mu \) be partitions such that \( l(\lambda) \leq n \) and \( l(\mu) \leq n-1 \).

**Lemma 2.4.**

1. If \( l(\lambda) < n \) and \( l(\mu) = n-1 \),

\[
[\hat{\lambda}_{D(n)} \cdot \mu_{D(n-1)}^+] = [\hat{\lambda}_{D(n)} \cdot \mu_{D(n-1)}^-].
\]

2. If \( l(\lambda) = n \) and \( l(\mu) < n-1 \),

\[
[\hat{\lambda}_{D(n)}^+ \cdot \mu_{D(n-1)}] = [\hat{\lambda}_{D(n)}^- \cdot \mu_{D(n-1)}].
\]

3. If \( l(\lambda) = n \) and \( l(\mu) = n-1 \),

\[
[\hat{\lambda}_{D(n)}^+ \cdot \mu_{D(n-1)}^-] = [\hat{\lambda}_{D(n)}^- \cdot \mu_{D(n-1)}^+].
\]

**Proof.** Let \( \sigma \) be the involutive outer automorphism of \( SO(2n, \mathbb{C}) \) defined by

\[
\sigma : SO(2n, \mathbb{C}) \ni x \mapsto \sigma(x) = \sigma_0 x \sigma_0^{-1} \in SO(2n, \mathbb{C}),
\]
where \( \sigma_0 \) is the \( 2n \times 2n \) matrix given by

\[
\sigma_0 = \begin{pmatrix}
1 & & & \\
& \ddots & & \\
& & 1 & 0 \\
& & 0 & 1 \\
& & & \ddots \\
& & & & 1 \\
& & & & & \ddots \\
& & & & & & 1 \\
& & & & & & & \ddots \\
& & & & & & & & 1
\end{pmatrix}.
\]
Then the action of this automorphism $\sigma$ on the characters of $G$ and $H$ is given by

$$\sigma(\chi_{D(n)}) = \lambda_{D(n)}$$
if $l(\lambda) < n$

$$\sigma(\lambda_{D(n)}^{\pm}) = \lambda_{D(n)}^{\mp}$$
if $l(\lambda) = n$

$$\sigma(z^k \times \mu_{D(n-1)}) = z^k \times \mu_{D(n-1)}$$
if $l(\mu) < n - 1$

$$\sigma(z^k \times \mu_{D(n-1)}^{\pm}) = z^k \times \mu_{D(n-1)}^{\mp}$$
if $l(\mu) = n - 1$.

So this lemma can be shown by applying $\sigma$ to (2.1) and (2.2).

For convenience, we write

$$[\lambda_{D(n)}; \mu_{D(n-1)}]$$

$$= \begin{cases} 
[\lambda_{D(n)}; \mu_{D(n-1)}] & (l(\lambda) < n, l(\mu) < n - 1) \\
[\lambda_{D(n)}; \mu_{D(n-1)}^{\mp}] & (l(\lambda) < n, l(\mu) = n - 1) \\
[\lambda_{D(n)}^{\pm}; \mu_{D(n-1)}] + [\lambda_{D(n)}^{\mp}; \mu_{D(n-1)}] & (l(\lambda) = n, l(\mu) < n - 1) \\
[\lambda_{D(n)}^{\pm}; \mu_{D(n-1)}] - [\lambda_{D(n)}^{\mp}; \mu_{D(n-1)}] & (l(\lambda) = n, l(\mu) = n - 1). 
\end{cases}$$

And we put, if $l(\lambda) = n$ and $l(\mu) = n - 1$,

$$[\lambda_{D(n)}; \mu_{D(n-1)}] = [\lambda_{D(n)}^{\pm}; \mu_{D(n-1)}] - [\lambda_{D(n)}^{\mp}; \mu_{D(n-1)}].$$

Then, from (2.1), (2.2), and Lemma 2.4, we have

$$\lambda_{D(n)} = \sum_{l(\mu) \leq n - 1} [\lambda_{D(n)}; \mu_{D(n-1)}] \cdot \mu_{D(n-1)}$$

$$\tilde{\lambda}_{D(n)} = \sum_{l(\mu) = n - 1} [\tilde{\lambda}_{D(n)}; \mu_{D(n-1)}] \cdot \mu_{D(n-1)}.$$ 

Let $h_k(x, x^{-1})$ be the $k$th complete symmetric polynomial in $x$ and $x^{-1}$; that is, for $k > 0$,

$$h_k(x, x^{-1}) = x^k + x^{k-2} + \ldots + x^{-k+2} + x^{-k}.$$ 

Note that $h_0(x, x^{-1}) = 1$ and $h_k(x, x^{-1}) = 0$ if $k < 0$. For an integer vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we denote by $h_{\alpha}$ and $x^\alpha$ the column vectors $(h_{\alpha_1}(x, x^{-1}), \ldots, h_{\alpha_n}(x, x^{-1}))$ and $(x^{\alpha_1}, \ldots, x^{\alpha_n})$, respectively. And we put $\alpha^* = (\alpha_1 - 1, \alpha_3 - 2, \ldots, \alpha_n - n + 1)$ and $1 = (1, 1, \ldots, 1)$. Then the following lemmas can be obtained from Propositions 2.2 and 2.3 by manipulation similar to [6].
LEMMA 2.5.
$$[\lambda^+_{D(n)} : \mu_{D(n-1)}](z) = \det(h_{\lambda^*+\mu_1}, h_{\lambda^*+(\mu_2-1)}, \ldots,$$
$$h_{\lambda^*+(\mu_{n-2}+n-3)}, z^{\lambda^*+(n-1)}+z^{(-1)\lambda^*-n+1}).$$

LEMMA 2.6. If $l(\lambda) = n$ and $l(\mu) = n - 1$, then we have
$$[\lambda^+_{D(n)} : \mu_{D(n-1)}](z) = \det(h_{\lambda^*+\mu_1}, h_{\lambda^*+(\mu_2-1)}, \ldots,$$
$$h_{\lambda^*+(\mu_{n-2}+n-2)}), z^{\lambda^*+(n-1)}-z^{(-1)\lambda^*-n+1}).$$

Note that $[\lambda^+_{D(n)} : \mu_{D(n-1)}] = [\lambda^+_{D(n)} : \mu_{D(n-1)}] = 0$ unless $\lambda \geq \mu$.

LEMMA 2.7. If $l(\lambda) = n$ and $l(\mu) < n - 1$, then
$$[\lambda^+_{D(n)} : \mu_{D(n-1)}](z) = \det(h_{\lambda^*+\mu_1}, h_{\lambda^*+(\mu_2-1)}, \ldots,$$
$$h_{\lambda^*+(\mu_{n-2}+n-2)}), z^{\lambda^*+(n-1)}+z^{(-1)\lambda^*-n+1}).$$

Proof: Since $\mu_{n-1} = 0$, we have
$$[\lambda^+_{D(n)} : \mu_{D(n-1)}] = [\lambda^+_{D(n)} : \mu_{D(n-1)}]$$
$$\frac{1}{2} \det(h_{\lambda^*+\mu_1}, h_{\lambda^*+(\mu_2-1)}, \ldots,$$
$$h_{\lambda^*+(\mu_{n-2}+n-3)}), z^{\lambda^*+(n-1)}+z^{(-1)\lambda^*-n+1}).$$

By subtracting the $(n-1)$st column multiplied by $z-z^{-1}$ from the $n$th column, we see that
$$[\lambda^+_{D(n)} : \mu_{D(n-1)}] = \det(h_{\lambda^*+\mu_1}, h_{\lambda^*+(\mu_2-1)}, \ldots,$$
$$h_{\lambda^*+(\mu_{n-2}+n-3)}), z^{\lambda^*+(n-1)}+z^{(-1)\lambda^*-n+1}).$$

In the similar way, we can show that $[\lambda^+_{D(n)} : \mu_{D(n-1)}] = 0$ if $l(\mu) < n - 2$.

LEMMA 2.8. If $l(\lambda) = n$ and $l(\mu) = n - 1$, then
$$[\lambda^+_{D(n)} : \mu^+_D(n-1)] = [\lambda^+_{D(n)} : \mu^+_D(n-1)] = z^{(n-1)} \det(h_{\lambda^*+\mu^*_i}, h_{\lambda^*+(\mu_2-1)}),$${n-1} \det(h_{\lambda^*+\mu^*_i}, h_{\lambda^*+(\mu_2-1)}), \ldots,$$
$$h_{\lambda^*+(\mu_{n-2}+n-3)}), z^{\lambda^*+(n-1)}+z^{(-1)\lambda^*-n+1}).$$

where $\tilde{\mu} = (\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_{n-1})$.
DEFINITION. Let \( \lambda \) and \( \mu \) be partitions such that \( l(\lambda) \leq n \), \( l(\mu) \leq n - 1 \), and \( \lambda \geq \mu \). A skew \( SO(2n)/SO(2n-2) \)-tableau of skew shape \( \lambda/\mu \) is a map \( T: Y(\lambda) \rightarrow \{n, \bar{n}\} \) satisfying the same conditions as in the definition of \( SO(2n) \)-tableau. Let \( \text{Tab}(\lambda_{D(n)}/\mu_{D(n-1)}) \) be the set of all skew \( SO(2n)/SO(2n-2) \)-tableaux of skew shape \( \lambda/\mu \).

Comparing \([5, 1.5], (5.12)\) and the above determinantal expressions in Lemmas 2.5, 2.7, and 2.8, we have

**Lemma 2.9.** (1) If \( l(\lambda) < n \) and \( l(\mu) < n - 1 \), then

\[
[\lambda_{D(n)}; \mu_{D(n-1)}](x_n) = \sum_{T \in \text{Tab}(\lambda_{D(n)}/\mu_{D(n-1)})} x^T.
\]

(2) If \( l(\lambda) < n \) and \( l(\mu) = n - 1 \), then

\[
[\lambda_{D(n)}; \mu_{D(n-1)}^\mathbb{C}](x_n) = \sum_{T \in \text{Tab}(\lambda_{D(n)}/\mu_{D(n-1)})} x^T.
\]

(3) If \( l(\lambda) = n \) and \( l(\mu) < n - 1 \), then

\[
[\lambda_{D(n)}^\mathbb{C}; \mu_{D(n-1)}](x_n) = \sum_{T \in \text{Tab}(\lambda_{D(n)}/\mu_{D(n-1)})} x^T.
\]

(4) If \( l(\lambda) = n \) and \( l(\mu) = n - 1 \), then

\[
[\lambda_{D(n)}^\mathbb{C}; \mu_{D(n-1)}^\mathbb{C}](x_n) = \sum_{T \in \text{Tab}(\lambda_{D(n)}/\mu_{D(n-1)})} x^T.
\]

**Proof of Theorem 2.1.** We prove Theorem 2.1 by using induction on \( n \). First we consider the case where \( l(\lambda) < n \). Then, by (2.1), the induction hypothesis, and Lemma 2.9, we have

\[
\lambda_{D(n)} = \sum_{l(\mu) < n - 1} \left( \sum_{(T, e) \in \text{Tab}(\mu_{D(n-1)})} x^T \right) \left( \sum_{T' \in \text{Tab}(\lambda_{D(n)}/\mu_{D(n-1)})} x^{T'} \right) + \sum_{l(\mu) = n - 1} \left( \sum_{(T, e) \in \text{Tab}(\mu_{D(n-1)}^\mathbb{C})} x^T \right) \left( \sum_{T' \in \text{Tab}(\lambda_{D(n)}/\mu_{D(n-1)})} x^{T'} \right) + \sum_{l(\mu) = n - 1} \left( \sum_{(T, e) \in \text{Tab}(\mu_{D(n-1)}^\mathbb{C})} x^T \right) \left( \sum_{T' \in \text{Tab}(\lambda_{D(n)}/\mu_{D(n-1)})} x^{T'} \right).
\]
For each term of the right hand side, we define a signed $SO(2n)$-tableau $(\tilde{T}, \tilde{\varepsilon})$ as follows. Let $(T, \varepsilon) \in \text{Tab}(\mu_{D(n-1)})$ (or $\text{Tab}(\mu_{D(n-1)}^\pm)$) and $T' \in \text{Tab}(\lambda_{D(n)}/\mu_{D(n-1)})$. Then we put

$$\tilde{T}(i, j) = \begin{cases} T(i, j) & \text{if } (i, j) \in Y(\mu) \\ T'(i, j) & \text{if } (i, j) \in Y(\lambda) - Y(\mu) \end{cases}$$

and $\tilde{\varepsilon}_i = \varepsilon_i$ for $1 \leq i \leq n - 1$. Since $\tilde{T}$ has at most $n - 1$ rows, the signature $\tilde{\varepsilon}_n$ is uniquely determined by the compatibility with $\tilde{T}$. The collection of $(\tilde{T}, \tilde{\varepsilon})$ thus obtained from all $(T, \varepsilon)$ and $T'$ coincides with $\text{Tab}(\lambda_{D(n)})$. And we have $x^{\tilde{T}} = x^T \cdot x^{T'}$, so that

$$\lambda_{D(n)} = \sum_{(\tilde{T}, \tilde{\varepsilon}) \in \text{Tab}(\lambda_{D(n)})} x^{\tilde{T}}.$$

Next we consider the character $\lambda^+_{D(n)}$ where $l(\lambda) = n$. Then, by (2.2), the induction hypothesis and Lemma 2.9, we have

$$\lambda^+_{D(n)} = \sum_{l(\mu) < n - 1} \left( \sum_{(T, \varepsilon) \in \text{Tab}(\mu_{D(n-1)})} x^T \right) \sum_{T' \in \text{Tab}(\lambda_{D(n)}/\mu_{D(n-1)})} x^{T'}$$

$$+ \sum_{l(\mu) = n - 1} \left( \sum_{(T, \varepsilon) \in \text{Tab}(\mu_{D(n-1)})} x^T \right) \sum_{T' \in \text{Tab}(\lambda_{D(n)}/\mu_{D(n-1)})} x^{T'}$$

$$+ \sum_{l(\mu) = n - 1} \left( \sum_{(T, \varepsilon) \in \text{Tab}(\mu_{D(n-1)})} x^T \right) \sum_{T' \in \text{Tab}(\lambda_{D(n)}/\mu_{D(n-1)})} x^{T'}.$$

For each term of the right hand side, we define a signed $SO(2n)$-tableau $(\tilde{T}, \tilde{\varepsilon})$ as follows. Let $\mu$ be a partition with $l(\mu) = n - 2$ and take $(T, \varepsilon) \in \text{Tab}(\mu_{D(n-1)})$ and $T' \in \text{Tab}(\lambda_{D(n)}/\mu_{D(n-1)})$. (As noted after Lemma 2.7, $\text{Tab}(\lambda_{D(n)}/\mu_{D(n-1)}) = \emptyset$ if $l(\mu) < n - 2$.) Noting that $T'(n - 1, 1) = n$ and $T'(n, 1) = \bar{n}$, we put

$$\tilde{T}(i, j) = \begin{cases} T(i, j) & \text{if } (i, j) \in Y(\mu) \\ T'(i, j) & \text{if } (i, j) \in Y(\lambda) - Y(\mu) \end{cases}$$

and $\tilde{\varepsilon}_i = \varepsilon_i$ for $1 \leq i \leq n - 1$. Then we see that $(\tilde{T}, \tilde{\varepsilon}) \in \text{Tab}(\lambda^+_{D(n)})$. Let $\mu$ be a partition with $l(\mu) = n - 1$ and take $(T, \varepsilon) \in \text{Tab}(\mu^+_{D(n-1)})$ (resp. $\text{Tab}(\mu_{D(n-1)})$) and $T' \in \text{Tab}(\lambda_{D(n)}/\mu_{D(n-1)})$ such that $T(n, 1) = n$ (resp. $\bar{n}$). Then we put
The collection of $(\tilde{T}, \tilde{\varepsilon})$ thus obtained from all $(T, \varepsilon)$ and $T'$ coincides with $\text{Tab}(\lambda^+_{D(n)})$. And we have $x^T - x^T \cdot x^{T'}$, so that

$$\lambda^+_{D(n)} = \sum_{(T, \varepsilon) \in \text{Tab}(\lambda^+_{D(n)})} x^T.$$ 

Similar argument holds for the character $\lambda_{D(n)}$. 

3. Ordinary Insertion Algorithm and $SO(2n)$-Tableau

In this section we study some properties of $SO(2n)$-tableau which will be needed in the later sections.

In the following we identify a tableau $T: \text{Y}(\lambda) \rightarrow \Gamma_n$ with its extension $T: \mathbb{N} \times \mathbb{N} \rightarrow \Gamma_n \cup \{\infty\}$ by setting

$$T(i, j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \infty & \text{if } (i, j) \in P \times P - Y(\lambda). \end{cases}$$

Here, by convention, $0 < 1 < \bar{1} < \cdots < n < \bar{n} < \infty$.

Let $T$ be a semi-standard tableau of shape $\lambda$ on $\Gamma_n$, $\gamma \in \Gamma_n$ and $k \in P$. Now we define the map $T' = \text{Ins}(T, \gamma, k): \mathbb{N} \times \mathbb{N} \rightarrow \Gamma_n \cup \{\infty\}$ as follows. Let $l$ be the integer such that $T(k, l - 1) \leq \gamma < T(k, l)$. Then

$$T'(i, j) = \begin{cases} \gamma & \text{if } (i, j) = (k, l) \\ T(i, j) & \text{otherwise.} \end{cases}$$

We denote by bump$'(T, \gamma, k)$ the bumped element $T(k, l)$. Now we assume that $T' = \text{Ins}(T, \gamma, k)$ is a semi-standard tableau, i.e.,

$$T'(i, j) \leq T'(i, j + 1) \quad \text{and} \quad T'(i, j) < T'(i + 1, j)$$

unless the both sides are 0 or $\infty$. Then we call $T'$ the tableau obtained by inserting $\gamma$ into the $k$th row of $T$. This algorithm is a fundamental step for the insertion algorithm.

**Lemma 3.1.** Let $T$ be a semi-standard tableau on $\Gamma_n$, $\gamma \in \Gamma_n$, and $k \in P$.

1. If $k = 1$, then $\text{Ins}(T, \gamma, k)$ is a semi-standard tableau.
(2) If \( T' = \text{Ins}(T, \gamma, k) \) is semi-standard and \( \gamma' = \text{bump}^j(T, \gamma, k) \in \Gamma_n \), then \( \text{Ins}(T', \gamma', k + 1) \) is also a semi-standard tableau.

Now we study the properties of \( \text{Ins}(T, \gamma, k) \) where \( T \) is an \( SO(2n) \)-tableau.

**Lemma 3.2.** Let \( T \) be an \( SO(2n) \)-tableau, \( \gamma \in \Gamma_n \), and \( k \in \mathbb{P} \). If \( T' = \text{Ins}(T, \gamma, k) \) is an \( SO(2n) \)-tableau and if \( \gamma' = \text{bump}^j(T, \gamma, k) < k + 1 \), then

\[
\gamma = k, \quad \text{bump}^j(T, \gamma, k) = \bar{k}, \quad \text{and} \quad T(k, 1) = \bar{k}.
\]

**Proof.** Since \( T' \) is an \( SO(2n) \)-tableau, we have \( \gamma = T'(k, l) \geq k \). On the other hand \( \gamma < T(k, l) = \gamma' < k + 1 \) from the definition of \( T' \). Hence we see that \( \gamma = k \) and \( \gamma' = \bar{k} \). If we assume that \( T(k, 1) \neq \bar{k} \), we have \( T(k, 1) = \cdots = T(k, l - 1) = k \) and \( T(k, l) = \bar{k} \). Then \( T'(k, l) = k \) and \( T'(k - 1, l) = T(k - 1, l) = k \). This contradicts the assumption that \( T' \) is semi-standard. So we have \( T(k, 1) = \bar{k} \). (In this case, \( l = 1 \).) \( \blacksquare \)

**Lemma 3.3.** Let \( T \) be an \( SO(2n) \)-tableau, \( \gamma \in \Gamma_n \), and \( k \in \mathbb{P} \). Suppose that \( \gamma \geq k \) and that \( T' = \text{Ins}(T, \gamma, k) \) is a semi-standard tableau. If \( T' \) is not an \( SO(2n) \)-tableau, then there are three possible cases as follows:

(1) \( \gamma = T(k, l - 1) = T(k + 1, 1) = \cdots = T(k + 1, l - 1) = k + 1 \), \( T(k + 1, l) = \bar{k} + 1 \), and \( T(k, l) = k + 1 \)

\[
T = \begin{array}{cccc}
& & & \\
k + 1 & \cdots & k + 1 & \bar{k} + 1 \\
\end{array}
\]

\[
T' = \begin{array}{cccc}
& & & \\
k + 1 & \cdots & k + 1 & \bar{k} + 1 \\
\end{array}
\]

(2) \( \gamma = k \), \( T(k + 1, 1) = \cdots = T(k + 1, l - 1) = k + 1 \), \( T(k + 1, l) = \bar{k} + 1 \), \( T(k, l) = k + 1 \), and \( T(k - 1, l) = k \)

\[
T = \begin{array}{cccc}
& & & \\
k + 1 & \cdots & k + 1 & \bar{k} + 1 \\
\end{array}
\]

\[
T' = \begin{array}{cccc}
& & & \\
k + 1 & \cdots & k + 1 & \bar{k} + 1 \\
\end{array}
\]
(3) $\gamma = k, \ T(k, 1) = \ldots = T(k, l - 1) = k, \ T(k, l) \geq k + 1, \text{ and } T(k - 1, l) \leq k - 1$ (or $k = 1$).

$$
\begin{align*}
T &= T(k - 1, l) \\
    & \quad \ k \quad \cdots \quad k \quad T(k, l) \\
T' &= T(k - 1, l) \\
    & \quad \ k \quad \cdots \quad k \quad k
\end{align*}
$$

**Proof.** Since $T'$ is semi-standard and $\gamma \leq k$, $T'$ does not satisfy the condition

$$(D4)_k \quad \text{if} \quad T'(k, 1) = k, \text{ then } T'(k, j) = k \text{ implies } T'(k - 1, j) = k, \text{ or}$$

$$(D4)_{k+1} \quad \text{if} \quad T'(k + 1, 1) = k + 1, \text{ then } T'(k + 1, j) = k + 1 \text{ implies } T'(k, j) = k + 1.$$

It is easy to see that the situation (3) occurs if $(D4)_k$ does not hold for $T'$ and that (1) and (2) occur if $(D4)_{k+1}$ does not hold for $T'$. 

We give the inverse of the above insertion algorithm. For a semi-standard tableau $S$, $\beta \in \Gamma_n \cup \{\infty\}$ and $k \in \mathbb{P}$, we define a map $S' = Del(S, \beta, k) : \mathbb{N} \times \mathbb{N} \rightarrow \Gamma_n \cup \{\infty\}$ as follows. Let $l$ be the integer $j$ such that $S(k, j) < \beta < S(k, j + 1)$. Then we put

$$S'(i, j) = \begin{cases} 
\beta & \text{if } (i, j) = (k, l) \\
S(i, j) & \text{otherwise.}
\end{cases}$$

And we put $bump^D(S, \beta, k) = S(k, l)$. The proofs of the following lemmas are routine, so we omit them.

**Lemma 3.4.** Let $S$ be a semi-standard tableau of shape $\lambda, \beta \in \Gamma_n \cup \{\infty\}$ and $k \in \mathbb{P}$.

1. If $\beta = \infty$ and $\lambda_k > \lambda_{k+1}$, then $Del(S, \beta, k)$ is a semi-standard tableau.

2. If $S' = Del(S, \beta, k)$ is a semi-standard tableau and $k \geq 2$, then $Del(S', \beta', k - 1)$ ($\beta' = bump^D(S, \beta, k)$) is also a semi-standard tableau.

**Lemma 3.5.** Let $T$ be a semi-standard tableau, $\gamma \in \Gamma_n$, and $k \in \mathbb{P}$. Then we have

$Del(Ins(T, \gamma, k), \gamma', k) = T, \quad Ins(Del(T, \gamma, k), \gamma'', k) = T,$

where $\gamma' = bump'(T, \gamma, k)$ and $\gamma'' = bump''(T, \gamma, k)$. 

Lemma 3.6. Let $T$ be a semi-standard tableau, $\gamma \in \Gamma_n$, and $k \in \mathbb{P}$. The following conditions are equivalent:

1. $\text{Ins}(T, \gamma, k + 1)$ is semi-standard.
2. $\text{Del}(T, \gamma, k)$ is semi-standard.

Lemma 3.7. Let $S$ be an $SO(2n)$-tableau, $\beta \in \Gamma_n \cup \{\infty\}$, $k \in \mathbb{P}$. If $\beta \geq k + 1$ and $S' = \text{Del}(S, \beta, k)$ is semi-standard, then $S'$ is an $SO(2n)$-tableau.

4. Punctured $SO(2n)$-Tableau

Definition. Let $\lambda$ be a partition and $(k, l) \in Y(\lambda)$. A punctured tableau of shape $\lambda$ with empty cell in position $(k, l)$ is a map $T: Y(\lambda) - \{(k, l)\} \to \Gamma_n$. Then we often say that $T(k, l)$ is the empty cell. If the empty cell is at the corner of the Young diagram of $\lambda$, we will identify the punctured tableau with an ordinary tableau of shape $\mu$ where $Y(\mu) = Y(\lambda) - \{(k, l)\}$.

Definition. A punctured tableau $T$ of shape $\lambda$ with empty cell in position $(k, l)$ is called a punctured $SO(2n)$-tableau if it satisfies the following conditions:

1. The entries increase weakly in each row.
2. The entries increase strictly in each column.
3. $T(i, j) \geq i$ if $(i, j) \in Y(\lambda) - \{(k, l)\}$.
4. If $T(i, 1) = i$ and $T(i, j) = i$, then either
   a) $T(i - 1, j) = i$ or
   b) $T(i - 1, j)$ is the empty cell and $T(i - 1, j + 1) \geq i$.
5. If $T(k, l - 1) = k$ and $T(k, l + 1) = \tilde{k}$, then $T(k - 1, l) = k$.
6. If $T(k, l + 1) = k$, then $\tilde{k}$ does not appear in the $k$th row.

A semi-standard punctured tableau is a punctured tableau satisfying the conditions (D'1) and (D'2).

We denote the empty cell by $\square$. For example,

$$
\begin{array}{ccc}
1 & \square & 2 \\
2 & \square & 2
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
1 & 2 & 2 \\
2 & \square & 2
\end{array}
$$

are punctured $SO(4)$-tableaux, but

$$
\begin{array}{ccc}
1 & \square & 1 \\
2 & 2 & \square
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
1 & 1 & 2 \\
2 & \square & 2
\end{array}
$$

are not.
DEFINITION. Let $T$ be a punctured tableau of shape $\lambda$ with empty cell in position $(k, l)$. Then we define a punctured tableaux $A(T)$ and $B(T)$ as follows. If the empty cell is at the corner of $Y(\lambda)$, then $A(T)$ is undefined. Otherwise we define $A(T)$ to be the tableau obtained by interchanging the empty cell $(k, l)$ with $T(k, l + 1)$ or $T(k + 1, l)$ according as $T(k, l + 1) < T(k + 1, l)$ or $T(k, l + 1) \geq T(k + 1, l)$. If the empty cell is in position $(1, 1)$, then $B(T)$ is undefined. Otherwise we define $B(T)$ to be the tableau obtained by interchanging the empty cell $(k, l)$ with $T(k, l - 1)$ or $T(k - 1, l)$ according as $T(k, l - 1) > T(k - 1, l)$ or $T(k, l - 1) \leq T(k - 1, l)$.

For example,

$$T = \begin{array}{ccc} 1 & 2 & 2 \\ \textbullet & 2 & \textbullet' \\ 2 & \textbullet & 2' \end{array}, \quad A(T) = \begin{array}{ccc} 1 & 2 & 2 \\ 2 & \textbullet & 2' \\ 2 & 2 & \textbullet \end{array}, \quad B(T) = \begin{array}{ccc} 1 & \textbullet & 2 \\ 2 & 2 & \textbullet \\ 2 & 2 & \textbullet' \end{array}.$$

**Lemma 4.1.** Let $T$ be a semi-standard punctured tableau. If $A(T)$ (resp. $B(T)$) is defined, then $A(T)$ (resp. $B(T)$) is also semi-standard.

*Proof.* See [1, Lemma 2].

**Lemma 4.2.** Let $T$ be a semi-standard tableau.

1. If $B(T)$ is defined, then $AB(T) = T$.
2. If $A(T)$ is defined, then $BA(T) = T$.

*Proof.* Clear from the definition.

**Lemma 4.3.** If $T$ is a punctured $SO(2n)$-tableau, then $A(T)$ is also a punctured $SO(2n)$-tableau.

*Proof.* It is clear that $A(T)$ satisfies the condition (D'3). So it is enough to show that $A(T)$ satisfies the conditions (D'4), (D'5), and (D'6).

First we show that $A(T)$ satisfies the condition (D'4). Suppose that $A(T)(i, 1) = i$, $A(T)(i, j) = i$. It is clear that $A(T)(i - 1, j) = i$, except for the following three cases:

(a) The empty cell of $A(T)$ is in position $(i - 1, j)$.

(b) The empty cell slides from the $(i - 1, j)$-cell of $T$ to the $(i - 1, j + 1)$-cell of $A(T)$.

(c) The empty cell of $T$ is in position $(i, j)$.

In the case (a), we can see that $T(i - 1, j) = i$, hence $A(T)(i - 1, j + 1) = T(i - 1, j + 1) \geq i$. In the case (b), we have $i \leq T(i - 1, j + 1) < T(i, j) = T(i - 1, j + 1)$ so that $A(T)(i - 1, j) = T(i - 1, j + 1) = i$. In the case (c), it follows from (D'4) and (D'5) that $T(i, j - 1) = T(i - 1, j) = i$ or $T(i, j - 1) = T(i, j + 1) = i$, $T(i - 1, j - 1) = T(i - 1, j) = i$. 


Next we consider (D'5). Suppose that \( A(T)(k, l) \) is the empty cell and that \( A(T)(k, l-1) = \tilde{k} \) and \( A(T)(k, l+1) = \tilde{k} \). Then it follows from (D'6) that the empty cell of \( T \) is in position \((k-1, l)\). Hence we have

\[
k \leq k, l \leq T(k-1, l) < T(k, l+1) = \tilde{k},
\]

so that \( A(T)(k-1, l) = \tilde{k} \).

Finally we check the condition (D'6). Now we assume that \( A(T)(k, l) \) is the empty cell and that \( A(T)(k, l+1) = \tilde{k} \). In this case we have that \( T(k, l) \leq \tilde{k} = A(T)(k, l+1) > A(T)(k-1, l+1) = T(k-1, l+1) \). Hence the empty cell of \( T \) is in position \((k, l-1)\) and \( T(k, l) = \tilde{k} \). By the condition (D'6) for \( T \), we see that \( k \) does not appear in the \( k \)th row. ❄

**Lemma 4.4.** Let \( T \) be a punctured \( \text{SO}(2n) \)-tableau with empty cell in position \((k, l)\) where \((k, l) \neq (1, 1)\). If \( B(T) \) is not a punctured \( \text{SO}(2n) \)-tableau, then either

\[
\begin{align*}
(1) & \quad l \geq 2, \quad T(k+1, l) = \cdots = T(k+1, l-1) = k+1, \quad T(k+1, l) = k+1, \quad T(k-1, l) \leq k \quad \text{or} \quad k = 1, \\
(2) & \quad l = 1, \quad T(k-1, l) \leq k-1.
\end{align*}
\]

**Proof.** Let \( T \) be a punctured \( \text{SO}(2n) \)-tableau with empty cell in position \((k, l)\). Then it is easy to show the following claims.

**Claim 1.** If \( l \geq 2 \), then \( B(T) \) satisfies the condition (D'3).

**Claim 2.** \( B(T) \) satisfies the conditions (D'5) and (D'6).

Now assume that \( B(T) \) is not a punctured \( \text{SO}(2n) \)-tableau. If \( l = 1 \), then \( B(T) \) satisfies all conditions of \( \text{SO}(2n) \)-tableau except for (D'3). So we have \( T(k-1, l) \leq k-1 \). Suppose \( l \geq 2 \). Then, by the above claims, \( B(T) \) does not satisfy (D'4). Hence there are two possible cases:

\[
\begin{align*}
(\text{a}) & \quad B(T)(i, 1) = \cdots = B(T)(i, j-1) = i, \quad B(T)(i, j) = i, \quad \text{and} \quad B(T)(i-1, j) \leq i-1 \quad \text{for some} \quad i, \\
(\text{b}) & \quad B(T)(i, 1) = \cdots = B(T)(i, j-1) = i, \quad B(T)(i, j) = i, \quad \text{and} \quad B(T)(i-1, j+1) \leq i-1 \quad \text{and the} \quad (i-1, j)\text{-cell is the empty cell for some} \quad i.
\end{align*}
\]

In the case (a), we see that the empty cell of \( T \) must be in position \((i-1, j)\) because \( T \) is an \( \text{SO}(2n) \)-tableau. So (2) holds for this case. In the case (b), we have \( T(i, 1) = i, \quad T(i, j) = i, \quad \text{and} \quad T(i-1, j) \leq i-1 \). This contradicts the fact that \( T \) is an \( \text{SO}(2n) \)-tableau. ❄
DEFINITION. Let $T$ be a punctured $SO(2n)$-tableau. A signature vector $\varepsilon = (\varepsilon_0; \varepsilon_1, \ldots, \varepsilon_n)$ is compatible if it satisfies the following conditions:

(S'1) If $C(T) \cap \{i, i\} = \emptyset$, then $\varepsilon_i = 1$.

(S'2) If $C(T) \cap \{i, i\} = \{i\}$, then $\varepsilon_i = 1$.

(S'3) If $C(T) \cap \{i, i\} = \{i\}$, then $\varepsilon_i = -1$.

(S'4) If $C(T) \cap \{i, i\} = \{i, i\}$ and if $T(i, 1) \neq i$, then $\varepsilon_i = 1$.

(S'5) If $l = 1$ and $T(k-1, 1) \leq k-1$, then $T(k, 2) = k$ (resp. $T(k, 2) = k$) implies $\varepsilon_0 = 1$ (resp. $\varepsilon_0 = -1$).

(S'6) If $l = 1$ and $T(k-1, 1) \geq k+1$, then $\varepsilon_0 = 1$.

(S'7) If $l \geq 2$, then $\varepsilon_0 = 1$.

We call $\varepsilon_0$ the signature of empty cell.

For example, the possible signature vectors compatible with

\[
\begin{array}{cccc}
1 & 2 & & \\
4 & 4 & & \\
\hline
\end{array}
\]

are $\varepsilon = (1; -1, 1, 1, 1)$ and $(1; -1, 1, 1, -1)$.

For a (punctured) $SO(2n)$-tableau $T$ and $1 \leq k \leq n$, we write

$$
\varepsilon_k(T) = \begin{cases} 
1 & \text{if } C(T) \cap \{k, \bar{k}\} = \emptyset \\
-1 & \text{if } C(T) \cap \{k, \bar{k}\} = \{k\} \\
1 & \text{if } C(T) \cap \{k, \bar{k}\} = \{\bar{k}\} \text{ and } T(k, 1) \neq \bar{k}.
\end{cases}
$$

If $C(T) \cap \{k, \bar{k}\} = \{k, \bar{k}\}$ and $T(k, 1) = \bar{k}$, then $\varepsilon_k(T)$ is not defined.

5. INSERTION ALGORITHM FOR $SO(2n)$

The main theorem of this paper is the following.

**Theorem 5.1.** Let $\lambda$ be a partition with $l(\lambda) \leq n$.

(1) If $l(\lambda) < n$, there exists a weight-preserving bijection

$$
I(\lambda_{D(n)}): \text{Tab}(\lambda_{D(n)}) \times \Gamma_n \rightarrow \bigcup_{\mu \vdash \lambda \text{ or } \mu \vdash \lambda} \text{Tab}(\mu_{D(n)}).
$$
(2) If \( l(\lambda) = n \), there exists a weight-preserving bijection

\[
I(\tilde{\lambda}_{D(n)}): \text{Tab}(\tilde{\lambda}_{D(n)}) \times \Gamma_n \rightarrow \bigg\{ \bigg( \bigcup_{\mu \triangleright \lambda; \mu \neq \lambda} \text{Tab}(\mu_{D(n)}) \bigg) \bigg| \bigg( \bigcup_{\mu \triangleright \tilde{\lambda}; \mu \neq \tilde{\lambda}} \text{Tab}(\mu_{D(n)}) \bigg) \bigg| \bigg( \bigcup_{\mu \triangleright \lambda; \mu \neq \lambda} \text{Tab}(\mu_{D(n)}) \bigg) \bigg| \bigg( \bigcup_{\mu \triangleright \tilde{\lambda}; \mu \neq \tilde{\lambda}} \text{Tab}(\mu_{D(n)}) \bigg) \bigg) \bigg( \lambda_n \geq 2 \bigg),
\]

where \( \tilde{\lambda} = (\lambda_1, ..., \lambda_{n-1}) \).

Combining this theorem with Theorem 2.1, we obtain

**Corollary 5.2.** Let \( \lambda \) be a partition with \( l(\lambda) \leq n \).

1. If \( l(\lambda) < n \), then we have

\[
\lambda_{D(n)} \times (x_1 + x_1^{-1} + \cdots + x_n + x_n^{-1}) = \sum_{\mu \triangleright \lambda; \mu \neq \lambda} \mu_{D(n)}.
\]

2. If \( l(\lambda) = n \), then we have

\[
\tilde{\lambda}_{D(n)} \times (x_1 + x_1^{-1} + \cdots + x_n + x_n^{-1}) = \bigg( \bigg( \bigcup_{\mu \triangleright \tilde{\lambda}; \mu \neq \tilde{\lambda}} \mu_{D(n)} \bigg) \bigg) \bigg( \bigcup_{\mu \triangleright \lambda; \mu \neq \lambda} \mu_{D(n)} \bigg) \bigg) \bigg( \bigcup_{\mu \triangleright \tilde{\lambda}; \mu \neq \tilde{\lambda}} \mu_{D(n)} \bigg) \bigg) \bigg( \lambda_n = 1 \bigg),
\]

where \( \tilde{\lambda} = (\lambda_1, ..., \lambda_{n-1}) \).

In this section, we describe an algorithm which gives a map \( I(\tilde{\lambda}_{D(n)}) \) or \( I(\tilde{\lambda}^\pm_{D(n)}) \) in the above theorem. This algorithm is called the insertion algorithm for \( SO(2n, \mathbb{C}) \). The inverse map \( D(\tilde{\lambda}_{D(n)}) \) (resp. \( D(\tilde{\lambda}^\pm_{D(n)}) \)) of \( I(\tilde{\lambda}_{D(n)}) \) (resp. \( I(\tilde{\lambda}^\pm_{D(n)}) \)) is given in the next section.

In the following, we fix an irreducible character \( \chi = \tilde{\lambda}_{D(n)} \) or \( \tilde{\lambda}^\pm_{D(n)} \). And we put

\[
L(\chi) - \text{Tab}(\chi) \times \Gamma_n
\]

\[
R(\chi) = \bigg\{ \bigg( \bigcup_{\mu \triangleright \lambda; \mu \neq \lambda} \text{Tab}(\mu_{D(n)}) \bigg) \bigg| \bigg( \bigcup_{\mu \triangleright \tilde{\lambda}; \mu \neq \tilde{\lambda}} \text{Tab}(\mu_{D(n)}) \bigg) \bigg| \bigg( \bigcup_{\mu \triangleright \lambda; \mu \neq \lambda} \text{Tab}(\mu_{D(n)}) \bigg) \bigg| \bigg( \bigcup_{\mu \triangleright \tilde{\lambda}; \mu \neq \tilde{\lambda}} \text{Tab}(\mu_{D(n)}) \bigg) \bigg) \bigg( \lambda_n = 0 \bigg),
\]

\[
\bigg( \bigcup_{\mu \triangleright \lambda; \mu \neq \lambda} \text{Tab}(\mu_{D(n)}) \bigg) \bigg| \bigg( \bigcup_{\mu \triangleright \tilde{\lambda}; \mu \neq \tilde{\lambda}} \text{Tab}(\mu_{D(n)}) \bigg) \bigg| \bigg( \bigcup_{\mu \triangleright \lambda; \mu \neq \lambda} \text{Tab}(\mu_{D(n)}) \bigg) \bigg| \bigg( \bigcup_{\mu \triangleright \tilde{\lambda}; \mu \neq \tilde{\lambda}} \text{Tab}(\mu_{D(n)}) \bigg) \bigg) \bigg( \lambda_n = 1 \bigg),
\]

\[
\bigg( \bigcup_{\mu \triangleright \lambda; \mu \neq \lambda} \text{Tab}(\mu_{D(n)}) \bigg) \bigg| \bigg( \bigcup_{\mu \triangleright \tilde{\lambda}; \mu \neq \tilde{\lambda}} \text{Tab}(\mu_{D(n)}) \bigg) \bigg| \bigg( \bigcup_{\mu \triangleright \lambda; \mu \neq \lambda} \text{Tab}(\mu_{D(n)}) \bigg) \bigg| \bigg( \bigcup_{\mu \triangleright \tilde{\lambda}; \mu \neq \tilde{\lambda}} \text{Tab}(\mu_{D(n)}) \bigg) \bigg) \bigg( \lambda_n \geq 2 \bigg).
\]
DEFINITION. Let $\lambda$ be a partition with $l(\lambda) \leq n$. Let $\mathcal{X}_q(\lambda)$ be the set of all quartets $(T, \varepsilon, \gamma, k)$ of an $SO(2n)$-tableau $T$ of shape $\lambda$, a signature vector $\varepsilon$ compatible with $T$, $\gamma \in I_n$ and $k \in \mathbb{P}$ such that $\text{Ins}(T, \gamma, k)$ is a semi-standard tableau and that $\gamma \geq k$. Let $\mathcal{X}_\infty(\lambda)$ be the set of all quartets $(T, \varepsilon, \infty, k)$ of an $SO(2n)$-tableau $T$ of shape $\mu$, a signature vector $\varepsilon$ compatible with $T$, and $k \in \mathbb{P}$ such that $\lambda \prec \mu$ and that the cell of $Y(\mu) - Y(\lambda)$ is in the $(k - 1)$st row. Let $\mathcal{X}_p(\lambda)$ be the set of all pairs $(T, \varepsilon)$ of a punctured $SO(2n)$-tableau $T$ and a signature vector $\varepsilon$ compatible with $T$. And we put

$$\mathcal{X}(\lambda) = \mathcal{X}_1(\lambda) \bigsqcup \mathcal{X}_\infty(\lambda) \bigsqcup \mathcal{X}_p(\lambda).$$

Let us fix a signed $SO(2n)$-tableau $(T, \varepsilon) \in \text{Tab}(\chi)$ of shape $\lambda$ and $\gamma \in I_n$. For $(T, \varepsilon)$ and $\gamma$, we construct a sequence $X^{(0)}, X^{(1)}, \ldots, X^{(N)}$ of elements $X^{(i)}$ of $\mathcal{X}(\lambda)$ inductively until either

(a) $X^{(N)} \in \mathcal{X}_\infty(\lambda)$ or

(b) $X^{(N)} = (T^{(N)}, \varepsilon^{(N)})$ is an element of $\mathcal{X}_p(\lambda)$ such that the empty cell of $T^{(i)}$ is at the corner of $Y(\lambda)$ and that, if $l(\lambda) < n$, the signature $\varepsilon_0^{(N)}$ of empty cell is equal to 1.

And we define $\mathcal{I}(\chi)(T, \varepsilon, \gamma)$ as follows. If $X^{(N)} = (T^{(N)}, \varepsilon^{(N)}, \infty, k^{(N)}) \in \mathcal{X}_\infty(\lambda)$, then $\mathcal{I}(\chi)(T, \varepsilon, \gamma) = (T^{(N)}, \varepsilon^{(N)})$. If $X^{(N)} = (T^{(N)}, \varepsilon^{(N)}) \in \mathcal{X}_p(\lambda)$ ($\varepsilon^{(N)} = (\varepsilon_0^{(N)}, \varepsilon_1^{(N)}, \ldots, \varepsilon_n^{(N)})$), then $\mathcal{I}(\chi)(T, \varepsilon, \gamma) = (S, \delta)$ where $S$ is a tableau obtained by removing the empty cell from $T^{(i)}$ and $\delta = (\varepsilon_1^{(N)}, \ldots, \varepsilon_n^{(N)})$.

First we define

$$X^{(0)} = (T, \varepsilon, \gamma, 1).$$

Next we construct $X^{(i)}$ when $X^{(0)}, \ldots, X^{(i-1)}$ have been defined. Suppose that $X^{(i-1)} = (T^{(i-1)}, \varepsilon^{(i-1)}, \gamma^{(i-1)}, k^{(i-1)}) \in \mathcal{X}_1(\lambda)$. We put $k = k^{(i-1)}$, $T' = \text{Ins}(T^{(i-1)}, \gamma^{(i-1)}, k)$, and $\gamma' = \text{bump}'(T^{(i-1)}, \gamma^{(i-1)}, k)$.

Then we have the following three cases.

Case 1. $T'$ is an $SO(2n)$-tableau and $\gamma' \geq k + 1$.

Case 2. $T'$ is an $SO(2n)$-tableau and $\gamma' \leq k$.

Case 3. $T'$ is not an $SO(2n)$-tableau.

In Case 1, we define

$$T^{(i)} = T', \quad \gamma^{(i)} = \gamma', \quad k^{(i)} = k + 1.$$ 

If $\gamma^{(i-1)} = \tilde{a}$ and $\gamma' = \tilde{b}$ (where $\sim$ may be either blanks or bars), then we put

$$\varepsilon^{(i)} = \begin{cases} \varepsilon^{(i-1)} & \text{if } j = a, b, \\ \varepsilon^{(i-1)}(T^{(i)}) & \text{if } j = b. \end{cases}$$
The signature $\varepsilon_{i}^{(i)}$ is uniquely determined by the compatibility with $T^{(i)}$ except for the case where

$$T^{(i)}(k-1, 1) = T^{(i-1)}(k-1, 1) = k, \quad T^{(i)}(k, 1) = \gamma^{(i-1)} = k.$$ 

In this case we put $\varepsilon_{k}^{(i)} = 1$.

In Case 2, by Lemma 3.1, we have

$$\gamma^{(i-1)} = k, \quad \gamma' = \bar{k}, \quad T^{(i-1)}(k, 1) = \bar{k}.$$

Let us divide this case into two cases.

**Case 2.1.** $\prod_{j=1}^{k} c_{j} = \prod_{j=1}^{k-1} c_{j}$ and $T^{(i-2)}$ ($i \geq 2$) satisfies

$$T^{(i-2)}(k-1, 1) = k, \quad T^{(i-2)}(k, 1) = T^{(i-2)}(k, 2) = \bar{k}.$$

**Case 2.2.** Otherwise.

In Case 2.1, we see that the entries in position $(k-1, 1)$, $(k-1, 2)$, $(k, 1)$, and $(k, 2)$ of $T^{(i-2)}$, $T^{(i-1)}$, and $T'$ are

$$T^{(i-2)} = \begin{array} {ccc} k & k & \ast \\ k & k' & \bar{k} \end{array}, \quad T^{(i-1)} = \begin{array} {ccc} k & k \\ k & k' \end{array}, \quad T' = \begin{array} {ccc} \ast & k \\ k & \bar{k} \end{array}.$$ 

Then we define $T^{(i)}$ to be the punctured tableau obtained from $T'$ by puncturing the $(k-1, 2)$-cell of $T'$.

$$T^{(i)} = \begin{array} {ccc} \ast & \square \\ k & \bar{k} \end{array}.$$ 

As for the signature vector $\varepsilon^{(i)}$, we put

$$\varepsilon_{j}^{(i)} = \begin{cases} 1 & \text{if } j = 0 \\ 1 & \text{if } j = k \\ \varepsilon_{j}^{(i-1)} & \text{otherwise}. \end{cases}$$

In Case 2.2, let $T^{(i)}$ be the punctured tableau obtained from $T'$ by puncturing the $(k, 1)$-cell. And we put

$$\varepsilon_{j}^{(i)} = \begin{cases} \prod_{j=1}^{k} c_{j} / \prod_{j=1}^{k-1} c_{j}^{(i-1)} & \text{if } j = 0 \\ 1 & \text{if } j = k \\ \varepsilon_{j}^{(i-1)} & \text{otherwise}. \end{cases}$$

In Case 3, from Lemma 3.2, it is enough to consider the following three cases.
Case 3.1. The $k$th and $(k+1)$st rows of $T^{(i-1)}$ satisfy
\[
T^{(i-1)}(k+1, 1) = \ldots = T^{(i-1)}(k + 1, l - 1) = k + 1,
T^{(i-1)}(k + 1, l) = T^{(i-1)}(k + 1, l + 1) = k + 1,
T^{(i-1)}(k, l) = k + 1
\]
for some $l \geq 2$, and either
(a) $\gamma^{(i-1)} = T^{(i-1)}(k, l - 1)$ or
(b) $\gamma^{(i-1)} = \bar{k}$ and $T^{(i-1)}(k - 1, l) = k$.

Case 3.2. The $k$th and $(k+1)$st rows of $T^{(i-1)}$ satisfy
\[
T^{(i-1)}(k + 1, 1) = \ldots = T^{(i-1)}(k + 1, l - 1) = k + 1,
T^{(i-1)}(k + 1, l) = \bar{k} + 1, \quad T^{(i-1)}(k + 1, l + 1) \geq k + 2,
T^{(i-1)}(k, l) = k + 1
\]
for some $l \geq 2$, and either
(a) $\gamma^{(i-1)} = T^{(i-1)}(k, l - 1)$ or
(b) $\gamma^{(i-1)} = \bar{k}$ and $T^{(i-1)}(k - 1, l) = k$.

Case 3.3. $\gamma^{(i-1)} = \bar{k}$ and the $(k - 1)$st and $k$th rows of $T^{(i-1)}$ satisfy
\[
T^{(i-1)}(k, 1) = \ldots = T^{(i-1)}(k, l - 1) = k,
T^{(i-1)}(k, l) \geq k + 1,
T^{(i-1)}(k - 1, l) \leq \bar{k} - 1 \quad (\text{or } k = 1).
\]
for some $l \geq 2$.

In Case 3.1, the $k$th and $(k+1)$st rows of $T^{(i-1)}$ and $T'$ are
\[
T^{(i-1)} = \begin{array}{cccc}
      k + 1 & k + 1 \\
      k + 1 & \bar{k} + 1 & k + 1
\end{array},
T' = \begin{array}{cccc}
      k + 1 & \ldots & \gamma^{(i-1)} & k + 1 \\
      k + 1 & \ldots & \bar{k} + 1 & \bar{k} + 1
\end{array}.
\]
So we define $T^{(i)}$ to be the punctured tableau obtained from $T'$ by puncturing the $(k, l + 1)$-cell and changing the entry in the $(k + 1, l)$-cell into $k + 1$.
\[
T^{(i)} = \begin{array}{cccc}
      k + 1 & \ldots & \bar{k} + 1 & \bar{k} + 1 \\
      k + 1 & \ldots & \gamma^{(i-1)} & k + 1
\end{array}.
\]
And we put

$$
e^j_{(i)} = \begin{cases} 1 & \text{if } j = 0 \\ e^j_{(i-1)} & \text{otherwise.} \end{cases}$$

In Case 3.2, the $k$th and $(k + 1)$st rows of $T^{(i-1)}$ and $T'$ are

$$
T^{(i-1)} = \begin{bmatrix} k+1 & k+1 & \cdots & k+1 \\ k+1 & k+1 & \cdots & k+1 \\ \vdots & \vdots & \ddots & \vdots \\ k+1 & k+1 & \cdots & k+1 \\ k+1 & k+1 & \cdots & k+1 \\ \end{bmatrix} T^{(i-1)}(k, l+1) \\
T' = \begin{bmatrix} \gamma^{(i-1)} & \gamma^{(i-1)} & \cdots & \gamma^{(i-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \end{bmatrix} T^{(i-1)}(k, l+1) \\
\end{array}$$

We define $T^{(i)}$ to be the punctured tableau obtained from $T'$ by puncturing the $(k + 1, 1)$-cell and changing the entry in the $(k + 1, 1)$ cell into $k + 1$.

$$
T^{(i)} = \begin{bmatrix} \gamma^{(i-1)} & \gamma^{(i-1)} & \cdots & \gamma^{(i-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \end{bmatrix} T^{(i-1)}(k, l+1) \\
\end{array}$$

And we put

$$
e^j_{(i)} = \begin{cases} 1 & \text{if } j = 0 \\ e^j_{(i-1)} & \text{otherwise.} \end{cases}$$

In Case 3.3, the $(k - 1)$st and the $k$th rows of $T^{(i-1)}$ and $T'$ are

$$
T^{(i-1)} = \begin{bmatrix} k-1 & k-1 & \cdots & k-1 \\ k-1 & k-1 & \cdots & k-1 \\ \vdots & \vdots & \ddots & \vdots \\ k-1 & k-1 & \cdots & k-1 \\ k-1 & k-1 & \cdots & k-1 \\ \end{bmatrix} T^{(i-1)}(k-1, l) \\
T' = \begin{bmatrix} k-1 & k-1 & \cdots & k-1 \\ k-1 & k-1 & \cdots & k-1 \\ \vdots & \vdots & \ddots & \vdots \\ k-1 & k-1 & \cdots & k-1 \\ k-1 & k-1 & \cdots & k-1 \\ \end{bmatrix} T^{(i-1)}(k-1, l) \\
\end{array}$$

In this case let $T^{(i)}$ be the punctured tableau obtained from $T^{(i-1)}$ by puncturing the $(k, 1)$-cell of $T^{(i-1)}$.

$$
T^{(i)} = \begin{bmatrix} k-1 & k-1 & \cdots & k-1 \\ k-1 & k-1 & \cdots & k-1 \\ \vdots & \vdots & \ddots & \vdots \\ k-1 & k-1 & \cdots & k-1 \\ k-1 & k-1 & \cdots & k-1 \\ \end{bmatrix} T^{(i-1)}(k-1, l) \\
\end{array}$$

And we put

$$
e^j_{(i)} = \begin{cases} 1 & \text{if } j = 0 \\ e^j_{(i-1)} & \text{otherwise.} \end{cases}$$

Now we turn to the case where $X^{(i-1)} = (T^{(i-1)}, e^{(i-1)}) \in \mathfrak{X}_p(\lambda)$. Suppose that $T^{(i-1)}(k, l)$ is the empty cell. Then we consider the following three cases.
Case 4. \( l = 1 \) and either of the following conditions holds for \( X^{(i-1)} \).

- (a) \( T^{(i-1)}(k-1, 1) \leq \tilde{k}, \ T^{(i-1)}(k, 2) < T^{(i-1)}(k+1, 1), \) and \( e_0^{(i-1)}e_a^{(i-1)} = -e_d(A(T^{(i-1)})) \) if \( T^{(i-1)}(k, 2) = \tilde{a} \).
- (b) \( e_0^{(i-1)} = -1 \) and \( T^{(i-1)}(k, 2) \geq T^{(i-1)}(k+1, 1) \geq k+2 \).

Case 5. \( l = 1 \) and \( X^{(i-1)} \) satisfies the following conditions:

- (a) There exists an integer \( m \geq k + 2 \) such that \( T^{(i-1)}(m-1, 1) = m, \ T^{(i-1)}(m, 1) = \bar{m} \).
- (b) \( T^{(i-1)}(k-1, 1) \geq k + 1 \) and \( T^{(i-1)}(k, 2) < T^{(i-1)}(k+1, 1) \).
- (c) \( e_0^{(i-1)}e_a^{(i-1)} = -e_d(A(T^{(i-1)})) \) if \( T^{(i-1)}(k, 2) = \tilde{a} \).

Case 6. Otherwise.

In Case 4, we construct an element \( X^{(i)} = (T^{(i)}, e^{(i)}, \gamma^{(i)}, k^{(i)}) \) of \( X_1(\lambda) \) or \( X_{\infty}(\lambda) \) as

\[
T^{(i)}(p, q) = \begin{cases} 
  k + 1 & \text{if } (p, q) = (k, 1) \\
  \overline{k+1} & \text{if } (p, q) = (k+1, 1) \\
  T^{(i-1)}(p, q) & \text{otherwise}
\end{cases}
\]

\[
e^{(i)}_j = \begin{cases} 
  -1 & \text{if } j = k+1 \\
  e_b(T^{(i)}) & \text{if } j = b \\
  e^{(i-1)}_j & \text{otherwise}
\end{cases}
\]

\[
\gamma^{(i)} = T^{(i-1)}(k+1, 1)
\]

\[
k^{(i)} = k^{(i-1)} + 2,
\]

where \( T^{(i-1)}(k+1, 1) = \bar{b} \).

In Case 5, let \( m \) be the minimum integer such that \( m \geq k + 2 \) and that \( T^{(i-1)}(m-1, 1) = m, \ T^{(i-1)}(m, 1) = \bar{m} \). And, if \( T^{(i-1)}(k, 2) = \tilde{a} \), we put

\[
T^{(i)} = A(T^{(i-1)})
\]

\[
e^{(i)}_j = \begin{cases} 
  e_d(T^{(i)}) & \text{if } j = a \\
  -e^{(i-1)}_m & \text{if } j = m \\
  e^{(i-1)}_j & \text{otherwise}
\end{cases}
\]

In Case 6, we put

\[
T^{(i)} = A(T^{(i-1)})
\]

and define \( e^{(i)} \) as follows.
Case 6.1. The empty cell of $T^{(i-1)}$ is not in the first column. Then we put $\varepsilon^{(i)} = \varepsilon^{(i-1)}$.

Case 6.2. $T^{(i-1)}(k, 1)$ and $T^{(i)}(k, 2)$ are the empty cells. In this case, if $T^{(i-1)}(k, 2) = T^{(i)}(k, 1) = \hat{a}$, we put

$$
\varepsilon^{(i)}_j = \begin{cases} 
1 & \text{if } j = 0 \\
\varepsilon^{(i-1)}_j & \text{if } j \neq 0, a.
\end{cases}
$$

And the signature $\varepsilon^{(i)}_a$ is uniquely determined by the compatibility with $T^{(i)}$ except for the following two cases:

1. $T^{(i-1)}(k-1, 1) = k$, $T^{(i-1)}(k, 2) = k$
2. $T^{(i-1)}(k, 2) = k+1$, $T^{(i-1)}(k+1, 1) = \overline{k+1}$.

In these cases we define

1. $\varepsilon^{(i)}_k = \varepsilon^{(i-1)}_k \cdot \varepsilon^{(i-1)}_0$
2. $\varepsilon^{(i)}_{k+1} = \varepsilon^{(i-1)}_{k+1} \cdot \varepsilon^{(i-1)}_0$,

respectively.

Case 6.3. $T^{(i-1)}(k, 1)$ and $T^{(i)}(k+1, 1)$ are the empty cells. We put $\varepsilon^{(i)} = \varepsilon^{(i-1)}$ except for the case where

$$
T^{(i-1)}(k-1, 1) = k + 1, \quad T^{(i-1)}(k+1, 1) = \overline{k+1},
$$

$$
\varepsilon^{(i-1)}_0 = 1, \quad \varepsilon^{(i-1)}_{k+1} = -1.
$$

In this exceptional case we put

$$
\varepsilon^{(i)}_j = \begin{cases} 
-1 & \text{if } j = 0 \\
1 & \text{if } j = k+1 \\
\varepsilon^{(i-1)}_j & \text{otherwise}.
\end{cases}
$$

This completes the definition of the insertion algorithm and the map $I(\lambda_{\Delta_{(n)}})$ or $I(\lambda_{\Delta_{(n)}}^\pm)$. In order to see that the map $I(\lambda_{\Delta_{(n)}})$ or $I(\lambda_{\Delta_{(n)}}^\pm)$ is well-defined and weight-preserving, it is enough to show the following lemmas.

**Lemma 5.3.** For $0 \leq i \leq N$, we have $X^{(i)} \in X(\lambda)$.

*Proof.* We will check that $X^{(i)} \in X(\lambda)$ only in Case 2.2 and Case 4. The other cases are similar. In Case 2.2, we have $T^{(i)}(k-1, 1) \leq \overline{k-1}$ and $T^{(i)}(k, 2) \geq \overline{k}$. Hence we have to show that $T^{(i)}(k, 2) = \overline{k}$ implies $\varepsilon^{(i)}_0 = -1$. If $T^{(i)}(k-1, 2) = k$ and $T^{(i)}(k, 2) = \overline{k}$, then we can see that $i \geq 2$ and

$$
T^{(i-2)}(k, 1) = T^{(i-2)}(k, 2) = k, \quad T^{(i-2)}(k, 1) = T^{(i-2)}(k, 2) = \overline{k}.
$$
Since it is not Case 2.1, we have $e_0^{(i)} = -1$. If $T^{(i)}(k - 1, 2) \leq k - 1$ and $T^{(i)}(k, 2) = k$, then we see that $T^{(i-1)}(k - 1, 1) \leq k - 1$ and $e_j^{(i-1)} = e_j$ for all $j$. Hence we have $e_0^{(i)} = e_k^{(i-1)} = -1$.

In Case 4, we have to show that $T^{(i-1)}(k, 2) \geq k + 1$ and $T^{(i-1)}(k + 1, 1) \geq k + 2$. If $T^{(i-1)}(k, 2) = k$, it follows from the condition (S'5) that $e_0^{(i-1)} = 1$. If $T^{(i-1)}(k, 2) = k$, we have $T^{(i-1)}(k - 1, 1) \leq k$ and $e_0^{(i-1)} = 1$. This contradicts $e_0^{(i-1)} \cdot e_k^{(i-1)} = -e_k(A(T^{(i-1)}))$. Therefore $T^{(i-1)}(k, 2) \geq k + 1$. But it follows from the condition $e_0^{(i-1)} \cdot e_k^{(i-1)} = -e_k(A(T^{(i-1)}))$ that $(T^{(i-1)}(k, 2), T^{(i-1)}(k + 1, 1)) \neq (k + 1, k + 1)$. So we have $T^{(i-1)}(k + 1, 1) \geq k + 2$.

The following three lemmas can be easily checked.

**Lemma 5.4.** For an element $Z = (U, \eta, \beta, k) \in X_i(\lambda) \cup X_{\infty}(\lambda)$ or $Z = (U, \eta) \in X_p(\lambda)$ and $\alpha \in \Gamma_n$, let $m_\alpha(Z)$ be the multiplicity $\alpha$ in $U$ and $\beta$. Then, for $1 \leq k \leq n$,

$$m_k(X^{(i-1)}) = m_k(X^{(i-1)}) = m_k(X^{(i)}) - m_k(X^{(i)}).$$

In particular, the map $I(\chi)$ is weight-preserving.

**Lemma 5.5.** If $X^{(i-1)} \in X_i(\lambda)$ and $X^{(i)} \in X_p(\lambda)$, then we have

$$\prod_{j=1}^n e_j = \prod_{j=0}^{i-1} e_j^{(i)}.$$

**Lemma 5.6.** If $X^{(i-1)}$, $X^{(i)} \in X_p(\lambda)$, then we have

$$\prod_{j=1}^n e_j = \prod_{j=0}^{i} e_j^{(i)}$$

except for the case where $T^{(i-1)}(k, 1)$ is the empty cell and $X^{(i-1)}$ satisfies the following conditions:

(a) There exists no integer $m \geq k + 2$ such that $T^{(i-1)}(m - 1, 1) = m$,

(b) $T^{(i-1)}(k - 1, 1) \geq k + 1$ and $T^{(i-1)}(k, 2) < T^{(i-1)}(k + 1, 1)$

(c) $e_0^{(i-1)} \cdot e_0^{(i-1)} = -e_0(A(T^{(i-1)}))$ if $T^{(i-1)}(k, 2) = k$.

**Lemma 5.7.** Let $l(T, e, \gamma) = (S, \delta)$, $\delta = (\delta_0, ..., \delta_n)$ and $\mu$ be the shape of $S$. If $l(\lambda) = l(\mu) = n$, then we have

$$\prod_{j=1}^n e_j = \prod_{j=1}^n \delta_j$$

hence $I(\lambda^{\mu}_{D(n)})$ is well-defined.
\textbf{Proof.} If \( l(\lambda) = n \) and \( T^{(i-1)}(k-1, 1) \geq k + 1 \), then there exists an integer \( m \geq k + 2 \) such that \( T^{(i-1)}(m - 1, 1) = m \), \( T^{(i-1)}(m, 1) = m \). Hence this lemma follows from Lemmas 5.5 and 5.6. \hfill \blacksquare

We conclude this section with examples.

\textbf{Example 1.} Let \( \lambda = (2, 2, 1), n \geq 3 \), and

\[
\begin{pmatrix}
1 & 2 \\
3 & 3 \\
\end{pmatrix}, \quad \varepsilon = (-1, 1, \varepsilon_3), \quad \gamma = 1.
\]

(Here we omit the signatures \( \varepsilon_4 = \cdots = \varepsilon_n = 1 \).) Then the sequence constructed in the insertion algorithm is

\[
X^{(0)} = \begin{pmatrix}
1 & 2 \\
3 & 3, \quad (-1, 1, \varepsilon_3), \quad 1, \quad 1 \\
\end{pmatrix} \quad \text{(Case 2.2)}
\]

\[
X^{(1)} = \begin{pmatrix}
2 & \Box \\
3 & 3, \quad (-1; 1, 1, \varepsilon_3) \\
\end{pmatrix} \quad \text{(Case 5)}
\]

\[
X^{(2)} = \begin{pmatrix}
3 & 3, \quad (1; 1, 1, -\varepsilon_3) \\
\end{pmatrix} \quad \text{(Case 6)}
\]

\[
X^{(3)} = \begin{pmatrix}
3 & \Box, \quad (1; 1, 1, -\varepsilon_3) \\
\end{pmatrix}.
\]

Hence, if \( \chi = \lambda_{D(n)} \) or \( \lambda^+_{D(n)} \), we have

\[
I(\chi)(T, \varepsilon, \gamma) = \begin{pmatrix}
2 & 3 \\
3 & (1, 1, -\varepsilon_3) \\
\end{pmatrix}.
\]

\textbf{Example 2.} Let \( \lambda, n, T, \) and \( \varepsilon \) be as in Example 1. And we take \( \gamma = 1 \). In this case,
If $n = 3$, then we have

$$I(\lambda_{D(3)})_{\text{Case 1}}(T, \varepsilon, \gamma) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ (-1, 1, 1, 1) \end{pmatrix}.$$ 

If $n \geq 4$, according to Case 5,

$$I(\chi)(T, \varepsilon, \gamma) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ (-1, 1, 1, 1) \end{pmatrix} & \text{if } \varepsilon = 1 \\ \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 4 & (-1, 1, 1, 1, -1) \end{pmatrix} & \text{if } \varepsilon = -1 \end{cases}.$$ 

**Example 3.** In this example, $\lambda = (2, 2, 2)$, $n \geq 3$, and

$$T = \begin{pmatrix} 1 & 2 \\ 3 & 3 \\ (-1, 1, \varepsilon_3), \gamma = 1 \end{pmatrix}.$$
Then

\[
X^{(0)} = \begin{pmatrix}
1 & 2 \\
3 & 3, \quad (-1, 1, \varepsilon_3), & 1 \\
3 & 3 \\
\end{pmatrix}
\]  (Case 1)

\[
X^{(1)} = \begin{pmatrix}
1 & 1 \\
3 & 3, \quad (-1, 1, \varepsilon_3), & 2 \\
3 & 3 \\
\end{pmatrix}
\]  (Case 1)

\[
X^{(2)} = \begin{pmatrix}
2 & 3, \quad (-1, -1, \varepsilon_3), & 3 \\
3 & 3 \\
\end{pmatrix}
\]  (Case 2).

If \(\varepsilon_3 = 1\), then according to Case 2.2,

\[
X^{(3)} = \begin{pmatrix}
1 & 1 \\
2 & 3, \quad (-1; -1, -1, 1) \\
\square & 3 \\
\end{pmatrix}
\]

\[
X^{(4)} = \begin{pmatrix}
1 & 1 \\
2 & 3, \quad (1; -1, -1, 1) \\
\square & 3 \\
\end{pmatrix}
\]

hence we have

\[
I(\chi)(T, \varepsilon, \gamma) = \begin{pmatrix}
1 & 1 \\
2 & 3, \quad (-1, -1, -1) \\
3 \\
\end{pmatrix}
\]

If \(\varepsilon_3 = -1\), then according to Case 2.1,

\[
X^{(3)} = \begin{pmatrix}
1 & 1 \\
2 & \square, \quad (1; -1, -1, 1) \\
3 & 3 \\
\end{pmatrix}
\]

\[
X^{(4)} = \begin{pmatrix}
1 & 1 \\
2 & 3, \quad (1; -1, -1, 1) \\
3 & \square \\
\end{pmatrix}
\]
hence we have

$$I(\chi)(T, e, \gamma) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 3 & (-1, -1, 1) \end{pmatrix}. $$

6. Deletion Algorithm for $SO(2n, \mathbb{C})$

In this section we describe the algorithm, called the deletion algorithm, which gives the inverse map $D(\chi)$ of $I(\chi)$, where $\chi = \lambda_{D(n)}$ or $\lambda_{D(n)}^\perp$.

**DEFINITION.** Let $T$ be a (punctured) $SO(2n)$-tableau. A partial signature vector compatible with $T$ is a vector $e = (e_1, \ldots, e_n) \in \{-1, 0, 1\}^n$ or $e = (e_0; e_1, \ldots, e_n) \in \{-1, 0, 1\}^{n+1}$ where $e_0 \neq 0$) satisfying the same conditions as in the definition of compatible signature vectors. (See Sections 2 and 4.) And, using partial signature vectors instead of signature vectors, we define the sets $\mathcal{K}_*(\lambda)$, $\mathcal{X}_*(\lambda)$, and $\mathcal{X}_p(\lambda)$ in the same manner as $\mathcal{K}_*(\lambda)$, $\mathcal{X}_*(\lambda)$, and $\mathcal{X}_p(\lambda)$.

Let us fix an element $(S, \delta)$ of $R(\chi)$. For this pair, we construct a sequence $Y(0), Y(1), \ldots, Y(M)$ of elements $Y(i) \in \mathcal{K}_*(\lambda)$ inductively until $Y(M)$ is an element of $\mathcal{K}_*(\lambda)$ of the form $\lambda = (S(M), \delta(M), b(M), 1)$. The definition of $D(\chi)(S, \delta)$ will be given afterward.

First we define $Y(0)$ as follows. Let $\mu$ be the shape of $S$. If $\mu \triangleright \lambda$ and the cell of $Y(\mu) - Y(\lambda)$ is in the $k$th row, then we define $Y(0) = (S(0), \delta(0), \beta(0), 1)$ as $Y(0) = (S(0), \delta(0), \beta(0), 1)$ a $\mathcal{K}_*(\lambda)$. If $\mu \triangleleft \lambda$, then let $S(0)$ be the punctured tableau of shape $\lambda$ obtained by appending to $S$ the empty cell at the cell of $Y(\lambda) - Y(\mu)$. And we put

$$\delta(0) = \begin{cases} 1 & \text{if } l(\lambda) < n \\ \epsilon \prod_{i=1}^n \delta_i & \text{if } l(\lambda) = n, \end{cases}$$

where $\epsilon = 1$ or $-1$ according as $\chi = \lambda_{D(n)}^+$ or $\lambda_{D(n)}^-$. Then $\delta(0) = (\delta(0), \delta(1), \ldots, \delta(n))$ and $Y(0) = (S(0), \delta(0)) \in \mathcal{K}_p(\lambda)$.

Next we define $Y(i)$ when $Y(0), Y(1), \ldots, Y(i-1)$ have been constructed. Suppose that $Y(i-1) = (S(i-1), \delta(i-1), \beta(i-1), k(i-1))$ is an element of $\mathcal{K}_*(\lambda)$ or $\mathcal{X}_*(\lambda)$. We put $k = k(i-1)$, $S' = \text{Del}(S(i-1), \beta(i-1), k - 1)$, and $\beta' = \text{bump}(S(i-1), \beta(i-1), k - 1)$. Then we consider the following two cases:

**Case 1.** $\beta' = k(i-1) - 1$ is bumped from the $(k(i-1), 1)$-cell of $S(i-1)$ and $S(i-1)(k-2, 1) = k - 1, \delta_k(i-1) = -1$.  

Case 2. Otherwise.

In Case 1, we define $S^{(i)}$ to be the punctured tableau obtained from $S'$ by puncturing the $(k - 2, 1)$-cell. If $\beta^{(i - 1)} = \tilde{b}$, then we put

$$\delta^{(i)}_j = \delta^{(i - 1)}_j$$

for $j \neq 0, b$

and determine $\delta^{(i)}_b$ by the compatibility with $S^{(i)}$ except for the case where

$$S^{(i)}(k, 1) = \tilde{k}, \quad S^{(i)}(k - 1, 1) = \beta^{(i - 1)} = k.$$

In this case, we put $\delta^{(i)}_k = 0$. As for the signature $\delta^{(i)}_0$ of the empty cell, we put

$$\delta^{(i)}_0 = \left\{ \begin{array}{ll} -\delta^{(i)}_a \cdot \varepsilon_u(A(S^{(i)})) & \text{if } S^{(i)}(k - 2, 2) < S^{(i)}(k - 1, 1) \\ -1 & \text{if } S^{(i)}(k - 2, 2) \geq S^{(i)}(k - 1, 1), \end{array} \right.$$

where $S^{(i)}(k - 2, 2) = \tilde{a}$.

In Case 2, we put

$$S^{(i)} = S' = \text{Del}(S^{(i - 1)}, \beta^{(i - 1)}, k - 1)$$

$$\beta^{(i)} = \beta' = \text{bump}(S^{(i - 1)}, \beta^{(i - 1)}, k - 1)$$

$$k^{(i)} = k - 1.$$

If $\beta^{(i - 1)} = \tilde{b}$ and $\beta^{(i)} = \tilde{a}$, then we put

$$\delta^{(i)}_j = \left\{ \begin{array}{ll} \varepsilon_u(S^{(i)}) & \text{if } j = a \\ \delta^{(i - 1)}_j & \text{if } j \neq a, b. \end{array} \right.$$

And $\delta^{(i)}_b$ is determined by the compatibility with $S^{(i)}$ except for the case where

$$S^{(i)}(k, 1) = \tilde{k}, \quad S^{(i)}(k - 1, 1) = \beta^{(i - 1)} = k.$$

In this case we define $\delta^{(i)}_k = 0$.

Next we consider the case where $Y^{(i - 1)} = (S^{(i - 1)}, \delta^{(i - 1)}) \in \tilde{X}_p(\lambda)$. In this case, we put $S'' = B(S^{(i - 1)})$. From Lemma 4.4, it is enough to consider the following three cases:

Case 3. $S''$ is an $SO(2n)$-tableau.

Case 4. $S^{(i - 1)}(k, 1)$ is the empty cell and $S^{(i - 1)}(k - 1, 1) \leq k - 1$ (or $k = 1$).

Case 5. $S^{(i - 1)}(k, l)$ ($l \geq 2$) is the empty cell and

$$S^{(i - 1)}(k + 1, 1) = \cdots = S^{(i - 1)}(k + 1, l - 1) = k + 1$$

$$S^{(i - 1)}(k + 1, l) = k + 1,$$

$$S^{(i - 1)}(k - 1, 1) \leq k \quad \text{or} \quad k = 1.$$
In Case 3, we define \( S^{(i)} = \hat{S}^{(i-1)} = B(S^{(i-1)}) \). The signature vector \( \delta^{(i)} \) is defined as follows:

**Case 3.1.** The empty cell of \( S^{(i)} \) is not in the first column. In this case we put \( \delta^{(i)} = \delta^{(i-1)} \).

**Case 3.2.** \( S^{(i-1)}(k, 2) \) and \( S^{(i)}(k, 1) \) are the empty cells. This case is divided as follows:

**Case 3.2.1.** \( S^{(i-1)}(k - 1, 1) \geq k + 1 \) and \( Y^{(i-1)} \) satisfies the following conditions:

(a) There exists an integer \( m \geq k + 2 \) such that \( S^{(i)}(m - 1, 1) = m \), \( S^{(i)}(m, 1) = \bar{m} \)

(b) \( S^{(i)}(k, 2) < S^{(i)}(k + 1, 1) \)

(c) \( \delta^{(i-1)}_a = -\varepsilon_a(S^{(i)}) \) if \( S^{(i)}(k, 2) = \bar{a} \).

**Case 3.2.2.** \( S^{(i-1)}(k - 1, 1) \geq k + 1 \) and \( Y^{(i-1)} \) satisfies the following conditions:

(a) There exists no integer \( m \geq k + 2 \) such that \( S^{(i)}(m - 1, 1) = m \), \( S^{(i)}(m, 1) = \bar{m} \)

(b) \( S^{(i)}(k, 2) < S^{(i)}(k + 1, 1) \)

(c) \( \delta^{(i-1)}_a = -\varepsilon_a(S^{(i)}) \) if \( S^{(i)}(k, 2) = \bar{a} \).

**Case 3.2.3.** Otherwise.

In Case 3.2.1, let \( m \geq k + 2 \) be the minimum integer such that \( S^{(i-1)}(m - 1, 1) = m \) and \( S^{(i-1)}(m, 1) = \bar{m} \). If \( S^{(i)}(k, 2) = S^{(i-1)}(k, 1) = \bar{a} \), we put

\[
\delta^{(i)}_j = \begin{cases} 
\varepsilon_a(S^{(i)}) & \text{if } j = a \\
-\delta^{(i-1)}_m & \text{if } j = m \\
\delta^{(i-1)}_j & \text{otherwise.}
\end{cases}
\]

In Case 3.2.2, we put

\[
\delta^{(i)}_j = \begin{cases} 
\varepsilon_a(S^{(i)}) & \text{if } j = a \\
\delta^{(i-1)}_j & \text{otherwise,}
\end{cases}
\]

where \( S^{(i)}(k, 2) = S^{(i-1)}(k, 1) = \bar{a} \).

In Case 3.2.3, we put

\[
\delta^{(i)}_j = \begin{cases} 
\delta^{(i-1)}_a \cdot \varepsilon_a(S^{(i)}) & \text{if } j = 0 \\
\varepsilon_a(S^{(i)}) & \text{if } j = a \\
\delta^{(i-1)}_j & \text{otherwise,}
\end{cases}
\]

where \( S^{(i)}(k, 2) = S^{(i-1)}(k, 1) = \bar{a} \).
Case 3.3. \( S^{(i-1)}(k+1, 1) \) and \( S^{(i)}(k, 1) \) are the empty cells. Then we put \( \delta^{(i)} = \delta^{(i-1)} \) except for the case where
\[
S^{(i-1)}(k-2, 1) = k \quad S^{(i-1)}(k-1, 1) = \bar{k} \quad \delta^{(i-1)}_0 = -1.
\]
In this case we put
\[
\delta^{(i)}_j = \begin{cases} 
1 & \text{if } j = 0 \\
-1 & \text{if } j = k \\
\delta^{(i-1)}_j & \text{otherwise.}
\end{cases}
\]

In Case 4, we construct an element \( Y^{(i)} = (S^{(i)}, \delta^{(i)}, \beta^{(i)}, k^{(i)}) \) of \( \mathfrak{X}_i(\lambda) \). Let us divide this case into five cases. Here we note that \( S^{(i-1)}(k, 2) \neq \bar{k} \), if the signature \( \delta^{(i-1)}_0 \) of empty cell is equal to 1.

Case 4.1. \( \delta^{(i-1)}_0 = 1 \), \( S^{(i-1)}(k, 2) \geq k + 1 \), and \( S^{(i-1)}(k-1, 2) \geq k \). Then we put
\[
S^{(i)}(p, q) = \begin{cases} 
k & \text{if } (p, q) = (k, 1) \\
S^{(i-1)}(p, q) & \text{otherwise}
\end{cases}
\]
\[
\delta^{(i)}_j = \begin{cases} 
-1 & \text{if } j = k \\
\delta^{(i-1)}_j & \text{otherwise}
\end{cases}
\]
\[
\beta^{(i)} = k
\]
\[
k^{(i)} = k.
\]

Case 4.2. \( \delta^{(i-1)}_0 = 1 \), \( S^{(i-1)}(k, 2) \geq k + 1 \), and \( S^{(i-1)}(k-1, 2) \leq \bar{k-1} \).
\[
S^{(i)}(p, q) = \begin{cases} 
k & \text{if } (p, q) = (k, 1) \\
S^{(i-1)}(p, q) & \text{otherwise}
\end{cases}
\]
\[
\delta^{(i)}_j = \begin{cases} 
1 & \text{if } j = k \\
\delta^{(i-1)}_j & \text{otherwise}
\end{cases}
\]
\[
\beta^{(i)} = \bar{k}
\]
\[
k^{(i)} = k.
\]

If \( \delta^{(i-1)}_0 = 1 \) and \( S^{(i-1)}(k, 2) = k \), let \( l \) be the integer such that \( S^{(i-1)}(k, l) = k \) and \( S^{(i-1)}(k, l+1) \neq k \). Then we have \( S^{(i-1)}(k, l+1) \geq k + 1 \) from the definition of punctured \( SO(2n) \)-tableaux.

Case 4.3. \( \delta^{(i-1)}_0 = 1 \), \( S^{(i-1)}(k, 2) = k \), and \( S^{(i-1)}(k-1, l+1) \geq k \).
\[
S^{(i)}(p, q) = \begin{cases} 
k & \text{if } (p, q) = (k-1, l), (k, 1) \\
\bar{k} & \text{if } (p, q) = (k, l) \\
S^{(i-1)}(p, q) & \text{otherwise}
\end{cases}
\]
\[ \delta_j^{(i)} = \begin{cases} 1 & \text{if } j = k \\ \delta_j^{(i-1)} & \text{otherwise} \end{cases} \]

\[ \beta^{(i)} = S^{(i-1)}(k - 1, l) \]

\[ k^{(i)} - k - 1. \]

**Case 4.4.** \( \delta_0^{(i-1)} = 1, \) \( S^{(i-1)}(k, 2) = k, \) and \( S^{(i-1)}(k - 1, l + 1) \leq k - 1 \) (or \( k = 1 \)).

\[ S^{(i)}(p, q) = \begin{cases} k & \text{if } (p, q) = (k, 1), (k, l) \\ S^{(i-1)}(p, q) & \text{otherwise} \end{cases} \]

\[ \delta_j^{(i)} = \begin{cases} 1 & \text{if } j = k \\ \delta_j^{(i-1)} & \text{otherwise} \end{cases} \]

\[ \beta^{(i)} = k \]

\[ k^{(i)} = k. \]

**Case 4.5.** \( \delta_0^{(i-1)} = -1. \)

In this case we note that \( S^{(i-1)}(k, 2) \geq k. \) So we define

\[ S^{(i)}(p, q) = \begin{cases} k & \text{if } (p, q) = (k, 1) \\ S^{(i-1)}(p, q) & \text{otherwise} \end{cases} \]

\[ \delta_j^{(i)} = \begin{cases} -1 & \text{if } j = k \\ \delta_j^{(i-1)} & \text{otherwise} \end{cases} \]

\[ \beta^{(i)} = k \]

\[ k^{(i)} = k. \]

In Case 5, we construct an element \( Y^{(i)} = (S^{(i)}, \delta^{(i)}, \beta^{(i)}, k^{(i)}) \) of \( \mathcal{X}_\lambda(\lambda). \) Let us divide this case into two cases.

**Case 5.1.** \( l = 2. \)

\[ S^{(i)}(p, q) = \begin{cases} k + 1 & \text{if } (p, q) = (k + 1, 1) \\ k + 1 & \text{if } (p, q) = (k, 2) \\ S^{(i-1)}(p, q) & \text{otherwise} \end{cases} \]

\[ \delta_j^{(i)} = \begin{cases} -1 & \text{if } j = k + 1 \\ \delta_j^{(i-1)} & \text{otherwise} \end{cases} \]

\[ \beta^{(i)} = k + 1 \]

\[ k^{(i)} = k + 1. \]
Case 5.2. \( l \geq 3 \).

\[
S^{(i)}(p, q) = \begin{cases} 
  k + 1 & \text{if } (p, q) = (k + 1, l - 1) \\
  k + 1 & \text{if } (p, q) = (k, l) \\
  S^{(i-1)}(p, q) & \text{otherwise}
\end{cases}
\]

\[
\delta_j^{(i)} = \delta_j^{(i-1)}
\]

\[
\beta^{(i)} = S^{(i-1)}(k, l - 1)
\]

\[
k^{(i)} = k.
\]

Now we have already constructed the sequence \( Y^{(0)}, \ldots, Y^{(M)} \) of elements of \( \tilde{\mathcal{H}}(\lambda) \) such that \( Y^{(M)} = (S^{(M)}, \delta^{(M)}, \beta^{(M)}, 1) \in \tilde{\mathcal{H}}_l(\lambda) \). Then \( D(S, \delta) = (T, \varepsilon, \gamma) \) is defined as follows. We put \( T = S^{(M)} \) and \( \gamma = \beta^{(M)} \). Now we define the signature vector \( \varepsilon \). (Note that \( \delta_k^{(M)} \) may be 0.) If \( \delta_k^{(M)} \neq 0 \), then we put \( \varepsilon_k = \delta_k^{(M)} \). Let \( k_1, \ldots, k_l \) \((k_1 < k_2 < \cdots < k_l)\) be the integers such that \( \delta_k^{(M)} = 0 \). The signature \( \varepsilon_{k_j} \) is determined inductively on \( j \). Suppose that \( \varepsilon_{k_1}, \ldots, \varepsilon_{k_{j-1}} \) is determined. Let \( k = k_j \) and \( i \) be the minimum integer such that \( \delta_k^{(i)} = 0 \). Then it follows from the above algorithm that \( Y^{(i-2)} \in \tilde{\mathcal{H}}_{\rho}(\lambda) \).

Now we define

\[
\varepsilon_k = \left( \prod_{t=0}^{k-1} \delta_t^{(i-2)} \right) \left( \prod_{t=1}^{k-1} \varepsilon_t \right).
\]

This completes the definition of the map \( D = D(\lambda_{D(n)}) \) or \( D(\lambda_{\rho_{D(n)}}) \).

Now we define

\[
\delta_j^{(i)} = \begin{cases} 
  \delta_j^{(i)} & \text{if } \delta_j^{(i)} \neq 0 \\
  \varepsilon_j & \text{if } \delta_j^{(i)} = 0
\end{cases}
\]

and \( \bar{Y}^{(i)} \) to be the element obtained from \( Y^{(i)} \) by replacing the partial signature vector \( \delta^{(i)} \) with \( \delta^{(i)} \). Then it follows from the following lemmas that \( D(\bar{\lambda}) \) is well-defined. These lemmas can be checked case by case.

**Lemma 6.1.** For \( 0 \leq i \leq M \), we have

\[
Y^{(i)} \in \tilde{\mathcal{H}}(\lambda), \quad \bar{Y}^{(i)} \in \mathcal{X}(\lambda).
\]

**Proof.** It follows from Lemma 3.6 that \( S^{(i)} \) is semi-standard. Then it is clear that \( S^{(i)} \) is a (punctured) \( SO(2n) \)-tableau. \( \square \)

**Lemma 6.2.** If \( Y^{(i-1)} \in \tilde{\mathcal{H}}_{\rho}(\lambda) \) and \( Y^{(i)} \in \tilde{\mathcal{H}}_{\lambda}(\lambda) \), then we have

\[
\prod_{j=0}^{n} \delta_j^{(i-1)} = \prod_{j=1}^{n} \varepsilon_j.
\]
Lemma 6.3. If $Y^{(i-1)}$, $Y^{(i)} \in \tilde{X}_\mu(\lambda)$, then we have

$$\prod_{j=0}^n \delta_j^{(i-1)} = \prod_{j=0}^n \delta_j^{(i)}$$

except for Case 3.2.2.

Lemma 6.4. If $l(\lambda) = n$, then

$$\prod_{j=1}^n \delta_j^{(0)} = \prod_{j=1}^n \varepsilon_j,$$

where the product is taken over $0 \leq j \leq n$ if $\mu \preceq \lambda$ and over $1 \leq j \leq n$ if $\mu \succ \lambda$.

Proof. If $\mu \succ \lambda$, then no bumping occurs in the first column, so that $\delta_j = \varepsilon_j$ for all $j$. If $\mu \preceq \lambda$, then it follows from Lemmas 6.2 and 6.3 that

$$\prod_{j=0}^n \delta_j^{(0)} = \prod_{j=1}^n \varepsilon_j.$$ 

Proof of Theorem 5.1. We fix an irreducible character $\chi = \lambda_{D(n)}$ or $\lambda_{D(n)}^\pm$. For $(T, \varepsilon, \gamma) \in L(\chi)$, let $X^{(0)}$, ..., $X^{(N)}$ be the sequence of elements of $X(\lambda)$ constructed in the insertion algorithm. For $(S, \delta) \in R(\chi)$, let $Y^{(0)}$, ..., $Y^{(M)}$ be the sequence of elements of $X(\lambda)$ constructed in the deletion algorithm, and $\bar{Y}^{(0)}$, ..., $\bar{Y}^{(M)}$ be the sequence of elements of $X(\lambda)$ obtained as above by replacing the partial signature $\delta^{(i)}$ with $\delta^{(i)}$. Now we remark the following correspondence.

<table>
<thead>
<tr>
<th>Insertion Algorithm</th>
<th>Deletion Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>Case 2</td>
</tr>
<tr>
<td>Case 2.1</td>
<td>Case 5.1</td>
</tr>
<tr>
<td>Case 2.2</td>
<td>Case 4.1, 4.5</td>
</tr>
<tr>
<td>Case 3.1</td>
<td>Case 5.2</td>
</tr>
<tr>
<td>Case 3.2</td>
<td>Case 4.3</td>
</tr>
<tr>
<td>Case 3.3</td>
<td>Case 4.2, 4.4</td>
</tr>
<tr>
<td>Case 4</td>
<td>Case 1</td>
</tr>
<tr>
<td>Case 5</td>
<td>Case 3.2.1</td>
</tr>
<tr>
<td>Case 6.1</td>
<td>Case 3.1</td>
</tr>
<tr>
<td>Case 6.2</td>
<td>Case 3.2.2, 3.2.3</td>
</tr>
<tr>
<td>Case 6.3</td>
<td>Case 3.3</td>
</tr>
</tbody>
</table>

Then it follows from Lemma 3.5 and 4.2 that, if $I(\chi)(T, \varepsilon, \gamma) = (S, \delta)$ (or $D(\chi)(S, \delta) = (T, \varepsilon, \gamma)$), $M = N$ and $X^{(i)} = \bar{Y}^{(N-i+1)}$. Therefore we have

$$D(\chi) I(\chi)(T, \varepsilon, \gamma) = (T, \varepsilon, \gamma)$$

$$I(\chi) D(\chi)(S, \delta) = (S, \delta).$$
Finally we consider the irreducible decomposition of the tensor space \( W \otimes^k \) as \( SO(2n) \)-module, where \( W = \mathbb{C}^{2n} \) is the natural representation space of \( SO(2n, \mathbb{C}) \).

**Definition.** For an irreducible character \( \chi \) of \( SO(2n, \mathbb{C}) \), we denote by \( S(\chi) \) the set of all irreducible characters \( \eta \) appearing in the product \( \chi \cdot (1)_{D(n)} \). Here \( (1)_{D(n)} = x_1 + x_1^{-1} + \cdots + x_n + x_n^{-1} \) is the character of the natural representation.

Then, from Theorem 5.1 and Corollary 5.2, we can see that the map

\[
I(\chi): \text{Tab}(\chi) \times \Gamma_n \to \bigsqcup_{\eta \in S(\chi)} \text{Tab}(\eta)
\]

is a weight-preserving bijection. Moreover we have

**Proposition 6.5.** (1) If \( \chi = \lambda_{D(n)} \) (\( l(\lambda) \leq n - 2 \)), then

\[
S(\lambda_{D(n)}) = \{ \mu_{D(n)} : \mu \vdash \lambda \} \cup \{ \mu_{D(n)} : \mu \ll \lambda \}.
\]

(2) If \( \chi = \lambda_{D(n)} \) (\( l(\lambda) = n - 1 \)), then

\[
S(\lambda_{D(n)}) = \{ \mu_{D(n)} : \mu \vdash \lambda, l(\mu) = n - 1 \} \cup \{ \mu_{D(n)} : \mu \ll \lambda \}
\]

\[
\cup \{ v_{D(n)}^+, v_{D(n)}^- : v = (\lambda_1, \ldots, \lambda_{n-1}, 1) \}.
\]

(3) If \( \chi = \lambda_{D(n)}^{\pm} \) (\( l(\lambda) = n \)) and \( \lambda_n = 1 \), then

\[
S(\lambda_{D(n)}^{\pm}) = \{ \mu_{D(n)}^{\pm} : \mu \vdash \lambda \} \cup \{ \mu_{D(n)}^{\pm} : \mu \ll \lambda, l(\mu) = n \}
\]

\[
\cup \{ v_{D(n)} : v = (\lambda_1, \ldots, \lambda_{n-1}) \}.
\]

(4) If \( \chi = \lambda_{D(n)}^{\pm} \) (\( l(\lambda) = n \)) and \( \lambda_n \geq 2 \), then

\[
S(\lambda_{D(n)}^{\pm}) = \{ \mu_{D(n)}^{\pm} : \mu \vdash \lambda \} \cup \{ \mu_{D(n)}^{\pm} : \mu \ll \lambda \}.
\]

**Definition.** For an irreducible character \( \chi \) of \( SO(2n, \mathbb{C}) \) and \( k \in \mathbb{N} \), let \( \mathcal{M}^k(\chi) \) be the set of all sequence \( (\chi_0, \chi_1, \ldots, \chi_k) \) of the irreducible characters \( \chi_i \) of \( SO(2n, \mathbb{C}) \) satisfying the following conditions:

1. \( \chi_0 = \varnothing_{D(n)} \) (the trivial character).
2. \( \chi_k = \chi \).
3. \( \chi_i \in S(\chi_{i-1}) \) (\( 1 \leq i \leq k \)).

We note that \( \mathcal{M}^k(\lambda_{D(n)}) \) or \( \mathcal{M}^k(\lambda_{D(n)}^{\pm}) \) is empty unless \( |\lambda| \equiv k \) (mod 2) and \( |\lambda| \leq k \).

For a word \( w = y_1 y_2 \cdots y_k \in \Gamma_n^k \), we define \( (P_i, Q_i) \in \bigsqcup_{\chi} \mathcal{M}^k(\chi) \times \text{Tab}(\chi) \) inductively. First we put \( (P_0, Q_0) = (\varnothing_{D(n)}, \varnothing) \), where \( \varnothing \) is the unique
element of \( \text{Tab}(\emptyset_{D(n)}) \). If \( P_{i-1} = (\chi_0, \ldots, \chi_{i-1}) \) and \( Q_{i-1} \in \text{Tab}(\chi_{i-1}) \), then we put
\[
Q_i = I(\chi_{i-1})(Q_{i-1}, \gamma_i) \\
P_i = (\chi_0, \ldots, \chi_{i-1}, \chi_i),
\]
where \( Q_i \in \text{Tab}(\chi_i) \). Then we define
\[
P(w) = P_k, \\
Q(w) = Q_k.
\]

**Theorem 6.6.** The map
\[
I_n^k \rightarrow \bigsqcup \mathcal{M}(\chi) \times \text{Tab}(\chi) \\
w \mapsto (P(w), Q(w))
\]
is a weight-preserving bijection.

**Corollary 6.7.**
\[
(x_1 + x_1^{-1} + \cdots + x_n + x_n^{-1})^k = \sum_{\chi} \# \mathcal{M}(\chi) \cdot \chi
\]
where \( \chi \) runs over all irreducible characters of \( SO(2n, \mathbb{C}) \).

**References**