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A ranking model in uncertain, imprecise and multi-experts contexts: The application of evidence theory ☆

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ABSTRACT

We consider ranking problems where the actions are evaluated on a set of ordinal criteria. The evaluation of each alternative with respect to each criterion may be imperfect and is provided by one or several experts. We model each imperfect evaluation as a basic belief assignment (BBA). In order to rank the BBAs characterizing the performances of the actions according to each criterion, a new concept called RBBD and based on the comparison of these BBAs to ideal or nadir BBAs is proposed. This is performed using belief distances that measure the dissimilarity of each BBA to the ideal or nadir BBAs. A model inspired by Xu et al.'s method is also proposed and illustrated by a pedagogical example.

1. Introduction

Evidence theory [18], also called Dempster–Shafer theory or belief functions theory, is a convenient framework for modeling imperfect data and for combining information. This formalism has been widely used in many fields such as the classification [10], the data mining [25], the multicriteria decision analysis [2,4,5,23,29], etc. Within the latter field, authors generally distinguish three main problems: the choice, the sorting and the ranking [17].

The ranking problem refers to the ordering of a set of actions evaluated on several criteria in a partial or a total preorder. Within this context, evidence theory has been used for modeling three ranking procedures: Utkin's approach [23,24], the DS/AHP method [1–3] and the evidential reasoning algorithm [29–31]. The first approach allows to deal with comparisons of groups of actions and then to deduce a ranking based on the computation of the belief and plausibility functions of each alternative or of the possible rankings. The second method is an extension of the AHP (Analytic Hierarchy Process) approach which considers verbal judgements given by the decision maker on groups of actions compared to the actions set on each criterion. The third method is a procedure that considers imperfect evaluations of the actions on a set of ordinal criteria modeled by belief structures. However, the main drawback of this method is that the belief structures expressing the performances of the actions are defined using the same set of assessment grades on all the criteria. This can be not possible in some situations. Indeed, the decision maker can prefer to consider a set of assessment grades for each criterion rather than the same set for all the criteria.

In what follows, we will propose another ranking model based on evidence theory in a context different to those considered in Utkin's approach, the DS/AHP method and the evidential reasoning algorithm. We consider ranking problems where

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the actions are evaluated by ordinal criteria and where the evaluations are estimated by several experts. We assume that the assessment grades set used to evaluate the actions is not the same for all the criteria as in the evidential reasoning approach, i.e., each criterion has its own set of evaluation grades. Moreover, we assume that the experts agree on these sets and that the evaluations of the actions are imperfect. One of the main sources of this imperfection is the subjectiveness of the human behavior. For instance, in a problem of personnel recruiting, an expert may hesitate between two or more successive assessments grades when evaluating a candidate. He may be sure that this candidate has "good" or "very good" communication skills without being able to refine his judgment.

Information provided by several experts about the evaluation of an action on a given criterion has to be combined in order to reach a collective evaluation of this action that synthesizes the experts' opinions. Evidence theory offers several tools that allow combining the information issued from several experts such as Dempster's rule [7,18] and the normalized cautious rule [8]. These rules can be solutions for the aggregation of the experts' opinions in our problem. However, we will show that these combination operators do not usually respect the unanimity property which is a natural condition of an aggregation operator.

In this work, we will propose a model inspired by Xu et al.'s [26] that addresses ranking problems in the context described above. The imperfect evaluations will be modeled by evidential functions called BBAs (**B**asic **B**elief **A**ssignment). Moreover, we will propose a new concept called RBBD (**R**anking **B**BAs based on **B**elief **D**istances) that permits to obtain partial and total rankings of the evaluations on each criterion. We will show that this approach is coherent with the first belief dominance [4,6] which is a procedure allowing pairwise comparisons between the BBAs. We will also show that using the first belief dominance to rank alternatives can lead to poor results since an important number of incomparable BBAs can be induced. We will prove that this number is usually superior or equal to the one obtained by the RBBD concept which constitutes understandably an interesting advantage for the RBBD with regard to the first belief dominance. Finally, we will suggest in our model inspired by Xu et al.'s another manner for aggregating the different experts' opinions in order to avoid the drawback of Dempster's and the normalized cautious rules mentioned previously.

This paper is organized as follows: in Section 2 we introduce the key concepts of evidence theory, then the notion of RBBD is presented in Section 3 and a model inspired by Xu et al.'s method is proposed in Section 4. The model is illustrated by a pedagogical example in Section 5.

2. Evidence theory: some concepts

Evidence theory has been initially introduced Arthur Dempster in 1967 [7] and has been later developed by Glenn Shafer in 1976 as a general framework for modeling uncertainty [18]. This theory has been proposed as a generalization of the subjective probability theory and as a model that allows representing the total ignorance case. It has been also the starting point of many models such as the transferable belief model [20]. In this section, we will recall the basic concepts of this theory that are necessary to understand the rest of the paper.

2.1. Knowledge model

Basically, the imperfection in data within evidence theory is modeled by a BBA. This function is defined on a finite set of mutually exclusive and exhaustive hypotheses called the frame of discernment. Let Θ be this set and 2^Θ be the set of all subsets of Θ. A BBA [20] is a function m defined from 2^Θ to $[0,1]$ such as $\sum_{A\subseteq\Theta} m(A) = 1$.

The quantity $m(A)$, called the belief mass of subset A, represents the partial belief that A is true. When $m(A) \neq 0$, A is called focal set. Moreover, a BBA m is said to be:

- Normal if $\emptyset$ is not a focal set, i.e., $m(\emptyset) = 0$. The initial works [7,18] on evidence theory requires that $m(\emptyset) = 0$, but this condition is not imposed in the transferable belief model [20]. In this paper, we will only consider normal BBAs;
- Bayesian if all its focal sets are singletons;
- Dogmatic if Θ is not a focal set, i.e., $m(\Theta) = 0$;
- Vacuous if Θ is the only focal set, i.e., if $m(\Theta) = 1$ and $m(A) = 0$ for all $A \neq \Theta$. This type of BBA is used to represent the total ignorance case;
- Simple if $m(\Theta) = w$ and $m(A) = 1 - w$ for some $A \neq \Theta$ and $0 \leqslant w \leqslant 1$. When $w > 1$, m is not a BBA since it is no longer a function from 2^Θ to $[0,1]$. Such a function can be referred as an inverse simple BBA. Both simple and inverse simple BBAs are known as generalized simple BBAs and denoted as denoted A^w. w is interpreted as the weight of evidence [21] (these notions will be used later in the definition of the normalized cautious rule of combination).

A BBA can equivalently be represented by its associated belief (or credibility) and plausibility functions [18] defined respectively by the following formulas:

$$Bel(A) = \sum_{B\subseteq A} m(B) \tag{1}$$

$$Pl(A) = \sum_{A\cap B\neq\emptyset} m(B) \tag{2}$$

The quantity $Bel(A)$ measures the total belief that support completely A whereas the plausibility $Pl(A)$ quantifies the total belief that can potentially be placed in A, i.e., the belief that support completely and partially A. Of course, $Bel(A) \leqslant Pl(A)$ for all $A \subseteq \Theta$. Moreover, these two functions are connected by the equation $Pl(A) = 1 - Bel(\overline{A})$ where $\overline{A}$ denotes the complement of A.

Finally, there is a third function that can be deduced from the BBA called the commonality function. Formally, this function is defined as follows:

$$Q(A) = \sum_{A \subseteq B} m(B) \tag{3}$$

where $Q(A)$ is measure of the belief mass that can move freely to any element of A. This function has no real intuitive sense, but it is used in evidence theory to optimize computations and to prove and express theorems and properties. In particular, the notion of weight of evidence introduced above can be expressed using the commonality function using the following equation:

$$w(A) = \prod_{A \subseteq B} Q(B)^{(-1)^{|B|-|A|+1}} \tag{4}$$

2.2. Combination

The combination is an operation that constitutes a crucial component of evidence theory. Given several BBAs induced by several sources, it is usually required to aggregate them in order to yield a global BBA synthesizing the knowledge of the different sources. Within this context, several combinations rules have been proposed in evidence theory to aggregate independent and dependant sources. Among others, we can mention Dempster's rule [7,18], Dubois and Prade's rule [11], Yager's rule [27], the cautious rule (normalized and unnormalized) [8], etc. In this section, we will only describe Dempster's and the normalized cautious rules.

Dempster's rule, also known as the normalized conjunctive rule [19], has been the first combination operator proposed in evidence theory. This rule allows the combination of BBAs provided by independent sources, i.e., distinct BBAs. Let m_1 and m_2 be two distinct BBAs to combine, Dempster's rule is defined for all $A \subseteq \Theta$ as follows:

$$m(A) = (1-k)^{-1} \cdot \sum_{B \cap C = A} m_1(B) \cdot m_2(C) \tag{5}$$

where $k = \sum_{B \cap C = \emptyset} m_1(B) \cdot m_2(C)$, i.e., the belief mass that the combination assigns to the empty set. The coefficient k reflects the conflict between the sources whereas the quotient $(1-k)^{-1}$ is a normalization term guarantying that no belief is associated to empty set and that the total belief is equal to one.

Dempster's rule has several interesting mathematical properties. It can be proved to be both commutative and associative. Therefore, the combination result of several BBAs is independent of the order in which they are considered. This rule has been used in several applications for instance in the expert systems [1]. However, the main drawback of this operator is that it cannot be applied to combine BBAs given by dependant sources, i.e., nondistinct BBAs. To perform the combination in such situations, Denoeux has proposed the normalized cautious rule [8]. This operator, which combines nondogmatic BBAs, is based on the notion of weight of evidence described previously. In practice, the combination of two nondogmatic BBAs m_1 and m_2 using the normalized cautious rule is determined as follows:

- Compute the commonality functions Q_1 and Q_2 using Eq. (3).
- Compute the weights of evidence w_1 and w_2 that are obtained from the commonalities using Eq. (4).
- Determine the generalized simple BBAs $A^{\min(w_1(A), w_2(A))}$ for all $A \subset \Theta$ such that $\min(w_1(A), w_2(A)) \neq 1$.
- Combine the induced generalized simple BBAs using Dempster's rule.

The normalized cautious rule is also commutative and associative and it has been also used to combine expert opinions. The interested reader can refer to [12] that gives an application of this rule to climate sensitivity assessment.

3. RBBD concept

In this section, we will propose a new concept called RBBD in the context of multicriteria decision problems which allows ranking evaluations expressed by BBAs. The underlying idea of this approach has been inspired from a multicriteria ranking method called TOPSIS [13] which is based on the comparison of the actions to two referential solutions called ideal and nadir actions using Euclidean distances. In our approach, the BBAs are compared to ideal or nadir BBAs using distances called belief distances. Therefore, after presenting formally the problem, we will introduce the notion of belief distances. Then, we will define the ideal and nadir BBAs and we will describe how a partial or a total preorder of the BBAs can be obtained. Finally, we will discuss the properties of our technique.

3.1. Problem formulation

Before describing formally the problem, let us note that a part of the theoretical material presented in this paper has been already used in [4] including the notation described below. Other concepts will be defined after when introducing the belief distances and our model inspired by Xu et al.'s method.

We consider a ranking problem which can be represented by three elements: the actions, the ordinal criteria and the assessment grades sets related to the criteria. At first, we will consider situations where only one expert evaluates the alternatives. In what follows, let:

- $A = \{a_1, a_2, \ldots, a_n\}$ be the set of actions;
- $G = \{g_1, g_2, \ldots, g_q\}$ be the set of ordinal criteria;
- $X^h = \{x_1^h, x_2^h, \ldots, x_{r_h}^h\}$ be the assessment grades set of criterion g_h.

The n alternatives are evaluated, on each criterion g_h, using the r_h assessment grades x_j^h (with $j = 1, 2, \ldots, r_h$) which are required to be mutually exclusive and exhaustive. These grades constitute the frame of discernment in evidence theory and are defined such as $x_1^h \prec x_2^h \prec \cdots \prec x_{r_h}^h$, i.e., x_1^h is less preferred than x_2^h and so on.

The evaluation of each action a_i with respect to each criterion g_h is given by a normal BBA m_i^h defined on the set X^h. Since $x_1^h \prec x_2^h \prec \cdots \prec x_{r_h}^h$, the focal sets of any BBA defined on X^h should be either singletons or disjunctions of successive elements of X^h. In what follows, we will denote $S(X^h)$ the set of singletons and all the subsets constituted of successive elements of X^h. Therefore, m_i^h is defined formally as a function from $S(X^h)$ to $[0,1]$ such as $m_i^h(\emptyset) = 0$ and $\sum_{C \subseteq X^h} m_i^h(C) = 1$. Of course, when the expert is unable to express the assessment of an action a_i on a given criterion g_h, this is modeled using a vacuous BBA. In such case, the total belief mass is assigned to the frame X^h.

Finally, it is interesting to define the sets A_k^h and B_l^h before describing the RBBD concept. For all $h \in \{1, 2, \ldots, q\}$ and for all $k \in \{0, 1, \ldots, r_h\}$, let:

$$A_k^h = \begin{cases} \emptyset & \text{if } k = 0 \\ \{x_1^h, \ldots, x_k^h\} & \text{otherwise} \end{cases} \tag{6}$$

and let $\overrightarrow{S}(X^h)$ denote the set $\{A_1^h, A_2^h, \ldots, A_{r_h}^h\}$. Similarly, for all $h \in \{1, 2, \ldots, q\}$ and for all $l \in \{0, 1, \ldots, r_h\}$ such as $l = r_h - k$, let:

$$B_l^h = \begin{cases} \emptyset & \text{if } l = 0 \\ \{x_{r_h - l + 1}^h, \ldots, x_{r_h}^h\} & \text{otherwise} \end{cases} \tag{7}$$

and let $\overleftarrow{S}(X^h)$ denote the set $\{B_1^h, B_2^h, \ldots, B_{r_h}^h\} \cdot k$ and l represent respectively the number of elements of the sets A_k^h and B_l^h. Obviously, $|\overrightarrow{S}(X^h)| = |\overleftarrow{S}(X^h)| = r_h$, $\overline{A_k^h} = B_{r_h - k}^h = B_l^h$ for all $k \in \{0, 1, \ldots, r_h\}$ and $\overline{B_l^h} = A_{r_h - l}^h = A_k^h$ for all $l \in \{0, 1, \ldots, r_h\}$.

3.2. Belief distances

The term "distance" is not a new concept in evidence theory. Indeed, several distances that quantify the dissimilarity between BBAs have been defined. For instance, one can cite Tessem's distance [22], Jousselme et al.'s distance [15], Ristic and Smets' distance [16], etc. However, these distances do not seem to be well suited to measure the divergence between BBAs defined on a frame with ordered elements. For instance, let us consider three BBAs m_1^h, m_2^h and m_3^h that represent respectively the evaluations of three actions a_1, a_2 and a_3 on a given criterion g_h. Let $X^h = \{x_1^h, x_2^h, x_3^h\}$ be the set of assessment grades of g_h defined such as $x_1^h \prec x_2^h \prec x_3^h$ and let us suppose that $m_1^h(\{x_1^h\}) = 1$, $m_2^h(\{x_2^h\}) = 1$ and $m_3^h(\{x_3^h\}) = 1$. Using one of the distances cited previously leads to the following result: the distance between m_1^h and m_2^h $(d(m_1^h, m_2^h))$ is equal to the distance between m_1^h and m_3^h $(d(m_1^h, m_3^h))$. This result is understandably incoherent because $d(m_1^h, m_2^h)$ should be inferior to $d(m_1^h, m_3^h)$ since x_1^h is closer to x_2^h than to x_3^h in terms of preference. To avoid this type of counter-intuitive results, we will propose distance measures called "belief distances" that take into account the preference information on the elements of the frame. Before presenting these distances, let us define the notions of ascending and descending belief functions [4,6] which have been already used to define the first belief dominance approach (see formula 18).

Definition 1. The ascending belief function, denoted $\overrightarrow{Bel}_i^h$ and induced by m_i^h, is a function $\overrightarrow{Bel}_i^h : \overrightarrow{S}(X^h) \to [0,1]$ defined such as $\overrightarrow{Bel}_i^h(A_k^h) = \sum_{C \subseteq A_k^h} m_i^h(C)$ for all $A_k^h \in \overrightarrow{S}(X^h)$.

Definition 2. The descending belief function, denoted $\overleftarrow{Bel}_i^h$ and induced by m_i^h, is a function $\overleftarrow{Bel}_i^h : \overleftarrow{S}(X^h) \to [0,1]$ defined such as $\overleftarrow{Bel}_i^h(B_l^h) = \sum_{C \subseteq B_l^h} m_i^h(C)$ for all $B_l^h \in \overleftarrow{S}(X^h)$.

These two functions allow taking into account implicitly the fact that $x_1^h \prec x_2^h \prec \cdots \prec x_{r_h}^h$. Indeed, the former represents the beliefs of the nested sets $A_1^h, A_2^h, \ldots, A_{r_h}^h$, i.e., the sets $\{x_1^h\}, \{x_1^h, x_2^h\}, \ldots, \{x_1^h, \ldots, x_{r_h}^h\}$ whereas the latter represents the beliefs of the nested sets $B_1^h, B_2^h, \ldots, B_{r_h}^h$, i.e., the sets $\{x_{r_h}^h\}, \{x_{r_h-1}^h, x_{r_h}^h\}, \ldots, \{x_1^h, \ldots, x_{r_h}^h\}$. Of course, since x_1^h and $x_{r_h}^h$ are respectively the worst and the best assessment grades of X^h, the more the values of $\overrightarrow{Bel}_i^h$ for all $A_k^h \in \overrightarrow{S}(X^h)$ decrease and those of $\overleftarrow{Bel}_i^h$ for all $B_l^h \in \overleftarrow{S}(X^h)$ increase, the better is the BBA.

Basically, the intuition behind the proposed belief distances is that two BBAs will be as close as their ascending belief functions and their descending belief functions are alike. Therefore, in order to quantify the dissimilarity between two BBAs, we should define a distance measure between their ascending belief functions and another one between their descending belief functions. We will call these distances respectively ascending and descending belief distances.

Definition 3. Let $\overrightarrow{Bel}_i^h$ and $\overrightarrow{Bel}_{i'}^h$ be two ascending belief functions related respectively to two BBAs m_i^h and $m_{i'}^h$, the ascending belief distance is defined as follows:

$$d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{i'}^h\right) = \sum_{k=1}^{r_h} \left|\overrightarrow{Bel}_i^h(A_k^h) - \overrightarrow{Bel}_{i'}^h(A_k^h)\right| \tag{8}$$

Definition 4. Let $\overleftarrow{Bel}_i^h$ and $\overleftarrow{Bel}_{i'}^h$ be two descending belief functions related respectively to two BBAs m_i^h and $m_{i'}^h$, the descending belief distance is defined as follows:

$$d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{i'}^h\right) = \sum_{l=1}^{r_h} \left|\overleftarrow{Bel}_i^h(B_l^h) - \overleftarrow{Bel}_{i'}^h(B_l^h)\right| \tag{9}$$

Of course, the ascending belief distance is a metric, i.e., it satisfies the following axioms:

- The non-negativity: $d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{i'}^h\right) \geqslant 0$.
- The symmetry: $d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{i'}^h\right) = d\left(\overrightarrow{Bel}_{i'}^h, \overrightarrow{Bel}_i^h\right)$.
- The non-degeneracy: $d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{i'}^h\right) = 0 \Longleftrightarrow \overrightarrow{Bel}_i^h = \overrightarrow{Bel}_{i'}^h$.
- The triangle inequality: $d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{i'}^h\right) \leqslant d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{i''}^h\right) + d\left(\overrightarrow{Bel}_{i''}^h, \overrightarrow{Bel}_{i'}^h\right)$.

Similarly, the descending belief distance verifies the above axioms and therefore it is a metric. The proofs are trivial and obvious. Thus, we will not give them in this paper.

3.3. Ideal and nadir BBAs

As mentioned above, the fundamental idea of the RBBD concept is that the BBAs representing the evaluations of the actions on each criterion are compared to ideal or nadir BBAs. In what follows, we will define these two particular BBAs.

The ideal BBA is the best BBA among all the BBAs that can be defined on the set $S(X^h)$ whereas the nadir BBA is the worst one among these BBAs. They represent respectively the best and the worst evaluations that can be reached on a criterion g_h. Since x_1^h and $x_{r_h}^h$ are respectively the worst and the best assessment grades related to g_h, therefore the ideal and nadir BBAs can be defined respectively as follows:

Definition 5. The ideal BBA on a criterion g_h is a function m_{ideal}^h from $S(X^h)$ to $[0,1]$ such as $m_{ideal}^h(\{x_{r_h}^h\}) = 1$.

Definition 6. The nadir BBA on a criterion g_h is a function m_{nadir}^h from $S(X^h)$ to $[0,1]$ such as $m_{nadir}^h(\{x_1^h\}) = 1$.

The comparison of the BBAs to the ideal and nadir BBAs is based on the notion of belief distances defined above. The dissimilarity of a given BBA m_i^h to the ideal BBA m_{ideal}^h is measured through the two ascending and descending belief distances $d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{ideal}^h\right)$ and $d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{ideal}^h\right)$. Since m_{ideal}^h is a particular type of BBA, these belief distances can respectively be expressed as follows:

$$d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{ideal}^h\right) = \left(\sum_{k=1}^{r_h} \overrightarrow{Bel}_i^h(A_k^h)\right) - 1 \tag{10}$$

$$d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{ideal}^h\right) = r_h - \sum_{l=1}^{r_h} \overleftarrow{Bel}_i^h(B_l^h) \tag{11}$$

In the same way, the comparison of m_i^h to the nadir BBA m_{nadir}^h is performed using the two ascending and descending belief distances $d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{nadir}^h\right)$ and $d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{nadir}^h\right)$. Since m_{nadir}^h is a particular case of BBA, these belief distances can respectively be written as:

$$d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{nadir}^h\right) = r_h - \sum_{k=1}^{r_h} \overrightarrow{Bel}_i^h(A_k^h) \tag{12}$$

$$d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{nadir}^h\right) = \left(\sum_{l=1}^{r_h} \overleftarrow{Bel}_i^h(B_l^h)\right) - 1 \tag{13}$$

The proofs of formulas 10, 11, 12 and 13 are given in Appendix A (see proofs 1, 2, 3 and 4, resp.).

3.4. Partial and total preorders of the BBAs

Based on the belief distances between the BBAs and the ideal BBA on each criterion g_h, it is possible to obtain two rankings of the BBAs representing the evaluations of the actions on g_h. The first one is associated to the ascending belief distances $d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{ideal}^h\right)$ whereas the second one is related to the descending belief distances $d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{ideal}^h\right)$. Of course, the lower the values of these belief distances, the better m_i^h. Similarly, the comparison of the BBAs to the nadir BBA allows obtaining two rankings related respectively to the ascending and descending belief distances $d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{nadir}^h\right)$ and $d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{nadir}^h\right)$: the higher their values, the better m_i^h.

From formulas 10 to 13, it is easy to see that the ascending belief distances to the ideal and nadir BBAs are linearly linked (idem for the descending belief distances to the ideal and nadir BBAs). Since the belief distances to the ideal BBA are to minimize and those to the nadir BBA are to be maximize, one can deduce that the rankings related to $d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{ideal}^h\right)$ and $d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{nadir}^h\right)$ are the same (idem for the rankings related to $d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{ideal}^h\right)$ and $d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{nadir}^h\right)$). Therefore, the comparison to the ideal and nadir BBAs can be reduced to a comparison to one of them.

When comparing the BBAs to the ideal BBA, it is possible to deduce a partial preorder of these BBAs called the RBBD I ranking. This preorder is obtained as the intersection of the two rankings related to $d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{ideal}^h\right)$ and $d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{ideal}^h\right)$. Three preference situations can be distinguished in this context between the BBAs on g_h: the preference (P_h) (P_h^{-1} for the inverse), the indifference (I_h) and the incomparability (J_h). The latter appears between two BBAs when it is impossible to express indifference or preference between them. Formally, these relations can be expressed as follows:

$$\begin{cases} m_i^h P_h m_{i'}^h \iff \begin{cases} d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{ideal}^h\right) \leqslant d\left(\overrightarrow{Bel}_{i'}^h, \overrightarrow{Bel}_{ideal}^h\right) \text{ and } d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{ideal}^h\right) < d\left(\overleftarrow{Bel}_{i'}^h, \overleftarrow{Bel}_{ideal}^h\right) \\ d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{ideal}^h\right) < d\left(\overrightarrow{Bel}_{i'}^h, \overrightarrow{Bel}_{ideal}^h\right) \text{ and } d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{ideal}^h\right) \leqslant d\left(\overleftarrow{Bel}_{i'}^h, \overleftarrow{Bel}_{ideal}^h\right) \end{cases} \\ m_i^h I_h m_{i'}^h \iff d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{ideal}^h\right) = d\left(\overrightarrow{Bel}_{i'}^h, \overrightarrow{Bel}_{ideal}^h\right) \text{ and } d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{ideal}^h\right) = d\left(\overleftarrow{Bel}_{i'}^h, \overleftarrow{Bel}_{ideal}^h\right) \\ m_i^h J_h m_{i'}^h \iff \begin{cases} d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{ideal}^h\right) < d\left(\overrightarrow{Bel}_{i'}^h, \overrightarrow{Bel}_{ideal}^h\right) \text{ and } d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{ideal}^h\right) > d\left(\overleftarrow{Bel}_{i'}^h, \overleftarrow{Bel}_{ideal}^h\right) \\ d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{ideal}^h\right) > d\left(\overrightarrow{Bel}_{i'}^h, \overrightarrow{Bel}_{ideal}^h\right) \text{ and } d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{ideal}^h\right) < d\left(\overleftarrow{Bel}_{i'}^h, \overleftarrow{Bel}_{ideal}^h\right) \end{cases} \end{cases} \tag{14}$$

Using the belief distances to the nadir BBA, the RBBD I ranking can be obtained as the intersection of the two rankings related to $d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{nadir}^h\right)$ and $d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{nadir}^h\right)$. In this case, the relations P_h, I_h and J_h can be deduced as follows:

$$\begin{cases} m_i^h P_h m_{i'}^h \Longleftrightarrow \begin{cases} d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{nadir}^h\right) \geqslant d\left(\overrightarrow{Bel}_{i'}^h, \overrightarrow{Bel}_{nadir}^h\right) \text{ and } d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{nadir}^h\right) > d\left(\overleftarrow{Bel}_{i'}^h, \overleftarrow{Bel}_{nadir}^h\right) \\ d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{nadir}^h\right) > d\left(\overrightarrow{Bel}_{i'}^h, \overrightarrow{Bel}_{nadir}^h\right) \text{ and } d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{nadir}^h\right) \geqslant d\left(\overleftarrow{Bel}_{i'}^h, \overleftarrow{Bel}_{nadir}^h\right) \end{cases} \\ m_i^h I_h m_{i'}^h \Longleftrightarrow d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{nadir}^h\right) = d\left(\overrightarrow{Bel}_{i'}^h, \overrightarrow{Bel}_{nadir}^h\right) \text{ and } d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{nadir}^h\right) = d\left(\overleftarrow{Bel}_{i'}^h, \overleftarrow{Bel}_{nadir}^h\right) \\ m_i^h J_h m_{i'}^h \Longleftrightarrow \begin{cases} d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{nadir}^h\right) > d\left(\overrightarrow{Bel}_{i'}^h, \overrightarrow{Bel}_{nadir}^h\right) \text{ and } d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{nadir}^h\right) < d\left(\overleftarrow{Bel}_{i'}^h, \overleftarrow{Bel}_{nadir}^h\right) \\ d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{nadir}^h\right) < d\left(\overrightarrow{Bel}_{i'}^h, \overrightarrow{Bel}_{nadir}^h\right) \text{ and } d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{nadir}^h\right) > d\left(\overleftarrow{Bel}_{i'}^h, \overleftarrow{Bel}_{nadir}^h\right) \end{cases} \end{cases} \tag{15}$$

In addition to the partial preorder, it is possible to obtain a total preorder of the BBAs on each criterion g_h, i.e., a complete ranking without incomparabilities. This preorder, called the RBBD II ranking, can be deduced based on the belief distances to the ideal BBA or to the nadir BBA. When the BBAs are compared to the ideal BBA, the total preorder is established based on a global score α_i^h computed for each BBA m_i^h as follows:

$$\alpha_i^h = d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{ideal}^h\right) + d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{ideal}^h\right) \tag{16}$$

Of course, the lower α_i^h, the better m_i^h. Using the belief distances to the nadir BBA, the RBBD II ranking is deduced using a global score β_i^h defined for each BBA m_i^h as follows:

$$\beta_i^h = d\left(\overrightarrow{Bel}_i^h, \overrightarrow{Bel}_{nadir}^h\right) + d\left(\overleftarrow{Bel}_i^h, \overleftarrow{Bel}_{nadir}^h\right) \tag{17}$$

Obviously, the higher β_i^h, the better m_i^h. Moreover, it is easy to deduce from formulas 10 to 13 that $\alpha_i^h = -\beta_i^h - 2 + 2.r_h$. Therefore, α_i^h and β_i^h are linearly linked.

3.5. Properties

The RBBD concept verifies two important properties: the dominance and the stability of the preferences. The first property is a natural condition that our procedure should satisfy. Within evidence theory, the dominance or the non-dominance of a BBA to another is determined using the first belief dominance concept (FBD) (let us note that similar approaches to this concept called credal orderings have been proposed by Thierry Denoeux [9]). This approach, which is a generalization of the first stochastic dominance, is defined as follows:

$$m_i^h \mathrm{FBD} m_{i'}^h \Longleftrightarrow \begin{cases} \overrightarrow{Bel}_i^h(A_k^h) \leqslant \overrightarrow{Bel}_{i'}^h(A_k^h) \text{ for all } A_k^h \in \overrightarrow{S}(X^h) \\ \overleftarrow{Bel}_i^h(B_l^h) \geqslant \overleftarrow{Bel}_{i'}^h(B_l^h) \text{ for all } B_l^h \in \overrightarrow{S}(X^h) \end{cases} \tag{18}$$

The first belief dominance allows performing pairwise comparisons between the BBAs. Two situations can be identified when using this concept:

- FBD when m_i^h dominates $m_{i'}^h$, i.e., if $m_i^h \mathrm{FBD} m_{i'}^h$.
- $\overline{\mathrm{FBD}}$ when m_i^h does not dominate $m_{i'}^h$, i.e., $m_i^h \overline{\mathrm{FBD}} m_{i'}^h$.

Of course, when $m_i^h \overline{\mathrm{FBD}} m_{i'}^h$ and $m_{i'}^h \overline{\mathrm{FBD}} m_i^h$, m_i^h and $m_{i'}^h$ are incomparable according to this concept. Moreover, let us note that this approach has been used recently in a multicriteria choice model inspired by ELECTRE I to compare evaluations expressed by BBAs [4].

Proposition 1. *If $m_i^h \mathrm{FBD} m_{i'}^h$, then $m_i^h P_h m_{i'}^h$ or $m_i^h I_h m_{i'}^h$.*

This proposition means that if a BBA m_i^h dominates a BBA $m_{i'}^h$ according to the first belief dominance, the rank of m_i^h may not be worse than the rank of $m_{i'}^h$ using the RBBD concept. In other words, m_i^h should be preferred or indifferent to $m_{i'}^h$. The proof is detailed in Appendix A (proof 5).

The second characteristic of the RBBD concept is the stability of the preferences. This property means that the ranks of two BBAs m_i^h and $m_{i'}^h$ may not be reversed when a third BBA is removed or added to the initial set of BBAs. Our approach does not suffer, therefore, from the rank reversal phenomenon. This property is natural since all the BBAs are compared to a referential BBA (ideal or nadir), i.e., the rank of a BBA does not depend of the remaining BBAs that need to be ranked.

Before ending this section, it is worth mentioning that it is possible to use the first belief dominance concept to obtain a ranking of the BBAs. For that purpose, we apply at first this approach in order to compare all the pairs of the BBAs. These comparisons are then represented by a graph that synthesises all the relations (FBD or $\overline{\text{FBD}}$) holding between the BBAs. Finally, based on this graph, the ranking of the BBAs is deduced. However, the main drawback of using the first belief dominance to rank the BBAs is the number of incomparabilities between the BBAs that can be important is some situations (this problem is well known in the multicriteria analysis). This number is always superior or equal to the one obtained by the RBBD concept (RBBD I ranking). This result is a direct consequence of Proposition 1. The proof is given in Appendix A (proof 6). The following example illustrates a case where the first belief dominance and the RBBD concepts are used to rank a set of BBAs.

Example 1. Let us consider a multicriteria problem where five actions are evaluated on a set of ordinal criteria and where the evaluations are expressed by BBAs. In this example, we will only consider the evaluations on criterion g_1 given in Table 1. Let $x_1^1,\ x_2^1$ and x_3^1 be the assessment grades of g_1 defined such as $x_1^1 \prec x_2^1 \prec x_3^1$.

The first belief dominance is applied to rank these BBAs. Therefore, we compute at first the ascending and descending belief functions of each BBA and we determine the observed belief dominances according to this concept for each pair of alternatives. The results are illustrated in Table 2. Then, we build the graph synthesizing all the relations between the BBAs representing the evaluations and we deduce the ranking of these BBAs. Fig. 1 gives this preorder. As can be noticed, the eval-

Table 1
BBAs characterizing the actions performances on criterion g_1.

	a_1	a_2	a_3	a_4	a_5
BBAs	$m_1^1(\{x_1^1\}) = 0.4$ $m_1^1(\{x_2^1\}) = 0.3$ $m_1^1(\{x_3^1\}) = 0.2$ $m_1^1(\{x_1^1, x_2^1, x_3^1\}) = 0.1$	$m_2^1(\{x_1^1\}) = 0.2$ $m_2^1(\{x_1^1, x_2^1, x_3^1\}) = 0.8$	$m_3^1(\{x_1^1\}) = 0.2$ $m_3^1(\{x_2^1\}) = 0.2$ $m_3^1(\{x_3^1\}) = 0.5$ $m_3^1(\{x_1^1, x_2^1, x_3^1\}) = 0.1$	$m_4^1(\{x_1^1\}) = 0.1$ $m_4^1(\{x_2^1\}) = 0.2$ $m_4^1(\{x_3^1\}) = 0.3$ $m_4^1(\{x_1^1, x_2^1, x_3^1\}) = 0.4$	$m_5^1(\{x_1^1\}) = 0.2$ $m_5^1(\{x_1^1, x_2^1\}) = 0.2$ $m_5^1(\{x_1^1, x_2^1, x_3^1\}) = 0.6$

Table 2
Observed belief dominances between the evaluations.

	a_1	a_2	a_3	a_4	a_5
a_1	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$
a_2	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	FBD
a_3	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$	FBD
a_4	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	–	FBD
a_5	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	–

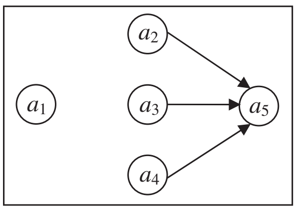

Fig. 1. Ranking of the evaluations obtained by the first belief dominance.

Table 3
Ascending and descending belief distances and global scores with regard to the ideal BBA.

	a_1	a_2	a_3	a_4	a_5
$d\left(\overrightarrow{Bel_i^1}, \overrightarrow{Bel_{ideal}^1}\right)$	1.1	0.4	0.6	0.4	0.6
$d\left(\overleftarrow{Bel_i^1}, \overleftarrow{Bel_{ideal}^1}\right)$	1.3	2	0.8	1.2	2
α_i^1	2.4	2.4	1.4	1.6	2.6

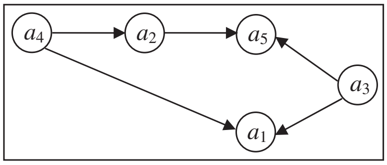

Fig. 2. RBBD I ranking of the evaluations.

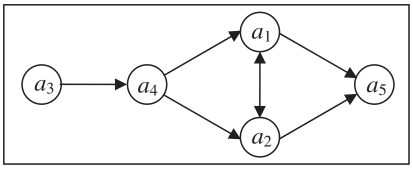

Fig. 3. RBBD II ranking of the evaluations.

Table 4
Ascending and descending belief distances and global scores with regard to the nadir BBA.

	a_1	a_2	a_3	a_4	a_5
$d\left(\overrightarrow{Bel_i^1}, \overrightarrow{Bel_{nadir}^1}\right)$	0.9	1.6	1.4	1.6	1.4
$d\left(\overleftarrow{Bel_i^1}, \overleftarrow{Bel_{nadir}^1}\right)$	0.7	0	1.2	0.8	0
β_i^1	1.6	1.6	2.6	2.4	1.4

uations of a_2, a_3 and a_4 dominate the one of a_5 on criterion g_1 and the evaluation of a_1 is incomparable to those of a_2, a_3 and a_4. Thus, the best evaluations on g_1 are those of a_1, a_2, a_3 and a_4. Moreover, we can easily identify 7 incomparable pairs of evaluations which are those of a_1 and a_2, a_1 and a_3, a_1 and a_4, a_1 and a_5, a_2 and a_3, a_2 and a_4 and finally a_3 and a_4.

The RBBD concept has been also used to rank the BBAs given in Table 1. For that purpose, we determine at first the ideal BBA. Since x_3^1 is the best assessment grade related to criterion g_1, the ideal BBA is therefore: $m_{ideal}^1(\{x_3^1\}) = 1$. Then, we compute the ascending and descending belief distances of each BBA with regard to the ideal BBA. Table 3 gives the values of these distances. Finally, we determine the two rankings associated to the ascending and descending belief distances. The intersection of these two rankings is the RBBD I ranking. Fig. 2 illustrates this partial preorder. As can be remarked, the evaluation of a_4 is preferred to those of a_1, a_2 and a_5 on criterion g_1 and the evaluation of a_3 is incomparable to the one of a_4. Thus, the best evaluations on g_1 are those of a_3 and a_4. Furthermore, the number of incomparable pairs of evaluations is 4. These pairs are the following: a_1 and a_2, a_1 and a_5, a_2 and a_3 and finally a_3 and a_4. It is clear therefore that the number of incomparable evaluations obtained by the RBBD concept is inferior to the one obtained by the first belief dominance approach.

Let us suppose now that a total preorder of the evaluations should be determined. Thus, we can use the RBBD II ranking. For that purpose, we compute the global scores of the BBAs which are given in Table 3 (last line). Then, we determine the total preorder as shown in Fig. 3. As can be seen, the evaluations of actions a_1 and a_2 are indifferent. The best and the worst evaluations are respectively those of a_3 and a_5.

Finally, let us note that it is possible to determine the RBBD I and II rankings based on the comparisons of the BBAs to the nadir BBA. Of course, since x_1^1 is the worst assessment grade on g_1, the nadir BBA is nothing else than $m_{nadir}^1(\{x_1^1\}) = 1$. Table 4 gives the ascending and descending belief distances of the BBAs with regard the nadir BBA as well as the global scores.

4. The model

In this section, we will propose a model inspired by a multicriteria procedure called Xu et al.'s method [26] to rank alternatives in uncertain, imprecise and multi-experts contexts. At first, we will describe briefly Xu et al.'s procedure. Then, we will present the steps of our model.

4.1. Xu et al.'s method

Xu et al.'s method is a ranking approach which uses partial or total preorders of the evaluations of the actions on each criterion. These single-criterion preorders reflect situations of preference (P_h) $(P_h^{-1}$ for the inverse), indifference (I_h) and incomparability (J_h) between the actions on each criterion g_h. In what follows, let $R_h(a_i, a_{i'})$ be the preference relation holding between two actions a_i and $a_{i'}$ on g_h.

Basically, the underlying idea of this method is that an action a_i performs better if more relations $a_i P_h a_{i'}$ and fewer relations $a_i P_h^{-1} a_{i'}$ hold for $i' \neq i$. In order to determine the performance of a_i, Xu et al. have incorporated in their approach the notion of distance between the preference relations. Briefly, this concept allows quantifying the degree of divergence between each pair of preference relations. The numerical values of this distance have been determined on the basis of several conditions suggested in [14]. For instance, P_h and P_h^{-1} are considered as the most discordant relations and J_h is viewed as equidistant from I_h, P_h and P_h^{-1}. Table 5 gives the distance values between each pair of preference relations. Let us note that Xu et al.'s method considers only the distances with regard to P_h and P_h^{-1}, i.e., the distances between P_h and each $R_h(a_i, a_{i'})$ denoted $\Delta(P_h, R_h(a_i, a_{i'}))$ and the distances between P_h^{-1} and each $R_h(a_i, a_{i'})$ denoted $\Delta(P_h^{-1}, R_h(a_i, a_{i'}))$. Of course, the less $\Delta(P_h, R_h(a_i, a_{i'}))$ and the more $\Delta(P_h^{-1}, R_h(a_i, a_{i'}))$, the better a_i.

The distances $\Delta(P_h, R_h(a_i, a_{i'}))$ and $\Delta(P_h^{-1}, R_h(a_i, a_{i'}))$ allow respectively building two total preorders O_1 and O_2 which are then combined to determine the global ranking of the actions denoted O. The total preorder O_1 is built iteratively as follows. In the first step, a distance that describes the dominating character of a_i on each criterion g_h is computed. This distance, denoted $d^{P_h}(a_i)$, is determined as the sum of all $\Delta(P_h, R_h(a_i, a_{i'}))$ for all $i' \neq i$:

$$d^{P_h}(a_i) = \sum_{i' \neq i} \Delta(P_h, R_h(a_i, a_{i'})) \tag{19}$$

Then, the distance that represents the dominating character of a_i on all the criteria is deduced as follows:

$$d^{O_1}(a_i) = \sum_{h=1}^{q} w_h . d^{P_h}(a_i) \tag{20}$$

where w_h denotes the weight of criterion g_h. Of course, the less this distance, the better a_i. Based on the values of $d^{O_1}(a_i)$, we determine the best action(s), i.e. the action(s) that has (have) the minimum of these values. In the second step, the above distances are recomputed but only for the actions that are not yet ranked. These distances are determined without taking into account the action(s) ranked in the first step. Then, the best alternative(s) is (are) determined as previously. This procedure continues as describe above until all the actions are ranked. Let us note that two actions are indifferent in O_1if their related distances are equal and represent the minimal values of the distances computed on a given iteration.

In the same way, the total preorder O_2 is built iteratively, but instead of considering the dominating character of a_i, we consider its dominated intensity. For that purpose, we determine the distance $d^{P_h^{-1}}(a_i)$ that describes this character on each criterion g_h. Formally:

$$d^{P_h^{-1}}(a_i) = \sum_{i' \neq i} \Delta(P_h^{-1}, R_h(a_i, a_{i'})) \tag{21}$$

Then, we compute the distance that represents the dominated character of a_i on all the criteria using the following formula:

$$d^{O_2}(a_i) = \sum_{h=1}^{q} w_h . d^{P_h^{-1}}(a_i) \tag{22}$$

Of course, the best alternative(s) is (are) the one (those) that has (have) the maximum of the values of $d^{O_2}(a_i)$. Moreover, two actions are indifferent in O_2 if their related distances are equal and represent the maximal values of the distances on a given iteration.

Finally, the global preorder O is determined as the intersection of O_1 and O_2. In this context, three preference situations can be deduced between the actions: the preference (P) (P^{-1} for the inverse), the indifference (I) and the incomparability (J). The intersection is carried using the following principles:

- $a_i P a_{i'}$ in O if $a_i P a_{i'}$ in both O_1 and O_2, or if $a_i P a_{i'}$ in one preorder and $a_i I a_{i'}$in the other;
- $a_i I a_{i'}$ in O if $a_i I a_{i'}$ in both O_1 and O_2;
- $a_i J a_{i'}$ in O if $a_i P a_{i'}$ in one preorder and $a_i P^{-1} a_{i'}$ in the other.

Table 5
Numerical values of the distances between the preference relations.

	I_h	P_h	J_h	P_h^{-1}
I_h	$\Delta(I_h, I_h) = 0$	$\Delta(I_h, P_h) = 1$	$\Delta(I_h, J_h) = 4/3$	$\Delta(I_h, P_h^{-1}) = 1$
P_h	$\Delta(P_h, I_h) = 1$	$\Delta(P_h, P_h) = 0$	$\Delta(P_h, J_h) = 4/3$	$\Delta(P_h, P_h^{-1}) = 5/3$
J_h	$\Delta(J_h, I_h) = 4/3$	$\Delta(J_h, P_h) = 4/3$	$\Delta(J_h, J_h) = 0$	$\Delta(J_h, P_h^{-1}) = 4/3$
P_h^{-1}	$\Delta(P_h^{-1}, I_h) = 1$	$\Delta(P_h^{-1}, P_h) = 5/3$	$\Delta(P_h^{-1}, J_h) = 4/3$	$\Delta(P_h^{-1}, P_h^{-1}) = 0$

4.2. Steps of the model

In what follows, we will describe our model inspired by Xu et al.'s method. At first, we assume that the experts are non-equivalent in their importance within the group. We further assume that several experts express their assessments for a set of actions with respect to a set of criteria. This information is provided on the form of BBAs. Additionally, we assume that each expert gives his own values to the criteria weights. In what follows, let:

- w_f be the importance of expert E_f within the group (with $f = 1,2,\ldots,s$).
- $w_{h|f}$ be the weight of criterion g_h given by expert E_f.
- $m^h_{i|f}$ be the BBA that represents the evaluation of action a_i according to criterion g_h and given by expert E_f.

The steps of the proposed model are the following. At first, the RBBD concept is applied by each expert to rank his individual BBAs on each criterion. Based on the obtained single-criterion preorders of the evaluations, each expert determines his individual global ranking of the actions as in Xu et al.'s method. The same procedure used in this method to obtain the individual global rankings is then applied to aggregate them in order to obtain the collective global ranking of the actions.

4.2.1. Ranking the individual BBAs

In the first step of the model, the individual BBAs given by each expert and characterizing the actions performances on each criterion are ranked using the RBBD concept. Two single-criterion preorders of the evaluations can be deduced: the RBBD I and II rankings. As mentioned before, the former gives a partial preorder of the evaluations on each criterion according to each expert and allows taking into account situations of incomparability. The latter gives a total preorder of the evaluations on each criterion according to each expert, i.e., a ranking from the best to the worst or vice versa.

4.2.2. Determining the individual global rankings

Once the single-criterion preorders of the evaluations are determined by each expert, we aggregate them in order to determine the individual global ranking of the actions. The aggregation is performed as in Xu et al.'s method. For that purpose, each expert determines at first the two individual total preorders O_1^f and O_2^f. Let $R_h^f(a_i, a_{i'}) \in \{P_h, P_h^{-1}, I_h, J_h\}$ be the preference relation between a_i and $a_{i'}$ observed on the single-criterion preorder of g_h given by expert E_f. The preorder O_1^f is built iteratively using the distance that quantifies the individual dominating character of a_i on all the criteria defined as follows:

$$d^{O_1^f}(a_i) = \sum_{h=1}^{q} w_{h|f}.d_f^{P_h}(a_i) \tag{23}$$

where $d_f^{P_h}(a_i) = \sum_{i' \neq i} \Delta(P_h, R_h^f(a_i, a_{i'}))$ is a distance that describes the individual dominating character of a_i on criterion g_h. In the same way, the preorder O_2^f is established iteratively on the basis of the distance representing the individual dominated character of a_i on all the criteria and given by the following formula:

$$d^{O_2^f}(a_i) = \sum_{h=1}^{q} w_{h|f}.d_f^{P_h^{-1}}(a_i) \tag{24}$$

where $d_f^{P_h^{-1}}(a_i) = \sum_{i' \neq i} \Delta(P_h^{-1}, R_h^f(a_i, a_{i'}))$ is a distance that describes the individual dominated character of a_i on g_h. Then, the two total preorders O_1^f and O_2^f are combined in order to obtain the individual global ranking of the actions O^f.

Finally, it is worth mentioning that, at this step, each expert has the possibility to choose between the two single-criterion preorders of the evaluations (i.e., the RBBD I or II rankings) before applying the aggregation procedure of Xu et al.'s method. The choice between them can be also imposed by the decision maker. Of course, using the RBBD II rankings can lead to a loss of useful information about incomparabilities. That is why it is preferable to use the RBBD I rankings, i.e., the single-criterion partial preorders of the evaluations.

4.2.3. Determining the collective global ranking

In this step, the individual global rankings are aggregated in order to obtain the collective global ranking of the actions. For that purpose, we propose to use the same procedure used in Xu et al.'s method in the aggregation of the single-criterion preorders. We should, therefore, determine at first the two collective total preorders O_1 and O_2. In what follows, let $R^f(a_i, a_{i'}) \in \{P, P^{-1}, I, J\}$ be the preference relation between a_i and $a_{i'}$ observed on the individual global ranking O^f.

The collective total preorder O_1 is built iteratively using a distance that quantifies the collective dominating character of a_i according to all experts. More formally:

$$d^{O_1}(a_i) = \sum_{f=1}^{s} w_f.d_f^P(a_i) \tag{25}$$

where $d_f^P(a_i) = \sum_{i' \neq i} \Delta(P, R^f(a_i, a_{i'}))$ is a distance that quantifies the individual dominating character of a_i according to expert E_f. The distance $d^{O_1}(a_i)$ is computed at each iteration for the actions not yet ranked and without considering the actions that are ranked in the previous steps. Similarly, the collective total preorder O_2 is built iteratively, but instead of considering the collective dominating character of a_i according to all experts, we consider its collective dominated intensity measured by the following distance:

$$d^{O_2}(a_i) = \sum_{f=1}^{s} w_f . d_f^{P^{-1}}(a_i) \tag{26}$$

where $d_f^{P^{-1}}(a_i) = \sum_{i' \neq i} \Delta(P^{-1}, R^f(a_i, a_{i'}))$ is a distance that quantifies the individual dominated character of a_i according to expert E_f. Finally, the collective global preorder O is determined as the intersection of O_1 and O_2 using the principles given at the end of Section 4.1.

Before ending this section, let us note that another manner for the aggregation of the different opinions of the experts is the use of a combination rule offered by evidence theory to aggregate the individual BBAs given by the experts. Let m_i^h be the BBA resulting from the combination of $m_{i|1}^h, m_{i|2}^h, \ldots$ and $m_{i|s}^h$. Obviously, m_i^h represents the collective evaluation of action a_i according to criterion g_h.

The combination rule used for the aggregation of the individual BBAs should be commutative and associative. Therefore:

- If the BBAs are distinct, we suggest Dempster's rule of combination to take into account independencies between experts;
- If the BBAs are nondistinct, we suggest the normalized cautious rule of combination to take into account dependencies between experts.

Once the collective BBAs are determined, the RBBD concept is used to rank these BBAs. Two collective single-criterion preorders of the evaluations can be deduced: the RBBD I and II rankings. These preorders (RBBD I or II) are then aggregated as in Xu et al.'s method in order to obtain the collective global ranking of the actions. However, both combination rules that we have suggested do not respect in some situations the unanimity property which is a natural condition of an aggregation operator. Formally, this property means that for all $h \in \{1,2,\ldots,q\}$:

- If $m_{i|f}^h P_h m_{i'|f}^h$ for all $f \in \{1,2,\ldots,s\}$, then $m_i^h P_h m_{i'}^h$.
- If $m_{i|f}^h P_h^{-1} m_{i'|f}^h$ for all $f \in \{1,2,\ldots,s\}$, then $m_i^h P_h^{-1} m_{i'}^h$.
- If $m_{i|f}^h I_h m_{i'|f}^h$ for all $f \in \{1,2,\ldots,s\}$, then $m_i^h I_h m_{i'}^h$.
- If $m_{i|f}^h J_h m_{i'|f}^h$ for all $f \in \{1,2,\ldots,s\}$, then $m_i^h J_h m_{i'}^h$.

That is why we have adopted the procedure used in Xu et al.'s method for the aggregation.

Two counter-examples to the unanimity property are introduced below (see Examples 2 and 3). In the first one, we have used Dempster's rule to combine distinct BBAs. In the second one, we have used the normalized cautious rule to aggregate nondistinct BBAs.

Example 2. Let us consider a multicriteria problem where two actions are evaluated by two experts and where the evaluations are expressed by BBAs. In this example, we will consider only the evaluations on criterion g_1. We will assume that the assessment grades related to g_1 are $x_1^1,\ x_2^1,\ x_3^1$ and x_4^1 defined such as $x_1^1 \prec x_2^1 \prec x_3^1 \prec x_4^1$. Furthermore, we will suppose that each expert gives the evaluations without interacting with the other. Therefore, the experts are independent and their induced BBAs are distinct.

Dempster's rule of combination is used in this example to aggregate the BBAs induced by the experts for each action on criterion g_1. The objective is to yield a combined BBA that represents a collective evaluation of each action on criterion g_1. Then, the RBBD concept is applied to rank the BBAs given by each expert and to rank the combined BBAs. Table 6 gives these BBAs and their RBBD I and II rankings. These preorders have been obtained on the basis of the ascending and descending

Table 6
Counter-example to the unanimity property: case of distinct BBAs.

	a_1	a_2	RBBD I ranking	RBBD II ranking
The BBAs given by expert 1	$m_{1\|1}^1(\{x_2^1, x_3^1\}) = 0.5$ $m_{1\|1}^1(\{x_4^1\}) = 0.5$	$m_{2\|1}^1(\{x_2^1\}) = 0.33$ $m_{2\|1}^1(\{x_3^1\}) = 0.67$	$m_{1\|1}^1 P_1 m_{2\|1}^1$	$m_{1\|1}^1 P_1 m_{2\|1}^1$
The BBAs given by expert 2	$m_{1\|2}^1(\{x_2^1, x_3^1\}) = 1$	$m_{2\|2}^1(\{x_1^1, x_2^1\}) = 0.5$ $m_{2\|2}^1(\{x_2^1, x_3^1\}) = 0.5$	$m_{1\|2}^1 P_1 m_{2\|2}^1$	$m_{1\|2}^1 P_1 m_{2\|2}^1$
The combined BBAs given by the two experts	$m_1^1(\{x_2^1, x_3^1\}) = 1$	$m_2^1(\{x_2^1\}) = 0.5$ $m_2^1(\{x_3^1\}) = 0.5$	$m_1^1 J_1 m_2^1$	$m_1^1 I_1 m_2^1$

belief distances with regard to the ideal BBA. Table 7 illustrates the values of these distances used to deduce the RBBD I ranking and the global scores used to obtain the RBBD II ranking. For instance, the ranking of the combined BBAs is determined as follows:

- $d\left(\overrightarrow{Bel_1^1}, \overrightarrow{Bel_{ideal}^1}\right) < d\left(\overrightarrow{Bel_2^1}, \overrightarrow{Bel_{ideal}^1}\right)$ and $d\left(\overleftarrow{Bel_1^1}, \overleftarrow{Bel_{ideal}^1}\right) > d\left(\overleftarrow{Bel_2^1}, \overleftarrow{Bel_{ideal}^1}\right)$, thus m_1^1 and m_2^1 are incomparable according to the RBBD I ranking;
- $\alpha_1^1 = \alpha_2^1$, thus m_1^1 and m_2^1 are indifferent according to the RBBD II ranking.

As can be noticed, the experts agree that the evaluation of a_1 is preferred to the one of a_2 on criterion g_1 according to the RBBD I and II rankings. However, when we apply the RBBD concept to rank the combined BBAs, we obtain: the evaluations of a_1 and a_2 are incomparable on g_1 according to the RBBD I ranking whereas they are indifferent on g_1 according to the RBBD II ranking. Therefore, Dempster's rule does not respect the unanimity property.

Example 3. Let us consider a multicriteria problem where two actions are evaluated by two experts and where the evaluations are expressed by BBAs. In this example, we will consider only the evaluations on criterion g_1. We will assume that the assessment grades related to g_1 are x_1^1 and x_2^1 defined such as $x_1^1 \prec x_2^1$. Moreover, we will assume that the experts interact between them when they give the evaluations. Thus, the experts are dependent and their induced BBAs are nondistinct.

The normalized cautious rule of combination is used in this example to aggregate the BBAs induced by the experts for each action on criterion g_1. The objective is to yield a combined BBA that represents a collective evaluation of each action on criterion g_1. Then, the RBBD concept is applied to rank the BBAs given by each expert and to rank the combined BBAs. Table 8 gives these BBAs and their RBBD I and II rankings. Table 9 illustrates the details about the way we have obtained

Table 7
Ascending and descending belief distances and global scores (case of distinct BBAs).

		a_1	a_2
The BBAs given by expert 1	$d\left(\overrightarrow{Bel_{i\|1}^1}, \overrightarrow{Bel_{ideal\|1}^1}\right)$	0.5	1.33
	$d\left(\overleftarrow{Bel_{i\|1}^1}, \overleftarrow{Bel_{ideal\|1}^1}\right)$	1	1.33
	$\alpha_{i\|1}^1$	1.5	2.66
The BBAs given by expert 2	$d\left(\overrightarrow{Bel_{i\|2}^1}, \overrightarrow{Bel_{ideal\|2}^1}\right)$	1	1.5
	$d\left(\overleftarrow{Bel_{i\|2}^1}, \overleftarrow{Bel_{ideal\|2}^1}\right)$	2	2.5
	$\alpha_{i\|2}^1$	3	4
The combined BBAs given by the two experts	$d\left(\overrightarrow{Bel_i^1}, \overrightarrow{Bel_{ideal}^1}\right)$	1	1.5
	$d\left(\overleftarrow{Bel_i^1}, \overleftarrow{Bel_{ideal}^1}\right)$	2	1.5
	α_i^1	3	3

Table 8
Counter-example to the unanimity property: case of nondistinct BBAs.

	a_1	a_2	RBBD I ranking	RBBD II ranking
The BBAs given by expert 1	$m_{1\|1}^1(\{x_1^1\}) = 0.4$ $m_{1\|1}^1(\{x_1^1, x_2^1\}) = 0.6$	$m_{2\|1}^1(\{x_1^1\}) = 0.6$ $m_{2\|1}^1(\{x_2^1\}) = 0.2$ $m_{2\|1}^1(\{x_1^1, x_2^1\}) = 0.2$	$m_{1\|1}^1 J_1 m_{2\|1}^1$	$m_{1\|1}^1 I_1 m_{2\|1}^1$
The BBAs given by expert 2	$m_{1\|2}^1(\{x_1^1\}) = 0.5$ $m_{1\|2}^1(\{x_2^1\}) = 0.2$ $m_{1\|2}^1(\{x_1^1, x_2^1\}) = 0.3$	$m_{2\|2}^1(\{x_1^1\}) = 0.3$ $m_{2\|2}^1(\{x_1^1, x_2^1\}) = 0.7$	$m_{1\|2}^1 J_1 m_{2\|2}^1$	$m_{1\|1}^1 I_1 m_{2\|1}^1$
The combined BBAs given by the two experts	$m_1^1(\{x_1^1\}) = 0.5$ $m_1^1(\{x_2^1\}) = 0.2$ $m_1^1(\{x_1^1, x_2^1\}) = 0.3$	$m_2^1(\{x_1^1\}) = 0.6$ $m_2^1(\{x_2^1\}) = 0.2$ $m_2^1(\{x_1^1, x_2^1\}) = 0.2$	$m_1^1 P_1 m_2^1$	$m_1^1 P_1 m_2^1$

Table 9
Combined BBAs deduced by using the normalized cautious rule.

	a_1					a_2				
	$Q^1_{1\|1}$	$Q^1_{1\|2}$	$w^1_{1\|1}$	$w^1_{1\|2}$	w^1_1	$Q^1_{2\|1}$	$Q^1_{2\|2}$	$w^1_{2\|1}$	$w^1_{2\|2}$	w^1_2
$\{x^1_1\}$	1	0.8	0.6	0.375	0.375	0.8	1	0.25	0.7	0.25
$\{x^1_2\}$	0.6	0.5	1	0.6	0.6	0.4	0.7	0.5	1	0.5
$\{x^1_1, x^1_2\}$	0.6	0.3	–	–	–	0.2	0.7	–	–	–
	The combined BBA m^1_1 is obtained by combining two simple BBAs $\{x^1_1\}^{0.375}$ and $\{x^1_2\}^{0.6}$ using the Dempster's rule. This leads to the following BBA: $m^1_1(\{x^1_1\}) = 0.5$; $m^1_1(\{x^1_2\}) = 0.2$; $m^1_1(\{x^1_1, x^1_2\}) = 0.3$					The combined BBA m^1_2 is obtained by combining two simple BBAs $\{x^1_1\}^{0.25}$ and $\{x^1_2\}^{0.5}$ using the Dempster's rule. This leads to the following BBA: $m^1_2(\{x^1_1\}) = 0.6$; $m^1_2(\{x^1_2\}) = 0.2$; $m^1_2(\{x^1_1, x^1_2\}) = 0.2$				

Table 10
Ascending and descending belief distances and global scores (case of nondistinct BBAs).

		a_1	a_2
The BBAs given by expert 1	$d\left(\overrightarrow{Bel}^1_{i\|1}, \overrightarrow{Bel}^1_{ideal\|1}\right)$	0.4	0.6
	$d\left(\overleftarrow{Bel}^1_{i\|1}, \overleftarrow{Bel}^1_{ideal\|1}\right)$	1	0.8
	$\alpha^1_{i\|1}$	1.4	1.4
The BBAs given by expert 2	$d\left(\overrightarrow{Bel}^1_{i\|2}, \overrightarrow{Bel}^1_{ideal\|2}\right)$	0.5	0.3
	$d\left(\overleftarrow{Bel}^1_{i\|2}, \overleftarrow{Bel}^1_{ideal\|2}\right)$	0.8	1
	$\alpha^1_{i\|2}$	1.3	1.3
The combined BBAs given by the two experts	$d\left(\overrightarrow{Bel}^1_i, \overrightarrow{Bel}^1_{ideal}\right)$	0.5	0.6
	$d\left(\overleftarrow{Bel}^1_i, \overleftarrow{Bel}^1_{ideal}\right)$	0.8	0.8
	α^1_i	1.3	1.4

the combined BBAs, i.e., the commonality functions, the weights of evidence and the simple BBAs. Moreover, Table 10 gives the values of the ascending and descending belief distances with regard to the ideal BBA used to deduce the RBBD I ranking and the global scores used to obtain the RBBD II ranking. For instance, the ranking of the combined BBAs is determined as follows: $d\left(\overrightarrow{Bel}^1_1, \overrightarrow{Bel}^1_{ideal}\right) < d\left(\overrightarrow{Bel}^1_2, \overrightarrow{Bel}^1_{ideal}\right)$ and $d\left(\overleftarrow{Bel}^1_1, \overleftarrow{Bel}^1_{ideal}\right) = d\left(\overleftarrow{Bel}^1_2, \overleftarrow{Bel}^1_{ideal}\right)$, thus m^1_1 is preferred to m^1_2 according to the RBBD I ranking. This result is of course verified according to the RBBD II ranking since $\alpha^1_1 < \alpha^1_2$.

As can be noticed, the experts agree that the evaluations of a_1 and a_2 are incomparable on g_1 according to the RBBD I ranking whereas they are indifferent on g_1 according to the RBBD II ranking. However, when we apply the RBBD concept to rank the combined BBAs, we obtain: the evaluation of a_1 is preferred to the one of a_2 on criterion g_1 according to the RBBD I and II rankings. Thus, the normalized cautious rule does not respect the unanimity property. Finally, let us note that the mean operator [28] which is also commonly used to aggregate nondistinct sources does not verify this property.

5. Illustrative example

In order to illustrate the model, let us consider the following example. A multinational group wants to construct a new hotel in a city where the group is not yet established. Five sites are considered which are evaluated on the basis of three ordinal criteria:

- The investment costs (including the land purchasing and the construction costs) (to be minimized).
- The annual operating costs (to be minimized).
- The facility of access to the hotel from the airport (to be maximized).

Table 11
Criteria and assessment grades.

Criterion	Assessment grades
g_1 "Investment costs"	x_1^1 "Very high" x_2^1 "High" x_3^1 "Average" x_4^1 "Low" x_5^1 "Very low"
g_2 "Annual operating costs"	x_1^2 "Very high" x_2^2 "High" x_3^2 "Average" x_4^2 "Low" x_5^2 "Very low"
g_3 "Facility of access to the hotel from the airport"	x_1^3 "Difficult" x_2^3 "Rather easy" x_3^3 "Very easy"

Table 12
Values of the criteria weights given by the experts.

	g_1	g_2	g_3
Expert 1	1/3	1/3	1/3
Expert 2	0.2	0.3	0.5
Expert 3	0.25	0.5	0.25

Table 11 presents the assessment grades associated to each criterion.

A decision for selecting a site has to be made based on the opinion of a committee composed of three experts referred to as experts 1, 2 and 3. The coefficients of their relative importance within the group are respectively 0.3, 0.4 and 0.3. We will assume that the experts use Xu et al.'s method for the decision and that each of them proposes his own values of the criteria weights. Table 12 gives these values. Moreover, we will suppose that each expert can express individually the assessments for all the sites with respect to all the criteria and that the evaluations are modeled by BBAs. Tables 13–15 present the BBAs characterizing the evaluations of the sites given respectively by experts 1, 2 and 3. For instance, the evaluations of site a_1 are established by the experts as follows:

- On criterion g_1, experts 1 and 3 hesitate between the third and the fourth assessment grades. They are sure that the investment costs in this site are average or low without being able to refine their judgment whereas expert 2 is sure that these costs are average.
- On criterion g_2, experts 1 and 3 hesitate between the first, the second and the third assessment grades whereas expert 2 hesitates between the third, the fourth and the fifth ones. The formers are sure that the annual operating costs are very high, high or average whereas the latter is sure that these costs are average, low or very low. The three experts are unable to refine their judgments.
- On criterion g_3, experts 1 and 3 are certain that the access to the hotel is difficult whereas expert 2 is unable to express his assessment on the site (total ignorance case). That is why the total mass is assigned to the set of assessment grades $X^3 = \{x_1^3, x_2^3, x_3^3\}$.

The RBBD concept is applied by each expert to rank, on each criterion, the evaluations of the sites. Two types of single-criterion preorders can be obtained: the RBBD I and II rankings. In what follows, we will consider the case where the experts use the RBBD I rankings (RBBD II rankings, resp.) when they apply Xu et al.'s method. Our objective is to compare the results that will be deduced from these two types of rankings.

Based on the RBBD I (RBBD II, resp.) ranking of the evaluations obtained by each expert, the individual global rankings of the sites are deduced as in Xu et al.'s method. For that purpose, each expert builds at first the two individual total preorders O_1^f and O_2^f (with $f = 1, 2, \ldots, s$) which are then combined in order to obtain the individual global ranking of the sites O^f. Figs. 4–6 (8–10, resp.) illustrate the RBBD I (RBBD II, resp.) ranking of the evaluations and the individual total and global preorders of the sites given respectively by experts 1, 2 and 3. Appendixes B, C and D give the belief distances and the global scores with regard to the ideal and nadir BBAs which are used respectively by experts 1, 2 and 3 to determine the RBBD I and II rankings of the sites evaluations (let us recall that the comparison to the ideal BBA leads to the same ranking obtained when comparing with regard to the nadir BBA). For instance, let us focus ourselves on the results given by Fig. 4. It is easy to deduce from the first three lines of this figure that:

Table 13
BBAs characterizing the evaluations of the sites given by expert 1.

	g_1	g_2	g_3
a_1	$m^1_{1\|1}(\{x^1_3\}) = 0.8$ $m^1_{1\|1}(\{x^1_3, x^1_4\}) = 0.2$	$m^2_{1\|1}(\{x^2_1\}) = 0.1$ $m^2_{1\|1}(\{x^2_1, x^2_2, x^2_3\}) = 0.9$	$m^3_{1\|1}(\{x^3_1\}) = 1$
a_2	$m^1_{2\|1}(\{x^1_1\}) = 0.6$ $m^1_{2\|1}(\{x^1_1, x^1_2\}) = 0.4$	$m^2_{2\|1}(\{x^2_4\}) = 0.6$ $m^2_{2\|1}(\{x^2_4, x^2_5\}) = 0.4$	$m^3_{2\|1}(\{x^3_2\}) = 1$
a_3	$m^1_{3\|1}(\{x^1_2\}) = 1$	$m^2_{3\|1}(\{x^2_2\}) = 0.6$ $m^2_{3\|1}(\{x^2_3\}) = 0.4$	$m^3_{3\|1}(\{x^3_2\}) = 0.7$ $m^3_{3\|1}(\{x^3_2, x^3_3\}) = 0.3$
a_4	$m^1_{4\|1}(\{x^1_1, x^1_2, x^1_3, x^1_4\}) = 1$	$m^2_{4\|1}(\{x^2_1, x^2_2\}) = 0.7$ $m^2_{4\|1}(\{x^2_2, x^2_3\}) = 0.3$	$m^3_{4\|1}(\{x^3_2\}) = 0.8$ $m^3_{4\|1}(\{x^3_3\}) = 0.2$
a_5	$m^1_{5\|1}(\{x^1_2\}) = 1$	$m^2_{5\|1}(\{x^2_3\}) = 0.6$ $m^2_{5\|1}(\{x^2_3, x^2_4\}) = 0.4$	$m^3_{5\|1}(\{x^3_1\}) = 0.9$ $m^3_{5\|1}(\{x^3_1, x^3_2\}) = 0.1$

Table 14
BBAs characterizing the evaluations of the sites given by expert 2.

	g_1	g_2	g_3
a_1	$m^1_{1\|2}(\{x^1_3\}) = 1$	$m^2_{1\|2}(\{x^2_3, x^2_4\}) = 0.8$ $m^2_{1\|2}(\{x^2_5\}) = 0.2$	$m^3_{1\|2}(\{x^3_1, x^3_2, x^3_3\}) = 1$
a_2	$m^1_{2\|2}(\{x^1_1, x^1_2\}) = 1$	$m^2_{2\|2}(\{x^2_1, x^2_2, x^2_3, x^2_4, x^2_5\}) = 1$	$m^3_{2\|2}(\{x^3_2\}) = 1$
a_3	$m^1_{3\|2}(\{x^1_2\}) = 0.6$ $m^1_{3\|2}(\{x^1_3\}) = 0.4$	$m^2_{3\|2}(\{x^2_3\}) = 1$	$m^3_{3\|2}(\{x^3_2\}) = 0.7$ $m^3_{3\|2}(\{x^3_2, x^3_3\}) = 0.3$
a_4	$m^1_{4\|2}(\{x^1_1\}) = 1$	$m^2_{4\|2}(\{x^2_3\}) = 0.9$ $m^2_{4\|2}(\{x^2_3, x^2_4\}) = 0.1$	$m^3_{4\|2}(\{x^3_2\}) = 0.8$ $m^3_{4\|2}(\{x^3_2, x^3_3\}) = 0.2$
a_5	$m^1_{5\|2}(\{x^1_2\}) = 1$	$m^2_{5\|2}(\{x^2_3, x^2_4\}) = 1$	$m^3_{5\|2}(\{x^3_1\}) = 0.5$ $m^3_{5\|2}(\{x^3_2\}) = 0.5$

- The best evaluations on criterion g_1 according to expert 1 are those of a_1 and a_4 since the evaluation of a_1 is preferred to those of a_2, a_3 and a_5 and the evaluations of a_1 and a_4 are incomparable according to the RBBD I ranking. The worst evaluation is the one of a_2 (see the first line).
- The best evaluation on criterion g_2 according to expert 1 is the one of a_2 since it is preferred to those of a_1, a_3, a_4 and a_5. The worst evaluations are those of a_1 and a_4 which are incomparable according to the RBBD I ranking (see the second line).
- The best evaluations on criterion g_3 according to expert 1 are those of a_3 and a_4 since they are incomparable according to the RBBD I ranking and they are preferred to those of a_1, a_2 and a_5. The worst evaluation is the one of a_1 (see the third line).

The fourth and fifth lines of Fig. 4 give the individual total preorders deduced from the use of Xu et al.'s method in order to aggregate the RBBD I rankings of the sites evaluations (see the first three lines). As can be noticed, the best sites according to the first individual total preorder are a_2 and a_3 which are indifferent. The best ones according to the second total preorder are a_3 and a_4 which are also indifferent. In both preorders, the worst site is a_1. These preorders are combined in order to obtain the individual global ranking of the sites (see the sixth line). It is clear that the best and worst sites are respectively a_3 and a_1.

Finally, the individual global preorders of the sites O^1, O^2 and O^3 are aggregated in order to obtain a collective global preorder that synthesizes the opinions of experts 1, 2 and 3. This is performed using the same procedure of Xu et al.'s method applied above to obtain the individual global rankings. Therefore, two collective total preorders O_1 and O_2 are at first determined which are then combined to determine the collective global preorder O. Fig. 7 (11, resp.) gives the collective total and global preorders of the sites obtained in the case where the experts use the RBBD I (RBBD II, resp.) ranking of the evaluations in order to obtain the individual global rankings. For instance, the first and second lines of Fig. 7 (11, resp.) give the collective

Table 15
BBAs characterizing the evaluations of the sites given by expert 3.

	g_1	g_2	g_3
a_1	$m^1_{1\|3}(\{x^1_3\}) = 0.94$ $m^1_{1\|3}(\{x^1_3, x^1_4\}) = 0.06$	$m^2_{1\|3}(\{x^2_1\}) = 0.04$ $m^2_{1\|3}(\{x^2_3\}) = 0.6$ $m^2_{1\|3}(\{x^2_1, x^2_2, x^2_3\}) = 0.36$	$m^3_{1\|3}(\{x^3_1\}) = 1$
a_2	$m^1_{2\|3}(\{x^1_1\}) = 0.6$ $m^1_{2\|3}(\{x^1_1, x^1_2\}) = 0.4$	$m^2_{2\|3}(\{x^2_4\}) = 0.6$ $m^2_{2\|3}(\{x^2_4, x^2_5\}) = 0.4$	$m^3_{2\|3}(\{x^3_2\}) = 1$
a_3	$m^1_{3\|3}(\{x^1_2\}) = 1$	$m^2_{3\|3}(\{x^2_2\}) = 0.33$ $m^2_{3\|3}(\{x^2_3\}) = 0.67$	$m^3_{3\|3}(\{x^3_2\}) = 0.84$ $m^3_{3\|3}(\{x^3_2, x^3_3\}) = 0.16$
a_4	$m^1_{4\|3}(\{x^1_1\}) = 0.67$ $m^1_{4\|3}(\{x^1_1, x^1_2, x^1_3, x^1_4\}) = 0.33$	$m^2_{4\|3}(\{x^2_1, x^2_2\}) = 0.44$ $m^2_{4\|3}(\{x^2_2, x^2_3\}) = 0.19$ $m^2_{4\|3}(\{x^2_3\}) = 0.37$	$m^3_{4\|3}(\{x^3_2\}) = 0.9$ $m^3_{4\|3}(\{x^3_3\}) = 0.1$
a_5	$m^1_{5\|3}(\{x^1_2\}) = 1$	$m^2_{5\|3}(\{x^2_3\}) = 0.6$ $m^2_{5\|3}(\{x^2_3, x^2_4\}) = 0.4$	$m^3_{5\|3}(\{x^3_1\}) = 0.9$ $m^3_{5\|3}(\{x^3_2\}) = 0.05$ $m^3_{5\|3}(\{x^3_1, x^3_2\}) = 0.05$

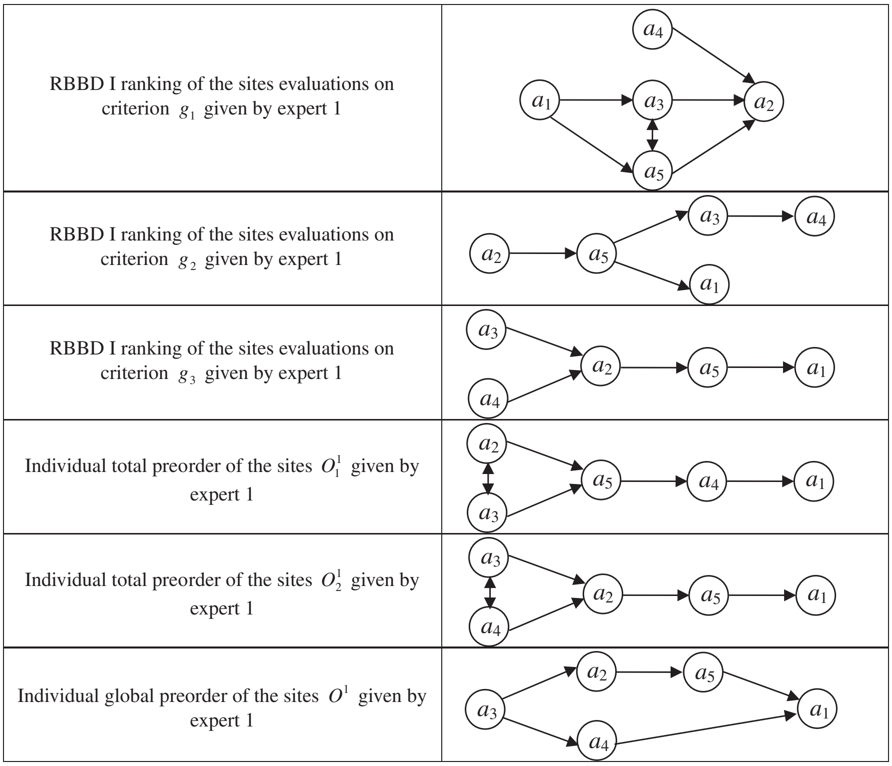

Fig. 4. Illustration of the different steps of Xu and Martel's method based on the RBBD I rankings of the sites evaluations given by expert 1.

total preorders of the sites which are obtained as in Xu et al.'s method by aggregating the individual global preorders of the sites given in the sixth lines of Figs. 4–6(8–10, resp.). These preorders are combined in order to obtain the collective global preorder of the sites which is given in the third line of Fig. 7 (11, resp.).

Based on the achieved results given in Figs. 7 and 11, it is easy to see that the best site is a_3 according to the collective global preorders of the sites obtained when the experts use the RBBD I or II rankings of the evaluations. The worst sites according to the collective global preorder of Fig. 7 are a_4 and a_5 and according to the one of Fig. 11 are a_1 and a_5. Moreover,

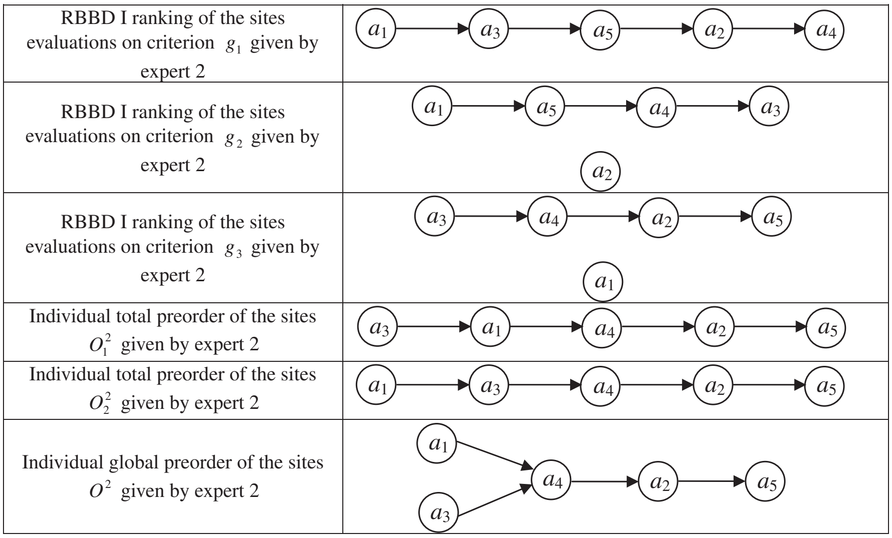

Fig. 5. Illustration of the different steps of Xu and Martel's method based on the RBBD I rankings of the sites evaluations given by expert 2.

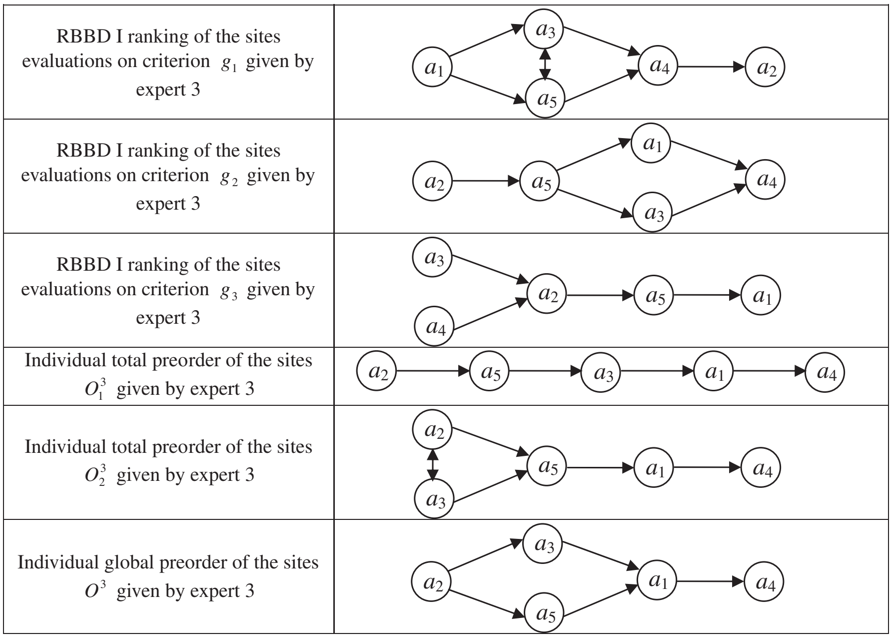

Fig. 6. Illustration of the different steps of Xu and Martel's method based on the RBBD I rankings of the sites evaluations given by expert 3.

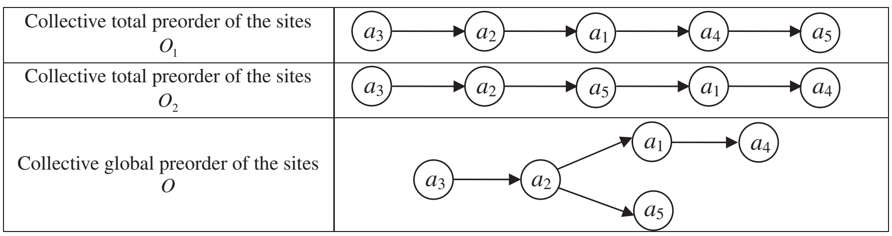

Fig. 7. Collective preorders of the sites in the case where the experts use RBBD I rankings of the evaluations.

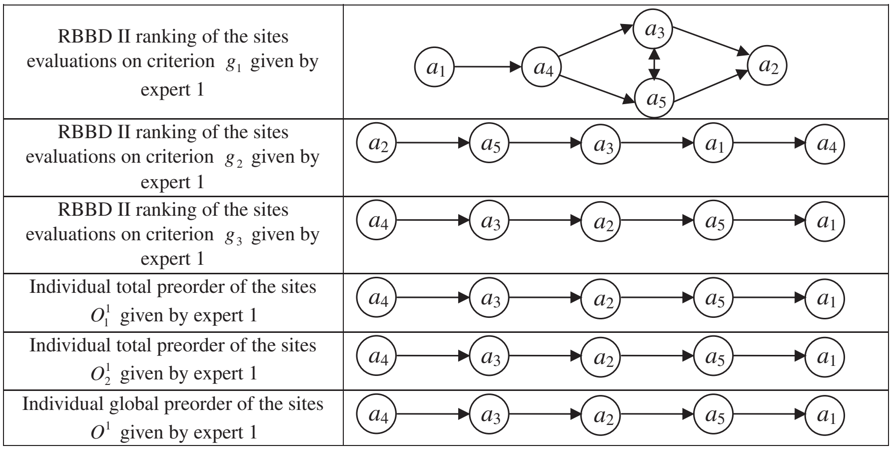

Fig. 8. Illustration of the different steps of Xu and Martel's method based on the RBBD II rankings of the sites evaluations given by expert 1.

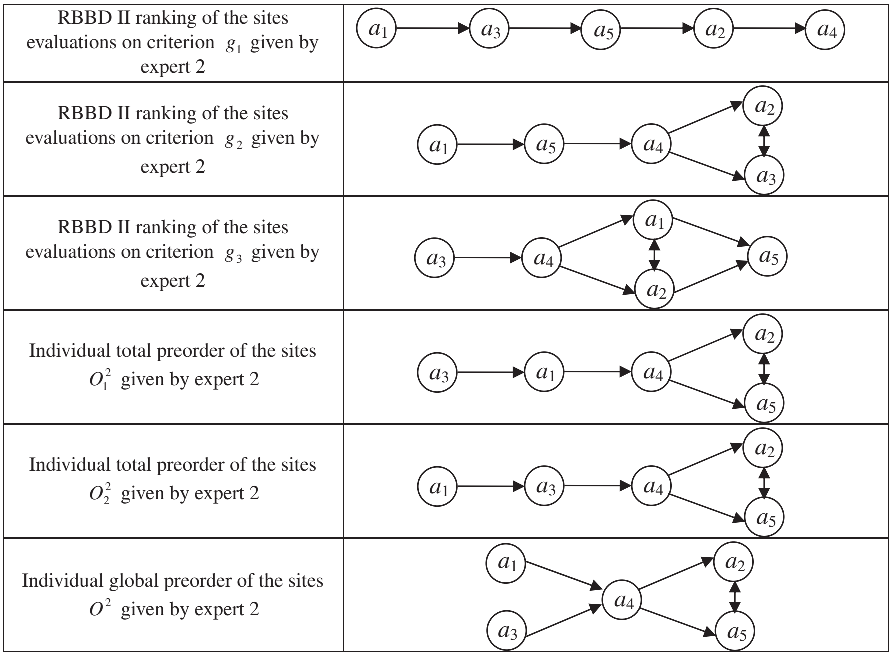

Fig. 9. Illustration of the different steps of Xu and Martel's method based on the RBBD II rankings of the sites evaluations given by expert 2.

it is easy to observe some differences between these two collective rankings on the preference relations between sites a_1 and a_4, sites a_2 and a_4 and sites a_4 and a_5. Indeed:

- a_1 (a_2, resp.) is preferred to a_4 according to the collective global preorder deduced from the aggregation of the individual global preorders that use RBBD I rankings whereas they are incomparable according to the one based on the RBBD II rankings.
- a_4 and a_5 are incomparable according to the collective global preorder deduced from the aggregation of the individual global preorders that use RBBD I rankings whereas a_4 is preferred to a_5 according to the one based on the RBBD II rankings.

6. Conclusion

In this paper, we have addressed ranking problem in uncertain, imprecise and multi-experts contexts. Evidence theory offers convenient tools to tackle such kind of problems. At first, the concept of BBA allows experts to express freely their assessments and even to represent the total ignorance. In order to rank the BBAs that represent the evaluations of the actions

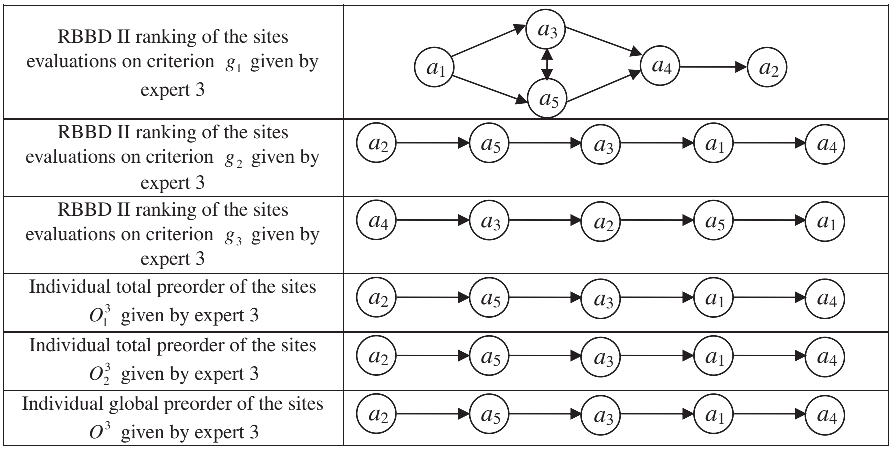

Fig. 10. Illustration of the different steps of Xu and Martel's method based on the RBBD II rankings of the sites evaluations given by expert 3.

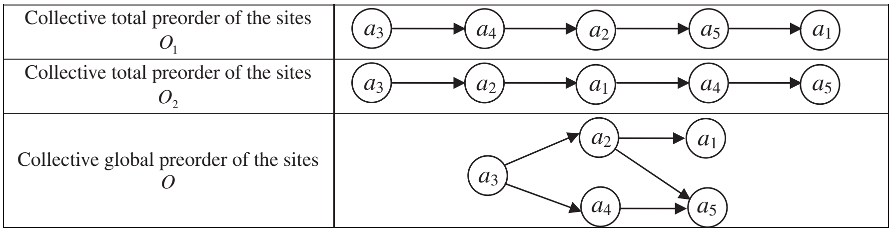

Fig. 11. Collective preorders of the sites in the case where the experts use RBBD II rankings of the evaluations.

on each criterion, the concept of RBBD has been introduced. This approach is based on the comparison of the BBAs to ideal or nadir BBAs using belief distances. Two rankings can be deduced called RBBD I and II which give respectively partial and total preorders of the BBAs. In addition, a model inspired by Xu et al.'s method has been proposed and illustrated on a pedagogical example.

We have shown that the RBBD concept is coherent with the first belief dominance approach. Moreover, we have proven that the number of incomparable pairs of BBAs in the RBBD I ranking is usually inferior or equal to the one in the ranking deduced by using the first belief dominance to rank the BBAs. At this point, let us note that the RBBD I ranking can be viewed as a mid-way between the RBBD II ranking and the preorder given by the first belief dominance. Indeed, whereas the former can lead to excessive rich results (a ranking without incomparabilities), the latter can lead on the contrary to poor results since an important number of incomparable pairs of BBAs can be induced.

Finally, we have illustrated the benefits of using evidence theory in ranking problems. Of course, there are still many directions for future research. Among others, we can mention the development of combination rules that respect the unanimity principle. Moreover, it is of course interesting to compare the results given by the RBBD I and II rankings. The RBBD II provides an "agreeable" ranking of the BBAs, but some useful information about incomparabilities gets lost.

Appendix A. Proofs

Proof 1. For $r_h = 1$, since $m^h_{ideal}(\{x^h_1\}) = 1$, $\overrightarrow{Bel}^h_{ideal}(A^h_1) = 1$. It follows that:

$$d\left(\overrightarrow{Bel}^h_i, \overrightarrow{Bel}^h_{ideal}\right) = \left|\overrightarrow{Bel}^h_i(A^h_1) - 1\right| = \overrightarrow{Bel}^h_i(A^h_1) - 1 = 0$$

For $r_h > 1$, since $m^h_{ideal}(\{x^h_{r_h}\}) = 1$, $\overrightarrow{Bel}^h_{ideal}(A^h_k) = 0$ for all $k = 1,\ldots,r_h - 1$ and $\overrightarrow{Bel}^h_{ideal}(A^h_{r_h}) = 1$. It follows that:

$$d\left(\overrightarrow{Bel}^h_i, \overrightarrow{Bel}^h_{ideal}\right) = \sum_{k=1}^{r_h-1}\left|\overrightarrow{Bel}^h_i(A^h_k) - 0\right| + \left|\overrightarrow{Bel}^h_i(A^h_{r_h}) - 1\right| = \left(\sum_{k=1}^{r_h}\overrightarrow{Bel}^h_i(A^h_k)\right) - 1$$

Proof 2. Since $m^h_{ideal}(\{x^h_{r_h}\}) = 1,\ \overleftarrow{Bel}^h_{ideal}(B^h_l) = 1$ for all $l = 1,\ldots,r_h$. It follows that:

$$d\left(\overleftarrow{Bel}^h_i, \overleftarrow{Bel}^h_{ideal}\right) = \sum_{l=1}^{r_h}\left|\overleftarrow{Bel}^h_i(B^h_l) - 1\right| = \sum_{l=1}^{r_h}\left(1 - \overleftarrow{Bel}^h_i(B^h_l)\right) = r_h - \sum_{l=1}^{r_h}\overleftarrow{Bel}^h_i(B^h_l)$$

Proof 3. Since $m^h_{nadir}(\{x^h_1\}) = 1,\ \overrightarrow{Bel}^h_{nadir}(A^h_k) = 1$ for all $k = 1,\ldots,r_h$. It follows that:

$$d\left(\overrightarrow{Bel}^h_i, \overrightarrow{Bel}^h_{nadir}\right) = \sum_{k=1}^{r_h}\left|\overrightarrow{Bel}^h_i(A^h_k) - 1\right| = \sum_{k=1}^{r_h}\left(1 - \overrightarrow{Bel}^h_i(A^h_k)\right) = r_h - \sum_{k=1}^{r_h}\overrightarrow{Bel}^h_i(A^h_k)$$

Proof 4. For $r_h = 1$, since $m^h_{nadir}(\{x^h_1\}) = 1,\ \overleftarrow{Bel}^h_{nadir}(B^h_1) = 1$. It follows that:

$$d\left(\overleftarrow{Bel}^h_i, \overleftarrow{Bel}^h_{nadir}\right) = \left|\overleftarrow{Bel}^h_i(B^h_1) - 1\right| = \overleftarrow{Bel}^h_i(B^h_1) - 1 = 0$$

For $r_h > 1$, since $m^h_{nadir}(\{x^h_1\}) = 1,\ \overleftarrow{Bel}^h_{nadir}(B^h_l) = 0$ for all $l = 1,\ldots,r_h - 1$ and $\overleftarrow{Bel}^h_{nadir}(B^h_{r_h}) = 1$. It follows that:

$$d\left(\overleftarrow{Bel}^h_i, \overleftarrow{Bel}^h_{nadir}\right) = \sum_{l=1}^{r_h-1}\left|\overleftarrow{Bel}^h_i(B^h_l) - 0\right| + \left|\overleftarrow{Bel}^h_i(B^h_{r_h}) - 1\right| = \left(\sum_{l=1}^{r_h}\overleftarrow{Bel}^h_i(B^h_l)\right) - 1$$

Proof 5. Since $m^h_i \mathrm{FBD} m^h_{i'},\ \overrightarrow{Bel}^h_i(A^h_k) \leqslant \overrightarrow{Bel}^h_{i'}(A^h_k)$ for all $A^h_k \in \overrightarrow{S}(X^h)$. Therefore:

$$\sum_{k=1}^{r_h}\overrightarrow{Bel}^h_i(A^h_k) \leqslant \sum_{k=1}^{r_h}\overrightarrow{Bel}^h_{i'}(A^h_k) \Longleftrightarrow \left(\sum_{k=1}^{r_h}\overrightarrow{Bel}^h_i(A^h_k)\right) - 1 \leqslant \left(\sum_{k=1}^{r_h}\overrightarrow{Bel}^h_{i'}(A^h_k)\right) - 1 \Longleftrightarrow d\left(\overrightarrow{Bel}^h_i, \overrightarrow{Bel}^h_{ideal}\right) \leqslant d\left(\overrightarrow{Bel}^h_{i'}, \overrightarrow{Bel}^h_{ideal}\right)$$

Similarly, since $m^h_i \mathrm{FBD} m^h_{i'},\ \overleftarrow{Bel}^h_i(B^h_l) \geqslant \overleftarrow{Bel}^h_{i'}(B^h_l)$ for all $B^h_l \in \overrightarrow{S}(X^h)$. Thus:

$$\sum_{l=1}^{r_h}\overleftarrow{Bel}^h_i(B^h_l) \geqslant \sum_{l=1}^{r_h}\overleftarrow{Bel}^h_{i'}(B^h_l) \Longleftrightarrow r_h - \sum_{l=1}^{r_h}\overleftarrow{Bel}^h_i(B^h_l) \leqslant r_h - \sum_{l=1}^{r_h}\overleftarrow{Bel}^h_{i'}(B^h_l) \Longleftrightarrow d\left(\overleftarrow{Bel}^h_i, \overleftarrow{Bel}^h_{ideal}\right) \leqslant d\left(\overleftarrow{Bel}^h_{i'}, \overleftarrow{Bel}^h_{ideal}\right)$$

Based on the definitions given in formula 14, it follows that if $m^h_i \mathrm{FBD} m^h_{i'}$, then $m^h_i P_h m^h_{i'}$ or $m^h_i I_h m^h_{i'}$ according to the RBBD concept. Let us note that the proof of this proposition can be performed based on the belief distances to the nadir BBA and the definitions given in formula 15.

Proof 6. According to Proof 5, if m^h_i FBD$m^h_{i'}$, $d\left(\overrightarrow{Bel}^h_i, \overrightarrow{Bel}^h_{ideal}\right) \leqslant d\left(\overrightarrow{Bel}^h_{i'}, \overrightarrow{Bel}^h_{ideal}\right)$ and $d\left(\overleftarrow{Bel}^h_i, \overleftarrow{Bel}^h_{ideal}\right) \leqslant d\left(\overleftarrow{Bel}^h_{i'}, \overleftarrow{Bel}^h_{ideal}\right)$. Therefore, if $m^h_i \overline{\mathrm{FBD}} m^h_{i'}$, we have one of the following cases:

- $d\left(\overrightarrow{Bel}^h_i, \overrightarrow{Bel}^h_{ideal}\right) \leqslant d\left(\overrightarrow{Bel}^h_{i'}, \overrightarrow{Bel}^h_{ideal}\right)$ and $d\left(\overleftarrow{Bel}^h_i, \overleftarrow{Bel}^h_{ideal}\right) > d\left(\overleftarrow{Bel}^h_{i'}, \overleftarrow{Bel}^h_{ideal}\right)$;
- $d\left(\overrightarrow{Bel}^h_i, \overrightarrow{Bel}^h_{ideal}\right) > d\left(\overrightarrow{Bel}^h_{i'}, \overrightarrow{Bel}^h_{ideal}\right)$ and $d\left(\overleftarrow{Bel}^h_i, \overleftarrow{Bel}^h_{ideal}\right) \leqslant d\left(\overleftarrow{Bel}^h_{i'}, \overleftarrow{Bel}^h_{ideal}\right)$;
- $d\left(\overrightarrow{Bel}^h_i, \overrightarrow{Bel}^h_{ideal}\right) \geqslant d\left(\overrightarrow{Bel}^h_{i'}, \overrightarrow{Bel}^h_{ideal}\right)$ and $d\left(\overleftarrow{Bel}^h_i, \overleftarrow{Bel}^h_{ideal}\right) < d\left(\overleftarrow{Bel}^h_{i'}, \overleftarrow{Bel}^h_{ideal}\right)$;
- $d\left(\overrightarrow{Bel}^h_i, \overrightarrow{Bel}^h_{ideal}\right) < d\left(\overrightarrow{Bel}^h_{i'}, \overrightarrow{Bel}^h_{ideal}\right)$ and $d\left(\overleftarrow{Bel}^h_i, \overleftarrow{Bel}^h_{ideal}\right) \geqslant d\left(\overleftarrow{Bel}^h_{i'}, \overleftarrow{Bel}^h_{ideal}\right)$.

Based on the definitions given in formula 14, it follows that if $m^h_i \overline{\mathrm{FBD}} m^h_{i'}$, then $m^h_i P_h m^h_{i'}$ or $m^h_i J_h m^h_{i'}$ or $m^h_i P_h^{-1} m^h_{i'}$ according to the RBBD concept. This result leads to deduce that when $m^h_i \overline{\mathrm{FBD}} m^h_{i'}$ and $m^h_{i'} \overline{\mathrm{FBD}} m^h_i$, $m^h_i P_h m^h_{i'}$ or $m^h_i J_h m^h_{i'}$ or $m^h_i P_h^{-1} m^h_{i'}$. In other words, when m^h_i and $m^h_{i'}$ are incomparable according to the first belief dominance, $m^h_i P_h m^h_{i'}$ or $m^h_i J_h m^h_{i'}$ or $m^h_i P_h^{-1} m^h_{i'}$ according to the RBBD concept. Therefore, the number of incomparabilities obtained by the first belief dominance approach is superior or equal to the one obtained by the RBBD concept (RBBD II).

Appendix B. Belief distances used by expert 1 to determine the RBBD I and II rankings of the evaluations

Tables 16–18

Table 16
Belief distances used by expert 1 on criterion g_1.

	a_1	a_2	a_3	a_4	a_5
$d\left(\overrightarrow{Bel}^1_{i\|1}, \overrightarrow{Bel}^1_{ideal\|1}\right)$	1.8	3.6	3	1	3
$d\left(\overleftarrow{Bel}^1_{i\|1}, \overleftarrow{Bel}^1_{ideal\|1}\right)$	2	4	3	4	3
$\alpha^1_{i\|1}$	3.8	7.6	6	5	6
$d\left(\overrightarrow{Bel}^1_{i\|1}, \overrightarrow{Bel}^1_{nadir\|1}\right)$	2.2	0.4	1	3	1
$d\left(\overleftarrow{Bel}^1_{i\|1}, \overleftarrow{Bel}^1_{nadir\|1}\right)$	2	0	1	0	1
$\beta^1_{i\|1}$	4.4	0.4	2	3	2

Table 17
Belief distances used by expert 1 on criterion g_2.

	a_1	a_2	a_3	a_4	a_5
$d\left(\overrightarrow{Bel}^2_{i\|1}, \overrightarrow{Bel}^2_{ideal\|1}\right)$	2.2	0.6	2.6	2.7	1.6
$d\left(\overleftarrow{Bel}^2_{i\|1}, \overleftarrow{Bel}^2_{ideal\|1}\right)$	4	1	2.6	3.7	2
$\alpha^2_{i\|1}$	6.2	1.6	5.2	6.4	3.6
$d\left(\overrightarrow{Bel}^2_{i\|1}, \overrightarrow{Bel}^2_{nadir\|1}\right)$	1.8	3.4	1.4	1.3	2.4
$d\left(\overleftarrow{Bel}^2_{i\|1}, \overleftarrow{Bel}^2_{nadir\|1}\right)$	0	3	1.4	0.3	2
$\beta^2_{i\|1}$	1.8	6.4	2.8	1.6	4.4

Table 18
Belief distances used by expert 1 on criterion g_3.

	a_1	a_2	a_3	a_4	a_5
$d\left(\overrightarrow{Bel}^3_{i\|1}, \overrightarrow{Bel}^3_{ideal\|1}\right)$	2	1	0.7	0.8	1.9
$d\left(\overleftarrow{Bel}^3_{i\|1}, \overleftarrow{Bel}^3_{ideal\|1}\right)$	2	1	1	0.8	2
$\alpha^3_{i\|1}$	4	2	1.7	1.6	3.9
$d\left(\overrightarrow{Bel}^3_{i\|1}, \overrightarrow{Bel}^3_{nadir\|1}\right)$	0	1	1.3	1.2	0.1
$d\left(\overleftarrow{Bel}^3_{i\|1}, \overleftarrow{Bel}^3_{nadir\|1}\right)$	0	1	1	1.2	0
$\beta^3_{i\|1}$	0	2	2.3	2.4	0.1

Appendix C. Belief distances used by expert 2 to determine the RBBD I and II rankings of the evaluations

Tables 19–21

Table 19
Belief distances used by expert 2 on criterion g_1.

	a_1	a_2	a_3	a_4	a_5
$d\left(\overrightarrow{Bel}^1_{i\|2}, \overrightarrow{Bel}^1_{ideal\|2}\right)$	2	3	2.6	4	3
$d\left(\overleftarrow{Bel}^1_{i\|2}, \overleftarrow{Bel}^1_{ideal\|2}\right)$	2	4	2.6	4	3
$\alpha^1_{i\|2}$	4	7	5.2	8	6
$d\left(\overrightarrow{Bel}^1_{i\|2}, \overrightarrow{Bel}^1_{nadir\|2}\right)$	2	1	1.4	0	1
$d\left(\overleftarrow{Bel}^1_{i\|2}, \overleftarrow{Bel}^1_{nadir\|2}\right)$	2	0	1.4	0	1
$\beta^1_{i\|2}$	4	1	2.8	0	2

Table 20
Belief distances used by expert 2 on criterion g_2.

	a_1	a_2	a_3	a_4	a_5
$d\left(\overrightarrow{Bel}^2_{i\|2}, \overrightarrow{Bel}^2_{ideal\|2}\right)$	0.8	0	2	1.9	1
$d\left(\overleftarrow{Bel}^2_{i\|2}, \overleftarrow{Bel}^2_{ideal\|2}\right)$	1.6	4	2	2	2
$\alpha^2_{i\|2}$	2.4	4	4	3.9	3
$d\left(\overrightarrow{Bel}^2_{i\|2}, \overrightarrow{Bel}^2_{nadir\|2}\right)$	3.2	4	2	2.1	3
$d\left(\overleftarrow{Bel}^2_{i\|2}, \overleftarrow{Bel}^2_{nadir\|2}\right)$	2.4	0	2	2	2
$\beta^2_{i\|2}$	5.6	4	4	4.1	5

Table 21
Belief distances used by expert 2 on criterion g_3.

	a_1	a_2	a_3	a_4	a_5
$d\left(\overrightarrow{Bel}^3_{i\|2}, \overrightarrow{Bel}^3_{ideal\|2}\right)$	0	1	0.7	0.8	1.5
$d\left(\overleftarrow{Bel}^3_{i\|2}, \overleftarrow{Bel}^3_{ideal\|2}\right)$	2	1	1	1	1.5
$\alpha^3_{i\|2}$	2	2	1.7	1.8	3
$d\left(\overrightarrow{Bel}^3_{i\|2}, \overrightarrow{Bel}^3_{nadir\|2}\right)$	2	1	1.3	1.2	0.5
$d\left(\overleftarrow{Bel}^3_{i\|2}, \overleftarrow{Bel}^3_{nadir\|2}\right)$	0	1	1	1	0.5
$\beta^3_{i\|2}$	2	2	2.3	2.2	1

Appendix D. Belief distances used by expert 3 to determine the RBBD I and II rankings of the evaluations

Tables 22–24

Table 22
Belief distances used by expert 3 on criterion g_1.

	a_1	a_2	a_3	a_4	a_5
$d\left(\overrightarrow{Bel}^1_{i\|3}, \overrightarrow{Bel}^1_{ideal\|3}\right)$	1.94	3.6	3	3.01	3
$d\left(\overleftarrow{Bel}^1_{i\|3}, \overleftarrow{Bel}^1_{ideal\|3}\right)$	2	4	3	4	3
$\alpha^1_{i\|3}$	3.94	7.6	6	7.01	6
$d\left(\overrightarrow{Bel}^1_{i\|3}, \overrightarrow{Bel}^1_{nadir\|3}\right)$	2.06	0.4	1	0.99	1
$d\left(\overleftarrow{Bel}^1_{i\|3}, \overleftarrow{Bel}^1_{nadir\|3}\right)$	2	0	1	0	1
$\beta^1_{i\|3}$	4.06	0.4	2	0.99	2

Table 23
Belief distances used by expert 3 on criterion g_2.

	a_1	a_2	a_3	a_4	a_5
$d\left(\overrightarrow{Bel}^2_{i\|3}, \overrightarrow{Bel}^2_{ideal\|3}\right)$	2.08	0.6	2.33	2.44	1.6
$d\left(\overleftarrow{Bel}^2_{i\|3}, \overleftarrow{Bel}^2_{ideal\|3}\right)$	2.8	1	2.33	3.07	2
$\alpha^2_{i\|3}$	4.88	1.6	4.66	5.51	3.6
$d\left(\overrightarrow{Bel}^2_{i\|3}, \overrightarrow{Bel}^2_{nadir\|3}\right)$	1.92	3.4	1.67	1.56	2.4
$d\left(\overleftarrow{Bel}^2_{i\|3}, \overleftarrow{Bel}^2_{nadir\|3}\right)$	1.2	3	1.67	0.93	2
$\beta^2_{i\|3}$	3.12	6.4	3.34	2.49	4.4

Table 24
Belief distances used by expert 3 on criterion g_3.

	a_1	a_2	a_3	a_4	a_5
$d\left(\overrightarrow{Bel}^3_{i\|3}, \overrightarrow{Bel}^3_{ideal\|3}\right)$	2	1	0.84	0.9	1.9
$d\left(\overleftarrow{Bel}^3_{i\|3}, \overleftarrow{Bel}^3_{ideal\|3}\right)$	2	1	1	0.9	1.95
$\alpha^3_{i\|3}$	4	2	1.84	1.8	3.85
$d\left(\overrightarrow{Bel}^3_{i\|3}, \overrightarrow{Bel}^3_{nadir\|3}\right)$	0	1	1.16	1.1	0.1
$d\left(\overleftarrow{Bel}^3_{i\|3}, \overleftarrow{Bel}^3_{nadir\|3}\right)$	0	1	1	1.1	0.05
$\beta^3_{i\|3}$	0	2	2.16	2.2	0.15

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