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Existence and roughness of exponential dichotomies of linear dynamic equations on time scales☆

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ABSTRACT

In this paper, we define the exponential dichotomy of linear dynamic equations on time scales, then we present perturbation theorems on the roughness of exponential dichotomy, and develop several explicit sufficient criteria for linear dynamic equations to have an exponential dichotomy. As applications of the criteria of exponential dichotomy, we derive some new sufficient conditions for the existence of periodic solutions of semi-linear dynamic equations and nonlinear dynamic equations on time scales.

1. Introduction

In the mathematical theory of dynamical systems, the well-known and established notion of an exponential dichotomy generalizes the concept of hyperbolicity from autonomous to nonautonomous linear systems and plays an important role in the analysis of nonautonomous dynamical systems. The development of exponential dichotomy of linear differential equations can be traced back to Perron [1] who initially introduced the terminology to study the problem of conditional stability of linear systems. Since then, exponential dichotomy has been extensively studied and applied in investigating differential equations; see, for example, Fink [2], Coppel [3], Chow [4], Pliss [5] et al. and references cited therein. Li [6] establishes analogous results for nonautonomous discrete-time dynamical systems, and the application of exponential dichotomy in investigating such systems then sees intensive development in the work of many authors [7,8,3,9,10]. Exponential dichotomy is very important in both theory and applications of the nonautonomous continuous and discrete dynamical systems.

Recently, Pötzsche [11,12] introduces the notion of the exponential dichotomy in the calculus on measure chains or time scales, which originates from [13,14] and allows a simultaneous treatment of differential equations, difference equations and dynamic equations on general time scales. With such a framework, many properties and applications of exponential dichotomies on measure chains or time scales have been discussed within a certain range such as the spectral notion [15], ordinary dichotomy [16], invariant manifolds [17–19], and the Hartman–Grobman theorems [20,21]. In fact, there are many aspects of exponential dichotomies on measure chains or time scales yet to be explored.

One of the most important and useful properties of exponential dichotomies in theory and applications is its roughness under perturbations. Roughly speaking, if a homogeneous linear dynamic equation has an exponential dichotomy, then all

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"neighboring" systems also have an exponential dichotomy with a similar projection. Roughness of exponential dichotomy was first proved by Massera and Schäffer [22], and since then has been extensively studied for continuous or discrete dynamical systems [7,4,3,23,5]. However, there are no similar results available for exponential dichotomy on general time scales yet. This is indeed the first motivation of the present paper. We will establish several new perturbation theorems on the roughness of exponential dichotomy in comparison to [24], and obtain a more accurate exponential estimate.

The periodic problem of dynamic equations on general time scales is a very interesting topic, which has been studied on the basis of several different approaches such as the coincidence degree theory [25,26], the Krasnosel'skiĭ fixed point theorem [27–30], the nonlinear Leray–Schauder alternative [31], the bounded solutions and the characteristic multipliers [32]. It is well known that the exponential dichotomy is one of the most important methods and tools in the study of periodic solutions of differential equations and difference equations. Therefore, it is reasonable to explore periodic solutions of dynamic equations on time scales with the help of exponential dichotomy. In order to carry out the analytical studies of dynamic equations on time scales, it is necessary to understand the conditions for the existence of an exponential dichotomy for linear dynamic equations on time scales. One can find sufficient conditions for the uniform exponential stability (as a special case of dichotomies) in [33] and sufficient dichotomy conditions for the time-invariant and periodic case in [11]. However, to the best of our knowledge, there are no sufficient criteria for the existence of exponential dichotomies for dynamic equations on general time scales. This is further motivation of this study.

In this paper, we define the exponential dichotomy of linear dynamic equations on time scales in Section 3 and discuss its roughness in Section 4. In Section 5, we establish several explicit sufficient criteria for linear dynamic equations to have an exponential dichotomy. As an application of exponential dichotomy and the results obtained in previous sections, we investigate the existence of periodic solutions of general high-dimensional semi-linear dynamic equations and nonlinear dynamic equations on time scales in Section 6.

2. Preliminaries

To make this paper self-contained, we will introduce some basic terminology and results of the calculus on time scales; details can be found in [13,14].

Let $\mathbb{T}$ be a *time scale*, i.e., an arbitrary nonempty closed subset of the real numbers in $\mathbb{R}$. Throughout this paper, the time scale $\mathbb{T}$ is assumed to be unbounded above and below.

Definition 2.1. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+ = [0, \infty)$ are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad \text{and} \quad \mu(t) = \sigma(t) - t \quad \text{for } t \in \mathbb{T},$$

respectively. If $\sigma(t) = t$, t is called right-dense (otherwise: right-scattered), and if $\rho(t) = t$, then t is called left-dense (otherwise: left-scattered).

Definition 2.2. Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) given by

$$f^{\Delta}(t) := \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

In this case, $f^{\Delta}(t)$ is called the delta (or Hilger) derivative of f at t. Moreover, f is said to be delta or Hilger differentiable on $\mathbb{T}$ if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$. Then we define

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for } s, r \in \mathbb{T}.$$

Definition 2.3. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limit exists (finite) at all left-dense points in $\mathbb{T}$.

The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ $(\mathbb{R}^{n\times n})$ will be denoted by $\mathrm{C}_{\mathrm{rd}}(\mathbb{T})$; meanwhile, the set of functions $f : \mathbb{T} \to \mathbb{R}$ $(\mathbb{R}^{n\times n})$ that are differential and whose derivatives are rd-continuous is denoted by $\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T})$. A function $p : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. An $n \times n$ matrix-valued function $A(t)$ on a time scale $\mathbb{T}$ is called *regressive* provided $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}$. The set of such regressive and rd-continuous functions is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})(\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n}))$. The set of all *regressive* functions on time scales forms an Abelian group under the addition $\oplus$ defined by $p \oplus q \triangleq p + q + \mu(t)pq$. Meanwhile, the additive inverse in this group is denoted by $\ominus p \triangleq -\frac{p}{1+\mu(t)p}$.

Definition 2.4. If $p \in \mathcal{R}$, then we define the exponential function by

$$e_p(t,s) = \begin{cases} \exp\left(\int_s^t p(\tau)\Delta\tau\right), & \mu(\tau) = 0; \\ \exp\left(\int_s^t \frac{1}{\mu(\tau)}\mathrm{Log}(1+p(\tau)\mu(\tau))\Delta\tau\right), & \mu(\tau) \neq 0; \end{cases} \qquad \text{for } s, t \in \mathbb{T},$$

where Log is the principal logarithm.

Theorem 2.1. *If $p \in \mathcal{R}$ and $t, s, r \in \mathbb{T}$, then*

$$e_p(t,t) \equiv 1, \qquad e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t), \qquad e_p(t,s)e_p(s,r) = e_p(t,r), \qquad [e_p(\cdot,s)]^{\Delta} = pe_p(\cdot,s).$$

Theorem 2.2. *Let $p \in \mathrm{C_{rd}}(\mathbb{T})$, $b \geq 0$, and $a \in \mathbb{R}$. Then*

$$p(t) \leq a + b\int_{t_0}^t p(\tau)\Delta\tau \quad \textit{for all } t \in \mathbb{T}$$

implies

$$p(t) \leq ae_b(t,t_0) \quad \textit{for all } t \in \mathbb{T}.$$

In this paper, we assume that there exists a positive number χ such that $\sup_{t\in\mathbb{T}} \mu(t) = \chi$. To facilitate the discussion below, we introduce some notation:

$$\mathbb{T}^+ = [0,\infty) \cap \mathbb{T}, \qquad \vartheta = \min\{[0,\infty) \cap \mathbb{T}\}, \qquad [a,b] = [a,b] \cap \mathbb{T} = \{t \in \mathbb{T} : a \leq t \leq b\}, \quad a, b \in \mathbb{R}.$$

The norm of $\mathbb{R}^n$ in the present paper is l^∞ norm, i.e., $|x| = \sup_i |x_i|,\ x \in \mathbb{R}^n$.

3. Exponential dichotomy on time scales

Consider the following linear dynamic equation on time scales:

$$x^{\Delta}(t) = A(t)x(t), \tag{3.1}$$

where $A(t) \in \mathcal{R}$ is an $n \times n$ matrix-valued function on $\mathbb{T}$.

Now we define exponential dichotomy on time scales for (3.1).

Definition 3.1. (3.1) is said to have an exponential dichotomy or to be exponentially dichotomous on $\mathbb{T}$ if there exist a projection matrix P (i.e., $P^2 = P$) on $\mathbb{R}^n$ and positive constants K_i and α_i, $i = 1, 2$, such that

$$\begin{aligned} &|X(t)PX^{-1}(s)| \leq K_1 e_{\ominus\alpha_1}(t,s), \quad t \geq s, \\ &|X(t)(I-P)X^{-1}(s)| \leq K_2 e_{\ominus\alpha_2}(s,t), \quad t \leq s, \end{aligned} \tag{3.2}$$

where $X(t)$ is a fundamental solution matrix of (3.1) and I is the identity matrix. When (3.2) holds with $\alpha_1 = \alpha_2 = 0$, (3.1) is said to possess an ordinary dichotomy.

Remark 3.1. If $\mathbb{T} = \mathbb{R}$, then Definition 3.1 agrees with the classical definition of exponential dichotomy for nonautonomous linear differential equations [3]. If $\mathbb{T} = \mathbb{Z}$, then (3.2) reduces to the usual dichotomy estimates [9,10] for the linear difference equation

$$\begin{aligned} &|X(t)PX^{-1}(s)| \leq K_1\left(\frac{1}{1+\alpha_1}\right)^{t-s}, \quad t \geq s, \\ &|X(t)(I-P)X^{-1}(s)| \leq K_2\left(\frac{1}{1+\alpha_2}\right)^{s-t}, \quad t \leq s. \end{aligned}$$

For the convenience of later discussion, we derive an equivalent definition of the exponential dichotomy for (3.1).

Lemma 3.1. *(3.1) has an exponential dichotomy if and only if $P(t) = X(t)PX^{-1}(t)$ is uniformly bounded and there exist positive constants B_i and α_i $(i = 1, 2)$ such that*

$$\begin{aligned} &|X(t)P\xi| \leq B_1 e_{\ominus\alpha_1}(t,s)|X(s)P\xi|, \quad t \geq s, \\ &|X(t)(I-P)\xi| \leq B_2 e_{\ominus\alpha_2}(s,t)|X(s)(I-P)\xi|, \quad t \leq s, \end{aligned} \tag{3.3}$$

where ξ is an arbitrary n-dimensional vector.

Proof. (Necessity) It is obvious that $P(t)$ is uniformly bounded if we set $s = t$ in (3.2). For any vector ξ, we have

$$|X(t)P\xi| \le |X(t)PX^{-1}(s)X(s)P\xi| \le K_1 e_{\ominus\alpha_1}(t,s)|X(s)P\xi| \quad \text{for } t \ge s.$$

Similarly, one can show that

$$|X(t)(I-P)\xi| \le K_2 e_{\ominus\alpha_2}(s,t)|X(s)(I-P)\xi| \quad \text{for } t \le s.$$

(Sufficiency) It follows that there is an $N_0 > 0$ such that

$$|P(t)| \le N_0, \qquad |I - P(t)| \le N_0 + 1.$$

Set $\xi = X^{-1}(s)x_0$, $x_0 \in \mathbb{R}^n$; then

$$|X(t)PX^{-1}(s)x_0| \le B_1 e_{\ominus\alpha_1}(t,s)|X(s)PX^{-1}(s)x_0| \le N_0 B_1 e_{\ominus\alpha_1}(t,s)|x_0| \quad \text{for } t \ge s.$$

Since x_0 is arbitrary, we thus conclude

$$|X(t)PX^{-1}(s)| \le K_1 e_{\ominus\alpha_1}(t,s), \quad t \ge s,$$

where $K_1 = N_0 B_1$. Similarly, one has

$$|X(t)(I-P)X^{-1}(s)| \le K_2 e_{\ominus\alpha_2}(s,t), \quad t \le s,$$

where $K_2 = (N_0 + 1)B_2$. ■

4. Roughness of exponential dichotomy

In order to study the roughness of exponential dichotomy, we consider the following linearly perturbed dynamic equation:

$$x^{\Delta}(t) = (A(t) + B(t))x(t), \tag{4.1}$$

where $A(t) + B(t) \in \mathcal{R}$. For an interval $\mathbb{J} \subseteq \mathbb{T}$, let $BC_{\mathrm{rd}} = BC_{\mathrm{rd}}\ (\mathbb{J}, \mathbb{R}^n)(BC_{\mathrm{rd}}(\mathbb{J}, \mathbb{R}^{n\times n}))$ and $BC = BC\ (\mathbb{J}, \mathbb{R}^n)(BC(\mathbb{J}, \mathbb{R}^{n\times n}))$ be the sets of the bounded rd-continuous functions and the bounded continuous functions on $\mathbb{J}$, respectively. It is easy to show that BC_{rd} and BC are both Banach spaces when endowed with the supremum norm $\|\cdot\|$.

To study the roughness of the exponential dichotomy of (3.1) under the perturbation $B(t)$, we are interested in finding appropriate conditions on $B(t)$ such that (4.1) is also exponentially dichotomous. In the following discussion of this section, the solution is in a Caratheodory sense. The corresponding required Lebesgue theory is due to [34]. In addition, one can find the treatment of the piecewise rd-continuous inhomogeneities in [11].

First, we show that (4.1) has an ordinary dichotomy on $\mathbb{J} = \mathbb{T}^+$ if (3.1) has an exponential dichotomy. In order to establish some useful lemmas, consider the nonhomogeneous linear dynamic equation

$$x^{\Delta}(t) = [A(t) + B(t)]x(t) + f(t) \tag{4.2}$$

where $A(t), A(t) + B(t) \in \mathcal{R}$ are $n \times n$ matrix-valued functions on $\mathbb{T}$, $f \in L^1$, and $L^1 = L^1(\mathbb{J}, \mathbb{R}^n)$ is the Banach space of Lebesgue integrable functions defined on $\mathbb{J}$.

Lemma 4.1. *Assume that (3.1) has an exponential dichotomy on $\mathbb{T}^+$, and $B(t) \in BC_{\mathrm{rd}}$ with*

$$\|B\| \left(\frac{K_1(1 + \alpha_1\chi)}{\alpha_1} + \frac{K_2}{\alpha_2} \right) < 1. \tag{4.3}$$

Then for each $f \in L^1$, (4.2) has a unique bounded solution.

Proof. For any given $y \in BC$, define the mapping T by

$$(Ty)(t) = \int_{\vartheta}^{t} X(t)PX^{-1}(\sigma(\tau))(B(\tau)y(\tau) + f(\tau))\Delta\tau - \int_{t}^{\infty} X(t)(I-P)X^{-1}(\sigma(\tau))(B(\tau)y(\tau) + f(\tau))\Delta\tau.$$

Obviously, $(Ty)(t)$ is continuous, and

$$\begin{aligned}
|(Ty)(t)| \le\ & \|B\|\,\|y\| \left(\int_{\vartheta}^{t} |X(t)PX^{-1}(\sigma(\tau))|\Delta\tau + \int_{t}^{\infty} |X(t)(I-P)X^{-1}(\sigma(\tau))|\Delta\tau \right) \\
& + \int_{\vartheta}^{t} |X(t)PX^{-1}(\sigma(\tau))||f(\tau)|\Delta\tau + \int_{t}^{\infty} |X(t)(I-P)X^{-1}(\sigma(\tau))||f(\tau)|\Delta\tau \\
\le\ & \|B\|\,\|y\| \left(K_1 \int_{\vartheta}^{t} e_{\ominus\alpha_1}(t,\sigma(\tau))\Delta\tau + K_2 \int_{t}^{\infty} e_{\ominus\alpha_2}(\sigma(\tau),t)\Delta\tau \right) + \max(K_1, K_2) \int_{\vartheta}^{\infty} |f(\tau)|\Delta\tau
\end{aligned}$$

$$
\begin{aligned}
&= \|B\|\,\|y\| \left(K_1 \int_{\vartheta}^{t} (1+\mu(\tau)\alpha_1) e_{\alpha_1}(\tau,t)\Delta\tau + K_2 \int_{t}^{\infty} (1 \ominus \alpha_2\mu(\tau)) e_{\ominus\alpha_2}(\tau,t)\Delta\tau \right) \\
&\quad + \max(K_1,K_2) \int_{\vartheta}^{\infty} |f(\tau)|\Delta\tau \\
&= \|B\|\,\|y\| \left(\frac{K_1(1+\alpha_1\chi)}{\alpha_1} \int_{\vartheta}^{t} \alpha_1 e_{\alpha_1}(\tau,t)\Delta\tau - \frac{K_2}{\alpha_2} \int_{t}^{\infty} \ominus\alpha_2 e_{\ominus\alpha_2}(\tau,t)\Delta\tau \right) + \max(K_1,K_2) \int_{\vartheta}^{\infty} |f(\tau)|\Delta\tau \\
&= \|B\|\,\|y\| \left(\frac{K_1(1+\alpha_1\chi)}{\alpha_1} + \frac{K_2}{\alpha_2} \right) + \max(K_1,K_2) \int_{\vartheta}^{\infty} |f(\tau)|\Delta\tau.
\end{aligned}
$$

Then T maps BC into BC. Moreover, for any $y_1, y_2 \in BC$, we have

$$
\|Ty_1 - Ty_2\| \le \|B\| \left(\frac{K_1(1+\alpha_1\chi)}{\alpha_1} + \frac{K_2}{\alpha_2} \right) \|y_1 - y_2\|.
$$

Therefore, T is a contraction mapping. By the Contraction Mapping Principle, there exists a unique fixed point $y(t) \in BC$ such that $y(t) = (Ty)(t)$, which is a bounded solution of (4.2). ■

Assume that U_1 is the subspace of $\mathbb{R}^n$ consisting of the initial values of all bounded solutions of (4.1) and U_2 is any fixed subspace of $\mathbb{R}^n$ supplementary to U_1 such that $\mathbb{R}^n$ can be written as the direct sum $\mathbb{R}^n = U_1 \bigoplus U_2$. With the help of Lebesgue theory on measure chains (see [34] or [11]), similar to what is done in Proposition 3.4 in [3], we claim that:

Lemma 4.2. *If (4.2) has a bounded solution for every function $f \in L^1$, then there exists a positive constant r such that, for every $f \in L^1$, the unique bounded solution $y(t)$ of (4.2) with $y(\vartheta) \in U_2$ satisfies $\|y\| \le r\|f\|_{L^1}$.*

Theorem 4.1. *Assume that (3.1) has an exponential dichotomy on $\mathbb{T}^+$ with $X(\vartheta) = I$, $A(t), B(t) \in BC_{\mathrm{rd}}$ and the inequality (4.3) is satisfied. Then (4.1) has an ordinary dichotomy on $\mathbb{T}^+$ with a projection Q similar to the projection P.*

Proof. Consider a matrix function $Z \in BC$ with $\|Z\| = \sup_{t\ge\vartheta} |Z(t)|$, and the mapping T defined by

$$
(TZ)(t) = X(t)P + \int_{\vartheta}^{t} X(t)PX^{-1}(\sigma(\tau))B(\tau)Z(\tau)\Delta\tau - \int_{t}^{\infty} X(t)(I-P)X^{-1}(\sigma(\tau))B(\tau)Z(\tau)\Delta\tau.
$$

It follows that

$$
|(TZ)(t)| \le K_1 + \|B\|\,\|Z\| \left(\frac{K_1(1+\alpha_1\chi)}{\alpha_1} + \frac{K_2}{\alpha_2} \right)
$$

and

$$
\|TZ_1 - TZ_2\| \le \|B\| \left(\frac{K_1(1+\alpha_1\chi)}{\alpha_1} + \frac{K_2}{\alpha_2} \right) \|Z_1 - Z_2\| \quad \text{for } Z_1, Z_2 \in BC.
$$

It is clear that $T : BC \to BC$ is a contraction mapping. Therefore, there exists a unique fixed point $Y_1(t)$ such that

$$
Y_1(t) = X(t)P + \int_{\vartheta}^{t} X(t)PX^{-1}(\sigma(\tau))B(\tau)Y_1(\tau)\Delta\tau - \int_{t}^{\infty} X(t)(I-P)X^{-1}(\sigma(\tau))B(\tau)Y_1(\tau)\Delta\tau. \tag{4.4}
$$

Obviously, $Y_1(t)$ is a bounded solution of (4.1). In addition, we also show that

$$
Y_1(t)P = X(t)P + \int_{\vartheta}^{t} X(t)PX^{-1}(\sigma(\tau))B(\tau)Y_1(\tau)P\Delta\tau - \int_{t}^{\infty} X(t)(I-P)X^{-1}(\sigma(\tau))B(\tau)Y_1(\tau)P\Delta\tau.
$$

Then $Y_1(t)P$ is also a fixed point of T, so we have $Y_1(t)P = Y_1(t)$. Now let $Q = Y_1(\vartheta)$; then we obtain $QP = Q$. Putting $t = s$ and multiplying both sides of (4.4) by $X(t)PX^{-1}(s)$, we then have

$$
X(t)PX^{-1}(s)Y_1(s) = X(t)P + \int_{\vartheta}^{s} X(t)PX^{-1}(\sigma(\tau))B(\tau)Y_1(\tau)\Delta\tau. \tag{4.5}
$$

If we set $t = s = \vartheta$ in the above identity, we conclude that $PQ = P$. It is straightforward to show that $Y_1(t)Q$ is also a fixed point of T such that $Y_1(t)Q = Y_1(t)$. Letting $t = \vartheta$, we then have $Y_1(\vartheta)Q = Y_1(\vartheta)$, that is, $Q^2 = Q$. These imply that Q is a projection. It follows from $QP = Q$ and $PQ = P$ that $L = I - P + Q$ is invertible, with inverse $L^{-1} = I + P - Q$ such that $Q = LPL^{-1}$. Therefore, the projection Q is similar to the projection P.

Assume that $Y(t)$ is a fundamental solution matrix of the system (4.1) with $Y(\vartheta) = I$; then we have $Y_1(t) = Y(t)Q$. We then can conclude that $Q[\mathbb{R}^n] \subseteq U_1$. Define

$$
G(t,s) = \begin{cases} Y(t)QY^{-1}(s), & \text{for } t > s \ge \vartheta, \\ -Y(t)(I-Q)Y^{-1}(s), & \text{for } s > t \ge \vartheta \end{cases}
$$

and consider $y(t) = \int_{\vartheta}^{\infty} G(t, \sigma(\tau))f(\tau)\Delta\tau$. For a fixed $t_1 \in \mathbb{T}^+$, we choose a function $f \in L^1$ which vanishes for $t \geq t_1$. Since $y(t) = Y(t)Q \int_{\vartheta}^{t_1} Y^{-1}(\sigma(\tau))f(\tau)\Delta\tau$ for $t \geq t_1$ and $y(\vartheta) = -(I - Q) \int_{\vartheta}^{t_1} Y^{-1}(\sigma(\tau))f(\tau)\Delta\tau \in U_2$, then one can show that $y(t) = \int_{\vartheta}^{t_1} G(t, \sigma(\tau))f(\tau)\Delta\tau$ is a bounded solution of (4.1). By Lemmas 4.1 and 4.2, we have $\|y\| \leq r\|f\|_{L^1}$. For any fixed point $s \in \mathbb{T}^+$, we have three cases as in the following.

Case 1. s is right-dense. Then there exist a sequence of time scale points $s_k \in \mathbb{T}$ $(s_k > s)$, $k \in \mathbb{N}$, such that $\lim_{k\to\infty} s_k = s$ from the right. Let $h_k = s_k - s$ and define f as

$$f(t) = \begin{cases} \xi, & \text{for } s \leq t \leq s + h_k, \\ 0, & \text{otherwise,} \end{cases}$$

where ξ is any fixed constant vector, $s \geq \vartheta$. Therefore,

$$|y(t)| = \left| \int_s^{s+h_k} G(t, \sigma(\tau))\xi\Delta\tau \right| \leq rh_k|\xi|.$$

Dividing by h_k and letting $k \to \infty$, for $t \neq s$, we have $|G(t, \sigma(s))\xi| = |G(t, s)\xi| \leq r|\xi|$. Then it follows from the arbitrariness of ξ that

$$|G(t, s)| \leq r. \tag{4.6}$$

Case 2. s is both right-scattered and left-scattered. Define f as

$$f(t) = \begin{cases} \xi, & \text{for } \rho(s) \leq t \leq s, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\|f\|_{L^1} = \mu(\rho(s))|\xi|$; we have

$$|y(t)| = \left| \int_{\rho(s)}^{s} G(t, \sigma(\tau))\xi\Delta\tau \right| \leq r\mu(\rho(s))|\xi|.$$

Dividing by $\mu(\rho(s))$ in the above inequality, we get

$$|G(t, \sigma(\rho(s)))\xi| = |G(t, s)| \leq r|\xi|. \tag{4.7}$$

Case 3. s is right-scattered and left-dense. Let f be the following function:

$$f(t) = \begin{cases} \xi, & \text{for } s \leq t \leq \sigma(s), \\ 0, & \text{otherwise.} \end{cases}$$

By $\|f\|_{L^1} = \mu(s)|\xi|$, we have

$$|y(t)| = \left| \int_s^{\sigma(s)} G(t, \sigma(\tau))\xi\Delta\tau \right| \leq r\mu(s)|\xi|.$$

Dividing by $\mu(s)$ in the above inequality, we get

$$|G(t, \sigma(s))\xi| = |G(t, s)(I + u(s)(A(s) + B(s)))^{-1}\xi| \leq r|\xi|.$$

Therefore,

$$|G(t, s)| \leq r(1 + \chi(\|A\| + \|B\|)). \tag{4.8}$$

From the definition of Q, (4.6), (4.7) and (4.8), it follows that

$$\begin{aligned} &|Y(t)QY^{-1}(s)| \leq r(1 + \chi(\|A\| + \|B\|)) \quad \text{for } t > s \\ &|Y(t)(I - Q)Y^{-1}(s)| \leq r(1 + \chi(\|A\| + \|B\|)) \quad \text{for } s < t \end{aligned} \tag{4.9}$$

hold and $Q(U) = U_1$. Meanwhile, from the continuity of $Y(t)$, it follows that (4.9) is also valid for $s = t$. The proof is complete. ■

In order to obtain roughness of the exponential dichotomy, we need the following estimates of dichotomy inequalities.

Lemma 4.3. *Assume that u is a bounded, positive and continuous function on $[t_0, \infty)$ such that*

$$u(t) \leq be_{\ominus\gamma_1}(t, t_0)u(t_0) + \delta b_1 \int_{t_0}^{t} e_{\ominus\gamma_1}(t, \tau)u(\tau)\Delta\tau + \delta b_2 \int_t^{\infty} e_{\ominus\gamma_2}(\tau, t)u(\tau)\Delta\tau, \tag{4.10}$$

where $b, b_1, b_2, \delta, \gamma_1, \gamma_2$ are all positive constants. If $\kappa = \delta\left(\frac{b_1}{\gamma_1} + \frac{b_2(1+\gamma_2\chi)}{\gamma_2}\right) < 1$, and $\theta_1 = 1 - \frac{\delta b_1}{\gamma_1(1-\kappa)} > 0$, then

$$u(t) \le \frac{b}{1-\kappa} u(t_0) e_{\{(\ominus\alpha_1)(\theta_1)\}}(t, t_0) \quad \text{for } t \ge t_0.$$

Proof. We first assume that $u(t) \to 0$ as $t \to \infty$ and define $\rho_1(t) = \max_{\tau\in[t,\infty)} u(\tau)$. Obviously, $\rho_1(t)$ is a decreasing function. One can easily show that there is a $t_1^* \ge t$ such that $\rho_1(t) = u(t_1^*)$ for each $t \ge t_0$. By (4.10), we have

$$u(t_1^*) \le b e_{\ominus\gamma_1}(t_1^*, t_0) u(t_0) + \delta b_1 \int_{t_0}^{t_1^*} e_{\ominus\gamma_1}(t_1^*, \tau) u(\tau)\Delta\tau + \delta b_2 \int_{t_1^*}^{\infty} e_{\ominus\gamma_2}(\tau, t_1^*) u(\tau)\Delta\tau,$$

which implies that

$$\begin{aligned}
\rho_1(t) &\le b e_{\ominus\gamma_1}(t_1^*, t_0) u(t_0) + \delta b_1 \int_{t_0}^{t_1^*} e_{\ominus\gamma_1}(t_1^*, \tau) u(\tau)\Delta\tau + \delta b_2 \int_{t_1^*}^{\infty} e_{\ominus\gamma_2}(\tau, t_1^*) u(\tau)\Delta\tau \\
&= b e_{\ominus\gamma_1}(t_1^*, t_0) u(t_0) + \delta b_1 \int_{t_0}^{t} e_{\ominus\gamma_1}(t_1^*, \tau) u(\tau)\Delta\tau + \delta b_1 \int_{t}^{t_1^*} e_{\ominus\gamma_1}(t_1^*, \tau) u(\tau)\Delta\tau + \delta b_2 \int_{t_1^*}^{\infty} e_{\ominus\gamma_2}(\tau, t_1^*) u(\tau)\Delta\tau \\
&\le b e_{\ominus\gamma_1}(t, t_0) u(t_0) + \delta b_1 \int_{t_0}^{t} e_{\ominus\gamma_1}(t, \tau) u(\tau)\Delta\tau + \delta\rho_1(t)\left(b_1 \int_{t}^{t_1^*} e_{\ominus\gamma_1}(t_1^*, \tau)\Delta\tau + b_2 \int_{t_1^*}^{\infty} e_{\ominus\gamma_2}(\tau, t_1^*)\Delta\tau\right) \\
&\le b e_{\ominus\gamma_1}(t, t_0) u(t_0) + \delta b_1 \int_{t_0}^{t} e_{\ominus\gamma_1}(t, \tau) u(\tau)\Delta\tau + \delta\rho_1(t)\left(\frac{b_1}{\gamma_1} + \frac{b_2(1+\gamma_2\chi)}{\gamma_2}\right).
\end{aligned}$$

Therefore, we have

$$(1-\kappa)u(t) \le (1-\kappa)\rho_1(t) \le b e_{\ominus\gamma_1}(t, t_0) u(t_0) + \delta b_1 \int_{t_0}^{t} e_{\ominus\gamma_1}(t, \tau) u(\tau)\Delta\tau.$$

Let $\phi_1(t) = e_{\gamma_1}(t, \vartheta) u(t)$; the above inequality can be written as

$$\phi_1(t) \le \frac{b}{1-\kappa}\phi_1(t_0) + \frac{\delta b_1}{1-\kappa}\int_{t_0}^{t} \phi_1(\tau)\Delta\tau.$$

According to Theorem 2.2, we have

$$\phi_1(t) \le \frac{b}{1-\kappa}\phi_1(t_0) e_{\{\gamma_1(1-\theta_1)\}}(t, t_0)$$

or

$$u(t) \le \frac{b}{1-\kappa} u(t_0) e_{\{(\ominus\gamma_1)(\theta_1)\}}(t, t_0).$$

Note that we have shown that the conclusion of Lemma 4.3 is true when $u(t) \to 0$. If $u(t)$ does not have this property, then we define a new function $u_\beta(t) = u(t) e_{\ominus\beta}(t, \vartheta)$, where $0 < \beta < \gamma_2$. It follows from the boundedness of $u(t)$ that $u_\beta(t) \to 0$ as $t \to \infty$. By the inequality (4.10), we have

$$u_\beta(t) \le b e_{\{(\ominus\gamma_1)\oplus(\ominus\beta)\}}(t, t_0) u_\beta(t_0) + \delta b_1 \int_{t_0}^{t} e_{\{(\ominus\gamma_1)\oplus(\ominus\beta)\}}(t, \tau) u_\beta(\tau)\Delta\tau + \delta b_2 \int_{t}^{\infty} e_{\{(\ominus\gamma_2)\oplus(\beta)\}}(\tau, t) u_\beta(\tau)\Delta\tau.$$

By arguments similar to those in the above discussion, for $\kappa_\beta = \delta\left(\frac{b_1}{\gamma_1+\beta} + \frac{b_2(1+\gamma_2\chi)}{\gamma_2-\beta}\right) < 1$ and $\theta_\beta = 1 - \frac{\delta b_1}{(\gamma_1\oplus\beta)(1-\kappa_\beta)} > 0$, we have

$$u_\beta(t) \le \frac{b}{1-\kappa_\beta} u_\beta(t_0) e_{\{(\ominus(\gamma_1\oplus\beta))(\theta_\beta)\}}(t, t_0) \quad \text{for } t \ge t_0.$$

By letting $\beta \to 0$, we show that $u(t) \le \frac{b}{1-\kappa} u(t_0) e_{\{(\ominus\gamma_1)(\theta_1)\}}(t, t_0)$ for all $t \ge t_0$. This completes the proof. ■

Lemma 4.4. *If u is a bounded, positive and continuous function on $[t_0, \infty)$ and*

$$u(t) \le b e_{\ominus\gamma_2}(s, t) u(s) + \delta b_1 \int_{t_0}^{t} e_{\ominus\gamma_1}(t, \tau) u(\tau)\Delta\tau + \delta b_2 \int_{t}^{s} e_{\ominus\gamma_2}(\tau, t) u(\tau)\Delta\tau, \quad s \ge t \ge t_0 \tag{4.11}$$

holds, where $b, b_1, b_2, \delta, \gamma_1, \gamma_2$ are all positive constants, then

$$u(t) \le \frac{b}{1-\kappa} u(s) e_{\{(\ominus\gamma_2)(\theta_2)\}}(s,t) \quad \text{for } s \ge t \ge t_0,$$

where $\kappa = \delta\left(\frac{b_1}{\gamma_1} + \frac{b_2(1+\gamma_2\chi)}{\gamma_2}\right) < 1$ *and* $\theta_2 = 1 - \frac{\delta b_2}{\gamma_2(1-\kappa)} > 0$.

Proof. Define $\rho_2(t) = \max_{\tau\in[t_0,t]} u(\tau)$. It is clear that $\rho_2(t)$ is an increasing function and there exists a t_2^* such that $\rho_2(t) = u(t_2^*)$ for each $t \ge t_0$. It follows from (4.11) that

$$\begin{aligned}
u(t_2^*) &\le b e_{\ominus\gamma_2}(s,t_2^*)u(s) + \delta b_1 \int_{t_0}^{t_2^*} e_{\ominus\gamma_1}(t_2^*,\tau)u(\tau)\Delta\tau + \delta b_2 \int_{t_2^*}^{s} e_{\ominus\gamma_2}(\tau,t_2^*)u(\tau)\Delta\tau \\
&= b e_{\ominus\gamma_2}(s,t_2^*)u(s) + \delta b_2 \int_{t}^{s} e_{\ominus\gamma_2}(\tau,t_2^*)u(\tau)\Delta\tau + \delta b_2 \int_{t_2^*}^{t} e_{\ominus\gamma_2}(\tau,t_2^*)u(\tau)\Delta\tau + \delta b_1 \int_{t_0}^{t_2^*} e_{\ominus\gamma_1}(t_2^*,\tau)u(\tau)\Delta\tau \\
&\le b e_{\ominus\gamma_2}(s,t)u(s) + \delta b_2 \int_{t}^{s} e_{\ominus\gamma_2}(\tau,t)u(\tau)\Delta\tau + \delta\rho_2(t)\left(b_2 \int_{t_2^*}^{t} e_{\ominus\gamma_2}(\tau,t_2^*)\Delta\tau + b_1 \int_{t_0}^{t_2^*} e_{\ominus\gamma_1}(t_2^*,\tau)\Delta\tau\right) \\
&\le b e_{\ominus\gamma_2}(s,t)u(s) + \delta b_2 \int_{t}^{s} e_{\ominus\gamma_2}(\tau,t)u(\tau)\Delta\tau + \delta\left(\frac{b_2(1+\gamma_2\chi)}{\gamma_2} + \frac{b_1}{\gamma_1}\right).
\end{aligned}$$

Then we get

$$(1-\kappa)u(t) \le (1-\kappa)\rho_2(t) \le b e_{\ominus\gamma_2}(s,t)u(s) + \delta b_2 \int_{t}^{s} e_{\ominus\gamma_2}(\tau,t)u(\tau)\Delta\tau.$$

Let $\phi_2(t) = e_{\ominus\gamma_2}(t,\vartheta)u(t)$; it is not difficult to show that

$$\phi_2(t) \le \frac{b}{1-\kappa}\phi_2(s) e_{\{\gamma_2(1-\theta_2)\}}(s,t)$$

or

$$u(t) \le \frac{b}{1-\kappa} u(s) e_{\{(\ominus\gamma_2)(\theta_2)\}}(s,t). \quad \blacksquare$$

Theorem 4.2. *Assume that (3.1) has an exponential dichotomy on $\mathbb{T}^+$ with $X(\vartheta) = I$, and $B(t) \in BC_{\mathrm{rd}}$ satisfies*

$$\kappa^* = \|B\|\left(\frac{K_1(1+\alpha_1\chi)}{\alpha_1} + \frac{K_2(1+\alpha_2\chi)}{\alpha_2}\right) < 1.$$

Then the perturbed equation (4.1) also has an exponential dichotomy for its fundamental matrix $Y(t)$ with $Y(\vartheta) = I$ as follows:

$$\begin{aligned}
|Y(t)QY^{-1}(s)| &\le \frac{K_1(K_1+K_2)}{(1-2\zeta)(1-\kappa^*)} e_{(\ominus\alpha_1)(\theta_1^*)}(t,s) \quad \text{for } t \ge s \ge \vartheta, \\
|Y(t)(I-Q)Y^{-1}(s)| &\le \frac{K_2(K_1+K_2)}{(1-2\zeta)(1-\kappa^*)} e_{(\ominus\alpha_2)(\theta_2^*)}(s,t) \quad \text{for } s \ge t \ge \vartheta,
\end{aligned} \tag{4.12}$$

where $\zeta = \max\{\zeta_1, \zeta_2\}$ with $1 - 2\zeta > 0$,

$$\zeta_1 = \|B\|\left(\frac{K_1K_2(1+\alpha_2\chi)}{\alpha_2(1-\kappa^*)}\right), \qquad \zeta_2 = \|B\|\left(\frac{K_1K_2(1+\alpha_1\chi)}{\alpha_1(1-\kappa^*)}\right),$$

$$\theta_1^* = 1 - \|B\|\left(\frac{K_1(1+\alpha_1\chi)}{\alpha_1(1-\kappa^*)}\right) > 0, \qquad \theta_2^* = 1 - \|B\|\left(\frac{K_2}{\alpha_2(1-\kappa^*)}\right) > 0.$$

Moreover, the projection Q is similar to the projection P.

Proof. By an argument similar to that the proof of Theorem 4.1, we find the projection Q, similar to the projection P, and $Y_1(t) = Y(t)Q$. By (4.4) and (4.5), we obtain

$$Y_1(t) = X(t)PX^{-1}(s)Y_1(s) + \int_s^t X(t)PX^{-1}(\sigma(\tau))B(\tau)Y_1(\tau)\Delta\tau - \int_t^\infty X(t)(I-P)X^{-1}(\sigma(\tau))B(\tau)Y_1(\tau)\Delta\tau. \tag{4.13}$$

On the other hand, we put $Y_2(t) = Y(t)(I - Q)$ such that $Y(t) = Y_1(t) + Y_2(t)$. It follows from the variation of constants formula that

$$Y_2(t) = X(t)(I - Q) + \int_{\vartheta}^{t} X(t)X^{-1}(\sigma(\tau))B(\tau)Y_2(\tau)\Delta\tau.$$

Since $(I - P)(I - Q) = I - Q$, we then put $t = s$ and multiply by $X(t)(I - P)X^{-1}(s)$ on both sides of the above identify such that

$$X(t)(I - P)X^{-1}(s)Y_2(s) = X(t)(I - Q) + \int_{\vartheta}^{s} X(t)(I - P)X^{-1}(\sigma(\tau))B(\tau)Y_2(\tau)\Delta\tau.$$

Hence, we easily see that

$$Y_2(t) = X(t)(I - P)X^{-1}(s)Y_2(s) + \int_{\vartheta}^{t} X(t)PX^{-1}(\sigma(\tau))B(\tau)Y_2(\tau)\Delta\tau - \int_{t}^{s} X(t)(I - P)X^{-1}(\sigma(\tau))B(\tau)Y_2(\tau)\Delta\tau. \tag{4.14}$$

For any vector $\xi \in \mathbb{R}^n$, by (4.13) and (4.14), one can reach

$$\begin{aligned}
|Y_1(t)\xi| &\le K_1 e_{\ominus\alpha_1}(t,s)|Y_1(s)\xi| + \|B\|K_1 \int_{\vartheta}^{t} e_{\ominus\alpha_1}(t,\sigma(\tau))|Y_1(\tau)\xi|\Delta\tau + \|B\|K_2 \int_{t}^{\infty} e_{\ominus\alpha_2}(\sigma(\tau),t)|Y_1(\tau)\xi|\Delta\tau \\
&\le K_1 e_{\ominus\alpha_1}(t,s)|Y_1(s)\xi| + \|B\|K_1(1+\alpha_1\chi)\int_{\vartheta}^{t} e_{\ominus\alpha_1}(t,\tau)|Y_1(\tau)\xi|\Delta\tau \\
&\quad + \|B\|K_2 \int_{t}^{\infty} e_{\ominus\alpha_2}(\tau,t)|Y_1(\tau)\xi|\Delta\tau, \quad \text{for } t \ge s \ge \vartheta
\end{aligned}$$

and

$$\begin{aligned}
|Y_2(t)\xi| &\le K_2 e_{\ominus\alpha_2}(s,t)|Y_2(s)\xi| + \|B\|K_1 \int_{\vartheta}^{t} e_{\ominus\alpha_1}(t,\sigma(\tau))|Y_2(\tau)\xi|\Delta\tau + \|B\|K_2 \int_{t}^{s} e_{\ominus\alpha_2}(\sigma(\tau),t)|Y_2(\tau)\xi|\Delta\tau \\
&\le K_2 e_{\ominus\alpha_2}(s,t)|Y_1(s)\xi| + \|B\|K_1(1+\alpha_1\chi)\int_{\vartheta}^{t} e_{\ominus\alpha_1}(t,\tau)|Y_2(\tau)\xi|\Delta\tau \\
&\quad + \|B\|K_2 \int_{t}^{s} e_{\ominus\alpha_2}(\tau,t)|Y_2(\tau)\xi|\Delta\tau, \quad \text{for } s \ge t \ge \vartheta.
\end{aligned}$$

It follows from Lemmas 4.3 and 4.4 that

$$\begin{aligned}
|Y_1(t)\xi| &\le \frac{K_1}{1-\kappa^*} e_{(\ominus\alpha_1)(\theta_1^*)}(t,s)|Y_1(s)\xi| \quad \text{for } t \ge s \ge \vartheta, \\
|Y_2(t)\xi| &\le \frac{K_2}{1-\kappa^*} e_{(\ominus\alpha_2)(\theta_2^*)}(s,t)|Y_2(s)\xi| \quad \text{for } s \ge t \ge \vartheta.
\end{aligned} \tag{4.15}$$

In order to complete this proof, it is only necessary to show that $Y(t)QY^{-1}(t)$ is uniformly bounded from Lemma 3.1. Multiplying by $X(t)(I - P)X^{-1}(t)$ on both sides of (4.13), and by the first inequality of (4.15), we have

$$\begin{aligned}
|X(t)(I-P)X^{-1}(t)Y_1(t)\xi| &= \left| -\int_{t}^{\infty} X(t)(I-P)X^{-1}(\sigma(\tau))B(\tau)Y_1(\tau)\xi\Delta\tau \right| \\
&\le \|B\| \left(\frac{K_1K_2}{1-\kappa^*}\right)|Y_1(t)\xi| \int_{t}^{\infty} e_{(\ominus\alpha_1)(\theta_1^*)}(\tau,t)e_{\ominus\alpha_2}(\sigma(\tau),t)\Delta\tau \\
&\le \|B\| \left(\frac{K_1K_2}{1-\kappa^*}\right)|Y_1(t)\xi| \int_{t}^{\infty} e_{((\ominus\alpha_1)(\theta_1^*))\oplus(\ominus\alpha_2)}(\tau,t)\Delta\tau \\
&\le \|B\| \left(\frac{K_1K_2(1+\alpha_2\chi)}{\alpha_2(1-\kappa^*)}\right)|Y_1(t)\xi| = \zeta_1|Y_1(t)\xi|.
\end{aligned} \tag{4.16}$$

Similarly, it follows from (4.14) and the second inequality of (4.15) that we have

$$\begin{aligned}
|X(t)PX^{-1}(t)Y_2(t)\xi| &= \left| \int_{\vartheta}^{t} X(t)PX^{-1}(\sigma(\tau))B(\tau)Y_2(\tau)\xi\Delta\tau \right| \\
&\le \|B\| \left(\frac{K_1K_2}{1-\kappa^*}\right)|Y_2(t)\xi| \int_{\vartheta}^{t} e_{(\ominus\alpha_2)(\theta_2^*)}(t,\tau)e_{\ominus\alpha_1}(t,\sigma(\tau))\Delta\tau
\end{aligned}$$

$$\leq \|B\| \left(\frac{K_1K_2(1+\alpha_1\chi)}{\alpha_1(1-\kappa^*)} \right) |Y_2(t)\xi| \int_{\vartheta}^{t} e_{((\ominus\alpha_2)(\theta_2^*))\oplus(\ominus\alpha_1)}(t,\tau)\Delta\tau$$
$$\leq \|B\| \left(\frac{K_1K_2(1+\alpha_1\chi)}{\alpha_1(1-\kappa^*)} \right) |Y_2(t)\xi| = \zeta_2|Y_2(t)\xi|. \tag{4.17}$$

It is straightforward to show that

$$Y(t)QY^{-1}(t) - X(t)PX^{-1}(t) = X(t)(I-P)X^{-1}(t)Y(t)QY^{-1}(t) - X(t)PX^{-1}(t)Y(t)(I-Q)Y^{-1}(t),$$
$$Y(t)QY^{-1}(t) - X(t)PX^{-1}(t) = X(t)(I-P)X^{-1}(t) - Y(t)(I-Q)Y^{-1}(t).$$

Since ξ is arbitrary in (4.16) and (4.17), we then conclude that

$$|Y(t)QY^{-1}(t)| \leq \zeta_1|Y(t)QY^{-1}(t)| + \zeta_2|Y(t)(I-Q)Y^{-1}(t)| + |X(t)PX^{-1}(t)|,$$
$$|Y(t)(I-Q)Y^{-1}(t)| \leq \zeta_1|Y(t)QY^{-1}(t)| + \zeta_2|Y(t)(I-Q)Y^{-1}(t)| + |X(t)(I-P)X^{-1}(t)|.$$

Therefore, we have

$$|Y(t)QY^{-1}(t)| + |Y(t)(I-Q)Y^{-1}(t)| \leq 2\zeta(|Y(t)QY^{-1}(t)| + |Y(t)(I-Q)Y^{-1}(t)|) + K_1 + K_2,$$

that is,

$$\max\{|Y(t)QY^{-1}(t)|, |Y(t)(I-Q)Y^{-1}(t)|\} \leq \frac{K_1+K_2}{1-2\zeta}.$$

Then (4.12) follows immediately from 3.1. This completes the proof. ■

Remark 4.1. By carrying through arguments similar to those above, we conclude that Theorem 4.2 also holds if the regressivity is weakened to the regularity, and the growth rates are nonconstant and are not separated by 0. Note that the exponential dichotomy defined in [24] is more general than Definition 3.1 and Theorem 4.2 corresponds to Theorem 2.4 in [24]. When Theorem 2.4 in [24] reduces to the case explored here, Theorem 4.2 improves Theorem 2.4. Therefore, Theorem 4.2 extends the special case of [24].

Next we will discuss the relation between the solutions of Eq. (3.1) and the perturbed equation (4.2). The following lemma can be obtained similarly to Proposition 2.2 in [3].

Lemma 4.5. *If (3.1) has a dichotomy (ordinary dichotomy or exponential dichotomy) on $\mathbb{T}^+$ with $X(\vartheta) = I$, then there exists a dichotomy (ordinary dichotomy or exponential dichotomy) with a projection P_0 such that $P_0[\mathbb{R}^n] = U_0$, where U_0 is a subspace of $\mathbb{R}^n$ consisting of the initial values of all solutions of (3.1) which tend to zero as $t \to \infty$.*

Theorem 4.3. *If (3.1) has an ordinary dichotomy on $\mathbb{T}^+$, and $B(t), f(t) \in L^1$, then there exists a one-to-one mapping for the bounded solutions between (3.1) and (4.2) such that the difference between corresponding solutions tends to zero as $t \to \infty$.*

Proof. According to Lemma 4.5, (3.1) has an ordinary dichotomy with a projection P_0 such that $P_0[\mathbb{R}^n] = U_0$, i.e., $|X(t)P_0\xi| \to 0$ as $t \to \infty$ for any vector $\xi \in \mathbb{R}^n$. Choose a $t_N \in \mathbb{T}^+$ large enough that $\hbar = \max\left\{\frac{K_1(1+\alpha_1\chi)}{\alpha_1}, \frac{K_2}{\alpha_2}\right\} \int_{t_N}^{\infty} |B(\tau)|\Delta\tau < 1$. For $y(t) \in BC$, define a mapping T as follows:

$$(Ty)(t) = \int_{t_N}^{t} X(t)P_0X^{-1}(\sigma(\tau))(B(\tau)y(\tau) + f(\tau))\Delta\tau - \int_{t}^{\infty} X(t)(I-P_0)X^{-1}(\sigma(\tau))(B(\tau)y(\tau) + f(\tau))\Delta\tau.$$

By an argument similar to those in the proof of Lemma 4.1, it is clear that $T : BC \to BC$ and $\|Ty_1 - Ty_2\| \leq \hbar\|y_1 - y_2\|$. Assume that $x(t)$ is any bounded solution of (3.1); now we consider the following system:

$$y(t) = x(t) + (Ty)(t). \tag{4.18}$$

By the Contraction Mapping Principle, the system (4.18) has a unique bounded continuous solution $y(t)$. Obviously, the bounded solution $y(t)$ is a solution of (4.2). On the other hand, it is not difficult to show that $x(t) = y(t) - (Ty)(t)$ is a bounded solution of (3.1) if $y(t)$ is a bounded solution of (4.2). This implies that we construct a one-to-one mapping for the bounded solution between (3.1) and (4.2). In addition, for any $\epsilon > 0$, there exists a $t_\epsilon \geq t_N$ large enough that $\frac{K_2}{\alpha_2}\int_{t_\epsilon}^{\infty}[B(\tau)(|y(\tau)| + |f(\tau)|)]\Delta\tau \leq \epsilon$. Therefore, we have

$$|(Ty)(t)| \leq \left| X(t)P_0 \int_{t_N}^{t_\epsilon} X^{-1}(\sigma(\tau))[B(\tau)y(\tau) + f(\tau)]\Delta\tau \right| + \epsilon < 2\epsilon.$$

This shows that $y(t) - x(t) \to 0$ as $t \to \infty$. ■

Let $B(t) \equiv 0$ in the perturbed equation (4.2); then we have:

Corollary 4.1. *Assume that (3.1) has an ordinary dichotomy on $\mathbb{T}^+$; then the nonhomogeneous equation $y^\Delta(t) = A(t)y + f(t)$ has at least one bounded solution which tends to zero as $t \to \infty$ for each function $f \in L^1$.*

5. Sufficient criteria for being exponentially dichotomous

In this section, we establish explicit sufficient criteria for linear dynamic equations to have an exponential dichotomy. To facilitate the discussion below, we assume that $K = \max\{K_1, K_2\}$ and $\alpha = \min\{\alpha_1, \alpha_2\}$ in Definition 3.1; then (3.2) still holds for K and α. In order to obtain our main conclusion, we need the following lemma.

Lemma 5.1. *Let X be a fundamental matrix solution of (3.1). If:*

(i) *there exist positive constants B and α such that*

$$\begin{aligned} &|X(t)P\xi| \le Be_{\ominus\alpha}(t,s)|X(s)P\xi|, \quad t \ge s, \\ &|X(t)(I-P)\xi| \le Be_{\ominus\alpha}(s,t)|X(s)(I-P)\xi|, \quad t \le s, \end{aligned} \tag{5.1}$$

where P is a projector, and ξ is an arbitrary n-dimensional vector;

(ii) *there exist $C \ge 1$ and $\beta > 0$ such that*

$$|X(t)X^{-1}(s)| \le Ce_{\beta}(t,s), \quad t \ge s, \tag{5.2}$$

then $P(t) = X(t)PX^{-1}(t)$ and $I - P(t)$ are uniformly bounded.

Proof. If $P = 0$ or $P = I$, the conclusion is obvious. So, we assume that $P \ne 0, I$. By (5.1), for any fixed $t \in \mathbb{T}$, we choose a positive constant $h > 0$ such that $t + h \in \mathbb{T}$. Then

$$\begin{aligned} &|X(t+h)PX^{-1}(t)| \le Be_{\ominus\alpha}(t+h,t)|X(t)PX^{-1}(t)| = Be_{\ominus\alpha}(t+h,t)|P(t)|, \\ &|X(t+h)(I-P)X^{-1}(t)| \ge B^{-1}e_{\alpha}(t+h,t)|X(t)(I-P)X^{-1}(t)| = B^{-1}e_{\alpha}(t+h,t)|I-P(t)|. \end{aligned}$$

Then for a fixed constant $\gamma_0 > 0$, we choose an $h_0 > 0$, $t + h_0 \in \mathbb{T}$ such that

$$B^{-1}e_{\alpha}(t+h_0,t) - Be_{\ominus\alpha}(t+h_0,t) \ge \gamma_0 > 0.$$

This means that

$$\left|\frac{X(t+h_0)(I-P)X^{-1}(t)}{|I-P(t)|} + \frac{X(t+h_0)PX^{-1}(t)}{|P(t)|}\right| \ge \gamma_0.$$

From (5.2) and for the above h_0, one has

$$\begin{aligned} \left|\frac{P(t)}{|P(t)|} + \frac{I-P(t)}{|I-P(t)|}\right| &= \left|X(t)X^{-1}(t+h_0)\left[\frac{X(t+h_0)(I-P)X^{-1}(t)}{|I-P(t)|} + \frac{X(t+h_0)PX^{-1}(t)}{|P(t)|}\right]\right| \\ &\ge \frac{\gamma_0}{C}e_{\ominus\beta}(t+h_0,t) \ge \frac{\gamma_0}{C}\mathrm{e}^{-\beta h_0} := N > 0. \end{aligned}$$

Therefore,

$$\min\left\{\left|\frac{P(t)}{|P(t)|} + \frac{I-P(t)}{|I-P(t)|}\right|\right\} \ge N > 0 \quad \text{for all } t \in \mathbb{T}.$$

Note that

$$\left|\frac{\xi_1}{|\xi_1|} + \frac{\xi_2}{|\xi_2|}\right| \cdot \max\{|\xi_1|, |\xi_2|\} \le 2|\xi_1 + \xi_2|, \quad \xi_1 \ne 0,\ \xi_2 \ne 0.$$

Then we have

$$\max\{|P(t)|, |I-P(t)|\} \le \frac{2}{N}|P(t) + [I-P(t)]| = \frac{2}{N} \quad \text{for all } t \in \mathbb{T}.$$

The proof is complete. ■

Theorem 5.1. *If $A(t)$ is a uniformly bounded rd-continuous $n \times n$ matrix-valued function on $\mathbb{T}$, and there is a $\delta > 0$ such that*

$$|a_{ii}(t)| - \sum_{j\ne i}|a_{ij}(t)| - \frac{1}{2}\mu(t)\left(\sum_{j=1}^{n}|a_{ij}(t)|\right)^2 \ge 2\delta + \delta^2\mu(t), \quad t \in \mathbb{T},\ i = 1, 2, \ldots, n, \tag{5.3}$$

then (3.1) is an exponential dichotomy on $\mathbb{T}$.

Proof. From (5.3), one can see that, for each $i = 1, 2, \ldots, n$, a_{ii} keeps constant sign on $\mathbb{T}$. Without loss of any generality, we assume that there is an integer $1 \le k \le n$ such that

$$a_{ii} \begin{cases} > 0, & 1 \le i \le k, \\ < 0, & n \ge i > k. \end{cases}$$

Let $x(t)$ be a nontrivial solution of (3.1); then we have

$$\begin{aligned} \frac{1}{2}(|x_i(t)|^2)^\Delta &= \frac{1}{2}(x_i(t) + x_i^\sigma(t))x_i^\Delta(t) = \frac{1}{2}(2x_i(t) + \mu(t)x_i^\Delta(t))x_i^\Delta(t) \\ &= x_i(t)x_i^\Delta(t) + \frac{1}{2}\mu(t)(x_i^\Delta(t))^2 \\ &= a_{ii}(t)|x_i(t)|^2 + \sum_{j\ne i} a_{ij}(t)x_i(t)x_j(t) + \frac{1}{2}\mu(t)\left(\sum_{j=1}^n a_{ij}(t)x_j(t)\right)^2, \quad t \in \mathbb{T}. \end{aligned} \tag{5.4}$$

First, we show that $|x(t)|$ does not have any local maximum on $\mathbb{T}$. Otherwise, assume that $|x(t)|$ has a local maximum at some $s \in \mathbb{T}$. Let $|x(s)| = |x_i(s)|$ for some $1 \le i \le n$; by (5.4), one has

$$\begin{aligned} &\left[a_{ii}(s) - \sum_{j\ne i}|a_{ij}(s)| - \frac{1}{2}\mu(s)\left(\sum_{j=1}^n a_{ij}(s)\right)^2\right]|x_i(s)|^2 \le \frac{1}{2}(|x_i(s)|^2)^\Delta \\ &\quad \le \left[a_{ii}(s) + \sum_{j\ne i}|a_{ij}(s)| + \frac{1}{2}\mu(s)\left(\sum_{j=1}^n a_{ij}(s)\right)^2\right]|x_i(s)|^2. \end{aligned}$$

Then, from (5.3), we have

$$\frac{1}{2}(|x_i(s)|^2)^\Delta \ge (2\delta + \delta^2\mu(s))|x_i(s)|^2 > 0. \tag{5.5}$$

This is a contradiction. Therefore, $|x(t)|$ does not have any local maximum on $\mathbb{T}$.

Secondly, for any given $s \in \mathbb{T}$, we show that $|x(t)|$ is strictly increasing for all $t \ge s$, $t \in \mathbb{T}$, if and only if there is some $i \le k$ such that $|x(s)| = |x_i(s)|$. In fact, if there is an $i_0 \le k$ and $s \in \mathbb{T}$ such that $|x(s)| = |x_{i_0}(s)|$, then it follows from (5.5) that $|x_{i_0}(s)| < |x_{i_0}(s+h)|$ for sufficiently small h with $s + h \in \mathbb{T}$. Then

$$|x(s)| = |x_{i_0}(s)| < |x_{i_0}(s+h)| \le |x(s+h)|.$$

If there are $t_1, t_2 \in \mathbb{T}$, $t_2 > t_1 \ge s$ such that $|x(t_1)| \ge |x(t_2)|$, then we conclude that $|x(t)|$ has a local maximum on $(s, t_2) \subset \mathbb{T}$. This contradiction shows that, for any $t_1, t_2 \in \mathbb{T}$, $t_2 > t_1 \ge s$, $|x(t_2)| > |x(t_1)|$.

Conversely, assume that $|x(t)|$ is strictly increasing and $|x(s)| \ne |x_i(s)|$ for all $i \le k$, $s \in \mathbb{T}$; thus, for sufficiently small $h > 0$ and $s + h \in \mathbb{T}$, we find an $i_0 > k$ such that $|x(s+h)| = |x_{i_0}(s+h)|$. Like in the above arguments, we have

$$\frac{1}{2}(|x_{i_0}(s+h)|^2)^\Delta \le -(2\delta + \delta^2\mu(s+h))|x_{i_0}(s+h)|^2 < 0. \tag{5.6}$$

One can show $|x_{i_0}(s+h)| < |x_{i_0}(s)|$ from (5.6), so we have $|x(s+h)| < |x(s)|$ and this contradicts the fact that $|x(t)|$ is strictly increasing.

Next we show that there is a k-dimensional subspace V_1 of $\mathbb{R}^n$ such that $|x(t)|$ is strictly increasing when $x(t) \in V_1$. Let $X(t)$ be a fundamental matrix solution of (3.1) and let V_1 be a k-dimensional subspace defined by $\xi \in \mathbb{R}^n$, $\xi = (\xi_1, \xi_2, \ldots, \xi_k, 0, \ldots, 0)^T$. Select a sequence $\{s_m\} \subset \mathbb{T}$ such that $s_m \to -\infty$ as $m \to \infty$. Define $U_m = X^{-1}(s_m)V_1$; then U_m is a k-dimensional subspace of $\mathbb{R}^n$. Let $\xi_m^1, \xi_m^2, \ldots, \xi_m^k$ denote a standard orthogonal basis of U_m. By the relative compactness of the unit sphere, there exists a sequence of integers $\{m_v\}$ ($m_v \to +\infty$ as $v \to +\infty$) and orthogonal unit vectors $\xi^1, \xi^2, \ldots, \xi^k$ such that $\xi_{m_v}^i \to \xi^i$ as $v \to +\infty$ ($i = 1, 2, \ldots, k$). Let $x^j(t)$ be a solution of (3.1) with the initial conditions $x^j(\vartheta) = \xi^j$, $j = 1, 2, \ldots, k$; then $x^1(t), x^2(t), \ldots, x^k(t)$ are linearly independent. Consider any nontrivial solution of (3.1) of the form

$$x(t) = a_1x^1(t) + a_2x^2(t) + \cdots + a_kx^k(t), \quad a_i \in \mathbb{R}\ (i = 1, 2, \ldots, k).$$

We will show that $|x(t)|$ is strictly increasing on $\mathbb{T}$. If $x_{m_v}(t)$ is a solution of (3.1) according to the initial value $x_{m_v}(\vartheta) = \sum_{j=1}^k a_j\xi_{m_v}^j(t)$, then $x_{m_v}(t) \to x(t)$ as $v \to +\infty$ for every $t \in \mathbb{T}$. Since $x_{m_v}(\vartheta) \in U_{m_v}$, we have $x_{m_v}(s_{m_v}) \in V_1$ and $|x_{m_v}(t)|$ is strictly increasing for $t \ge s_{m_v}$. For any $t_1, t_2 \in \mathbb{T}$ and $t_2 > t_1$, we have $|x_{m_v}(t_2)| > |x_{m_v}(t_1)|$ for sufficiently large v; furthermore, $|x(t_2)| \ge |x(t_1)|$. Since t_1, t_2 are arbitrary, we have shown that $|x(t)|$ is nondecreasing on $\mathbb{T}$. If $|x(t)|$ is not strictly increasing on $\mathbb{T}$, then there exists $t_1 < t_2$ such that $x(t_1) = x(t_2)$ and hence there will exist an interval $I \subset \mathbb{T}$ such

that $|x(t)|$ is constant on I. Then $|x(t)|$ would admit a local maximum and we reach a contradiction. Therefore, $|x(t)|$ is strictly increasing on $\mathbb{T}$. This implies that there is a k-dimensional subspace V_1 such that $|x(t)|$ is strictly increasing as $x(t) \in V_1$.

If $x(t) \in V_1$ is a nontrivial solution of (3.1), then there is an $i \le k$ such that $|x(\tau)| = |x_i(\tau)|$ for $\tau \in \mathbb{T}$. If τ is right-scattered, from (5.5), it follows that

$$\begin{aligned}(2\delta + \delta^2\mu(\tau))(|x(\tau)|^2) &= (2\delta + \delta^2\mu(\tau))(|x_i(\tau)|) \le (|x_i(\tau)^2|)^{\Delta} \\ &= \frac{|x_i(\sigma(\tau))|^2 - |x_i(\tau)|^2}{\mu(\tau)} \\ &\le \frac{|x(\sigma(\tau))|^2 - |x(\tau)|^2}{\mu(\tau)} = (|x(\tau)|^2)^{\Delta}. \end{aligned} \tag{5.7}$$

If τ is right-dense, then we have

$$\begin{aligned}(2\delta + \delta^2\mu(\tau))(|x(\tau)|^2) &= (2\delta + \delta^2\mu(\tau))(|x_i(\tau)|) \le (|x_i(\tau)^2|)^{\Delta} \\ &= \lim_{h\to 0^+} \frac{|x_i(\tau+h)|^2 - |x_i(\tau)|^2}{h} \\ &\le \lim_{h\to 0^+} \frac{|x(\tau+h)|^2 - |x(\tau)|^2}{h} = (|x(\tau)|^2)^{\Delta}. \end{aligned} \tag{5.8}$$

From (5.7) and (5.8), we conclude that

$$(2\delta + \delta^2\mu(\tau))(|x(\tau)|^2) \le (|x(\tau)|^2)^{\Delta}, \quad \forall \tau \in \mathbb{T}. \tag{5.9}$$

Integrating both sides of (5.9) from s to t, one has

$$|x(t)| \ge e_{\delta}(t,s)|x(s)|, \quad t \ge s.$$

By an argument similar to that above, for $x(t) \in \mathbb{R}^n/V_1$, we have

$$|x(t)| \le e_{\ominus\delta}(t,s)|x(s)|, \quad t \ge s.$$

Let

$$x(t) = X(t)PX^{-1}(s)x_0, \qquad x(s) = X(s)PX^{-1}(s)x_0, \quad i > k$$

and

$$x(t) = X(t)(I-P)X^{-1}(s)x_0, \qquad x(s) = X(s)(I-P)X^{-1}(s)x_0, \quad i \le k,$$

where P is a projector and x_0 is an arbitrary initial value of (3.1). Since $A(t)$ is uniformly bounded, there is a constant $N_1 > 0$ such that $\sup_{t\in\mathbb{T}} |A(t)| \le N_1$, so we have

$$|X(t)X^{-1}(s)| \le e_{N_1}(t,s), \quad t \ge s.$$

Then, by Lemmas 3.1 and 5.1, it is easy to show that (3.1) is an exponential dichotomy. This proves the theorem. ■

Since $A^T(t)$ and $A(t)$ share the same eigenvalues, it follows from a similar argument that we have:

Theorem 5.2. *If $A(t)$ is a uniformly bounded rd-continuous $n \times n$ matrix-valued function on $\mathbb{T}$, and there is a $\delta_0 > 0$ such that*

$$|a_{ii}(t)| - \sum_{j\ne i} |a_{ji}(t)| - \frac{1}{2}\mu(t)\left(\sum_{i=1}^{n} |a_{ji}(t)|\right)^2 \ge 2\delta_0 + \delta_0^2\mu(t), \quad t \in \mathbb{T},\ i = 1,2,\ldots,n, \tag{5.10}$$

then (3.1) is exponentially dichotomous on $\mathbb{T}$.

Now we consider another type of linear dynamic equation:

$$x^{\Delta}(t) = A(t,u(t))x(t), \tag{5.11}$$

where $u \in \mathrm{C}(\mathbb{T},\mathbb{R}^n)$, $A(t,u(t)) \in \mathcal{R}$ is an $n \times n$ real-valued matrix function on $\mathbb{T}$.

Theorem 5.3. *Assume that $A(t,u)$ is uniformly bounded in $\mathbb{T} \times S$, where $S \subset \mathbb{R}^n$ is compact. Moreover, there exists a symmetric, nonsingular $n \times n$ matrix-valued function $H(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^{n\times n})$ satisfying:*

(i) *there is a positive constant ρ such that $|H(t)| \le \rho^2$;*
(ii) *the norms of all the eigenvalues $\lambda_i(t)$ of $H(t)$ are larger than some positive constant η, i.e., $|\lambda_i(t)| \ge \eta > 0,\ i = 1,2,\ldots,n$;*

(iii) *there is a* $\nu > 0$ *such that all the eigenvalues* $\lambda_i^*(t,u)$ *of the matrix*

$$N(t,u) = A^T(t,u)H(t) + [1+\mu(t)A^T(t,u)][H(t)^\Delta + H(\sigma(t))A(t,u)]$$

are smaller than $-[2\nu\rho + \nu^2\mu(t)]$, *i.e.,* $\lambda_i^*(t,u) \le -[2\nu\rho + \nu^2\mu(t)] < 0$ $(i = 1, 2, \ldots, n)$; *then* (5.11) *is exponentially dichotomous on* $\mathbb{T}$, *that is, there exists a projector* P *and positive constants* α *and* K *such that*

$$|X_u(t)PX_u(s)| \le Ke_{\ominus\alpha}(t,s), \quad t \ge s; \qquad |X_u(t)(I-P)X_u(s)| \le Ke_{\ominus\alpha}(s,t), \quad t \le s,$$

where $X_u(t)$ *is a fundamental matrix solution of* (5.11) *with* $X_u(\vartheta) = I$.

Proof. Define $V(t,x) = x^T(t)H(t)x(t)$; then

$$|V(t,x)| \le \rho^2|x(t)|^2 \quad \text{for } t \in \mathbb{T}. \tag{5.12}$$

Assume that $x(t)$ is a solution of (5.11) for $t \ge t_0$ with $x(t_0) = x_0$; then we have

$$\begin{aligned} V^\Delta(t,x) &= (x^T(t))^\Delta H(t)x(t) + x^T(\sigma(t))H^\Delta(t)x(t) + x^T(\sigma(t))H(\sigma(t))x^\Delta(t) \\ &= x^T(t)A^T(t,u)H(t)x(t) + [x^T(t) + \mu(t)(x^T(t))^\Delta]H^\Delta(t)x(t) + [x^T(t) + \mu(t)(x^T(t))^\Delta]H(\sigma(t))A(t,u)x(t) \\ &= x^T(t)A^T(t,u)H(t)x(t) + x^T(t)[1+\mu(t)A^T(t,u)]H^\Delta(t)x(t) + x^T(t)[1+\mu(t)A^T(t,u)]H(\sigma(t))A(t,u)x(t) \\ &= x^T(t)\{A^T(t,u)H(t) + [1+\mu(t)A^T(t,u)][H^\Delta(t) + H(\sigma(t))A(t,u)]\}x(t) \\ &\le -[2\nu\rho + \nu^2\mu(t)]|x(t)|. \end{aligned} \tag{5.13}$$

Without loss of generality, for $H(t)$, one can assume that

$$\lambda_i \le -\eta < 0, \quad i = 1, 2, \ldots, k; \qquad \lambda_j \ge \eta > 0, \quad j = k+1, k+2, \ldots, n,$$

and then there exist subspaces V_1 and V_2 such that $\mathbb{R}^n = V_1 \bigoplus V_2$. Moreover,

$$V(t,x_0) \le -\eta|x_0|^2, \quad x_0 \in V_1; \qquad V(t,x_0) \ge \eta|x_0|^2, \quad x_0 \in V_2.$$

Given $x_0 \in V_1$, from (5.13), one can show that the solution $x(t) = x(t,t_0,x_0)$ of (5.11) satisfies

$$V(t,x) \le V(t_0,x_0) \le -\eta|x_0|^2 \quad \text{for } t \ge t_0. \tag{5.14}$$

By carrying through arguments similar to those in Theorem 5.1, we prove that there exists a k-dimensional subspace Q_1 in $\mathbb{R}^n$ such that $V(t,x(t)) \le 0$ as $x(t) \in Q_1$ for any $t \in \mathbb{T}$. Since $V(t,x)$ is a quadratic type, we have

$$V(t,x) \le -\eta|x(t)|^2 \quad \text{for } x(t) \in Q_1. \tag{5.15}$$

Note that $V(t,x) \le 0$ for $x(t) \in Q_1$; from (5.12) and (5.13), we get

$$V^\Delta(t,x) \le -(2\nu\rho + \nu^2\mu(t))|x(t)|^2 \le -\frac{[2\nu\rho + \nu^2\mu(t)]}{\rho^2}|V(t,x)| = \left(\frac{\nu}{\rho} \oplus \frac{\nu}{\rho}\right)V(t,x), \tag{5.16}$$

for $x(t) \in Q_1$. Integrating (5.16) from s to t, for $t \ge s$, we obtain

$$V(t,x(t)) \le e_{\frac{\nu}{\rho}\oplus\frac{\nu}{\rho}}(t,s)V(s,x(s)). \tag{5.17}$$

From (5.12), (5.15) and (5.17), we have

$$|x(t)|^2 \ge \frac{|V(t,x(t))|}{\rho^2} \ge \frac{e_{\frac{\nu}{\rho}\oplus\frac{\nu}{\rho}}(t,s)|V(s,x(s))|}{\rho^2} \ge \frac{\eta}{\rho^2}e_{\frac{\nu}{\rho}\oplus\frac{\nu}{\rho}}(t,s)|x(s)|^2 \quad \text{for } t \ge s;$$

that is

$$|x(s)| \le \frac{\rho}{\sqrt[2]{\eta}}e_{\ominus\frac{\nu}{\rho}}(t,s)|x(t)| \quad \text{for } t \ge s.$$

Hence, we have

$$|x(t)| \le \frac{\rho}{\sqrt[2]{\eta}}e_{\ominus\frac{\nu}{\rho}}(s,t)|x(s)| \quad \text{for } s \ge t.$$

By carrying through arguments similar to the above, it is not difficult to show that there exists an $n-k$-dimensional subspace Q_2 of $\mathbb{R}^n$ such that, for $x(t) \in Q_2$,

$$|x(t)| \le \frac{\rho}{\sqrt[2]{\eta}}e_{\ominus\frac{\nu}{\rho}}(t,s)|x(s)| \quad \text{for } t \ge s,$$

where $Q_1 \bigoplus Q_2 = \mathbb{R}^n$, $Q_1 \bigcap Q_2 = \emptyset$.

Since $A(t,u)$ is uniformly bounded in $\mathbb{T}\times S$, there is a positive constant N_2 such that $\sup_{t\in\mathbb{T},u\in S}|A(t,u)|\le N_2$. Therefore, for all $s\in\mathbb{T}$, we have $|X_u(t)X_u^{-1}(s)|\le e_{N_2}(t,s)$. Let

$$x(t)=X(t)PX^{-1}(s)x_0,\qquad x(s)=X(s)PX^{-1}(s)x_0\quad \text{for } x_0\in Q_2$$

and

$$x(t)=X(t)(I-P)X^{-1}(s)x_0,\qquad x(s)=X(s)(I-P)X^{-1}(s)x_0\quad \text{for } x_0\in Q_1,$$

where P is a projector and x_0 is an arbitrary initial value of (5.11). By Lemmas 3.1 and 5.1, we conclude that (5.11) is exponentially dichotomous. This completes the proof of the theorem. ■

6. Application

In this section, we will explore the existence of periodic solutions of dynamic equations on time scales by exponential dichotomy. In the rest of this paper, the time scale $\mathbb{T}$ is assumed to be *ω-periodic*, i.e., $t\in\mathbb{T}$ implies $t\pm\omega\in\mathbb{T}$. Then, from [30], we know that

$$\sigma(t\pm\omega)=\sigma(t)\pm\omega,\qquad \rho(t\pm\omega)=\rho(t)\pm\omega,\qquad \mu(t\pm\omega)=\mu(t).$$

Consider the semi-linear and nonlinear dynamic equations

$$x^{\Delta}=A(t)x+f(t,x) \tag{6.1}$$

and

$$x^{\Delta}=A(t,x)x+f(t,x), \tag{6.2}$$

where $A(t), A(t,x(t))\in\mathcal{R}$ are ω-periodic $n\times n$ real-valued matrix functions on $\mathbb{T}$, and $f\in C_{rd}(\mathbb{T}\times\mathbb{R}^n,\mathbb{R}^n)$ is also ω-periodic in t. According to Section 8 in [13], it is easy to show that the solutions of (6.1) and (6.2) exist in the future on $\mathbb{T}$.

Before we investigate the existence of periodic solutions of (6.1) and (6.2), we should make some necessary preparations.

Lemma 6.1. *If g is ω-periodic rd-continuous function on $\mathbb{T}$, then*

$$\int_t^{t+\omega} g(s)\Delta s=\int_{\vartheta}^{\vartheta+\omega} g(s)\Delta s,\quad t\in\mathbb{T}.$$

Proof. Let $t=\vartheta+n\omega+r$, $0\le r<\omega$, $n\in\mathbb{Z}$; then we can show that

$$\begin{aligned}\int_t^{t+\omega} g(s)\Delta s&=\int_{\vartheta+n\omega+r}^{\vartheta+n\omega+r+\omega} g(s)\Delta s=\int_{\vartheta+r}^{\vartheta+r+\omega} g(s)\Delta s\\&=\int_{\vartheta}^{\vartheta+\omega} g(s)\Delta s+\int_{\vartheta+\omega}^{\vartheta+r+\omega} g(s)\Delta s-\int_{\vartheta}^{\vartheta+r} g(s)\Delta s\\&=\int_{\vartheta}^{\vartheta+\omega} g(s)\Delta s.\quad ■\end{aligned}$$

Lemma 6.2. *For any $t\in\mathbb{T}$ and $\alpha>0$, the following inequalities hold on $\mathbb{T}$:*

$$\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\Delta s\le\frac{R_1}{1-e_{\ominus\alpha}(\vartheta+\omega,\vartheta)},\qquad \int_t^{+\infty} e_{\ominus\alpha}(\sigma(s),t)\Delta s\le\frac{R_2}{1-e_{\ominus\alpha}(\vartheta+\omega,\vartheta)},$$

where

$$R_1=\int_{\vartheta}^{\vartheta+\omega}(1+\mu(s)\alpha)\Delta s,\qquad R_2=\int_{\vartheta}^{\vartheta+\omega}(1+\mu(s)\alpha)^{-1}\Delta s.$$

Proof. First, we have

$$e_{\ominus\alpha}(t,\sigma(s))=e_{\alpha}(\sigma(s),t)=(1+\mu(s)\alpha)e_{\alpha}(s,t)=(1+\mu(s)\alpha)e_{\ominus\alpha}(t,s).$$

It follows from the assumption and Lemma 6.1 that

$$\int_{t-(n+1)\omega}^{t-n\omega}(1+\mu(s)\alpha)e_{\ominus\alpha}(t,s)\Delta s\le e_{\ominus\alpha}(t,t-n\omega)\int_{t-(n+1)\omega}^{t-n\omega}(1+\mu(s)\alpha)\Delta s=e_{\ominus\alpha}(t,t-n\omega)R_1.$$

Hence

$$\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\Delta s = \sum_{n=0}^{+\infty}\int_{t-(n+1)\omega}^{t-n\omega}(1+\mu(s)\alpha)e_{\ominus\alpha}(t,s)\Delta s \le \sum_{n=0}^{+\infty} e_{\ominus\alpha}(t,t-n\omega)R_1$$
$$= \frac{R_1}{1-e_{\ominus\alpha}(\vartheta+\omega,\vartheta)}.$$

For the second inequality, we have

$$e_{\ominus\alpha}(\sigma(s),t) = (1+u(s)\cdot\ominus\alpha)e_{\ominus\alpha}(s,t) = (1+\mu(s)\alpha)^{-1}e_{\ominus\alpha}(s,t).$$

Carrying through arguments similar to those above, one can easily show that

$$\int_{t}^{+\infty} e_{\ominus\alpha}(s,t)\Delta s \le \frac{R_2}{1-e_{\ominus\alpha}(\vartheta+\omega,\vartheta)}. \quad \blacksquare$$

Lemma 6.3. *If $X(t)$ is a fundamental matrix solution of (3.1), then $X(t+\omega)$ is also a fundamental solution; moreover, for $t,s\in\mathbb{T}$,*

$$\begin{aligned} &X(t+\omega)PX^{-1}(s+\omega) = X(t)PX^{-1}(s)\\ &X(t+\omega)(I-P)X^{-1}(s+\omega) = X(t)(I-P)X^{-1}(s). \end{aligned} \tag{6.3}$$

Proof. Since $\det X(t)\neq 0$ and $X^{\Delta}(t)=A(t)X(t)$, we conclude that

$$\det X(t+\omega)\neq 0, \qquad X(t+\omega)^{\Delta} = A(t+\omega)X(t+\omega) = A(t)X(t+\omega),$$

and thus $X(t+\omega)$ is also a fundamental matrix solution. Furthermore, it follows from properties of the fundamental matrix solution that there is an n-dimensional real vector number C_0 such that $X(t+\omega)=X(t)C_0$ for all $t\in\mathbb{T}$. Then

$$X(t+\omega)PX^{-1}(s+\omega) = X(t)C_0PC_0^{-1}X^{-1}(s) = X(t)PX^{-1}(s)$$

and

$$X(t+\omega)(I-P)X^{-1}(s+\omega) = X(t)C_0(I-P)C_0^{-1}X^{-1}(s) = X(t)(I-P)X^{-1}(s).$$

This proves the lemma. $\blacksquare$

Lemma 6.4. *Assume that (3.1) is exponentially dichotomous and $g(t)$ is an ω-periodic rd-continuous vector function on $\mathbb{T}$; then*

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))g(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))g(s)\Delta s$$

is an ω-periodic solution of $x^{\Delta}(t)=A(t)x(t)+g(t)$ on $\mathbb{T}$.

Proof. Obviously, the above-defined function $x(t)$ is a solution of $x^{\Delta}(t)=A(t)x(t)+g(t)$. Next, we show that $x(t)$ is ω-periodic. Since $g(t)$ is ω-periodic, by Lemma 6.3, for any $t\in\mathbb{T}$, we have

$$\begin{aligned} x(t+\omega) &= \int_{-\infty}^{t+\omega} X(t+\omega)PX^{-1}(\sigma(s))g(s)\Delta s - \int_{t+\omega}^{+\infty} X(t+\omega)(I-P)X^{-1}(\sigma(s))g(s)\Delta s\\ &= \int_{-\infty}^{t} X(t+\omega)PX^{-1}(\sigma(s+\omega))g(s+\omega)\Delta s - \int_{t}^{+\infty} X(t+\omega)(I-P)X^{-1}(\sigma(s+\omega))g(s+\omega)\Delta s\\ &= \int_{-\infty}^{t} X(t+\omega)PX^{-1}(\sigma(s)+\omega)g(s+\omega)\Delta s - \int_{t}^{+\infty} X(t+\omega)(I-P)X^{-1}(\sigma(s)+\omega)g(s+\omega)\Delta s\\ &= \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))g(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))g(s)\Delta s\\ &= x(t). \quad \blacksquare \end{aligned}$$

Similarly, we have:

Lemma 6.5. *Assume that (5.11) is exponentially dichotomous and $g(t)$ is an ω-periodic rd-continuous vector function on $\mathbb{T}$; then*

$$x_u(t) = \int_{-\infty}^{t} X_u(t)PX_u^{-1}(\sigma(s))g(s)\Delta s - \int_{t}^{+\infty} X_u(t)(I-P)X_u^{-1}(\sigma(s))g(s)\Delta s$$

is an ω-periodic solution of $x^{\Delta}(t)=A(t,u(t))x(t)+g(t)$ on $\mathbb{T}$.

Now we are in a position to give existence theorems for periodic solutions for (6.1) and (6.2).

Theorem 6.1. *Assume that (5.11) has an exponential dichotomy. Moreover, if there is an $M_1 > 0$ such that*

$$\frac{K(R_1+R_2)}{1-e_{\ominus\alpha}(\vartheta+\omega,\vartheta)}\sup_{t\in[\vartheta,\vartheta+\omega],|x|\le M_1}|f(t,x)|\le M_1,$$

where R_1 and R_2 are defined in Lemma 6.2 and α is the growth constant in the exponential dichotomy, then Eq. (6.2) has an ω-periodic solution.

Proof. Define

$$B=\{u\in \mathrm{C}(\mathbb{T}): u(t+\omega)=u(t),\ t\in\mathbb{T}\}.$$

It is not difficult to show that B is a Banach space endowed with the supremum norm $|u|=\sup_{t\in\mathbb{T}}|u(t)|$. Take

$$B_0=\{u\in B: |u|\le M_1\}.$$

Obviously, it is a closed convex subset of B. For $u\in B_0$, consider the linear dynamic equation

$$x^{\Delta}(t)=A(t,u(t))x(t)+f(t,u(t)). \tag{6.4}$$

Since (5.11) is exponentially dichotomous, by Lemma 6.5, there exists an ω-periodic solution $x_u(t)$ of Eq. (6.4) and it is given by

$$x_u(t)=\int_{-\infty}^{t}X_u(t)PX_u^{-1}(\sigma(s))f(s,u(s))\Delta s-\int_{t}^{+\infty}X_u(t)(I-P)X_u^{-1}(\sigma(s))f(s,u(s))\Delta s.$$

It follows from the exponential dichotomy and Lemma 6.2 that

$$\begin{aligned}
|x_u(t)| &= \left|\int_{-\infty}^{t}X_u(t)PX_u^{-1}(\sigma(s))f(s,u(s))\Delta s-\int_{t}^{+\infty}X_u(t)(I-P)X_u^{-1}(\sigma(s))f(s,u(s))\Delta s\right|\\
&\le \int_{-\infty}^{t}|X_u(t)PX_u^{-1}(\sigma(s))f(s,u(s))|\Delta s+\int_{t}^{+\infty}|X_u(t)(I-P)X_u^{-1}(\sigma(s))f(s,u(s))|\Delta s\\
&\le \left(\int_{-\infty}^{t}Ke_{\ominus\alpha}(t,\sigma(s))\Delta s+\int_{t}^{+\infty}Ke_{\ominus\alpha}(\sigma(s),t)\Delta s\right)\sup_{t\in[\vartheta,\vartheta+\omega],\|u\|\le M_1}|f(t,u)|\\
&\le \frac{K(R_1+R_2)}{1-e_{\ominus\alpha}(\vartheta+\omega,\vartheta)}\sup_{t\in[\vartheta,\vartheta+\omega],\|u\|\le M_1}|f(t,u)|\\
&\le M_1.
\end{aligned}$$

Therefore, we define a mapping $T: B_0\to B_0$ by $Tu(t)=x_u(t)$. For any sequence $\{u_n(t)\}\subseteq B_0$, by the above arguments, it is clear that $\{Tu_n(t)\}$ is uniformly bounded. Moreover, we have

$$\begin{aligned}
|x_{u_n}^{\Delta}(t)| &= |A(t,u_n(t))x_{u_n}(t)+f(t,u_n(t))|\\
&\le |A(t,u_n(t))||x_{u_n}(t)|+|f(t,u_n(t))|\\
&\le \left(N^*+\frac{1-e_{\ominus\alpha}(\vartheta+\omega,\vartheta)}{K(R_1+R_2)}\right)M_1,
\end{aligned}\tag{6.5}$$

where

$$N^*=\sup_{t\in[\vartheta,\vartheta+\omega],\|u_n\|\le M_1}|A(t,u_n(t))|.$$

Therefore, $\{Tu_n(t)\}$ is equicontinuous. By the Ascoli–Arzela theorem, $\{x_{u_n}(t)\}$ has a subsequence which converges uniformly on any compact set of $\mathbb{T}$. For simplicity, we still denote it by $\{Tu_n(t)\}$. Since $\{Tu_n(t)\}$ is continuous and ω-periodic, then $\{Tu_n(t)\}$ is uniformly convergent on $\mathbb{T}$. Now we claim that $T(B_0)$ is relatively compact in B_0.

Next, we show that T is continuous. Suppose $\{u_n(t)\}\subseteq B_0$ and $u_n(t)\to u(t)$ as $n\to+\infty$. Since $u_n(t)$ is continuous and ω-periodic, $u_n(t)$ uniformly converges to $u(t)$ on $\mathbb{T}$. Furthermore, $x_{u_n}(t)$ is continuous; it is not difficult to show that $x_{u_n}(t)$ converges to $x_u(t)$, namely, $Tu_n\to Tu$, which implies that T is a continuous mapping. Therefore, by Schauder's fixed point theorem, T has a fixed point in B_0, that is, there is a $u_0\in B_0$ such that $Tu_0=u_0$. Therefore, there exists an ω-periodic solution of (6.2). This completes the proof of the theorem. ■

Similarly, we have:

Theorem 6.2. *Assume that (3.1) is exponentially dichotomous. Moreover, there is an $M_2>0$ such that*

$$\frac{K(R_1+R_2)}{1-e_{\ominus\alpha}(\vartheta+\omega,\vartheta)}\sup_{t\in[\vartheta,\vartheta+\omega],\|x\|\le M_1}|f(t,x)|\le M_2,$$

where R_1 and R_2 are defined in Lemma 6.2 and α is the growth constant in the exponential dichotomy; then Eq. (6.1) has an ω-periodic solution.

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