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The spectrum for large set of disjoint incomplete Latin squares ☆

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Abstract

An incomplete Latin square $\mathrm{LS}(n+a,a)$ is a Latin square of order $n+a$ with a missing subsquare of order a. A large set of disjoint $\mathrm{LS}(n+a,a)$s, denoted by $\mathrm{LDILS}(n+a,a)$, consists of n disjoint $\mathrm{LS}(n+a,a)s$. About the existence of LDILS, Zhu, Wu, Chen and Ge have already obtained some results (see Wu and Zhu, Bull Inst. Combin. Appl., to appear.). In this paper, we introduce a kind of auxiliary design $\mathrm{LS}_m(n)$ and, using it, completely solve the existence problem of LDILS. The conclusion is that for any positive integer n and any integer a, $0 \leqslant a \leqslant n$, there exists an $\mathrm{LDILS}(n+a,a)$ if and only if $(n,a) \neq (2,1)$ and $(6,5)$. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

An $n \times k$ (or $k \times n$, $k \leqslant n$) array over Z_n is named *Latin rectangle* if no element of Z_n occurs more than once in any row or column. An $n \times n$ Latin rectangle is called *Latin square* and is denoted by $\mathrm{LS}(n)$. Two Latin squares $A=(a_{ij})_1^n$ and $B=(b_{ij})_1^n$ are *orthogonal* if $a_{ij}=a_{st}$ and $b_{ij}=b_{st}$ implies $i=s$ and $j=t$. If $A_1, A_2, \ldots, A_k$ are pairwise orthogonal Latin squares of order n, then they are denoted by $k\mathrm{MOLS}(n)$. An *orthogonal array* $\mathrm{OA}(k,n)$ is a $n^2 \times k$ array over Z_n having the property that in any two columns, each ordered pair of Z_n occurs exactly once. It is well known [3] that $k\mathrm{MOLS}(n)$ is equivalent to $\mathrm{OA}(k+2,n)$.

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An *incomplete Latin square*, ILS($n+a,a$), denotes a LS($n+a$) with a missing sub-LS(a). Without loss of generality, we may assume that the missing subsquare, or *hole*, is at the lower right corner. We say $a_{ij} \in$ ILS($n+a,a$) if the entry in the cell (i,j) is a_{ij}. Let A_1 and A_2 be two ILS($n+a,a$)s on the same symbol set. If $a^1_{ij} \neq a^2_{ij}$ for any $a^1_{ij} \in A_1$ and $a^2_{ij} \in A_2$ except for the cell (i,j) in hole, then we say that A_1 and A_2 are *disjoint*. r-DILS($n+a,a$) denotes r pairwise disjoint ILS($n+a,a$)s. n-DILS($n+a,a$) are called a *large set of disjoint incomplete Latin squares* and denoted by LDILS($n+a,a$).

LDILS are useful in the singular indirect product construction of maximum constant weight code and generalized Steiner triple system (see [1–7]). The existence of LDILS itself is also an interesting problem to investigate. Zhu, Wu, Chen and Ge gave some existence results for LDILS (see [8]), i.e. for any positive integer n and integer a, $0 \leqslant a \leqslant n$, there exists an LDILS($n+a,a$) if one of following conditions hold:

(1) $n \not\equiv 2 \pmod 4$,
(2) $n \equiv 2 \pmod 4$ and $2|a$,
(3) $n = 6m$, $gcd(m,6) = 1$ and $a \equiv 1 \pmod 2$, $a \notin \{6m-1, 6m-3\}$.

In this paper, we will prove the following conclusion.

Theorem 1.1. *For any given positive integer n and any integer a, $0 \leqslant a \leqslant n$, there exist* LDILS($n+a,a$) *if and only if* $(n,a) \neq (2,1)$ *and* $(6,5)$.

To get the complete solution, we introduce an auxiliary design LS$_m(n)$, where n and m are positive integers and $m \leqslant n$. An LS$_m(n)$ is an LDILS$(n+0,0) = \{A_k;\ k \in Z_n\}$, where $A_0, A_1, \ldots, A_{n-1}$ are disjoint Latin squares of order n over Z_n and each A_k has m disjoint transversals $A_k(1), A_k(2), \ldots, A_k(m)$ such that $\bigcup_{k \in Z_n} A_k(s)$ form a Latin square for each $s, 1 \leqslant s \leqslant m$.

An $(n,k;\lambda)$-*difference matrix*, denoted briefly by (n,k,λ)-DM, is a $k \times n\lambda$ matrix $D = (d_{ij})$ with entries from an Abelian group G of order n, such that for each $1 \leqslant i < j \leqslant k$, the set $\{d_{is} - d_{js}; 1 \leqslant s \leqslant n\lambda\}$ contains every element of G λ times.

As well, for convenience, we give the following definitions. Let positive integer $n = \prod_i p_i^{e_i}$, where p_i are distinct primes and e_i are positive integers. Denote $\langle n \rangle = \prod_{p_i^{e_i} \geqslant 4} p_i^{e_i}$ and call $\langle n \rangle$ *regular part* of n. If $\langle n \rangle = n$, then the integer n is named *regular*, or else *non-regular*.

2. Regular integer orders

Theorem 2.1. *For positive integers n and m, if there exists an* LS$_m(n)$ *then there exist* LDILS($n+a,a$) *for any integer a, $0 \leqslant a \leqslant m$.*

Proof. Let LS$_m(n) = \{A_k;\ k \in Z_n\}$, where m disjoint transversals of $A_k = (a^k_{ij})_{i,j \in Z_n}$ are $A_k(1), A_k(2), \ldots, A_k(m)$, $k \in Z_n$. And, $\bigcup_{k \in Z_n} A_k(s)$ form a Latin square of order n, for given s, $1 \leqslant s \leqslant m$. Let $X = Z_n \cup \{\infty_1, \infty_2, \ldots, \infty_a\}$, where $\infty_s \notin Z_n$, $1 \leqslant s \leqslant a$. Define

n squares $B_k = (b_{ij}^k)_{i,j\in X}$ of order $n+a$ over $X, k \in Z_n$, where each B_k has an empty subsquare of order a. For any $a_{ij}^k \in A_k$,

if $(i,j,a_{ij}^k) \in A_k(s)$, $1 \leqslant s \leqslant a$, then let $b_{ij}^k = \infty_s$ and $b_{i,\infty_s}^k = b_{\infty_s,j}^k = a_{ij}^k$;
if $(i,j,a_{ij}^k) \in A_k \setminus \bigcup_{s=1}^a A_k(s)$, then let $b_{ij}^k = a_{ij}^k$.

It is easy to see that $\{B_k;\ k \in Z_n\}$ form an LDILS$(n+a,a)$ over X. $\square$

Lemma 2.2 (Colbourn and Dinitz [3]).

(1) *There exists a* $(q,q;1)$-DM *for any prime power* q.
(2) *The existence of* $(n,k;\lambda)$-DM *implies the existence of* $(n,k';\lambda)$-DM, *where* $k' < k$.
(3) *If* $(n,k;1)$-DM *and* $(m,k;1)$-DM *both exist then* $(mn,k;1)$-DM *exists too.*

Theorem 2.3. *If there exists an* $(n,4;1)$-DM *then there exists an* $\mathrm{LS}_n(n)$.

Proof. Let

$$D = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ b_0 & b_1 & \cdots & b_{n-1} \\ c_0 & c_1 & \cdots & c_{n-1} \\ d_0 & d_1 & \cdots & d_{n-1} \end{pmatrix} = [a_i,b_i,c_i,d_i]_i$$

be an $(n,4;1)$-DM, where $a_i,b_i,c_i,d_i \in Z_n$ and $i \in Z_n$. Similarly, we can denote an $(n,3;1)$-DM by $[a_i,b_i,c_i]_i$. It is not difficult to see that

(D1) If $[a_i,b_i,c_i]_i$ is an $(n,3;1)$-DM, then $[a_i+x,b_i+y,c_i+z]_i$ is still an $(n,3;1)$-DM for given x,y,z which not depend on i.
(D2) If $[a_i,b_i,c_i]_i$ is an $(n,3;1)$-DM, then $\bigcup_{s\in Z_n}[a_i+s,b_i+s,c_i+s]_i = \bigcup_{i,s\in Z_n}(a_i+s,b_i+s,c_i+s)$ form an OA(3,n).
(D3) If $[a_i,b_i,c_i,d_i]_i$ is an $(n,4;1)$-DM, then $[a_i-d_i,b_i-d_i,c_i-d_i]_i$ is an $(n,3;1)$-DM.

By (D1) and (D2), for each $k \in Z_n$, the $n^2 \times 3$ array

$$A_k = \{(a_i+k+s, b_i+k+s, c_i+s);\ i,s \in Z_n\}$$

form an OA$(3,n)$ over Z_n. Equivalently, A_k can be regarded as a Latin square of order n. And, these A_k, $k \in Z_n$, form an LDILS$(n+0,0)$ since they are disjoint.

Furthermore, let

$$A_k(s) = \{(a_i-d_i+k+s, b_i-d_i+k+s, c_i-d_i+s);\ i \in Z_n\}, \quad k,s \in Z_n.$$

Of course, $A_k(s) \subset A_k$ for any $k,s \in Z_n$. Since $[a_i,b_i,c_i,d_i]_i$ is an $(n,4;1)$-DM,

$$\{a_i-d_i;\ i \in Z_n\} = \{b_i-d_i;\ i \in Z_n\} = \{c_i-d_i;\ i \in Z_n\} = Z_n.$$

Then, each $A_k(s)$ is a transversal of A_k. And, by (D1)–(D3), $\bigcup_{s\in Z_n} A_k(s) = A_k$, i.e. $A_k(0), A_k(1), \ldots, A_k(n-1)$ are n disjoint transversals of A_k. Define

$$B_s = \bigcup_{k\in Z_n} A_k(s) = \{(a_i-d_i+s+k, b_i-d_i+s+k, c_i-d_i+s);\ i,k \in Z_n\},$$
$$s \in Z_n.$$

By (D3) and (D1), $[a_i - d_i + s, b_i - d_i + s, c_i - d_i + s - k]_i$ is also an $(n, 3; 1)$-DM. Thus, by (D2), $\bigcup_{i,k\in Z_n}((a_i - d_i + s) + k, (b_i - d_i + s) + k, (c_i - d_i + s - k) + k)$ form an OA$(3, n)$. So, each B_s is a Latin square of order n. □

Theorem 2.4. *For any regular integer n and any integer a*, $0 \leqslant a \leqslant n$, *there exist* $\mathrm{LS}_n(n)$ *and* LDILS$(n + a, a)$.

Proof. Let $n = \prod_i q_i$, where q_i are distinct prime powers and $q_i \geqslant 4$. We have the following series of inferences, where L and T denote Lemma and Theorem.

$$\overset{\text{L2.2(1)}}{\Rightarrow} (q_i, q_i; 1)\text{-DM} \overset{\text{L2.2(3)}}{\Rightarrow} (q_i, 4; 1)\text{-DM} \overset{\text{L2.2(2)}}{\Rightarrow} (n, 4; 1)\text{-DM} \overset{\text{T2.3}}{\Rightarrow} \mathrm{LS}_n(n)$$
$$\overset{\text{T2.1}}{\Rightarrow} \mathrm{LDILS}(n + a, a). \qquad \square$$

3. Main theorems

Theorem 3.1. *For any positive integer n, there exists an* LDILS$(n + n - 1, n - 1)$ *if and only if there exists* 2MOLS(n).

Proof. Suppose there exist 2MOLS(n), then there exists an OA$(4, n)$, each row of which is denoted by (i, j, b_{ij}, c_{ij}), where $i, j, b_{ij}, c_{ij} \in Z_n$. Construct $n(2n - 1) \times (2n - 1)$ arrays $A_k = (a^k_{ij})_{i,j\in X}$ over $X = Z_n \cup \{\infty_1, \dots, \infty_{n-1}\}$, $k \in Z_n$, as follows. For any $i, j \in Z_n$, $c_{ij} = s$,

if $s \neq k$ then $a^k_{ij} = \infty_{s-k}$, where $s - k$ is modulo n;

if $s = k$ then $a^k_{ij} = b_{ij}$ and $a^k_{i,\infty_t} = a^k_{\infty_t,j} \equiv b_{ij} + t \pmod n$, $1 \leqslant t \leqslant n - 1$.
It is not difficult to see that each A_k is an ILS$(n + n - 1, n - 1)$ and these A_k are pairwise disjoint.

In order to prove the necessity, let us state an obvious proposition:

$(*)$ *Let T be an* $n \times (n - 1)$ *Latin rectangle over* Z_n, $a_0, a_1, \dots, a_{n-1} \in Z_n$ *and*

$$\bar{T} = \left(\begin{array}{c|c} a_0 & \\ a_1 & \\ \cdot & T \\ \cdot & \\ a_{n-1} & \end{array}\right)$$

if the elements in each row of $\bar{T}$ *are distinct, then* $\bar{T}$ *is a Latin square. Thereby*, $\{a_0, a_1, \dots, a_{n-1}\} = Z_n$. *Similarly, for such case that*

$$\bar{H} = \left(\frac{b_0 \;\; b_1 \quad \cdots \quad b_{n-1}}{H}\right),$$

where H is an $(n - 1) \times n$ *Latin rectangle over* Z_n.

Suppose $Y=\{\infty_1,\ldots,\infty_{n-1}\}$, $Y\cap Z_n=\phi$ and

$$\left\{\begin{pmatrix} S_k & A_k \\ B_k & \phi \end{pmatrix};\ k\in Z_n\right\}$$

be an LDILS$(n+n-1,n-1)$ over $Z_n\cup Y$, where ϕ is the hole on $Y\times Y$. Let $S_k=(s_{ij}^k)_{i,j\in Z_n}$, $k\in Z_n$. Obviously, for each $k\in Z_n$, the entries s_{ij}^k $(i,j\in Z_n)$ belonging to Z_n are distributed just in a set L_k of n cells, one from each row and column of S_k. Denote

$$L_k=\{(i,\sigma_k(i),x_i^k);\ i\in Z_n\}=\{(\sigma_k^{-1}(j),j,y_j^k);\ j\in Z_n\},\quad k\in Z_n,$$

where $x_i^k=s_{i,\sigma_k(i)}^k$, $y_j^k=s_{\sigma_k^{-1}(j),j}^k$ and σ_k is a permutation on Z_n. Below, let us prove four inferences (1)–(4).

(1) For any $k\neq k'\in Z_n$ and $i\in Z_n$, $\sigma_k(i)\neq\sigma_{k'}(i)$.

Otherwise, let $j=\sigma_k(i)=\sigma_{k'}(i)$, s_{ij}^k and $s_{ij}^{k'}$ both are in Z_n. Then, since there are exactly n s_{ij}^k in Z_n for each k, there is a position $(\bar{i},\bar{j})\in Z_n\times Z_n$ such that all $s_{\bar{i}\bar{j}}^k$, $k\in Z_n$, not belong to Z_n. It is impossible, since Y contains only $n-1$ elements.

(2) For each $k\in Z_n$, $\{x_i^k;\ i\in Z_n\}=Z_n$.

In fact, since the entries in each row of (S_k,A_k) are distinct, we can use the proposition $(*)$ to the matrix

$$\bar{A}_k=\left(\begin{array}{c|c} x_0^k & \\ x_1^k & \\ \vdots & A_k \\ x_{n-1}^k & \end{array}\right).$$

(3) For each $i\in Z_n$, $\{x_i^k;\ k\in Z_n\}=Z_n$.

Let $A_k=(a_{ij}^k)_{i\in Z_n,j\in Y}$, $k\in Z_n$. For each $i\in Z_n$, denote $C_i=(a_{ij}^k)_{k\in Z_n,j\in Y}$, a matrix consisted by the ith row of all A_k. Each C_i is an $n\times(n-1)$ Latin rectangle, since all A_k are pairwise disjoint. Use the proposition $(*)$ to the matrix

$$\bar{C}_i=\left(\begin{array}{c|c} x_i^0 & \\ x_i^1 & \\ \vdots & C_i \\ x_i^{n-1} & \end{array}\right).$$

(4) For each $j\in Z_n$, $\{y_j^k;k\in Z_n\}=Z_n$.

Let $B_k=(b_{ij}^k)_{i\in Y,j\in Z_n}$, $k\in Z_n$. For each $j\in Z_n$, denote $D_j=(b_{ij}^k)_{i\in Y,k\in Z_n}$. Use the proposition $(*)$ to the matrix

$$\bar{D}_j=\left(\begin{array}{c} y_j^0\ \ y_j^1\ \ \cdots\ \ y_j^{n-1} \\ \hline D_j \end{array}\right).$$

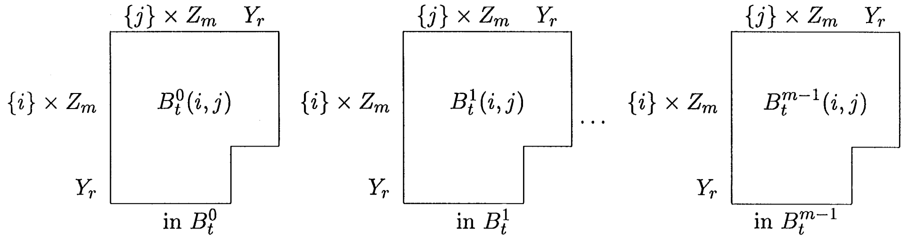
Lastly, regarding triple $(i,\sigma_k(i),x_i^k)$ as the entry x_i^k in the cell $(i,\sigma_k(i))$, the union of all L_k, $k\in Z_n$, form a matrix L of order n. There is no empty cell in L, since all

$(i, \sigma_k(i))$ are distinct by (1). Each row of L is a permutation of Z_n by (3). Also each column of L is a permutation of Z_n by (4). So, L is a Latin square of order n over Z_n. As well, by (2), these L_k $(k \in Z_n)$ are just n disjoint transversals of L. Such L is equivalent to 2MOLS(n). $\square$

Theorem 3.2. *Suppose $n, m, k, h_1, h_2, \ldots, h_k$ are positive integers. If there exist* LS$_k(n)$ *and* LDILS$(m + h_r, h_r)$ *for $1 \leqslant r \leqslant k$, then there exists an* LDILS$(mn + h, h)$, *where $h = \sum_{r=1}^{k} h_r$.*

Proof. Let $\{A_t; t \in Z_n\}$ be an LS$_k(n)$ over Z_n, where each Latin square A_t has k disjoint transversals $A_t(1), A_t(2), \ldots, A_t(k)$ such that $\bigcup_{t \in Z_n} A_t(r)$ form a Latin square of order n for any r, $1 \leqslant r \leqslant k$. Let $X = (\bigcup_{r=1}^{k} Y_r) \cup (Z_n \times Z_m)$, where $Y_r = \{\infty_{r,1}, \infty_{r,2}, \ldots, \infty_{r,h_r}\}$ and $Y_r \cap (Z_n \times Z_m) = \phi$, $1 \leqslant r \leqslant k$. Define mn $(mn + h) \times (mn + h)$ matrices B_t^s over X with a common hole of size h, $t \in Z_n$, $s \in Z_m$, as follows.

(1) If $(i, j, x) \in A_t(r)$, $1 \leqslant r \leqslant k$, and $\{B_t^s(i, j); s \in Z_m\}$ is an LDILS$(m + h_r, h_r)$ over $(\{x\} \times Z_m) \cup Y_r$, then each $B_t^s(i, j)$ is embedded in B_t^s as follows:



Here, each $B_t^s(i, j)$ is a submatrix of B_t^s, with a hole.

(2) If $(i, j, x) \in A_t \setminus \bigcup_{r=1}^{k} A_t(r)$ and $\{B_t^s(i, j); s \in Z_m\}$ is an LDILS$(m + 0, 0)$ over $\{x\} \times Z_m$, then we have still similar embedding, but Y_r is empty.

Lastly, we have $B_t^s = \bigcup_{i,j \in Z_n} B_t^s(i, j)$, $t \in Z_n$ and $s \in Z_m$. It is not difficult to see that $\{B_t^s; t \in Z_n, s \in Z_m\}$ form an LDILS$(mn + h, h)$ over X. $\square$

This theorem gives an important recurrent way to construct LDILS. But, it is a pity that Theorem 3.2 cannot be used for the case '$m = 2$ and h odd', since there exists no LDILS$(2 + 1, 1)$. In order to deal with this case we give another recurrence.

Theorem 3.3. *For positive integers n and m, $n \geqslant m \geqslant 2$, if there exists* LS$_m(n)$ *then there exist* LDILS$(2n + 2t + 1, 2t + 1)$ *for any $0 \leqslant t \leqslant m - 2$.*

Proof. Let $\{A_k; k \in Z_n\}$ be an LS$_m(n)$, where each $A_k = (a_{ij}^k)_{i,j \in Z_n}$ is a Latin square over Z_n and each A_k has m disjoint transversals $A_k(1), A_k(2), \ldots, A_k(m)$ such that all A_k are disjoint and $\bigcup_{k \in Z_n} A_k(s)$ form a Latin square of order n for any s, $1 \leqslant s \leqslant m$. Let $X = (Z_n \times Z_2) \cup \{\infty_1, \ldots, \infty_{2t+1}\}$, where each $\infty_i \notin Z_n \times Z_2$. Below, we will construct $2n$

matrices $B_k=(b^k_{ij})_{i,j\in X}$ and $C_k=(c^k_{ij})_{i,j\in X}$ over X, $k\in Z_n$. The constructing procedure is that for each entry $a^k_{ij}=x$ on A_k (denoted by $(i,j,x)\in A_k$), give a corresponding submatrix in B_k (and C_k).

$(i,j,x)\in A_k(1)$:

in B_k

	$(j,0)$	$(j,1)$	∞_1
$(i,0)$	∞_1	$(x,0)$	$(x,1)$
$(i,1)$	$(x,0)$	$(x,1)$	
∞_1	$(x,1)$		

in C_k

	$(j,0)$	$(j,1)$	∞_1
$(i,0)$	$(x,1)$	∞_1	$(x,0)$
$(i,1)$	∞_1	$(x,0)$	$(x,1)$
∞_1	$(x,0)$	$(x,1)$	

$(i,j,x)\in A_k(2)$:

in B_k

	$(j,0)$	$(j,1)$	∞_1
$(i,0)$	$(x,0)$	$(x,1)$	
$(i,1)$	$(x,1)$	∞_1	$(x,0)$
∞_1		$(x,0)$	

in C_k

	$(j,0)$	$(j,1)$
$(i,0)$	$(x,1)$	$(x,0)$
$(i,1)$	$(x,0)$	$(x,1)$

$(i,j,x)\in A_k(s+2)$: $(1\le s\le t)$

in B_k

	$(j,0)$	$(j,1)$	∞_{2s}	∞_{2s+1}
$(i,0)$	∞_{2s}	∞_{2s+1}	$(x,0)$	$(x,1)$
$(i,1)$	∞_{2s+1}	∞_{2s}	$(x,1)$	$(x,0)$
∞_{2s}	$(x,0)$	$(x,1)$		
∞_{2s+1}	$(x,1)$	$(x,0)$		

in C_k

	$(j,0)$	$(j,1)$	∞_{2s}	∞_{2s+1}
$(i,0)$	∞_{2s+1}	∞_{2s}	$(x,1)$	$(x,0)$
$(i,1)$	∞_{2s}	∞_{2s+1}	$(x,0)$	$(x,1)$
∞_{2s}	$(x,1)$	$(x,0)$		
∞_{2s+1}	$(x,0)$	$(x,1)$		

$(i,j,x)\in A_k\setminus\bigcup_{s=1}^{t+2}A_k(s)$:

in B_k

	$(j,0)$	$(j,1)$
$(i,0)$	$(x,0)$	$(x,1)$
$(i,1)$	$(x,1)$	$(x,0)$

in C_k

	$(j,0)$	$(j,1)$
$(i,0)$	$(x,1)$	$(x,0)$
$(i,1)$	$(x,0)$	$(x,1)$

It is not difficult to verify that the obtained $\{B_k,C_k;k\in Z_n\}$ form an LDILS$(2n+2t+1, 2t+1)$ over X. $\square$

4. Non-regular integer orders

In this section, we will discuss the case for non-regular integer. Obviously, besides $n=1,2,3,6$, there are three types of non-regular integers

(1) $n=2\langle n\rangle$, then $n\equiv 2,10,14,18,22,26,34 \pmod{36}$ and $n\geqslant 10$;
(2) $n=3\langle n\rangle$, then $n\equiv 3,12,15,21,24,33 \pmod{36}$ and $n\geqslant 12$;
(3) $n=6\langle n\rangle$, then $n\equiv 6,30 \pmod{36}$ and $n\geqslant 30$.

Firstly, we list two lemmas, where Lemma 4.1 is trivial and Lemma 4.2 is a deduction of Theorem 3.1 and the famous conclusion: ‘there exist 2MOLS(n) if and only if $n \neq 2$ and 6’.

Lemma 4.1. *For any positive integer n there exist* LDILS$(n+0,0)$ *and* LDILS$(n+n,n)$.

Lemma 4.2. *For any positive integer n there exists* LDILS$(n+n-1,n-1)$ *if and only if $n \neq 2,6$.*

Below, we will solve the existence of LDILS$(n+a,a)$ for $n=2,3,6$ and three types of non-regular integer n, respectively.

Theorem 4.3. *For $n=2,3$ or 6, $0 \leqslant a \leqslant n$, there exists* LDILS$(n+a,a)$ *except for $(n,a)=(2,1)$ or $(6,5)$.*

Proof. By Lemmas 4.1, 4.2 and a few of examples in [8], all conclusions can be obtained except LDILS$(6+3,3)$, which is listed as follows. The large set is over the set $\{A,B,C\} \cup Z_6$.

C	B	0	1	4	A	5	2	3
A	3	1	0	B	C	2	5	4
4	5	2	A	C	B	3	0	1
5	4	B	C	A	1	0	3	2
0	A	C	B	2	3	1	4	5
B	C	A	5	3	2	4	1	0
1	0	3	2	5	4			
3	2	5	4	1	0			
2	1	4	3	0	5			

4	5	2	A	C	B	3	0	1
5	4	B	C	A	1	0	3	2
0	A	C	B	2	3	1	4	5
B	C	A	5	3	2	4	1	0
C	B	0	1	4	A	5	2	3
A	3	1	0	B	C	2	5	4
2	1	4	3	0	5			
1	0	3	2	5	4			
3	2	5	4	1	0			

0	A	C	B	2	3	1	4	5
B	C	A	5	3	2	4	1	0
C	B	0	1	4	A	5	2	3
A	3	1	0	B	C	2	5	4
4	5	2	A	C	B	3	0	1
5	4	B	C	A	1	0	3	2
3	2	5	4	1	0			
2	1	4	3	0	5			
1	0	3	2	5	4			

3	2	A	C	B	5	0	1	4
2	B	C	A	5	4	1	0	3
A	C	B	3	1	0	4	5	2
C	A	3	2	0	B	5	4	1
B	1	5	4	A	C	2	3	0
1	0	4	B	C	A	3	2	5
4	5	0	1	2	3			
0	3	2	5	4	1			
5	4	1	0	3	2			

A	C	B	3	1	0	4	5	2
C	A	3	2	0	B	5	4	1
B	1	5	4	A	C	2	3	0
1	0	4	B	C	A	3	2	5
3	2	A	C	B	5	0	1	4
2	B	C	A	5	4	1	0	3
5	4	1	0	3	2			
4	5	0	1	2	3			
0	3	2	5	4	1			

B	1	5	4	A	C	2	3	0
1	0	4	B	C	A	3	2	5
3	2	A	C	B	5	0	1	4
2	B	C	A	5	4	1	0	3
A	C	B	3	1	0	4	5	2
C	A	3	2	0	B	5	4	1
0	3	2	5	4	1			
5	4	1	0	3	2			
4	5	0	1	2	3			

□

Theorem 4.4. *For positive integer n and integer a*, $n=2\langle n\rangle$ *and* $0\leqslant a\leqslant n$, *there exists* LDILS$(n+a,a)$.

Proof. Let $r=\langle n\rangle$, then r is regular, and there exists an $\mathrm{LS}_r(r)$ by Theorem 2.4.

(1) If $a=2t$, $0\leqslant t\leqslant r$, using Theorem 3.2 and taking $m=2$, $h_1=h_2=\cdots=h_t=2$, $h_{t+1}=\cdots=h_r=0$, we obtain LDILS$(2r+2t,2t)$ since LDILS$(2+2,2)$ exists.
(2) If $a=2t+1$ and $0\leqslant t\leqslant r-2$, there exists LDILS$(2r+2t+1,2t+1)$ by Theorem 3.3.
(3) If $a=2r-1$, there exists LDILS$(2r+2r-1,2r-1)$ by Lemma 4.2 since $2r\neq 2,6$. □

Theorem 4.5. *For positive integer n and integer a*, $n=3\langle n\rangle$ *and* $0\leqslant a\leqslant n$, *there exists* LDILS$(n+a,a)$.

Proof. Let $r=\langle n\rangle$, then r is regular, and there exists an $\mathrm{LS}_r(r)$ by Theorem 2.4. Furthermore, using Theorem 3.2 (taking $m=3$) and Theorem 4.3, LDILS$(3r+a,a)$ can be obtained, where

if $a=3t$, $0\leqslant t\leqslant r$, then take $h_1=\cdots=h_t=3$, $h_{t+1}=\cdots=h_r=0$;
if $a=3t+1$, $0\leqslant t\leqslant r-1$, then take $h_1=\cdots=h_t=3$, $h_{t+1}=1$, $h_{t+2}=\cdots=h_r=0$;
if $a=3t+2$, $0\leqslant t\leqslant r-1$, then take $h_1=\cdots=h_t=3$, $h_{t+1}=2$, $h_{t+2}=\cdots=h_r=0$. □

Theorem 4.6. *For positive integer n and integer a*, $n=6\langle n\rangle$ *and* $0\leqslant a\leqslant n$, *there exist* LDILS$(n+a,a)$.

Proof. Let $r=\langle n\rangle$, then there exists an $\mathrm{LS}_r(r)$ by Theorem 2.4. Still using Theorem 3.2 (taking $m=6$) and Theorem 4.3, LDILS$(6r+6t+s,\ 6t+s)$ can be obtained, where $0\leqslant s\leqslant 5$, $h_1=\cdots=h_t=6$ and

$$h_{t+1}=s,\ h_{t+2}=\cdots=h_r=0 \quad \text{for } 0\leqslant s\leqslant 4;$$
$$h_{t+1}=2,\ h_{t+2}=3,\ h_{t+3}=\cdots=h_r=0 \quad \text{for } s=5.$$

For the last case, i.e. $a=6t+5$, Theorem 3.2 can be used only for $t\leqslant r-2$. But, we can use Theorem 4.2 for $t=r-1$. □

5. Conclusion

Now, summarizing Theorems 2.4 and 4.3–4.6, we can obtain Theorem 1.1. Then, the existence problem for LDILS has been completely solved.

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References

[1] S. Blake-Wilson, K. Phelps, Constant weight codes and group divisible design, Designs, Codes and Cryptography 16 (1999) 11–28.
[2] K. Chen, G. Ge, L. Zhu, Generalized Steiner triple systems with group size five, J. Combin. Designs 7 (1999) 441–452.
[3] C.J. Colbourn, J.H. Dinitz (Eds.), The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, FL, 1996.
[4] T. Etzion, Optimal constant weight codes over Z_k and generalized designs, Discrete Math. 169 (1997) 55–82.
[5] K. Phelps, C. Yin, Generalized Steiner systems with block three and group size $g \equiv 3 \pmod 6$, J. Combin. Designs 5 (1997) 417–432.
[6] K. Phelps, C. Yin, Generalized Steiner systems with block three and group size four, Ars Combin. 53 (1999) 133–146.
[7] D. Wu, L. Zhu, G. Ge, Generalized Steiner triple systems with group size $g = 7, 8$, Ars Combin., to appear.
[8] D. Wu, L. Zhu, Large sets of disjoint incomplete Latin squares, Bull. Inst. Combin. Appl., to appear.