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$M^X/G/1$ unreliable retrial queue with option of additional service and Bernoulli vacation

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Abstract In this paper, retrial queue with unreliable server and bulk arrivals is investigated. The server is capable of providing *m*-optional services and any one of these available services, may be rendered to the customer after the first essential service if the customer opts for the same. It is assumed that the server may fail while rendering any phase of service and undergoes for the immediate repair. After the completion of the service of a customer, the server may either take a vacation for a random period or may continue to provide the service to the other customers waiting in the queue. The supplementary variables corresponding to service time, repair time and retrial time are incorporated to determine the queue size distribution. To examine the effect of different parameters on the performance measures of the system, the numerical illustration is given which is supported by numerical simulation and sensitivity analysis.

1. Introduction

In most of the literature of queueing theory, it is usual phenomenon that if the arriving customer in the system finds the server busy, then he cannot get the service immediately; in such situation, either he joins the queue to get the service or leaves the system forever. But, in many real world day-to-day activities, it is general practice that the customer again joins the system after some random time to get the service at the same service station. In such scenarios, the customers wait in a virtual buffer called retrial orbit and the queue with repeated attempts or queue of rejoining customers is formed which is known as retrial queue. To cite the retrial queueing situation, we consider the call center where the callers on finding the line busy may decide to join the retrial orbit and can try again and again after some time till finding the line free. Similar situation can be observed at the manufacturing system where the quality control engineer has to check the quality of the ready stock of the products, but on arrival if he finds that the server is not in the position to provide the service immediately because of its busy schedule then he may decide to join the virtual orbit and after some random period of time, he again comes back to check the quality of the stock of products if ready for

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checking otherwise again repeats his attempt till stock is ready for checking.

In day-to-day activities of the real world situations, it is a common assumption that the service station works on permanent basis without any interruption. In flexible manufacturing/production systems, the service station may breakdown and requires immediate attention of the repairman to get back in good condition so that the functioning of the system can be restored. Sometimes, the customers may be discouraged due to the service interruption. For the effective results in production systems, the manufacturer has to pay special attention toward the maintenance of the machines for the smooth functioning of the system. The server may take the regular vacation of fixed duration for the overhauling of the machines or optional vacation whenever required to correct some sudden faults in the machinery parts. In industrial queues, some units are kept in inactive mode i.e. automatically go on vacation and become active after certain duration i.e. vacation period so as to be ready to render the service.

In some congestion situations of service system, the provision of optional services on the demand of the customers is very realistic and is provided to the customers on demand after completion of the first essential phase of service. Such multiphase optional service systems are economic and time saving as the customer can receive the several phases of the service from the same server and this type of service facility can be realized in numerous real-life activities. For more clarity of the situation, we cite the example of the health care centers wherein the patients arrive for the routine checkup as regular service but some of the patients who are not satisfied with the regular checkup due to health problem or with the advice of the doctor, can go for additional checkup facilitated by the same doctor who desires to perform other tests viz. ECG, blood pressure checkup, etc.

In the congestion situations of retrial queues, several researchers have contributed significantly to develop stochastic models in different frameworks. Falin and Templeton [1] have discussed various models on retrial queues. In recent past, many researchers presented the important works on retrial queues with different variations [2–7]. Atencia and Moreno [8] discussed a queueing system with general retrial times under the assumption that the only unit at the head of the orbit is allowed to get another chance for the service. They obtained various performance indices to study the steady-state behavior of the system size distribution. The multi-server finite source retrial queue for limited capacity server was considered by Wuchner et al. [9] by incorporating the balking phenomenon. Purohit et al. [10] considered the $M/M/1$ queue with state dependent rates and constant retrial policy. In this work, they considered the threshold level to start the repair of the failed server. Using supplementary variable approach, the single server retrial queue with bulk input and balking was investigated by Bhagat and Jain [11]. In this paper, they assumed that second optional phase service is provided to the customers after availing first phase essential service. The queue size distribution and other performance indices to study the steady-state behavior for the non-markovian retrial queue with two phases of service under general vacation policy were studied by Kumar and Arumuganathan [12]. Choudhury and Ke [13] have analyzed the $M^X/G/1$ system with general retrial time and Bernoulli vacation schedule and presented the performance analysis of the system to study the steady state behavior. Singh et al. [14] have investigated the non-markovian queueing system with general service and state dependent arrival rates. In this investigation, they assumed that the server may breakdown at any stage of the service and undergoes for immediate repair. Recently, Zhang et al. [15] have considered the stochastic model for the unreliable server queue based on queueing characterization. They assumed that the server may fail at any stage of the service and the arriving customers decide to enter the retrial orbit or to follow the balking behavior based on the current information available in the system at their arrival. The non-markovian model of the queueing system with bulk arrival and general distribution of service time was studied by Singh et al. [16]. By incorporating some more realistic features namely the server vacation of a fixed duration and balking behavior of the customers, the mathematical analysis to study the steady state behavior of the system was done by using supplementary variable technique. Jain and Bhagat [17] have investigated the modified vacation policy for a retrial queue with bulk arrival and k-essential phases of service rendered to all customers by a server which is subject to breakdown and repair. Recently, Dimitriou [18] has analyzed a retrial queue to examine the performance characteristics of fault-tolerant system. The stochastic model of retrial queue for double orbit system with finite capacity and unreliable server was considered by Jain et al. [19]. In this study, they applied the matrix method to obtain the steady state results and threshold recovery parameter.

Choudhury and Deka [20] considered an unreliable server $M/G/1$ system where the customers join the system singly with homogeneous rate to get the two phases of services. In this investigation, they have obtained the probability-generating functions of the queue size distribution to evaluate the performance measures of interest. In industrial scenario, it is common practice that the units/jobs may join the service system in batches and the server is unreliable and may fail at any stage of the service. On the arrival, if the customers notice a long queue in the service system and find no scope of the immediate service, they immediately take the decision to join the retrial orbit to get the service in the next attempt. Beside this, in the real world congestion scenarios there are so many similar situations where the customers may not leave the system forever and would like to retry their attempts after random interval of time. Such important applications in real time situations motivated us to extend the model of Choudhury and Deka [20] by incorporating the retrial policy for batch arrival queue with the provision of essential service and m-optional services.

In many practical situations, the concept of retrial queues and the demand of optional services apart from the first phase essential service are normal features. Furthermore, the provision of vacation schedule of the server may be required for other secondary works, for example for maintenance work or to check the efficiency of the system. For many machining systems or servers, the rest is also needed for further smooth functioning which can be provided during the vacation period from the economic view point. In view of the above, the present investigation deals as the extension of Choudhury and Deka [20] by considering a single server queueing system with the provision to opt one of the m-optional services available in addition to first essential service rendered to all arriving customers. The aim of this investigation is to find the queue size and orbit size distributions which are further employed to obtain the cost analysis and other performance measures of the system. The outline of the remaining sections is as follows. By stating the requisite assumptions and notations, the mathematical model is described in Section 2. In Section 3, the supplementary variables corresponding to elapsed times of different processes such as service time, repair time and retrial time are considered and the governing equations are constructed. Section 4 contains the mathematical analysis of the system by using the probability reasoning. The stochastic decomposition is presented in Section 5. In Sections 6 and 7, the performance measures of the system and some special cases by setting the parameters are discussed. The numerical illustration for the sensitivity analysis of the model is considered in Section 8. Finally, the conclusion of the present investigation is drawn in the last section.

2. Model description

To analyze the more versatile service system of many realistic situations, we consider the multi-phase service system. It is assumed that the customers arrive in the batches of random size X with probability mass function $c_k = P(X = k);\ k \geqslant 1$ and follow Poisson process with rate λ. Since first phase service is compulsory to all arriving customers, as soon as essential service is completed either the customer may opt ith$(i = 1, 2, \ldots, m)$ optional service immediately with probability r_i or leave the system with probability $r_0 = 1 - \sum_{i=1}^{m} r_i$. It is also considered that after service completion of the customers, the server may take a vacation with probability p or may continue in the system for service with probability $(1 - p)$. Furthermore, on arrival if primary customer finds that the server is in busy state or on vacation state/broken down state, the customer may join the group of unsatisfied customers i.e. retrial orbit. Let the orbit size at time t be $N(t)$. The random variable R denotes the retrial time with distribution function $M(x)$ and its Laplace Stieltjes transforms $\overline{M}(.)$.

It is assumed that the inter arrival time and service time of essential/optional services are independent. Further, the service times of first essential service and ith$(i = 1, 2, \ldots, m)$ optional service are general distributed with distribution functions $B_0(x)$ and $B_i(x)$, their Laplace Stieltjes transforms $\overline{B}_0(.)$ and $\overline{B}_i(.)$, respectively. The general distribution functions of the vacation time, repair time of the server failed during essential service and repair time of the server failed during ith$(i = 1, 2, \ldots, m)$ service, respectively are denoted by $V(x)$, $G_0(x)$ and $G_i(x)$, with their respective Laplace Stieltjes transforms $\overline{V}(.)$, $\overline{G}_0(.)$ and $\overline{G}_i(.)$. The kth moment of repair time of the failed server during essential service and ith optional service are denoted by $g_0^{(k)}$ and $g_i^{(k)}$, respectively. At time t, the elapsed times of different processes such as retrial, service of essential service, service of ith$(i = 1, 2, \ldots, m)$ optional phase, vacation, repair if fails during the essential service, and repair of the server failed in the state of ith optional service are assumed as $R^0(t)$, $B_0^0(t)$, $B_i^0(t)$, $V^0(t)$, $G_0^0(t)$ and $G_i^0(t)$, respectively.

To specify the different server's state, the random variable $\gamma(t)$ is defined as:

$$\gamma(t) = \begin{cases} 0, & \text{if the server is idle with no customer in the system at time } t. \\ 1, & \text{if the server is idle during retrial duration at time } t. \\ 2, & \text{if the server is busy in rendering the essential service at time } t. \\ 2+i, & \text{if the server is busy in rendering the } i\text{th}(i = 1, 2, \ldots, m) \text{ optional service at time } t. \\ 3+m, & \text{if the server is on vacation at time } t. \\ 4+m, & \text{if the server is under repair when failed during the essential service at time } t. \\ 4+m+j, & \text{if the server is under repair when failed during the } j\text{th optional service at time } t. \end{cases}$$

For the analysis of the non-markovian model, the elapsed time $R^0(t)$, $B_0^0(t)$, $B_i^0(t)$, $V^0(t)$, $G_0^0(t)$ and $G_i^0(t)$ are introduced as supplementary variables to obtain the joint distribution of the number of customers in the orbit and in the queue by considering the bi-variate Markov process $\{N(t), Y(t)\}$ where $N(t) = 0, 1, 2, \ldots$; and

$Y(t) = 0,$ if $\gamma(t) = 0$; $Y(t) = R^0(t),$ if $\gamma(t) = 1$;
$Y(t) = B_0^0(t),$ if $\gamma(t) = 2$.
$Y(t) = B_i^0(t),$ if $\gamma(t) = 2 + i$ for $i = 1, 2, \ldots, m$;
$Y(t) = V^0(t),$ if $\gamma(t) = 3 + m$.
$Y(t) = G_0^0(t),$ if $\gamma(t) = 4 + m$; $Y(t) = G_i^0(t),$
if $\gamma(t) = 4 + m + i(i = 1, 2, \ldots, m)$.

The limiting probabilities are defined as

$$I_0 = \lim_{t\to\infty} Pr\{N(t) = 0,\, Y(t) = 0\}; \tag{1}$$

$$A_n(x) = \lim_{t\to\infty} Pr\{N(t) = n,\, Y(t) = R^0(t); x \leqslant R^0(t) \leqslant x + dx\}; \quad x > 0, n \geqslant 1, \tag{2}$$

$$P_n^{(0)}(x) = \lim_{t\to\infty} Pr\{N(t) = n,\, Y(t) = B_0^0(t); x \leqslant B_0^0(t) \leqslant x + dx\}; \quad x > 0, n \geqslant 1, \tag{3}$$

$$P_n^{(i)}(x) = \lim_{t\to\infty} Pr\{N(t) = n,\, Y(t) = B_i^0(t); x \leqslant B_i^0(t) \leqslant x + dx\}; \quad x > 0, n \geqslant 1, 1 \leqslant i \leqslant m, \tag{4}$$

$$V_n(y) = \lim_{t\to\infty} Pr\{N(t) = n,\, Y(t) = V^0(t); y \leqslant V^0(t) \leqslant y + dy\}; \quad y > 0, n \geqslant 0, \tag{5}$$

$$R_n^{(0)}(x, y) = \lim_{t\to\infty} Pr\{N(t) = n,\, Y(t) = R_0^0(t); y \leqslant R_0^0(t) \leqslant y + dy/B_0^0(t) = x\}; \quad (x, y) > 0, n \geqslant 1, \tag{6}$$

$$R_n^{(i)}(x, y) = \lim_{t\to\infty} Pr\{N(t) = n,\, Y(t) = R_i^0(t); y \leqslant R_i^0(t) \leqslant y + dy/B_i^0(t) = x\}; \quad (x, y) > 0, n \geqslant 1, 1 \leqslant i \leqslant m. \tag{7}$$

Also it is assumed that $M(0)=0$, $M(\infty)=1$, $V(0)=0$, $V(\infty)=1$, $B_i(0)=0$, $B_i(\infty)=1$, $G_i(0)=0$, $G_i(\infty)=1$; $G_i(y)$, $V(y)$ are continuous at $y=0$ and $B_i(x)$ is continuous at $x=0$, so that hazard rates corresponding to respective distributions are obtained as

$$k(x)dx=\frac{dM(x)}{1-M(x)},\mu_i(x)dx=\frac{dB_i(x)}{1-B_i(x)},v(y)dy=\frac{dV(y)}{1-V(y)},g_i(y)dy=\frac{dG_i(y)}{1-G_i(y)};0\leqslant i\leqslant m.$$

The probability generating functions with $|z|<1$ for $i=0,1,2,\ldots,m$, are defined as follows:

$$R^{(i)}(x,y,z)=\sum_{n=1}^{\infty}z^nR_n^{(i)}(x,y);\ R^{(i)}(x,0,z)=\sum_{n=1}^{\infty}z^nR_n^{(i)}(x,0);\ P^{(i)}(x,z)=\sum_{n=1}^{\infty}z^nP_n^{(i)}(x);$$

$$P^{(i)}(0,z)=\sum_{n=1}^{\infty}z^nP_n^{(i)}(0);\ V(y,z)=\sum_{n=0}^{\infty}z^nV_n(y);\ V(0,z)=\sum_{n=0}^{\infty}z^nV_n(0);\ A(x,z)=\sum_{n=1}^{\infty}z^nA_n(x).$$

3. Governing equations

By using the probability reasoning as discussed in Cox [21], the equations governing the model are formulated as follows:

$$\lambda I_0=q\left[r_0\int_0^\infty\mu_0(x)P_1^{(0)}(x)dx+\sum_{i=1}^{m}\int_0^\infty\mu_i(x)P_1^{(i)}(x)dx\right]+\int_0^\infty v(y)V_0(y)dy,\tag{8}$$

$$\frac{d}{dx}A_n(x)+[\lambda+k(x)]A_n(x)=0;\quad n\geqslant 1,\tag{9}$$

$$\frac{d}{dx}P_n^{(i)}(x)+[\lambda+\alpha_i+\mu_i(x)]P_n^{(i)}(x)=\lambda\sum_{j=1}^{n}c_jP_{n-j}^{(i)}(x)+\int_0^\infty g_i(y)R_n^{(i)}(x,y)dy;\quad (x,y)>0,0\leqslant i\leqslant m.\tag{10}$$

$$\frac{d}{dy}V_n(y)+[\lambda+v(y)]V_n(y)=\lambda\sum_{j=1}^{n}c_jV_{n-j}(y);\quad n\geqslant 0,y>0,\tag{11}$$

$$\frac{d}{dy}R_n^{(i)}(x,y)+[\lambda+g_i(y)]R_n^{(i)}(x,y)=\lambda\sum_{j=1}^{n}c_jR_{n-j}^{(i)}(x,y);\quad n\geqslant 1,(x,y)>0,0\leqslant i\leqslant m.\tag{12}$$

To solve Eqs. (9) and (10), the following boundary conditions at $x=0$ are considered.

$$A_n(0)=q\left[r_0\int_0^\infty\mu_0(x)P_{n+1}^{(0)}(x)dx+\int_0^\infty\mu_1(x)P_{n+1}^{(1)}(x)dx+\cdots+\int_0^\infty\mu_m(x)P_{n+1}^{(m)}(x)dx\right]+\int_0^\infty v(y)V_n(y)dy;\quad n\geqslant 1,\tag{13}$$

$$P_n^{(0)}(0)=\lambda\sum_{i=1}^{n}c_i\int_0^\infty A_{n-i}(x)dx+\lambda c_nI_0+\int_0^\infty k(x)A_n(x)dx;\quad n\geqslant 1,\tag{14}$$

$$P_n^{(i)}(0)=r_i\int_0^\infty\mu_0(x)P_n^{(0)}(x)dx;\quad n\geqslant 1,\ 1\leqslant i\leqslant m.\tag{15}$$

The boundary condition at $y=0$ for the solution of Eq. (11) is defined as follows:

$$V_n(0)=p\left[r_0\int_0^\infty\mu_0(x)P_{n+1}^{(0)}(x)dx+\sum_{i=1}^{m}\int_0^\infty\mu_i(x)P_{n+1}^{(i)}(x)dx\right];\quad n\geqslant 0\tag{16}$$

For the solution of Eq. (12), the boundary condition for fixed value of x, at $y=0$ is considered as

$$R_n^{(i)}(x;0)=\alpha_iP_n^{(i)}(x);\quad n\geqslant 1,\ i=0,1,2,\ldots,m\tag{17}$$

The normalizing condition can be stated as

$$I_0+\sum_{n=1}^{\infty}\sum_{i=0}^{m}\left[\int_0^\infty P_n^{(i)}(x)dx+\int_0^\infty\int_0^\infty R_n^{(i)}(x,y)dxdy\right]+\sum_{n=1}^{\infty}\int_0^\infty A_n(x)dx+\sum_{n=0}^{\infty}\int_0^\infty V_n(y)dy=1\tag{18}$$

4. Mathematical analysis

Following Choudhury and Deka [20], from Eqs. (9), (11) and (12), we get

$$A(x,z)=A(0,z)[1-M(x)]\exp(-\lambda x);\quad x>0,\tag{19}$$

$$V(y,z)=V(0,z)[1-V(y)]\exp(-a(z)y);\quad y>0,\tag{20}$$

$$R^{(i)}(x,y,z)=R^{(i)}(x,0,z)[1-G_i(y)]\exp(-a(z)y);\quad (x,y)>0,\ 0\leqslant i\leqslant m.\tag{21}$$

On multiplying Eq. (17) with suitable power of z, we get

$$R^{(i)}(x,0,z)=\alpha_iP^{(i)}(x,z);\quad i=0,1,2,\ldots,m\tag{22}$$

On solving Eq. (10) and using (21), we have

$$\frac{d}{dx}P^{(i)}(x,z)+(a(z)+\alpha_i+\mu_i(x))P^{(i)}(x,z)=R^{(i)}(x,0,z)\overline{G}_i(a(z));\quad 0\leqslant i\leqslant m.\tag{23}$$

By using Eq. (22) in (23) and on simplification, we get

$$P^{(i)}(x,z)=P^{(i)}(0,z)[1-B_i(x)]\exp(-\phi_i(z)x);\quad x>0,\ 0\leqslant i\leqslant m.\tag{24}$$

with $\phi_i(z)=a(z)+\alpha_i(1-\overline{G}_i(a(z)))$ and $a(z)=\lambda(1-X(z))$.

From Eqs. (21), (22) and (24), we have

$$R^{(i)}(x,y,z)=\alpha_iP^{(i)}(0,z)[1-B_i(x)]\exp(-\phi_i(z)x)[1-G_i(y)]\times\exp(-a(z)y);\quad (x,y)>0,\ 0\leqslant i\leqslant m.\tag{25}$$

On multiplying Eqs. (15) and (16) by suitable power of z and further solving, we get

$$P^{(i)}(0,z)=r_iP^{(0)}(0,z)\overline{B}_0(\phi_0(z));\quad 1\leqslant i\leqslant m.\tag{26}$$

$$V(0,z) = \frac{pP^{(0)}(0,z)\overline{B}_0(\phi_0(z))\left[r_0 + \sum_{i=1}^{m} r_i \overline{B}_i(\phi_i(z))\right]}{z} \tag{27}$$

Again, multiplying Eq. (13) by suitable power of z for $n \geqslant 1$ and using Eqs. (26) and (27), we have

$$A(0,z) = z^{-1}P^{(0)}(0,z)\overline{B}_0(\phi_0(z))\left\{r_0 + \sum_{i=1}^{m} r_i \overline{B}_i(\phi_i(z))\right\} \times \{q + p\overline{V}(a(z))\} - \lambda I_0 \tag{28}$$

Similarly, Eq. (14) gives

$$P^{(0)}(0,z) = \lambda I_0 X(z) + A(0,z)[\overline{M}(\lambda) + X(z)(1-\overline{M}(\lambda))] \tag{29}$$

Using Eqs. (28) and (29), we get

$$P^{(0)}(0,z) = \frac{zI_0 a(z)\overline{M}(\lambda)}{S(z)} \tag{30}$$

with

$$S(z) = \overline{B}_0(\phi_0(z))\left\{r_0 + \sum_{i=1}^{m} r_i \overline{B}_i(\phi_i(z))\right\}\{q + p\overline{V}(a(z))\}[\overline{M}(\lambda) + X(z)(1-\overline{M}(\lambda))] - z.$$

By using Eqs. (28) and (30), we get

$$A(0,z) = \frac{zI_0\left[z - \overline{B}_0(\phi_0(z))\left\{r_0 + \sum_{i=1}^{m} r_i \overline{B}_i(\phi_i(z))\right\}\{q + p\overline{V}(a(z))\}X(z)\right]}{S(z)} \tag{31}$$

Taking limit $z \to 1$, Eq. (30) yields

$$P^{(0)}(0,1) = \frac{\lambda I_0 E(X)\overline{M}(\lambda)}{(1-\rho-E(X)(1-\overline{M}(\lambda)))} \tag{32}$$

where

$$\rho = \lambda E(X)\{E(B_0)\left(1+\alpha_0 g_0^{(1)}\right) + \sum_{i=1}^{m} r_i E(B_i)\left(1+\alpha_i g_i^{(1)}\right) + pE(V)\}.$$

By taking the limit $z \to 1$ in Eqs. (19)–(21), (24)–(27) and using Eq. (32), we have

$$A(x,1) = \frac{\lambda I_0[E(X)-(1-\rho)][1-\overline{M}(\lambda)]\exp(-\lambda x)}{[1-\rho-E(X)(1-\overline{M}(\lambda))]} \tag{33}$$

$$P^{(0)}(x,1) = \frac{\lambda I_0 E(X)\overline{M}(\lambda)[1-B_0(x)]}{[1-\rho-E(X)(1-\overline{M}(\lambda))]}; \quad x>0, \tag{34}$$

$$P^{(i)}(x,1) = \frac{r_i\lambda I_0 E(X)\overline{M}(\lambda)[1-B_i(x)]}{[1-\rho-E(X)(1-\overline{M}(\lambda))]}; \quad x>0,\ 1 \leqslant i \leqslant m \tag{35}$$

$$V(y,1) = \frac{p\lambda I_0 E(X)\overline{M}(\lambda)[1-V(y)]}{[1-\rho-E(X)(1-\overline{M}(\lambda))]}; \quad y>0, \tag{36}$$

$$R^{(0)}(x,y,1) = \frac{\alpha_0\lambda I_0 E(X)\overline{M}(\lambda)[1-B_0(x)][1-G_0(y)]}{[1-\rho-E(X)(1-\overline{M}(\lambda))]}; \quad (x,y)>0. \tag{37}$$

$$R^{(i)}(x,y,1) = \frac{\alpha_i r_i\lambda I_0 E(X)\overline{M}(\lambda)[1-B_i(x)][1-G_i(y)]}{[1-\rho-E(X)(1-\overline{M}(\lambda))]}; \quad (x,y)>0. \tag{38}$$

Using Eqs. (33)–(38) in normalizing condition (18), we have

$$I_0 = \frac{[1-\rho-E(X)(1-\overline{M}(\lambda))]}{\overline{M}(\lambda)} \tag{39}$$

Lemma 1. *The necessary and sufficient condition for the system to be stable is given by inequality*

$$\rho + E(X)(1-\overline{M}(\lambda)) < 1. \tag{40}$$

where $\rho = \lambda E(X)\left\{E(B_0)\left(1+\alpha_0 g_0^{(1)}\right) + \sum_{i=1}^{m} r_i E(B_i)\left(1+\alpha_i g_i^{(1)}\right) + pE(V)\right\}$.

Proof. For proof see Appendix A. □

Theorem 1. *Under stability condition* $\rho + E(X)(1-\overline{M}(\lambda)) < 1$, *the probability generating functions of joint probability distributions of the server state and orbit size are given by*

$$A(x,z) = \frac{\lambda[1-\rho-E(X)(1-\overline{M}(\lambda))][1-M(x)]\exp(-\lambda x) \times \left[z - \overline{B}_0(\phi_0(z))\left\{r_0 + \sum_{i=1}^{m} r_i \overline{B}_i(\phi_i(z))\right\}\{q + p\overline{V}(a(z))\}X(z)\right]}{\overline{M}(\lambda)S(z)} \tag{41}$$

$$P^{(0)}(x,z) = \frac{[1-\rho-E(X)(1-\overline{M}(\lambda))]za(z)[1-B_0(x)]\exp(-\phi_0(z)x)}{S(z)} \tag{42}$$

$$P^{(i)}(x,z) = \frac{r_i[1-\rho-E(X)(1-\overline{M}(\lambda))]za(z)\overline{B}_0(\phi_0(z))[1-B_i(x)]\exp\{-\phi_i(z)x\}}{S(z)}, \quad 1 \leqslant i \leqslant m \tag{43}$$

$$V(y,z) = \frac{p[1-\rho-E(X)(1-\overline{M}(\lambda))]a(z)\overline{B}_0(\phi_0(z))\left\{r_0 + \sum_{i=1}^{m} r_i \overline{B}_i(\phi_i(z))\right\}[1-V(y)]\exp(-a(z)y)}{S(z)} \tag{44}$$

$$R^{(0)}(x,y,z)=\frac{\alpha_0[1-\rho-E(X)(1-\overline{M}(\lambda))]za(z)[1-B_0(x)]\exp\{-\phi_0(z)x\}[1-G_0(y)]\exp\{-a(z)y\}}{S(z)}, \tag{45}$$

$$R^{(i)}(x,y,z)=\frac{\alpha_i r_i[1-\rho-E(X)(1-\overline{M}(\lambda))]za(z)\overline{B}_0(\phi_0(z))[1-B_i(x)]\exp\{-\phi_i(z)x\}\times[1-G_i(y)]\exp\{-a(z)y\}}{S(z)},\quad 1\leqslant i\leqslant m \tag{46}$$

Proof. Using Eqs. (26), (30), (31) and (39) in Eqs. (19), (20) and (24), (25), we get the required results. □

Theorem 2. *The probability generating functions of the marginal probability distributions of the orbit size for different server states are*

$$A(z)=\frac{[1-\rho-E(X)(1-\overline{M}(\lambda))][1-\overline{M}(\lambda)]\times[z-\overline{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^m r_i\overline{B}_i(\phi_i(z))\}\{q+p\overline{V}(a(z))\}X(z)]}{\overline{M}(\lambda)S(z)} \tag{47}$$

$$P^{(0)}(z)=\frac{[1-\rho-E(X)(1-\overline{M}(\lambda))]za(z)[1-\overline{B}_0(\phi_0(z))]}{S(z)\phi_0(z)} \tag{48}$$

$$P^{(i)}(z)=\frac{r_i[1-\rho-E(X)(1-\overline{M}(\lambda))]za(z)\overline{B}_0(\phi_0(z))[1-\overline{B}_i(\phi_i(z))]}{S(z)\phi_i(z)},\quad 1\leqslant i\leqslant m. \tag{49}$$

$$V(z)=\frac{p[1-\rho-E(X)(1-\overline{M}(\lambda))]\overline{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^m r_i\overline{B}_i(\phi_i(z))\}[1-\overline{V}(a(z))]}{S(z)} \tag{50}$$

$$R^{(0)}(z)=\frac{\alpha_0 z[1-\rho-E(X)(1-\overline{M}(\lambda))][1-\overline{B}_0(\phi_0(z))][1-\overline{G}_0(a(z))]}{S(z)\phi_0(z)} \tag{51}$$

$$R^{(i)}(z)=\frac{\alpha_i r_i z[1-\rho-E(X)(1-\overline{M}(\lambda))]\overline{B}_0(\phi_0(z))[1-\overline{B}_i(\phi_i(z))][1-\overline{G}_i(a(z))]}{S(z)\phi_i(z)},\quad 1\leqslant i\leqslant m. \tag{52}$$

Proof. For proof see Appendix B. □

Theorem 3. *The probability generating functions of the queue size in the system and orbit at the arbitrary epoch, are*

$$P(z)=\frac{[1-\rho-E(X)(1-\overline{M}(\lambda))](1-z)\overline{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^m r_i\overline{B}_i(\phi_i(z))\}\{q+p\overline{V}(a(z))\}}{S(z)} \tag{53}$$

$$O(z)=\frac{[1-\rho-E(X)(1-\overline{M}(\lambda))](1-z)}{S(z)} \tag{54}$$

Proof. For proof see Appendix C. □

Theorem 4. *The probability generating function of the queue size at the departure epoch, is*

$$\pi(z)=\frac{[1-\rho-E(X)(1-\overline{M}(\lambda))](1-X(z))\overline{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^m r_i\overline{B}_i(\phi_i(z))\}\{q+p\overline{V}(a(z))\}}{E(X)S(z)} \tag{55}$$

Proof. For proof see Appendix D. □

5. Stochastic decomposition

The probability generating function given in Eq. (55) can be written as

$$\pi(z)=P(z)\cdot\frac{(1-X(z))}{E(X)(1-z)} \tag{56}$$

It is clear from Eq. (56) that the queue size distribution at departure epoch can be decomposed into following two parts:

(i) Probability generating function $P(z)$ as the stationary queue size distribution of unreliable bulk queue with retrial customers, J optional service, single optional vacation and repair of the failed server.
(ii) Term corresponding to the number of customers placed before a tagged customer.

The existence of stochastic decomposition for present system can be given as

$$P(z) = P_0(z)\chi(z) \tag{57}$$

where $P_0(z)$ is the PGF of the system size of an $M^X/G/1$ queue with unreliable server under Bernoulli vacation and its distribution can be obtained by setting $\overline{M}(\lambda) = 1$ in Eq. (53). Thus

$$P_0(z) = \frac{(1-\rho)(1-z)\overline{B}_0(\phi_0(z))\{r_0 + \sum_{i=1}^{m} r_i\overline{B}_i(\phi_i(z))\}\{q + p\overline{V}(a(z))\}}{[\overline{B}_0(\phi_0(z))\{r_0 + \sum_{i=1}^{m} r_i\overline{B}_i(\phi_i(z))\}\{q + p\overline{V}(a(z))\} - z]} \tag{58}$$

Also $\chi(z)$ is the PGF for additional system size due to retrial time and is given by

$$\chi(z) = \frac{[1-\rho-E(X)(1-\overline{M}(\lambda))][\overline{B}_0(\phi_0(z))\{r_0 + \sum_{i=1}^{m} r_i\overline{B}_i(\phi_i(z))\}\{q + p\overline{V}(a(z))\} - z]}{(1-\rho)S(z)} \tag{59}$$

6. Performance measures

In the previous sections, we have established the analytical results for the probability generating functions for the marginal and joint queue size distributions for the different system states which we can further use to evaluate some measures of performance as follows:

6.1. System state probabilities

The long run system state probabilities of the server being in different states can be obtained by taking limit $z \to 1$ in the marginal probability generating functions of the queue size distributions. Thus.

- The probability that the server is in idle state but is non empty

  $$P_N = \frac{(\rho + E(X) - 1)(1-\overline{M}(\lambda))}{\overline{M}(\lambda)},$$

- The probability that server is busy in rendering the essential service

  $$P_{B_0} = \lambda E(X)E(B_0),$$

- The probability that server is busy in rendering the ith$(i = 1, 2, \ldots, m)$ optional service

  $$P_{B_i} = r_i\lambda E(X)E(B_i),$$

- The probability that server is under vacation

  $$P_V = p\lambda E(X)E(V),$$

- The probability that the server failed during essential service is under repair

  $$P_{R_0} = \alpha_0\lambda E(X)E(B_0)g_0^{(1)},$$

- The probability that the server failed during the ith$(1 \leqslant i \leqslant m)$ optional service is under repair

  $$P_{R_i} = r_i\alpha_i\lambda E(X)E(B_i)g_i^{(1)},$$

- The probability that server is idle

  $$P_I = \frac{1-\rho-E(X)(1-\overline{M}(\lambda))}{\overline{M}(\lambda)}.$$

6.2. Mean queue length

(i) ***Mean number of customers in the orbit***
The mean number of customers in the orbit is obtained as

$$\begin{aligned}
L_0 &= \left.\frac{dO(z)}{dz}\right|_{z=1} \\
&= \frac{(\lambda E(X))^2\left\{2E(B_0)\left(1+\alpha_0 g_0^{(1)}\right)\sum_{i=1}^{m} r_i E(B_i)\left(1+\alpha_i g_i^{(1)}\right) + E\left(B_0^2\right)\left(1+\alpha_0 g_0^{(1)}\right)^2 + \alpha_0 g_0^{(2)}E(B_0) + \sum_{i=1}^{m} r_i\left[E\left(B_i^2\right)\left(1+\alpha_i g_i^{(1)}\right)^2 + \alpha_i g_i^{(2)}E(B_i)\right]\right\}}{2(1-\rho-E(X)(1-\overline{M}(\lambda)))} \\
&\quad + \frac{p(\lambda E(X))^2\left\{2E(V)\sum_{i=1}^{m} r_i E(B_i)\left(1+\alpha_i g_i^{(1)}\right) + 2E(B_0)\left(1+\alpha_0 g_0^{(1)}\right)E(V) + E(V^2)\right\}}{2(1-\rho-E(X)(1-\overline{M}(\lambda)))} + \frac{\rho E(X)(1-\overline{M}(\lambda)))}{(1-\rho-E(X)(1-\overline{M}(\lambda)))} + \frac{E(X^{(2)})}{2E(X)} \\
&\quad \times \frac{(\rho+E(X)(1-\overline{M}(\lambda)))}{(1-\rho-E(X)(1-\overline{M}(\lambda)))}
\end{aligned} \tag{60}$$

(ii) *Mean system size at arbitrary epoch*

The mean number of customers (L_q) at arbitrary epoch is

$$L_q = \left.\frac{dP(z)}{dz}\right|_{z=1}$$

$$L_q = \rho + \frac{(\lambda E(X))^2\left\{2E(B_0)\left(1+\alpha_0 g_0^{(1)}\right)\sum_{i=1}^m r_i E(B_i)\left(1+\alpha_i g_i^{(1)}\right) + E(B_0^2)\left(1+\alpha_0 g_0^{(1)}\right)^2 + \alpha_0 g_0^{(2)} E(B_0) + \sum_{i=1}^m r_i\left[E(B_i^2)\left(1+\alpha_i g_i^{(1)}\right)^2 + \alpha_i g_i^{(2)} E(B_i)\right]\right\}}{2(1-\rho-E(X)(1-\overline{M}(\lambda)))}$$
$$+\frac{p(\lambda E(X))^2\left\{2E(V)\sum_{i=1}^m r_i E(B_i)\left(1+\alpha_i g_i^{(1)}\right) + 2E(B_0)\left(1+\alpha_0 g_0^{(1)}\right)E(V) + E(V^2)\right\}}{2(1-\rho-E(X)(1-\overline{M}(\lambda)))}$$
$$+\frac{\rho E(X)(1-\overline{M}(\lambda)))}{(1-\rho-E(X)(1-\overline{M}(\lambda)))} + \frac{E(X^{(2)})}{2E(X)}\frac{(\rho+E(X)(1-\overline{M}(\lambda)))}{(1-\rho-E(X)(1-\overline{M}(\lambda)))} \quad (61)$$

(iii) *Mean system size at departure epoch*

The mean system size (L_D) at departure epoch can be determined by using

$$L_D = \left.\frac{d\pi(z)}{dz}\right|_{z=1}$$

$$L_D = \rho + \frac{(\lambda E(X))^2\left\{2E(B_0)\left(1+\alpha_0 g_0^{(1)}\right)\sum_{i=1}^m r_i E(B_i)\left(1+\alpha_i g_i^{(1)}\right) + E(B_0^2)\left(1+\alpha_0 g_0^{(1)}\right)^2 + \alpha_0 g_0^{(2)} E(B_0) + \sum_{i=1}^m r_i\left[E(B_i^2)\left(1+\alpha_i g_i^{(1)}\right)^2 + \alpha_i g_i^{(2)} E(B_i)\right]\right\}}{2(1-\rho-E(X)(1-\overline{M}(\lambda)))}$$
$$+\frac{p(\lambda E(X))^2\left\{2E(V)\sum_{i=1}^m r_i E(B_i)\left(1+\alpha_i g_i^{(1)}\right) + 2E(B_0)\left(1+\alpha_0 g_0^{(1)}\right)E(V) + E(V^2)\right\}}{2(1-\rho-E(X)(1-\overline{M}(\lambda)))}$$
$$+\frac{\rho E(X)(1-\overline{M}(\lambda)))}{(1-\rho-E(X)(1-\overline{M}(\lambda)))} + \frac{E(X^{(2)})}{2E(X)}\frac{(\rho+E(X)(1-\overline{M}(\lambda)))}{(1-\rho-E(X)(1-\overline{M}(\lambda)))} + \frac{E(X^{(2)})}{2E(X)} \quad (62)$$

From Eqs. (61) and (62) we can observe that $L_D = L_q + \frac{E(X^{(2)})}{2E(X)}$.

6.3. Reliability indices

The steady state availability A_v, which is the probability that the server is either busy in rendering service or in an idle state, is obtained as

$$A_v = \lim_{z\to 1}\left\{I_0 + P^{(0)}(z) + \sum_{i=1}^m P^{(i)}(z)\right\}$$

$$= \frac{1 - E(X)\left[\lambda\left\{E(B_0)\left(1+\alpha_0 g_0^{(1)}\right) + \sum_{i=1}^m r_i E(B_i)\left(1+\alpha_i g_i^{(1)}\right) + pE(V) + (1-\overline{M}(\lambda)) - \overline{M}(\lambda)\left(E(B_0) + \sum_{i=1}^m r_i E(B_i)\right)\right\}\right].}{\overline{M}(\lambda)} \quad (63)$$

The steady state failure frequency of the server is obtained as

$$F_f = \alpha_0\int_0^\infty P^{(1)}(x,1)dx + \sum_{i=1}^m \alpha_i\int_0^\infty P^{(i)}(x,1)dx$$
$$= \lambda E(X)\{\alpha_0 E(B_0) + \sum_{i=1}^m r_i\alpha_i E(B_i)\} \quad (64)$$

Theorem 5. *The expected lengths of busy period (T_b) and busy cycle (T_c) respectively, are*

$$E(T_b) = \frac{[1-\overline{M}(\lambda)](E(X)-1)}{\lambda E(X)[1-E(X)(1-\overline{M}(\lambda))-\rho]} + \frac{E(B_0)(1+\alpha_0 g_0^{(1)})+\sum_{i=1}^{m} r_i E(B_i)\left(1+\alpha_i g_i^{(1)}\right)+pE(V)}{[1-E(X)(1-\overline{M}(\lambda))-\rho]} \tag{65}$$

$$E(T_c) = \frac{\overline{M}(\lambda)}{\lambda E(X)[1-E(X)(1-\overline{M}(\lambda))-\rho]} \tag{66}$$

Proof. By using the argument of renewal theory [13], we have

$$E(T_b) = \frac{1}{\lambda E(X)}\left(\frac{1}{I_0}-1\right) \tag{67}$$

$$E(T_c) = \frac{1}{\lambda E(X) I_0}. \tag{68}$$

On substituting the results from (39) in (67) and (68), we get the results as given in Eqs. (65) and (66).

Further, if T_0 be the length of idle period then

$$E(T_c) = E(T_0) + E(T_b),$$

Thus, we get

$$E(T_0) = \frac{1}{\lambda E(X)} \qquad \square. \tag{69}$$

7. Special cases

In this section, some special cases are evaluated by setting the appropriate parameters to validate our results with the existing models.

Case (i): By setting $P(X=1)=1, r_1=1, m=1, \overline{M}(\lambda)=1$; Eq. (55) yields

$$\pi(z) = \frac{(1-\rho)(1-z)[\overline{B}_0(\phi_0(z))\overline{B}_1(\phi_1(z))\{q+p\overline{V}(\lambda(1-z))\}]}{[\overline{B}_0(\phi_0(z))\overline{B}_1(\phi_1(z))\{q+p\overline{V}(a_1(z))\}-z]} \tag{70}$$

where $\phi_i(z) = \lambda(1-z) + \alpha_i(1-\overline{G}_i(\lambda(1-z)))$, $i = 0, 1$. The above result matched with the results obtained by Chaudhury and Deka [20].

Case (ii): By setting $r_1 = 1, m = 1, \alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$, $\overline{M}(\lambda) = 1$; Eq. (55) gives

$$\pi(z) = \frac{(1-\rho)(1-z)[\overline{B}_0(\lambda(1-X(z)))\overline{B}_1(\lambda(1-X(z)))\{q+p\overline{V}(\lambda(1-X(z)))\}]}{[\overline{B}_0(\lambda(1-X(z)))\overline{B}_1(\lambda(1-X(z)))\{q+p\overline{V}(\lambda(1-X(z)))\}-z]} \tag{71}$$

The above result tallies with the results as given by Choudhury and Madan [22].

Case (iii): By setting $\alpha_0 = \alpha_1 = \cdots = \alpha_m = 0$, $p = 0$, $r_2 = \cdots = r_m = 0$, $P(X=1) = 1, \overline{M}(\lambda) = 1$; Eq. (55) converts to

$$\pi(z) = \frac{(1-\rho)(1-z)\overline{B}_0(\lambda(1-z))\{r_0+\overline{B}_1(\lambda(1-z))\}}{[\overline{B}_0(\lambda(1-z))\{r_0+\overline{B}_1(\lambda(1-z))\}-z]} \tag{72}$$

It is same as obtained by Jararha and Madan [23] and Medhi [24].

Case (iv): In case of retrial time distribution as exponential distribution with parameter $v > 0, \overline{M}(\lambda) = \frac{v}{\lambda+v}$. Now Eqs. (53)–(55) yield

$$P(z) = \frac{[(1-\rho)(\lambda+v)-E(X)\lambda](1-z)\overline{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m} r_i\overline{B}_i(\phi_i(z))\}\{q+p\overline{V}(a(z))\}}{\overline{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m} r_i\overline{B}_i(\phi_i(z))\}\{q+p\overline{V}(a(z))\}[v+X(z)\lambda]-z(\lambda+v)} \tag{73}$$

$$O(z) = \frac{[(1-\rho)(\lambda+v)-E(X)\lambda](1-z)}{\overline{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m} r_i\overline{B}_i(\phi_i(z))\}\{q+p\overline{V}(a(z))\}[v+X(z)\lambda]-z(\lambda+v)} \tag{74}$$

$$\pi(z) = \frac{[(1-\rho)(\lambda+v)-E(X)\lambda](1-X(z))\overline{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m} r_i\overline{B}_i(\phi_i(z))\}\{q+p\overline{V}(a(z))\}}{E(X)[\overline{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m} r_i\overline{B}_i(\phi_i(z))\}\{q+p\overline{V}(a(z))\}[v+X(z)\lambda]-z(\lambda+v)]} \tag{75}$$

Again, by setting $\alpha_0 = \alpha_1 = \cdots = \alpha_m = 0$, $p = 0, r_2 = \cdots = r_m = 0, P(X=1) = 1$ in Eq. (73), the above results coincide with those obtained by Choi et al. [25].

8. Numerical illustration

The queueing model developed in the present investigation has the potential applications in many congestion situations including the delay encountered in the quality control process of the electronic equipments in the production system. To elaborate the specific utility and economic issues, we cite the retrial queueing problem encountered in the quality check up process as follows: After manufacturing, the ready stock of the products i.e. electronic equipments arrive in bulk at the quality control department where checking is done by a quality control engineer in phases. After completing the first round checking, the engineer may either satisfy with the quality of the unit or may require some more checking/testing which can be performed in the optional phases as per requirement to ensure the quality of the unit. Due to hectic job of the checking phases, after completing the quality control process of an electronic unit, the quality engineer has the choice of either starting the job on the other unit or going for vacation. The quality engineer may not be available due to illness which reveals that he is unreliable and is subject to failure. In case, if the engineer is not in the position to perform his job due to either being busy in doing testing of the other unit or being on vacation/breakdown, the new ready stock may either wait in the queue or joins the virtual orbit. From the orbit after some random period of time, the electronic unit can again make attempt to find the engineer free for the quality checking. In order to examine the performance of the electronic equipments production system in the context of quality check up, various performance characteristics along with system state probabilities can be computed based on analytical results established in the previous section. In order to minimize the total cost involved, the optimal parameters should be determined. The sensitivity of the total cost with respect to various design parameters such as optional vacation probability, retrial rate, and admission rate, is also a key concern for the quality control engineer in order to provide the quality product. In the present section, we determine the optimal service rate by constructing the cost function in terms of different activities and associated cost elements. The extensive numerical experiment has been done to explore the sensitivity aspect of the concerned model.

For the computation of performance measures, the computer program is developed by coding it in MATLAB software. To observe the affects of the different parameters on various system performance measures, we consider that the input batch size of the customers follows geometric distribution. The first and second moments of batch size are considered as $E(X) = \frac{b}{a}$, $E(X^2) = \frac{b(1+b)}{a^2}$; $b = 1 - a$. The essential and optional service times are considered to be exponential distributed. For the computation purpose, the first and second moments are obtained by using $E(B_i) = \frac{1}{\mu_i}$, $E(B_i^2) = \frac{2}{\mu_i^2}$; $i = 0, 1, 2$ where μ_i denotes the rate of ith phase service. The vacation time distribution is assumed to be exponential with parameters v. The first and second moments of vacation time distribution are obtained as $E(V) = \frac{1}{v}$, $E(V^2) = \frac{2}{v^2}$. Furthermore, we assume that the repair time follows the exponential distribution with parameter g_i so that the first two moments become as $g_i^{(1)} = \frac{1}{g_i}$, $g_i^{(2)} = \frac{2}{g_i^2}$; $i = 0, 1, 2$. It is further assumed that the retrial time is considered as exponential distributed with parameter θ, such that $\overline{M}(\lambda) = \left(\frac{\theta}{\theta+\lambda}\right)$.

The interpretation of the results based on numerical illustration carried out for the different performance measures is as follows:

8.1. Queue length of the system

The single server can provide the service only one customer at a time, therefore rest of the arrivals either not join the system or form the queue in the system to get the service. The queue length of the customers is affected by various parameters of the system namely service rate of the server, arrival rate of the customers, retrial rate, etc. To discuss the effects of different parameters on the queue length, we provide the numerical results for which the default parameters are set as follows:

Table 1: $E(X) = 1, \mu_1 = \mu_2 = 3\mu_0, \alpha_0 = 0.1, \alpha_1 = \alpha_2 = 2\alpha_0, r_0 = r_1 = r_2 = 1/3, v = 20, \mu_0 = 5.5, m = 2, g_0^{(1)} = E(B_0)/5, g_1^{(1)} = E(B_1)/5, g_2^{(1)} = E(B_2)/5$.
Table 2: $E(X) = 1, \mu_1 = \mu_2 = 3\mu_0, \alpha_1 = \alpha_2 = 2\alpha_0, r_0 = r_1 = r_2 = 1/3, v = 20, m = 2, \lambda = 2.25, g_0^{(1)} = E(B_0)/5, g_1^{(1)} = E(B_1)/5, g_2^{(1)} = E(B_2)/5, \alpha_0 = 0.1$.
Table 3: $E(X) = 1, \mu_1 = \mu_2 = 3\mu_0, \alpha_1 = \alpha_2 = 2\alpha_0, r_0 = r_1 = r_2 = 1/3, v = 20, \lambda = 2.2, \mu_0 = 5.5, m = 2, g_0^{(1)} = E(B_0)/5, g_1^{(1)} = E(B_1)/5, g_2^{(1)} = E(B_2)/5$.
Tables 1 and 2 exhibit the effects of arrival rate (λ) and the service rates (μ_0) respectively, for the fixed values of other

Table 1 Effect of arrival rate (λ) on L_q.

λ	$p = 0.3$			$p = 0.5$			$p = 0.7$		
	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$
2.10	18.34	17.32	16.42	22.57	21.10	19.84	28.92	26.65	24.75
2.15	22.15	20.74	19.53	28.34	26.16	24.34	38.61	34.84	31.80
2.20	27.41	25.39	23.67	37.10	33.62	30.80	55.93	48.65	43.16
2.25	35.15	32.02	29.46	51.97	45.65	40.81	95.55	76.82	64.47
2.30	47.60	42.26	38.09	82.64	68.30	58.40	277.30	165.52	118.75

Table 2 Effect of service rate (μ_0) on L_q.

μ_0	$p = 0.3$			$p = 0.5$			$p = 0.7$		
	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$
5.25	53.77	47.07	41.96	101.28	80.58	67.16	650.26	251.69	157.43
5.50	35.15	32.02	29.46	51.97	45.65	40.81	95.55	76.82	64.47
5.75	26.35	24.47	22.88	35.38	32.21	29.61	52.50	46.05	41.12
6.00	21.22	19.94	18.82	27.06	25.09	23.42	36.62	33.24	30.49
6.25	17.87	16.91	16.06	22.06	20.68	19.48	28.36	26.22	24.41

parameters on the queue length (L_q). It is noted that the queue length (L_q) increases (decreases) with the growth of arrival rate (service rate) for the fixed values of optional vacation probabilities or retrial rates of the customers. Table 3 presents the queue length (L_q) by varying the failure rate (α_0) for the fixed values of optional vacation probability (p) or retrial rate (θ) of the customers. It is observed that the queue length (L_q) increases with the increment in the failure rate (α_0) which is as per our expectation.

8.2. Reliability indices

For the queueing system with unreliable server, the reliability measures also provide the information which is required for the improvement of the system. To justify and validate the analytical results of the model, the availability measure (A_v) and failure frequency (F_f) are obtained. For computation of these measures, the default parameters are considered as follows:

Table 4: $E(X) = 1, \mu_1 = \mu_2 = 3\mu_0, \alpha_1 = \alpha_2 = 2\alpha_0, r_0 = r_1 = r_2 = 1/3, v = 20, p = 0.5, g_0^{(1)} = E(B_0)/5, g_1^{(1)} = E(B_1)/5, g_2^{(1)} = E(B_2)/5$.

Table 5: $E(X) = 1, \mu_1 = \mu_2 = 3\mu_0, \alpha_1 = \alpha_2 = 2\alpha_0, v = 20, p = 0.5, \lambda = 2.25, g_0^{(1)} = E(B_0)/5, g_1^{(1)} = E(B_1)/5, g_2^{(1)} = E(B_2)/5$.

Table 4 demonstrates that A_v decreases but F_f increases with the growth of failure rate (α_0) and the same pattern we observe with the increase in the number of optional services (m). From Table 5, it is evident that as more customers arrive in the system, availability (A_v) decreases but failure frequency (F_f) increases. On the contrary, for the fixed arrival rate, A_v increases with the growth of the retrial rates (θ) of the customers. The system can be made more efficient by increasing the system efficiency in terms of enhancement in the service rate, which can reduce the failure frequency of the server.

8.2.1. Cost analysis

The total cost of the system depends on the cost incurred on different activities. In order to explore the effect of different parameters on the cost function and to determine the optimal service rate, the cost function is constructed as

$$\begin{aligned} TC &= C_h L_q + C_0 \frac{E(T_b)}{E(T_c)} + C_S \frac{1}{E(T_c)} + C_a \frac{E(T_0)}{E(T_c)} \\ &= C_h L_q + C_0(1 - I_0) + C_S \lambda E(X) I_0 + C_a I_0 \end{aligned}$$

Table 3 Effect of failure rate (α_0) on L_q.

α_0	$p = 0.3$			$p = 0.5$			$p = 0.7$		
	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$
0.1	27.41	25.39	23.67	37.10	33.62	30.80	55.93	48.65	43.16
0.3	28.61	26.43	24.59	39.20	35.36	32.27	60.56	52.17	45.95
0.5	29.91	27.54	25.57	41.52	37.27	33.88	65.96	56.20	49.10
0.7	31.31	28.75	26.62	44.12	39.38	35.65	72.35	60.85	52.66
0.9	32.83	30.05	27.75	47.02	41.72	37.58	80.04	66.28	56.75

Table 4 Effects of failure rate (α_0) and number of optional service (m) on A_v and F_f.

α_0	$m = 0$				$m = 1$				$m = 2$			
	$\theta = 3.5$		$\theta = 3.7$		$\theta = 3.5$		$\theta = 3.7$		$\theta = 3.5$		$\theta = 3.7$	
	A_v	F_f	A_v	F_f	A_v	F_f	A_v	F_f	A_v	F_f	A_v	F_f
0.1	0.642	0.041	0.658	0.041	0.598	0.055	0.617	0.055	0.583	0.059	0.603	0.059
0.3	0.637	0.123	0.654	0.123	0.593	0.164	0.611	0.164	0.578	0.177	0.597	0.177
0.5	0.632	0.205	0.649	0.205	0.587	0.273	0.606	0.273	0.572	0.296	0.592	0.296
0.7	0.628	0.286	0.644	0.286	0.582	0.382	0.601	0.382	0.567	0.414	0.586	0.414
0.9	0.623	0.368	0.639	0.368	0.576	0.491	0.595	0.491	0.561	0.532	0.581	0.532

Table 5 Effects of failure rate (α_0) and arrival rate (λ) on A_v and F_f.

α_0	$\lambda = 2.2$				$\lambda = 2.25$				$\lambda = 2.3$			
	$\theta = 3.5$		$\theta = 3.7$		$\theta = 3.5$		$\theta = 3.7$		$\theta = 3.5$		$\theta = 3.7$	
	A_v	F_f	A_v	F_f	A_v	F_f	A_v	F_f	A_v	F_f	A_v	F_f
0.1	0.600	0.058	0.619	0.058	0.584	0.059	0.603	0.059	0.566	0.060	0.586	0.060
0.3	0.595	0.173	0.614	0.173	0.578	0.177	0.597	0.177	0.560	0.181	0.581	0.181
0.5	0.590	0.289	0.608	0.289	0.572	0.296	0.592	0.296	0.554	0.302	0.575	0.302
0.7	0.584	0.404	0.603	0.404	0.567	0.414	0.586	0.414	0.549	0.423	0.569	0.423
0.9	0.579	0.520	0.598	0.520	0.561	0.532	0.581	0.532	0.543	0.544	0.564	0.544

where C_h is the holding cost per unit time; C_0 is the cost per unit time to keep the server on and in operations; C_S is setup cost per busy cycle and C_a is the startup cost per unit time for the preparation work of the server before starting the service. The default values of different costs elements are taken as $C_h = \$5$, $C_0 = \$100$, $C_a = \$100$, $C_s = \$1000$. The other default parameters for the numerical results displayed in Tables 6–8 are set as follows:

Table 6: $E(X) = 1, \mu_1 = \mu_2 = 3\mu_0, \alpha_0 = 0.1, \alpha_1 = \alpha_2 = 2\alpha_0, r_0 = r_1 = r_2 = 1/3, v = 20, \mu_0 = 5.5, m = 2, g_0^{(1)} = E(B_0)/5, g_1^{(1)} = E(B_1)/5, g_2^{(1)} = E(B_2)/5$.
Table 7: $E(X) = 1, \mu_1 = \mu_2 = 3\mu_0, \alpha_1 = \alpha_2 = 2\alpha_0, r_0 = r_1 = r_2 = 1/3, v = 20, \lambda = 2.2, \mu_0 = 5.5, m = 2, g_0^{(1)} = E(B_0)/5, g_1^{(1)} = E(B_1)/5, g_2^{(1)} = E(B_2)/5$.
Table 8: $E(X) = 1, \mu_1 = \mu_2 = 3\mu_0, \alpha_1 = \alpha_2 = 2\alpha_0, r_0 = r_1 = r_2 = 1/3, v = 20, m = 2, \lambda = 2.25, g_0^{(1)} = E(B_0)/5, g_1^{(1)} = E(B_1)/5, g_2^{(1)} = E(B_2)/5, \alpha_0 = 0.1$.

Table 6 depicts the variations of the total cost with λ and noted that the cost decreases first and then starts increasing with the growth of λ for different retrial rate (θ) and p. The convex nature of cost function with respect to arrival rate provides insight for the admission control policy. Further, variation seems prevalent with the increasing value of p and arrival rate which is due to the fact that with the growth in these parameters, the number of customers and completion period of the service become larger as such cost increases rapidly. Further for fixed values of λ, total cost increases (decreases) with the growth in retrial rate $\theta(p)$ but after certain value of λ, this opposite trend is seen. Further it is also evident that for fixed values of p, minimum total cost increases with the growth of θ.

Table 7 shows the effect of service rate (μ_0) on total cost (TC) with the variation in p and θ. From table, we observe that the total cost initially decreases with the growth in service rate (μ_0) which shows the convexity of the cost function with respect to μ_0. It is also noticed that the cost function decreases with the increase in retrial rates (θ) for different values of p. Such situations can be observed in many real life congestion situations and may be helpful in the modeling of the real time system. Further for fixed values of μ_0, total cost decreases (increases) with the increase in retrial rate $\theta(p)$ but after certain values of μ_0, this trend reveals a reverse pattern. Further from the table, it is also observed that for fixed values of p, minimum total cost decreases with the increase values of θ. Table 8 shows the change in the cost for the varying the values of failure rates (α_0) of the server for different values of p and θ.

The following default parameters are considered in order to find the optimal values of total cost as shown in Figs. 1–4.

Fig. 1: $E(X) = 1, \mu_1 = \mu_2 = 3\mu_0, \alpha_0 = 0.1, \alpha_1 = \alpha_2 = 2\alpha_0, r_0 = r_1 = r_2 = 1/3, v = 20, \mu_0 = 5.5, m = 2, p = 0.5, g_0^{(1)} = E(B_0)/5, g_1^{(1)} = E(B_1)/5, g_2^{(1)} = E(B_2)/5$.
Fig. 2: $E(X) = 1, \mu_1 = \mu_2 = 3\mu_0, \alpha_0 = 0.1, \alpha_1 = \alpha_2 = 2\alpha_0, r_0 = r_1 = r_2 = 1/3, v = 20, \lambda = 2.15, m = 2, p = 0.5, g_0^{(1)} = E(B_0)/5, g_1^{(1)} = E(B_1)/5, g_2^{(1)} = E(B_2)/5$.
Fig. 3: $E(X) = 1, \mu_1 = \mu_2 = 3\mu_0, \alpha_0 = 0.1, \alpha_1 = \alpha_2 = 2\alpha_0, r_0 = r_1 = r_2 = 1/3, v = 20, \mu_0 = 5.5, m = 2, \theta = 3.5, g_0^{(1)} = E(B_0)/5, g_1^{(1)} = E(B_1)/5, g_2^{(1)} = E(B_2)/5$.
Fig. 4: $E(X) = 1, \mu_1 = \mu_2 = 3\mu_0, \alpha_0 = 0.1, \alpha_1 = \alpha_2 = 2\alpha_0, r_0 = r_1 = r_2 = 1/3, v = 20, \lambda = 2.15, m = 2, \theta = 3.5, g_0^{(1)} = E(B_0)/5, g_1^{(1)} = E(B_1)/5, g_2^{(1)} = E(B_2)/5$.

Table 6 Effect of arrival rate (λ) on TC. The bold values indicate optimal values of total cost (TC).

λ	$p = 0.5$			$p = 0.7$		
	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$
2.10	563.07	573.98	584.93	524.27	531.87	540.32
2.15	541.23	549.93	559.33	**518.00**	**519.47**	**523.53**
2.20	**530.84**	**534.39**	540.10	546.19	531.58	524.76
2.25	547.40	538.24	**535.21**	682.14	611.81	572.13
2.30	639.33	591.58	564.71	1524.97	990.99	780.66

Table 7 Effect of service rate (μ_0) on TC. The bold values indicate optimal values of total cost (TC).

μ_0	$p = 0.5$			$p = 0.7$		
	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$
5.25	705.31	625.18	580.23	3367.06	1398.47	950.15
5.50	547.40	**538.24**	**535.21**	682.14	611.81	572.13
5.75	**545.36**	551.03	558.42	547.78	**537.99**	**534.54**
6.00	577.88	588.75	600.01	**542.53**	547.25	553.95
6.25	621.04	634.14	647.08	569.41	579.57	590.29

Table 8 Effect of failure rate (α_0) on TC. The bold values indicate optimal values of total cost (TC).

α_0	$p = 0.5$			$p = 0.7$		
	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$
0.1	530.84	534.39	540.10	546.19	531.58	524.76
0.3	529.36	531.26	535.77	557.34	537.33	526.99
0.5	**529.02**	528.98	532.10	572.39	545.62	530.99
0.7	530.02	527.69	529.19	592.39	557.03	537.11
0.9	532.59	527.54	527.15	618.85	572.36	545.82

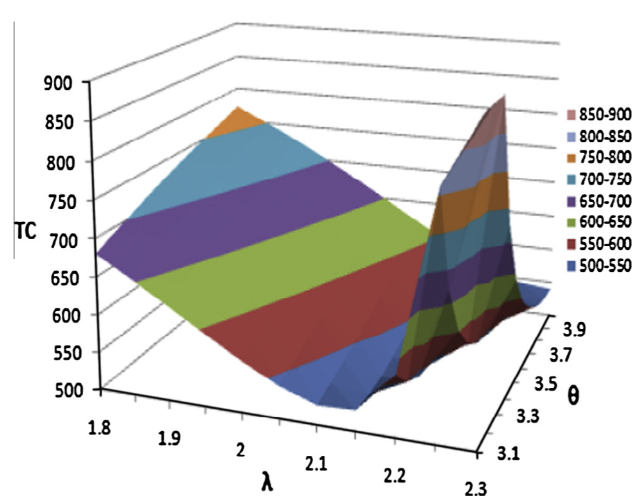

Figure 1 Effects of λ and θ on the total cost.

From Figs. 1 and 2, the optimal values for $(\lambda*, \theta*)$ and $(\mu_0*, \theta*)$ are obtained as $(2.1, 3.1)$ and $(5.5, 3.5)$, respectively with their optimal total costs \$525.00 and \$530.00 for fixed values of other parameters. Figs. 3 and 4 depict the optimal values for $(\lambda*, p*) = (2.1, 0.9)$ and $(\mu_0*, p*) = (6.0, 0.9)$; the corresponding optimal total cost $(TC*)$ is \$506.00 and \$528.00, respectively.

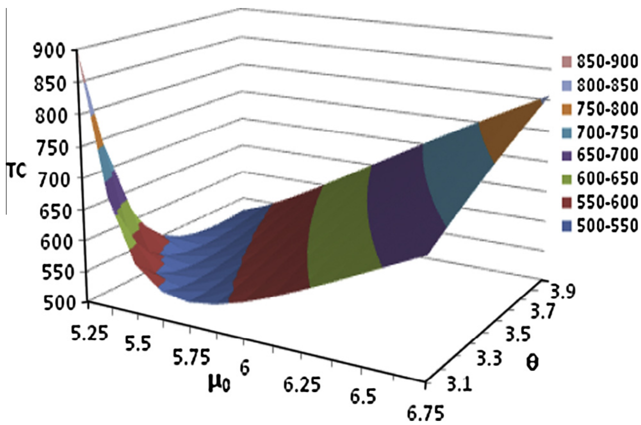

Figure 2 Effects of μ_0 and θ on the total cost.

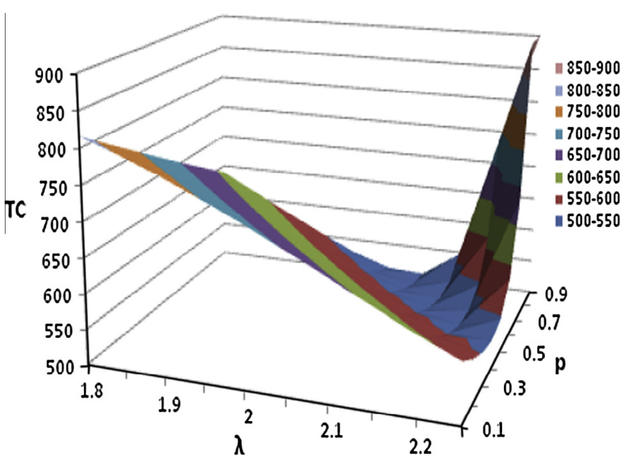

Figure 3 Effects of λ and p on the total cost.

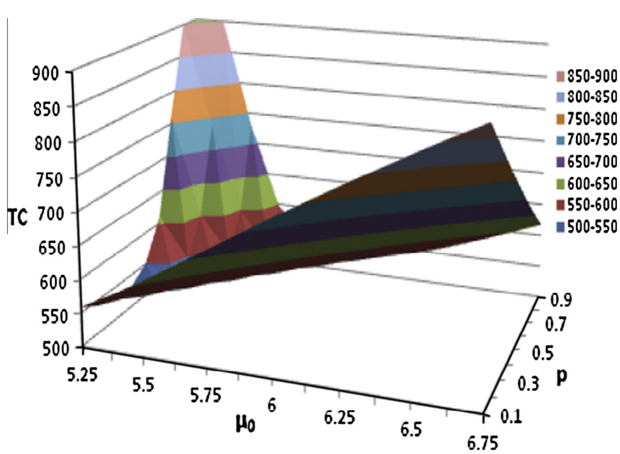

Figure 4 Effects of μ_0 and p on the total cost.

9. Conclusion

The retrial queueing system with unreliable server studied under Bernoulli vacation policy has the provision for the waiting of the customers in a virtual pool i.e. retrial orbit in order to try again in case if the server is busy. The incorporation of the provision of m-optional services apart from essential service makes the system closer to real life situations. Queueing characterization of such type of service facility may be helpful in reducing the congestion encountered at shopping malls, hospitals, and many other places and may also attract the customers for getting the service at one place as per requirement. We have provided explicit expressions for various performance measures which are validated by taking numerical illustration. The investigation done will provide insight to the concerned system designers and decision makers to develop more economic and better grade of service (GoS) based on the quantitative assessment of various performance characteristics of interest. This study can be further extended for the optimal N-policy or admission control based queueing system. The more realistic features viz. balking behavior of the customers and/or delayed repair of the server can also be incorporated for which work is in progress.

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Appendix A.

Proof of Lemma 1. It is evident that $\{X_n, n \in Z^+\}$ is an irreducible and periodic Markov chain. Following Foster's criterion on the mean drift (cf. Choudhury and Ke [13]), we can establish that an irreducible aperiodic Markov chain is positive recurrent if the conditions given by (40) are satisfied.

Appendix B.

Proof of Theorem 2. On integrating Eqs. (41)–(43) w.r.t. 'x' and using the result

$$\int_0^\infty e^{-sx}(1 - M(x))dx = \frac{1 - \overline{M}(s)}{s} \tag{B.1}$$

we get Eqs. (47)–(49).

Similarly on integrating Eq. (44) w.r.t. 'y' and using (B.1), we obtain Eq. (50). In the similar manner, from Eqs. (45) and (46) and using equation (B.1), we get the results given in Eqs. (51) and (52).

Appendix C.

Proof of Theorem 3. The probability generating function of the queue size distribution at arbitrary epoch is obtained by using Eqs. (47)–(52) in the relation

$$P(z) = I_0 + A(z) + \sum_{i=0}^{m} P^{(i)}(z) + V(z) + \sum_{i=0}^{m} R^{(i)}(z), \tag{C.1}$$

Further, the result given in Eq. (54) is obtained by using Eq. (53) in the relation

$$P(z) = O(z)\overline{B}_0(\phi_0(z))\left\{r_0 + \sum_{i=1}^{m} r_i \overline{B}_i(\phi_i(z))\right\}\{q + p\overline{V}(a(z))\}, \tag{C.2}$$

we obtain the result given in Eq. (54).

Appendix D.

Proof of Theorem 4. Let $\{\pi_j; j = 0, 1, 2, \ldots\}$ be the probability that there are j customers in the queue at a departure epoch. Then the queue size distribution at the departure epoch is determined using

$$\pi_j = k_0\left[q\left\{r_0\int_0^\infty \mu_0(x)P_{j+1}^{(0)}(x)dx + \int_0^\infty \mu_1(x)P_{j+1}^{(1)}(x)dx + \cdots + \int_0^\infty \mu_m(x)P_{j+1}^{(m)}(x)dx\right\} + \int_0^\infty v(y)V_j(y)dy\right] \tag{D.1}$$

where k_0 is the normalizing constant.

On multiplying Eq. (D.1) by z^j, using $\pi(z) = \sum_{j=0}^{\infty}\pi_j z^j$, and after some algebraic manipulation, we get

$$\pi(z) = \frac{k_0[1 - \rho - E(X)(1 - \overline{M}(\lambda))]a(z)\left[\overline{B}_0(\phi_0(z))\{r_0 + \sum_{i=1}^{m} r_i\overline{B}_i(\phi_i(z))\}\{q + p\overline{V}(a(z))\}\right]}{S(z)} \tag{D.2}$$

Using the condition $\pi(1) = 1$, we get

$$k_0 = \frac{1}{\lambda E(X)} \tag{D.3}$$

Eqs. (D.2) and (D.3) provide the required result.

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