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Total tightness implies Nash-solvability for three-person game forms

Endre Boros [a], Ondřej Čepek [b,c,*], Vladimir Gurvich [a]

[a] *RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway, NJ 08854, USA*

[b] *Department of Theoretical Informatics, Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic*

[c] *Institute of Finance and Administration - VŠFS, Estonská 500, 100 00, Praha 10, Czech Republic*

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ABSTRACT

It was recently shown that every totally tight two-person game form is acyclic, dominance-solvable, and hence, Nash-solvable too. In this paper, we exhibit an example showing that the first two implications fail for the three-person ($n = 3$) game forms. Yet, we show that the last one (total tightness implies Nash-solvability) still holds for $n = 3$ leaving the case $n > 3$ open.

1. Introduction

An n-person *game form* is a mapping $g : X = X_1 \times \cdots \times X_n \to A$, where X_i, $1 \le i \le n$, is the set of *strategies* of player i, and A is the set of *outcomes*. We will restrict our attention to finite game forms only, that is, we will assume that the sets X_i, $1 \le i \le n$, and A are all finite. Moreover, let $u_i : A \to R$, $1 \le i \le n$, be a real valued *utility function* (sometimes called *payoff function*) of player i, where $u_i(a)$ is interpreted as the profit of player i if the outcome $a \in A$ is realized. The vector $u = (u_1, \ldots, u_n)$ is called a *utility profile* and the pair (g, u) then defines a *normal form n-person game*.

In fact, in this paper, we are not interested in the numerical values that the utility functions assign to the individual outcomes. The only important information carried by a utility function, for the concepts studied in this paper, is the relative order of the outcomes. For outcomes $a, b \in A$, we say that a player i *(strictly) prefers* b to a and write $a \le_i b$ (respectively, $a < b$) when $u_i(a) \le u_i(b)$ (respectively, $u_i(a) < u_i(b)$). If both $a \le_i b$ and $b \le_i a$ hold (that is, $u_i(a) = u_i(b)$), we write $a =_i b$. Of course, by definition, both relations $<_i$ and $\le_i$ are transitive; moreover, $\le_i$ is complete, that is, any two outcomes are comparable.

Similar to the case of utilities, we use the preference orders of individual players to define a *preference profile* as a vector $p = (\le_1, \ldots, \le_n)$. Then, a game is defined by a pair (g, p).

An n-person game form g can be thought of as an n-dimensional matrix with entries from the set A. In accordance with this terminology, a vector $x = (x_1, \ldots, x_n) \in X$ is called a *strategy profile* (or just a *profile*, for short). A profile can be thought of as a position in the matrix, and $g(x)$ is then called an *entry* of the game form g, that is, an entry is an outcome for a given profile. A set of all profiles x with identical $x_j \in X_j$ for all $j \ne i$ (that is, only the i-th components may differ) will be referred to as a *line* in direction i. A set of all profiles x with identical $x_i \in X_i$ will be referred to as a *hyperplane* (perpendicular to the direction i). Using this matrix terminology, we will also say that profile x *dominates* profile y in *direction* i if $g(y) <_i g(x)$

* Corresponding author at: Department of Theoretical Informatics, Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic.
E-mail addresses: boros@rutcor.rutgers.edu (E. Boros), ondrej.cepek@mff.cuni.cz (O. Čepek), gurvich@rutcor.rutgers.edu (V. Gurvich).

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and $\forall j \neq i : y_j = x_j$ (both profiles are on the same line in direction i). Finally, we say that profile x is non-dominated in direction i if $g(y) \le_i g(x)$ holds for all y such that $\forall j \neq i : y_j = x_j$, i.e., if it is non-dominated on the line in direction i that "goes through" x.

Given a game (g, p), a profile $x = (x_1, \dots, x_n)$, is called a *Nash equilibrium* (NE, for short) if x is non-dominated in all directions, that is, if

$$g(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n) \le_i g(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \quad \text{for all } i,\ 1 \le i \le n,\ \text{and } y_i \in X_i.$$

This means than no player can achieve a strictly better outcome (according to his preferences) by changing his strategy if all other players keep their strategies unchanged.

A game form g is called *Nash solvable* if for every preference profile p the game (g, p) has an NE.

Quite obviously, given a game, it is not difficult to test in polynomial time whether it has an NE simply by inspecting all strategy profiles (checking if a given profile is an NE can be done in linear time with respect to $|X_1| + \cdots + |X_n|$ and the number of entries to check is $|X_1| \times \cdots \times |X_n|$). On the other hand, given a game form, it seems not to be easy to verify its Nash solvability. The number of all possible preference profiles to consider is $(|A|!)^n$ which gets computationally out of hand already for moderate values of $|A|$ and n. Unfortunately, no significantly faster algorithm than the brute force one is known for testing Nash solvability in the general n player setting. The above observation means that it is quite interesting to derive sufficient conditions for Nash solvability of game forms, especially those conditions that are easy to verify.

Let us for a while concentrate on the two-person case ($n = 2$), which is the most studied one. In [2], the following six classes of game forms were considered: tight (T), totally tight (TT), Nash-solvable (NS), dominance-solvable (DS), acyclic (AC), and assignable (AS). A two-person game form $g : X_1 \times X_2 \to A$ is *totally tight* (TT) if every 2×2 subform of g (which is a two-dimensional matrix in this case) contains a constant line (row or column). More precisely, let us call $g' : X_1' \times X_2' \to A$ to be a 2×2 restriction of g if $X_1' = \{x_1, x_1'\} \subseteq X_1$ and $X_2' = \{x_2, x_2'\} \subseteq X_2$ are 2-element subsets of X_1 and X_2. Then g is TT if and only if for every 2×2 restriction g' of g we have

$$g'(x_1, x_2) = g'(x_1, x_2'), \quad \text{or} \quad g'(x_1, x_2) = g'(x_1', x_2), \quad \text{or} \quad g'(x_1', x_2') = g'(x_1', x_2), \quad \text{or} \quad g'(x_1', x_2') = g'(x_1, x_2').$$

We defer the definitions of DS and AC to Section 5, and for the definitions of T and AS we refer the reader to [2], where more details on all six classes can be found. It was shown that for the two-person case the following implications hold

$$AS \Leftarrow TT \Leftrightarrow AC \Rightarrow DS \Rightarrow NS \Leftrightarrow T. \tag{1}$$

In fact, the last three implications, $DS \Rightarrow NS \Leftrightarrow T$, were obtained long ago; see [12,13,5,6,8], respectively. Furthermore, $AC \Rightarrow TT$ is obvious; see [10] or [2]. The remaining three implications $TT \Rightarrow AS$, $TT \Rightarrow DS$, and $TT \Rightarrow AC$ constitute the main results of [2] (note also that $TT \Rightarrow DS$ together with the trivial relationship $AC \Rightarrow TT$ implies $AC \Rightarrow DS$ as stated above in (1)). The first two of them were conjectured by Kukushkin ([10] and private communications); the last one was proven in [10] and independently in [2]. All three easily result from the recursive characterization of the TT two-person game forms obtained in [2]. Let us also mention that $TT \Rightarrow NS$ can be derived from Shapley's theorem [14], although it was not explicitly claimed there, as [14] deals with the zero-sum games and their saddle points, rather than with game forms and NS.

Out of the four sufficient conditions for NS in the two-player case (that is, TT, AC, DS, and T, the last one being not only sufficient but also necessary), only the property of TT is known to be testable in polynomial time with respect to the size of the two-dimensional matrix that defines the input game form. In fact, this task is trivial, since every 2×2 subform can be inspected in constant time, and there are only $O(|X_1|^2 \cdot |X_2|^2)$ different 2×2 subforms. (In contrast, verifying directly the definitions of DS or AC requires exponential time. Of course one can verify the AC property efficiently using the $TT \Leftrightarrow AC$ relation).

Testing T is more interesting. It was shown in [6] that this task is equivalent to testing whether certain two monotone Boolean functions associated with the game form constitute a dual pair. Thus, the complexity of testing T is the same as the complexity of dualization for monotone Boolean functions, which can be done (see [4]) in quasi-polynomial (that is, $N^{\log N}$, where N is the size of the input) time.

It should be also mentioned that, except for those given in (1), no other implications hold between the considered six classes of game forms; the corresponding examples, for all invalid implications, were given in [2]. Thus, the two-person case is fully studied with respect to the considered six classes, while much less is known for general, n-person, game forms.

Moulin [13] proved that for every n, $DS \Rightarrow NS$ and $DS \Rightarrow T$. Yet, tightness and Nash-solvability are no longer related, that is, both implications in $NS \Leftrightarrow T$ fail already for $n = 3$. In [6], $\Leftarrow$ was disproved, while $\Rightarrow$ mistakenly claimed; then, in [8] it was shown that both implications fail for $n \ge 3$.

Remark 1. For n person game forms, Nash-solvability results from tightness and the following extra assumption. An n-person game form $g : X_1 \times \cdots X_n \to A$ is called *rectangular* if for each outcome $a \in A$, its pre-image $g^{-1}(a)$ is a *box* of X, that is, $g^{-1}(a) = X_1' \times \cdots X_n'$, where $X_i' \subseteq X_i$ for $i = 1, \dots, n$. It was mentioned in Remark 3 of [6] that tightness is sufficient for Nash-solvability of a rectangular n-person game form g and, moreover, g is the normal form of a positional n-person game form modeled by a tree if and only if g is tight and rectangular. The last statement was proven in [7]; see also [9].

Remark 2. It is worth mentioning that game forms are a general concept. They can be viewed as discrete functions of n discrete arguments, which is a natural extension of the concept of Boolean functions. Interpretations of game forms in terms of games (or voting) theory are important but the same concept appears in many other contexts. In fact, among the six properties of game forms considered earlier in this section five are related to games, while assignability is not; it can be interpreted as a separability property; see [1].

The fact that T is no longer a sufficient condition for NS for $n > 2$ (if no additional assumptions are made about g, as in Remark 1) also dashes any hopes to generalize the dualization procedure in some clever way to obtain an effectively testable sufficient conditions for NS in the case of more than two players. Some other relations are known for $n > 2$. Of course, all the negative results (non-implications) carry over from the $n = 2$ case. On the positive side, acyclicity implies Nash-solvability ($AC \Rightarrow NS$) for any n, which trivially follows from the definitions of AC and NS (see Section 5 for the definition of AC).

The TT property can be generalized from the two-person case to n-person game form for $n > 2$. We will recall this definition introduced in [1] in the next section. It is known that total tightness implies assignability ($TT \Rightarrow AS$) for $n = 3$ (see [3] for a sketch of the proof). On the other hand, in this case total tightness implies neither acyclicity nor dominance-solvability. Both $TT \Rightarrow AC$ and $TT \Rightarrow DS$ fail already for $n = 3$, as shown by a single example given in Section 5. Moreover, the same example disproves a recent conjecture of [10] stating that a game form is acyclic if every subform of it is Nash solvable.[1] As the main result of this paper, we prove (in Section 4) the following theorem which provides a sufficient condition for Nash-solvability testable in polynomial time.

Theorem 3. *Total tightness implies Nash-solvability for three-person game forms.*

Whether $TT \Rightarrow NS$ for $n > 3$ remains an open question.

2. Total tightness for more than two players

Let $I = \{1, \ldots, n\}$ be a finite set of players whose subsets $K \subseteq I$ are called *coalitions*. Given an n-person game form $g : X \to A$ and a partition $I = K \cup \overline{K}$ of the players into two complementary coalitions, the two-person game form $g_K : X_K \times X_{\overline{K}} \to A$ is defined in the following way. The strategies of the first player are the elements of the Cartesian product $X_K = \prod_{i \in K} X_i$ and the strategies of the second player are the elements of the Cartesian product $X_{\overline{K}} = \prod_{i \in \overline{K}} X_i$. Naturally, for $x_K \in X_K$ and $x_{\overline{K}} \in X_{\overline{K}}$ we define $g_K(x_K, x_{\overline{K}}) = g(x)$ where the strategy profile x originates from x_K and $x_{\overline{K}}$ by concatenating them and reordering the coordinates according to the $X_1, \ldots, X_n$ order.

An n-person game form g is called *totally tight* if g_K is TT for all $K \subseteq I$. Furthermore, g is called *weakly totally tight (WTT)* if g_K is TT for all $K \subseteq I$ such that $|K| = 1$. Note that for a WTT game form g the entries in any 2×2 submatrix of $g_{\{i\}}$ can be geometrically thought of as the four intersections of two arbitrary distinct lines in direction i with two arbitrary distinct $(n-1)$-dimensional hyperplanes perpendicular to direction i. Let us also remark that although the two concepts (TT and WTT) coincide whenever $n \le 3$ (and hence there is no difference for the main result of the paper which deals with $n = 3$), it makes sense to differentiate these two concepts in the next section which assumes only WTT while its results are valid for all n.

3. Structural properties of minimal game forms

In this section, we shall consider the general n player case and assume that WTT does not imply NS. Thus we shall assume that there exist game forms which are WTT but not NS, and in particular we will be interested in those game forms which are minimal with this property. To this end let $g : X \to A$ be a game form which is assumed to be WTT but not NS. Introduce $d_i = |X_i|$, $1 \le i \le n$, and think of g as a $d_1 \times d_2 \times \cdots \times d_n$ matrix. Moreover, let us assume that g is minimal with the assumed property, i.e. that every proper submatrix of g is NS (such a proper submatrix is of course WTT as this property is hereditary). Finally, since g is assumed not to be NS, there exists at least one preference profile p, such that the game (g, p) has no NE. Let us for the rest of this section fix one such p.

We shall derive in this section several relatively strong properties which are valid for g. In the next section, we will then show that such properties cannot be fulfilled if $n = 3$, proving that in this case no such game form exists, and hence showing $WTT \Rightarrow NS$ for three players (which is in this case the same as $TT \Rightarrow NS$).

Lemma 4. *Game form g contains no constant hyperplane of dimension $n - 1$.*

Proof. Let us by contradiction assume that g contains a constant hyperplane H of dimension $n - 1$ and let g' originate from g by deleting H. By minimality of g we get that g' is NS and thus it contains an NE x (with respect to p). Now there are two cases to consider:

1. either x is an NE of g
2. or x is dominated by some y in H which implies that such a y is an NE of g.

In any case g has an NE with respect to p contradicting our assumptions. □

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[1] This observation was pointed to us by an anonymous referee.

Lemma 5. *Let us consider an arbitrary player i, the corresponding direction i in game form g, and the d_i hyperplanes $H_1, H_2, \ldots, H_{d_i}$ of dimension $n-1$ perpendicular to this direction. Then $d_i \le \prod_{j\neq i} d_j$ holds, and there are d_i distinct lines $\ell_1, \ell_2, \ldots, \ell_{d_i}$ in direction i and d_i pairwise different outcomes $a_1, a_2, \ldots, a_{d_i}$ such that*

(a) *for every j, we have $a_j = g(y^j)$ for the strategy profile y^j at the intersection of H_j with ℓ_j, and*
(b) *a_j is the unique maximum on ℓ_j with respect to the preferences of player i in preference profile p, and*
(c) *for every $j \neq k$ either $a_j <_i a_k$ or $a_j <_i a_k$.*

Proof. Let g^j originate from g by deleting H_j. By minimality of g we get that g^j is *NS* and thus it contains an NE x^j with respect to p. The only way to prevent the strategy profile x^j from being an NE of g is that the line in direction i containing x^j (let us denote this line by ℓ_j) intersects H_j in some y^j which strictly dominates x^j with respect to the preferences of player i in p. Since y^j strictly dominates x^j and x^j is non-dominated on ℓ_j in g^j, we get that $a_j = g(y^j)$ is a unique maximum on ℓ_j in g with respect to the preferences of player i. This finishes a proof of (a) and (b).

Let us consider lines ℓ_j and ℓ_k for $j \neq k$. Since ℓ_j attains its unique maximum in H_j and ℓ_k attains its unique maximum in H_k, these two lines cannot be the same. Thus the lines $\ell_1, \ell_2, \ldots, \ell_{d_i}$ are pairwise distinct, which proves that $d_i \le \prod_{j\neq i} d_j$ because the right hand side is equal to the number of all lines in direction i.

To prove (c) let us by contradiction assume that $a_j =_i a_k$ holds for some $j \neq k$. Let us consider the four entries in the intersections of ℓ_j and ℓ_k with H_j and H_k. Let us denote by c the outcome in the intersection of ℓ_j with H_k and by c' the outcome in the intersection of ℓ_k with H_j. Since a_j is a unique maximum on ℓ_j we get $c <_i a_j =_i a_k$ and since a_k is a unique maximum on ℓ_k we get $c' <_i a_k =_i a_j$. However, this implies $c \notin \{a_j, a_k\}$ and $c' \notin \{a_j, a_k\}$ and so the 2×2 matrix defined by these four entries (which is a submatrix of the game form $g_{\{i\}}$ as defined in Section 2) is not *TT* contradicting the assumption that g is *WTT*. Hence either $a_j <_i a_k$ or $a_k <_i a_j$ must hold which of course also implies that a_j and a_k are different outcomes. □

Since the unique maxima are pairwise different they can be strictly linearly ordered with respect to the preferences of player i. Henceforth, we assume each player's strategy set to be ordered accordingly and numbered $1, \ldots, d_i$ (which can be geometrically thought of as having the hyperplanes perpendicular to the direction of player i ordered in such a way that $a_1 <_i a_2 <_i \cdots <_i a_{d_i}$ holds for every player i). Now we are ready to prove the main theorem of this subsection which quite strongly specifies how the game form g must look like.

Theorem 6. *Let us consider player i, the corresponding direction in game form g, and d_i hyperplanes $H_1, H_2, \ldots, H_{d_i}$ of dimension $n-1$ perpendicular to direction i. Then there are d_i distinct lines $\ell_1, \ell_2, \ldots, \ell_{d_i}$ in direction i and d_i pairwise different outcomes $a_1 <_i a_2 <_i \cdots <_i a_{d_i}$ such that*

(a) *for every j, $1 \le j \le d_i$, outcome a_j is in the j-th entry on ℓ_j, a_j is the unique maximum on ℓ_j with respect to the preferences of player i, and*
(b) *every line ℓ in direction i contains outcomes $(a_1, a_2, \ldots, a_j, b, b, \ldots, b)$ in this order in hyperplanes $H_1, \ldots, H_{d_i}$ for some outcome $b = b^\ell$ dependent on ℓ and some index j, $0 \le j \le d_i$.*

In particular, for some lines ℓ we may have $j = 0$, that is, ℓ is a constant line $(b, b, \ldots, b)$. Furthermore, the special lines $\ell_1, \ell_2, \ldots, \ell_{d_i}$ have the property that ℓ_j, $1 \le j \le d_i$, is a line $(a_1, a_2, \ldots, a_j, b, b, \ldots, b)$ for some outcome $b <_i a_j$.

Proof. The fact that there exist d_i distinct lines $\ell_1, \ell_2, \ldots, \ell_{d_i}$ and d_i distinct values $a_1 <_i a_2 <_i \cdots <_i a_{d_i}$ such that a_j is the unique maximum on ℓ_j which lies in H_j (and thus is the j-th entry on ℓ_j) follows from Lemma 5 implying (a).

Now let us show that ℓ_j contains outcomes $(a_1, a_2, \ldots, a_j)$ in this order in hyperplanes $H_1, H_2, \ldots, H_j$. Let us fix j and pick $1 \le k < j$ arbitrarily (if such a k exists, the statement is trivial for $j = 1$). Now consider the four entries in the intersections of ℓ_j and ℓ_k with H_k and H_j. Let us denote by c the outcome in the intersection of ℓ_j with H_k and by c' the outcome in the intersection of ℓ_k with H_j. Since a_k is a unique maximum on ℓ_k and $a_k <_i a_j$, we get $c' \notin \{a_k, a_j\}$. Since a_j is a unique maximum on ℓ_j we get $c \neq a_j$. However, now to make the 2×2 matrix defined by these four entries *TT*, we must have $c = a_k$.

To prove (b) let us consider a line ℓ containing outcomes $(a_1, \ldots, a_j, b, \ldots)$, where the outcome b in position $j+1$ for some $0 \le j \le d_i - 2$ is the first one to differ from a_{j+1} (the statement is trivial if $j \ge d_i - 1$). We shall prove that every entry in ℓ with the i coordinate greater than $j+1$ is occupied by b. To this end let us fix $k > j+1$ arbitrarily and assume by contradiction that position k in ℓ is occupied by some $c \neq b$. Now consider the four entries in the intersections of ℓ and ℓ_k with H_{j+1} and H_k. These entries contain outcomes b and c in ℓ and a_{j+1} and a_k in ℓ_k. We have $b \neq c$, $b \neq a_{j+1}$, and $a_{j+1} \neq a_k$, and so $c = a_k$ must hold to make the 2×2 matrix defined by these four entries *TT*. On the other hand, replacing the role of ℓ_k by ℓ_{j+1} (i.e. considering the intersections of ℓ and ℓ_{j+1} with H_{j+1} and H_k) we get four outcomes b and c in ℓ and a_{j+1} and c' in ℓ_{j+1}, where $c' <_i a_{j+1}$ because a_{j+1} is the unique maximum on line ℓ_{j+1}. But now $b \neq c$, $b \neq a_{j+1}$, and $a_{j+1} \neq c'$, and so $c = c'$ must hold to make the 2×2 matrix defined by these four entries *TT*. This gives the desired contradiction because we have $c' <_i a_{j+1} <_i a_k$ and so $c = a_k$ and $c = c'$ cannot hold simultaneously. Note also that $b \neq a_{j+1}$ implies $\ell \notin \{\ell_{j+1}, \ell_k\}$ and so the proof works also for $\ell = \ell_j$. □

Theorem 6 immediately implies a simple corollary.

Corollary 7. *For every j, $1 \le j \le d_i$, the hyperplane H_j contains at least one outcome a_j which is the unique maximum on its corresponding line perpendicular to H_j.*

Theorem 6 works only with the lines in the direction which corresponds to player i but since i was picked arbitrarily the same statement is true for every direction. Now we are ready to disprove an existence of game form g for the case of three players.

4. Proof of Theorem 3

Let us consider the three player case with players A, B, C. Let us slightly change the notation. Let the minimal totally tight and not Nash-solvable game form g have dimensions $d_A \times d_B \times d_C$ (we may assume dimensions at least $2 \times 2 \times 2$ since reducing any dimension to 1 results in a two-person game form for which $TT \Rightarrow NS$ is known to hold, and hence no counterexample exists); let the hyperplanes perpendicular to the direction of player A be $H_1^A, \ldots, H_{d_A}^A$ (similarly $H_1^B, \ldots, H_{d_B}^B$ for player B and $H_1^C, \ldots, H_{d_C}^C$ for player C); let the preferences of player A among the outcomes be ordered by relation $<_A$ (similarly $<_B$ for player B and $<_C$ for player C); let the unique maxima on the perpendicular lines in hyperplanes $H_1^A, \ldots, H_{d_A}^A$ guaranteed by Theorem 6 be $a_1 <_A a_2 <_A \cdots <_A a_{d_A}$ ($b_1 <_B b_2 <_B \cdots <_B b_{d_B}$ for player B and $c_1 <_C c_2 <_C \cdots <_C c_{d_C}$ for player C). Let us also adopt the following terminology: if a hyperplane H_j^A contains an entry $e \neq a_j$ then by Theorem 6 all entries in the intersections of line ℓ perpendicular to H_j^A with the hyperplanes $H_{j+1}^A, \ldots, H_{d_A}^A$ must also be e, and we say that entry *eA-propagates* (similarly *B-propagates* for $e \neq b_j$ in H_j^B and *C-propagates* for $e \neq c_j$ in H_j^C).

In order to be able to refer to particular entries in g, we adopt a vector notation, where the strategy profile (x, y, z) is the intersection of H_x^A, H_y^B, and H_z^C. This notation can be extended to lines and hyperplanes. The line in the direction of player A denoted by $(*, y, z)$ is the intersection of H_y^B, and H_z^C (lines in directions of players B and C are denoted similarly), and hyperplane H_x^A can be denoted as $(x, *, *)$ (similarly for H_y^B, and H_z^C).

Now we shall go through a rather tedious case and subcase analysis, where in every subcase we shall arrive at conclusions which contradict the existence of the game form g. The contradiction always rests in deriving one of the following three facts:

- g contains a Nash equilibrium (NE for short) contradicting the choice of g, or
- g contains a constant hyperplane (CH for short) contradicting Lemma 4, or
- hyperplane H_j^A for some $1 \leq j \leq d_A$ contains no outcome a_j which is a unique maximum on its line in direction A, contradicting Corollary 7.

The first branching is done based on the mutual relations of outcomes a_1, b_1, c_1. Note that the arguments used in the first two cases (all three outcomes identical and all three outcomes different) can be easily generalized to n player game forms for arbitrary n, while the treatment of the last case (two outcomes identical and one outcome different) relies heavily on the three-dimensionality of the game form g.

Case I (*all identical*): $a_1 = b_1 = c_1$

In this case, we have just two possibilities. Either all three lines $(1, 1, *)$, $(1, *, 1)$, $(*, 1, 1)$ consist only of outcome $a_1 = b_1 = c_1$ in which case $(1, 1, 1)$ is an NE (which is a contradiction), or there is an outcome e different from $a_1 = b_1 = c_1$ on one of these lines. Since this case is symmetric, we may without loss of generality assume that e is in position $(j, 1, 1)$ for some index j. However, using Theorem 6 the line $(j, *, 1)$ in the direction of player B must be a constant e line (eB-propagates since it differs from b_1), which in turn implies that for every k the line $(j, k, *)$ in the direction of player C must be a constant e line (eC-propagates since it differs from c_1). This means that $(j, *, *)$ is a CH which is again a contradiction.

Case II (*all different*): $a_1 \neq b_1 \neq c_1 \neq a_1$

Consider the outcome e in position $(1, 1, 1)$. Clearly, e must be different from (at least) two outcomes from the set $\{a_1, b_1, c_1\}$. Due to symmetry, we may without loss of generality assume that $e \notin \{b_1, c_1\}$. By an identical argument as in the previous case (where we set $j = 1$) we get that $(1, *, *)$ is a CH.

Case III (*one different*): $a_1 \neq b_1 = c_1$

Consider the outcome e in position $(1, 1, 1)$. If $e \neq b_1 = c_1$ then the argument of Case II can be repeated and $(1, *, *)$ is a CH. Thus let us in the rest of this case assume that the outcome in position $(1, 1, 1)$ is $b_1 = c_1$. This also implies that this entry A-propagates and so $(*, 1, 1)$ is a constant b_1 line.

Now consider the outcome f in position $(1, 2, 1)$. If $f \notin \{a_1, c_1\}$ then f both A-propagates and C-propagates and so $(*, 2, *)$ is CH and if $f = c_1 = b_1$ then since $b_1 \neq b_2$ we get that b_1B-propagates and so $(1, *, 1)$ is a constant b_1 line. However, $b_1 \neq a_1$ now implies that each such b_1A-propagates and thus $(*, *, 1)$ is a CH. Therefore, we may in the rest of this case assume $f = a_1$ and moreover since this entry C-propagates we get that $(1, 2, *)$ is a constant a_1 line. By symmetry, also $(1, *, 2)$ is a constant a_1 line.

Furthermore, consider the outcome h in position $(1, 3, 1)$. If $h \neq b_1 = c_1$ then it C-propagates and hence $h = a_1$ must hold, since we already know that the entry $(1, 3, 2)$ is a_1. However, now either a_1 in position $(1, 2, 1)$ or a_1 in position $(1, 3, 1)$ B-propagates as a_1 cannot be simultaneously equal to both b_2 and b_3. If $h = b_1 = c_1$ then this entry B-propagates since $b_1 \neq b_3$. If we summarize these observations we get that the $(1, *, 1)$ line is either $\ell_a = (b_1, a_1, a_1, \ldots, a_1)$ or $\ell_b = (b_1, a_1, b_1, \ldots, b_1)$, and the same is true by symmetry for the $(1, 1, *)$ line.

Now we will branch again, this time based on the relation of outcome a_1 with respect to outcomes b_2, c_2.

Case A (*equal to both*): $a_1 = b_2 = c_2$

Note that in this case $b_1 <_B b_2 = a_1$ and also $c_1 <_C c_2 = a_1$ and hence a_1 dominates $b_1 = c_1$ in both directions in the hyperplane $(1, *, *)$. Let us distinguish two cases.

1. If the $(1, *, 1)$ line is $\ell_a = (b_1, a_1, a_1, \dots, a_1)$ (or the $(1, 1, *)$ line is $\ell_a = (b_1, a_1, a_1, \dots, a_1)$) then each such a_1C-propagates (or each such a_1B-propagates). In either case, it follows that the $(1, *, *) = H_1^A$ hyperplane contains only a_1 and b_1 entries and hence the unique a_1 maximum in H_1^A guaranteed by Corollary 7 is an NE.
2. If both $(1, *, 1)$ and $(1, 1, *)$ lines are $\ell_b = (b_1, a_1, b_1, \dots, b_1)$ then H_1^A may contain also other entries than a_1 and b_1 and we have to proceed differently. Consider the entry a_1 in position $(1, 2, 2)$. Since both $(1, 2, *)$ and $(1, *, 2)$ are constant a_1 lines, the $(1, 2, 2)$ entry is an NE unless it is dominated by some e (such that $a_1 <_A e$) in the direction of player A. Note that such e must be in position $(2, 2, 2)$ because $a_1 <_A a_2$ and so every non-dominating entry in position $(2, 2, 2)$ would A-propagate leaving the entry a_1 in position $(1, 2, 2)$ non-dominated. However, $a_1 <_A e$ implies $e \neq a_1 = b_2 = c_2$ and so e both B-propagates and C propagates. Thus H_2^A must contain e everywhere except in the $(2, 1, *)$ and $(2, *, 1)$ lines, which in turn implies that the only candidates for the unique a_1 maximum in H_1^A are the positions $(1, 2, 1)$ and $(1, 1, 2)$ (every other a_1 entry in H_1^A is dominated by one of the e entries in H_2^A). But now, such a unique a_1 maximum is an NE.

Case B (*equal to one*): $a_1 = b_2 \neq c_2$

Due to symmetry we may without a loss of generality consider the case $a_1 = b_2 \neq c_2$ (the case $a_1 = c_2 \neq b_2$ is symmetric). The fact $a_1 \neq c_2$ implies that the $(1, 1, *)$ line is $\ell_a = (b_1, a_1, a_1, \dots, a_1)$ and each a_1 in ℓ_aB-propagates. This means that H_1^A hyperplane must contain a_1 everywhere except in the $(1, *, 1)$ line (which may still be either $\ell_a = (b_1, a_1, a_1, \dots, a_1)$ or $\ell_b = (b_1, a_1, b_1, \dots, b_1)$). Note that $b_1 <_B b_2 = a_1$ and so if $b_1 <_C a_1$ holds then the unique a_1 maximum is an NE. So we may assume that $a_1 <_C b_1$ and distinguish the following two subcases.

1. Assume that the entry in position $(2, 2, 1)$ is $b_1 = c_1$. We may moreover assume that every entry e in the $(2, 2, *)$ line satisfies $a_1 <_A e$ (in particular $a_1 <_A b_1$) since otherwise the corresponding a_1 entry in $(1, 2, *)$ is an NE. That means that every entry in $(2, 2, *)$ is different from $a_1 = b_2$ and hence it B-propagates (in particular the $(2, *, 1)$ line is a constant b_1 line). The entries in $(2, 1, *)$ are either the same as the corresponding entries in $(2, 2, *)$ or they are equal to b_1 (any entry different from b_1 must B-propagate). That means that H_2^A consist completely of entries which dominate a_1 entries in H_1^A contradicting the existence of unique a_1 maximum in H_1^A.
2. Assume that the entry in position $(2, 2, 1)$ is $e \neq b_1 = c_1$. This implies that eC-propagates making $(2, 2, *)$ a constant e line. We must have $a_1 <_A e$ since otherwise any a_1 entry on the $(1, 2, *)$ line is an NE. Therefore $e \neq a_1 = b_2$ and so every e on the $(2, 2, *)$ line B-propagates making H_2^A an "almost" constant e hyperplane, the only exception being the $(2, 1, *)$ line. Note that the B-propagation of e in position $(2, 2, 1)$ forces the $(1, *, 1)$ line to be $\ell_a = (b_1, a_1, a_1, \dots, a_1)$ and not $\ell_b = (b_1, a_1, b_1, \dots, b_1)$ since b_1 in position $(1, 3, 1)$ would A-propagate (recall that $a_1 \neq b_1$) contradicting $e \neq b_1$. Thus we have a complete picture of H_1^A: it is a constant a_1 hyperplane with the exception of the b_1 entry in position $(1, 1, 1)$. To get a complete picture of H_2^A note that the $(2, 1, *)$ line may contain only e and b_1 entries as any entry different from b_1B-propagates into the constant e area of H_2^A. In fact there are three possibilities how the $(2, 1, *)$ line may look like.
   (a) The $(2, 1, *)$ line is $(b_1, e, \dots, e)$. In this case, there is no unique a_1 maximum in H_1^A as every a_1 entry in H_1^A is dominated by e.
   (b) The $(2, 1, *)$ line is $(b_1, b_1, \dots, b_1)$. Here either $e \leq_B b_1$, in which case the b_1 entry in position $(2, 1, 1)$ is an NE, or $b_1 <_B e$, in which case there are two more possibilities. Either $e = a_2$, in which case the unique a_2 maximum in H_2^A is an NE, or $e \neq a_2$, in which case every e entry in H_2^AA-propagates and therefore every such entry is an NE.
   (c) The $(2, 1, *)$ line is $(b_1, e, b_1, \dots, b_1)$. Note that in this case we may assume $d_C \geq 3$, since otherwise the $(2, 1, *)$ line is just (b_1, e) which falls under case (a) above. The fact that e does not C-propagate implies $e = c_2$, and hence $c_1 = b_1 <_C e$. Moreover, we may assume that $b_1 <_A a_1$ since otherwise there is no unique a_1 maximum in H_1^A (every entry in H_2^A is either e or b_1 and we already know that $a_1 <_A e$). The fact $b_1 <_C e$ implies that the e entry in the $(2, 1, 2)$ position is an NE unless $e = a_2$ and it is dominated by some entry f in the direction of player A. Note that such f must be in the $(3, 1, 2)$ position since a non-dominating entry in $(3, 1, 2)$ would have to be different from a_3 and hence it would A-propagate, leaving e non-dominated. Now on one hand we have that $c_2 = e <_A f$ implies that fC-propagates and so the entry in the $(3, 1, 3)$ position is f. On the other hand, $b_1 <_A a_1 <_A a_2$ implies that b_1 in the $(2, 1, 3)$ position A-propagates and so the entry in the $(3, 1, 3)$ position is b_1. However, $b_1 <_A a_1 <_A e <_A f$ holds implying $b_1 \neq f$, which is a contradiction.

   Note that the above three cases are indeed the only possibilities of how the mixture of e and b_1 entries on the $(2, 1, *)$ line may look like. If $(2, 1, 2)$ is $b_1 = c_1$ (implying $b_1 \neq c_2$) then this b_1 entry C-propagates resulting in the second case. If $(2, 1, 2)$ is e then either $e \neq c_2$ and this entry C-propagates (which is the first case), or $e = c_2$ in which case depending on the $(2, 1, 3)$ entry we get again the first case ($(2, 1, 3)$ is $e \neq c_3$) or the third case ($(2, 1, 3)$ is $b_1 \neq c_3$).

Case C (*different from both*): $a_1 \notin \{b_2, c_2\}$

The fact $a_1 \neq c_2$ implies that the $(1, 1, *)$ line is $\ell_a = (b_1, a_1, a_1, \dots, a_1)$ and each a_1B-propagates. Similarly, $a_1 \neq b_2$ implies that the $(1, *, 1)$ line is $\ell_a = (b_1, a_1, a_1, \dots, a_1)$ and each a_1C-propagates. Thus H_1^A is a constant a_1 hyperplane with the exception of the b_1 entry in position $(1, 1, 1)$. If $a_1 \leq_B b_1$ and $a_1 \leq_C b_1$ hold simultaneously, then the b_1 entry in position $(1, 1, 1)$ is an NE. If $b_1 \leq_B a_1$ and $b_1 \leq_C a_1$ hold simultaneously, then the unique a_1 maximum in H_1^A is an NE. This covers all the cases in which there are strict inequalities "in the same direction" and/or at least one equality between a_1 and b_1. The remaining two possibilities when the strict inequalities go "in the opposite directions" are symmetric, so let us in the rest of this case assume that $b_1 <_B a_1$ and $a_1 <_C b_1$. This implies that the unique a_1 maximum in H_1^A is an NE unless it appears in the $(1, 1, *)$ line. Let us distinguish two cases.

1. Let us assume that $a_1 \leq_A b_1$. Let the unique a_1 maximum in H_1^A be in the $(1, 1, i)$ position for some $i > 1$ and let e be the entry in the $(2, 1, i)$ position. Clearly, $e <_A a_1$ (since e is on a line with unique a_1 maximum) and hence $e <_A a_2$ and $e <_A b_1$. This implies that e both A-propagates and B-propagates which means that $(*, *, i) = H_i^C$ is an "almost" constant e hyperplane except for the $(1, *, i)$ constant a_1 line. However, this means that any such a_1 in the $(1, j, i)$ position for $j > 1$ is an NE.
2. Let us assume that $b_1 <_A a_1$. Let us denote the H_1^A hyperplane without the $(1, 1, *)$ line as region R_1 and the H_2^A hyperplane without the $(2, 1, *)$ line as region R_2. Every a_1 entry in R_1 is an NE unless it is dominated by some entry e in the direction of player A. Such e must be in R_2 since a non-dominating entry in R_2 would have to be different from a_2 and hence it would A-propagate, leaving the corresponding a_1 in H_1^A non-dominated. In particular, this means that R_2 contains no $b_1 = c_1$ entry which in turn implies that every entry in the $(2, j, 1)$ position for $j > 1$ C-propagates creating a constant line in R_2. Therefore, R_2 is fully specified by the $(2, *, 1)$ line, which by Theorem 6 has the form $(b_1, b_2, \dots, b_i, e, \dots, e)$ for some index $i \geq 1$ and some outcome $e \neq b_i$. Now we can distinguish two cases.
   - If $i \geq 2$ then every line in the direction of player B intersects R_2 in at least two different entries and hence the $(2, 1, *)$ line must be a constant b_1 line (any other entry would B-propagate). Hence every line in the direction of player B in H_2^A is a copy of $(2, *, 1)$. Let f be the maximum element with respect to $<_B$ in $(2, *, 1)$ (clearly either $f = b_i$ or $f = e$). Now either $f = a_2$ in which case the unique a_2 maximum in H_2^A is an NE, or $f \neq a_2$ in which case f A-propagates and any f entry in H_2^A is an NE.
   - If $i = 1$ then R_2 is a constant e region and there are three possibilities how the $(2, 1, *)$ line may look like, namely $(b_1, e, \dots, e)$, or $(b_1, b_1, \dots, b_1)$, or $(b_1, e, b_1, \dots, b_1)$ by the same argument as in the last paragraph of Case B. Moreover, note that in this case both hyperplane H_1^A and the three forms of H_2^A look exactly the same as in Case B, Subcase 2. Also the assumptions on mutual relations of outcomes are the same (in particular the assumptions $b_1 <_A a_1$ and $a_1 <_A e$) and hence the arguments from Case B, Subcase 2, paragraphs (a), (b), and (c) can be repeated word by word here. The fact $a_1 = b_2$ valid in Case B and not valid in Case C is only used in Case B, Subcase 2 to show that H_2^A contains e everywhere except on the $(2, 1, *)$ line (i.e. to show that R_2 is a constant e region), but it is never used in (a), (b) and (c) once the form of H_2^A is known.

Since the above case analysis covers all possible cases, this finishes the proof. □

5. On acyclicity and dominance solvability of *n*-person game forms

In this section, we present an example which shows that the implications $TT \Rightarrow AC$ and $TT \Rightarrow DS$, which are valid for 2-person game forms, fail already for three players.

Remark 8. This example also disproves a recent conjecture of [10] stating that a game form is acyclic if every subform of it is Nash solvable. Note that for games (not game forms) a similar statement was proved in [15]: a game need not be acyclic if every subgame of it has an NE.[2]

Let us start with the definition of AC. An n-person game is defined by a game form g and a preference profile p. Given a direction i and two profiles x and y such that $x_j = y_j$ for every $j \neq i$ (i.e. both entries are on the same line in direction i), the move from x to y is called an *improving move* in direction i if $g(x) <_i g(y)$ holds. A non-empty sequence of improving moves which starts and ends in the same entry is called an *improvement cycle* of the game. Clearly, if a game has an improvement cycle then it has one in which no two consecutive improving moves are in the same direction.

A game is called *acyclic* if it has no improvement cycle. Obviously, an acyclic game has a Nash equilibrium. Furthermore, a game form g is called *acyclic* (AC) if the obtained game is acyclic for any preference profile p. Hence, $AC \Rightarrow NS$. In order to show that a given game form g is not AC it suffices to show one particular preference profile p such that the resulting game (g, p) has an improvement cycle.

Fig. 1 shows a 3-person game form g of dimensions $3 \times 2 \times 2$ for players $\{A, B, C\}$ and three outcomes $\{a, b, c\}$. Player A chooses hyperplanes (strategies) perpendicular to left–right lines, B chooses hyperplanes perpendicular to up-down lines, and C chooses hyperplanes perpendicular to front-back lines. It is easy to verify that g is TT by checking all 2×2 submatrices of the game forms $g_{\{A\}}$, $g_{\{B\}}$, and $g_{\{C\}}$. On the other hand, if we set $b <_A c <_A a$ for player A, $a <_B c <_B b$ for player B, and $c <_C a <_C b$ for player C, then the resulting game contains an improvement cycle of length seven, which is marked by arrows on the edges in Fig. 1. Thus, TT does not imply AC for $n = 3$.

Remark 9. The concept of acyclicity can be naturally generalized by replacing individual improvements by coalitional ones. Very recently (see Theorem 4.2 in [11]) it was shown than an n-person game form is coalitional acyclic if and only if it is either dictatorial (there exists a single player that determines the outcome, i.e. the game form consists of parallel constant hyperplanes) or TT with just two outcomes.

[2] This remark was pointed to us by an anonymous referee.

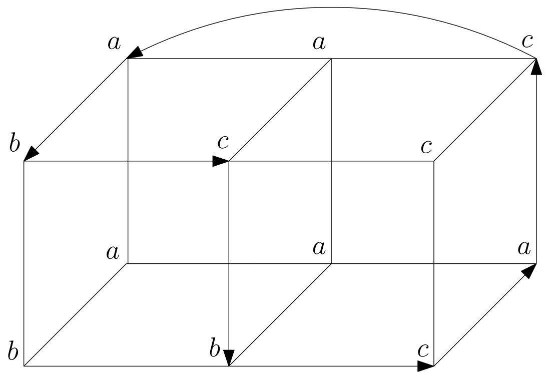

Fig. 1. A game form g contradicting $TT \Rightarrow AC$ and $TT \Rightarrow DS$.

Now let us define *DS*. Again, let us consider an n-person game defined by a game form g and a preference profile p. Given a direction i and two hyperplanes H_j and H_k perpendicular to direction i (these correspond to two different strategies of player i), we say that H_k is dominated by H_j if for every line ℓ in direction i (or in other words, for any fixed set of strategies of the remaining players) the intersection of ℓ with H_k (let us call this profile y) does not dominate the intersection of ℓ with H_j (let us call this profile x), i.e. if $g(y) \leq_i g(x)$. Let us consider a game form g' which we get from g by deleting a dominated hyperplane. Then we say that a game (g', p) was obtained from the game (g, p) by a *reduction step*. We say that a game is *dominance solvable* if there exists a sequence of reduction steps which reduces the game to a game with a single entry. Finally, a game form g is *dominance solvable* (*DS*) if the game (g, p) is dominance solvable for every preference profile p. Similarly as in the *AC* case this means that, in order to show that a given game form g is not *DS*, it suffices to show one particular preference profile p such that the resulting game (g, p) is not dominance solvable.

Again consider game form g from Fig. 1 and the same player preferences as before. It is easy to see that neither of the two hyperplanes perpendicular to direction B dominates the other, since there are arcs in both directions on the improvement cycle in between these two hyperplanes. The same is true for the two hyperplanes perpendicular to direction C. Thus the only reduction step possible is to remove the leftmost hyperplane perpendicular to direction A which is dominated by the middle hyperplane perpendicular to direction A. However, it is easy to verify that the resulting $2 \times 2 \times 2$ game form g' together with the player preferences now define a game in which no reduction step is possible. This proves that the original game is not dominance solvable, and thus *TT* does not imply *DS* for $n = 3$.

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