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# Analysis on the minimal representation of $O(p, q)$

## II. Branching laws

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### Abstract

This is a second paper in a series devoted to the minimal unitary representation of  $O(p, q)$ . By explicit methods from conformal geometry of pseudo Riemannian manifolds, we find the branching law corresponding to restricting the minimal unitary representation to natural symmetric subgroups. In the case of purely discrete spectrum we obtain the full spectrum and give an explicit Parseval–Plancherel formula, and in the general case we construct an infinite discrete spectrum.

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**Introduction**

This is the second in a series of papers devoted to the analysis of the minimal representation  $\varpi^{p,q}$  of  $O(p, q)$ . We refer to [15] for a general introduction; also the numbering of the sections is continued from that paper, and we shall refer back to sections there. However, the present paper may be read independently from [15], and its main object is to study the branching law for the minimal unitary representation  $\varpi^{p,q}$  from *analytic* and *geometric* point of view. Namely, we shall find by explicit means, coming from conformal geometry, the restriction of  $\varpi^{p,q}$  with respect to the symmetric pair

$$(G, G') = (O(p, q), O(p', q') \times O(p'', q'')).$$

If one of the factors in  $G'$  is compact, then the spectrum is discrete (see Theorem 4.2 also for an opposite implication), and we find the explicit branching law; when both factors are non-compact, there will still (generically) be an infinite discrete spectrum, which we also construct (conjecturally almost all of it; see Section 9.8). We shall see that the (algebraic) situation is similar to the theta-correspondence, where the metaplectic representation is restricted to analogous subgroups.

Let us here state the main results in a little more precise form, referring to Sections 8 and 9 for further notation and details.

**Theorem A** (The branching law for  $O(p, q) \downarrow O(p, q') \times O(q'')$ ; see Theorem 7.1). *If  $q'' \geq 1$  and  $q' + q'' = q$ , then the twisted pull-back  $\widetilde{\Phi}_1^*$  of the local conformal map  $\Phi_1$  between spheres and hyperboloids gives an explicit irreducible decomposition of the unitary representation  $\varpi^{p,q}$  when restricted to  $O(p, q') \times O(q'')$ :*

$$(\widetilde{\Phi}_1)^* : \varpi^{p,q}|_{O(p,q') \times O(q'')} \xrightarrow{\sim} \sum_{l=0}^{\infty} \oplus \pi_{+, l + \frac{q''}{2} - 1}^{p,q'} \boxtimes \mathcal{H}^l(\mathbb{R}^{q''}).$$

The representations appearing in the decompositions are in addition to usual spherical harmonics  $\mathcal{H}^l(\mathbb{R}^q)$  for compact orthogonal groups  $O(q)$ , also the representations  $\pi_{+, \lambda}^{p,q}$  for non-compact orthogonal groups  $O(p, q)$ . The latter ones

may be thought of as discrete series representations on hyperboloids

$$X(p, q) := \{x = (x', x'') \in \mathbb{R}^{p+q} : |x'|^2 - |x''|^2 = 1\}$$

for  $\lambda > 0$  or their analytic continuation for  $\lambda \leq 0$ ; they may be also thought of as cohomologically induced representations from characters of certain  $\theta$ -stable parabolic subalgebras. The fact that they occur in this branching law gives a different proof of the unitarizability of these modules  $\pi_{+, \lambda}^{p, q}$  for  $\lambda > -1$ , once we know  $\varpi^{p, q}$  is unitarizable (cf. Part I, Theorem 3.6.1). It might be interesting to remark that the unitarizability for  $\lambda < 0$  (especially,  $\lambda = -\frac{1}{2}$  in our setting) does not follow from a general unitarizability theorem on Zuckerman–Vogan’s derived functor modules [20], neither from a general theory of harmonic analysis on semisimple symmetric spaces. For another viewpoint from the orbit method, we refer to [11].

Our intertwining operator  $(\Phi_1)^*$  in Theorem A is derived from a conformal change of coordinate (see Section 6 for its explanation) and is explicitly written. Therefore, it makes sense to ask also about the relation of unitary inner products between the left- and the right-hand side in the branching formula. Here is an answer (see Theorem 8.6): We normalize the inner product  $\| \cdot \|_{\pi_{+, \lambda}^{p, q}}$  (see (8.4.2)) such that for  $\lambda > 0$ ,

$$\|f\|_{\pi_{+, \lambda}^{p, q}}^2 = \lambda \|f\|_{L^2(X(p, q))}^2 \quad \text{for any } f \in (\pi_{+, \lambda}^{p, q})_K.$$

**Theorem B** (The Parseval–Plancherel formula for  $O(p, q) \downarrow O(p, q') \times O(q'')$ ).

(1) *If we develop  $F \in \text{Ker } \tilde{\Delta}_M$  as  $F = \sum_l^\infty F_l^{(1)} F_l^{(2)}$  according to the irreducible decomposition in Theorem A, then we have*

$$\|F\|_{\varpi^{p, q}}^2 = \sum_{l=0}^\infty \|F_l^{(1)}\|_{\pi_{+, l + \frac{q''}{2} - 1}^{p, q'}}^2 \|F_l^{(2)}\|_{L^2(S^{q''-1})}^2.$$

(2) *In particular, if  $q'' \geq 3$ , then all of  $\pi_{+, l + \frac{q''}{2} - 1}^{p, q'}$  are discrete series for the hyperboloid  $X(p, q')$  and the above formula amounts to*

$$\|F\|_{\varpi^{p, q}}^2 = \sum_{l=0}^\infty \left( l + \frac{q''}{2} - 1 \right) \|F_l^{(1)}\|_{L^2(X(p, q'))}^2 \|F_l^{(2)}\|_{L^2(S^{q''-1})}^2.$$

The formula may be also regarded as an explicit unitarization of the minimal representation  $\varpi^{p, q}$  on the “hyperbolic space model” by means of the right side (for an abstract unitarization of  $\varpi^{p, q}$ , it suffices to choose a single pair  $(q', q'')$ ). We note that the formula was previously known in the case where  $(q', q'') = (0, q)$  (namely, when each summand in the right side is finite-dimensional) by Kostant, Binegar–Zierau by a different approach [1, 17]. The formula is new and seems to be particularly interesting even in the special case  $q'' = 1$ , where the minimal

representation  $\varpi^{p,q}$  splits into two irreducible summands when restricted to  $O(p, q - 1) \times O(1)$ .

In Theorem 9.1, we consider a more general setting and prove:

**Theorem C** (Discrete spectrum in the restriction  $O(p, q) \downarrow O(p', q') \times O(p'', q'')$ ). *The twisted pull-back of the locally conformal diffeomorphism also constructs*

$$\sum_{\lambda \in A'(p',q') \cap A'(q'',p'')} \pi_{+,\lambda}^{p',q'} \boxtimes \pi_{-,\lambda}^{p'',q''} \oplus \sum_{\lambda \in A'(q',p') \cap A'(p'',q'')} \pi_{-,\lambda}^{p',q'} \boxtimes \pi_{+,\lambda}^{p'',q''}$$

as a discrete spectrum in the branching law for the non-compact case.

Even in the special case  $(p'', q'') = (0, 1)$ , our branching formula includes a new and mysterious construction of the minimal representation on the hyperboloid as below (see Corollary 7.2.1): Let  $W^{p,r}$  be the set of  $K$ -finite vectors ( $K = O(p) \times O(r)$ ) of the kernel of the Yamabe operator

$$\text{Ker } \tilde{\Delta}_{X(p,r)} = \{f \in C^\infty(X(p, r)) : \Delta_{X(p,r)} f = \frac{1}{4}(p + r - 1)(p + r - 3)f\},$$

on which the isometry group  $O(p, r)$  and the Lie algebra of the conformal group  $O(p, r + 1)$  act. The following proposition is a consequence of Theorem 7.2.2 by an elementary linear algebra.

**Proposition D.** *Let  $m > 3$  be odd. There is a long exact sequence*

$$0 \rightarrow W^{1,m-1} \xrightarrow{\varphi_1} W^{2,m-2} \xrightarrow{\varphi_2} W^{3,m-3} \xrightarrow{\varphi_3} \dots \xrightarrow{\varphi_{m-2}} W^{m-1,1} \xrightarrow{\varphi_{m-1}} 0$$

such that  $\text{Ker } \varphi_p$  is isomorphic to  $(\varpi^{p,q})_K$  for any  $(p, q)$  such that  $p + q = m + 1$ .

We note that each representation space  $W^{p,q-1}$  is realized on a different space  $X(p, q - 1)$  whose isometry group  $O(p, q - 1)$  varies according to  $p$  ( $1 \leq p \leq m$ ). So, one may expect that only the intersections of adjacent groups can act (infinitesimally) on  $\text{Ker } \varphi_p$ . Nevertheless, a larger group  $O(p, q)$  can act on a suitable completion of  $\text{Ker } \varphi_p$ , giving rise to another construction of the minimal representation on the hyperboloid  $X(p, q - 1) = O(p, q - 1)/O(p - 1, q - 1)!$  We note that  $\text{Ker } \varphi_p$  is roughly half the kernel of the Yamabe operator on the hyperboloid (see Section 7.2 for details).

We briefly indicate the contents of the paper: In Section 4 we recall the relevant facts about discretely decomposable restrictions from [9,10], and apply the criteria to our present situation. In particular, we calculate the associated variety of  $\varpi^{p,q}$  as well as its asymptotic  $K$ -support introduced by Kashiwara–Vergne. Theorem 4.2 and Corollary 4.3 clarify the reason why we start with the subgroup  $G' = O(p, q') \times O(q'')$  (i.e.  $p'' = 0$ ). Section 5 contains the identification of the representations  $\pi_{+,\lambda}^{p,q}$  and  $\pi_{-,\lambda}^{p,q}$  of  $O(p, q)$  in several ways, namely as: derived functor modules, Dolbeault

cohomologies, eigenspaces on hyperboloids, and quotients or subrepresentations of parabolically induced modules. In Section 6 we give the main construction of embedding conformally a direct product of hyperboloids into a product of spheres; this gives rise to a canonical intertwining operator between solutions to the so-called Yamabe equation, studied in connection with conformal differential geometry, on conformally related spaces. Applying this principle in Section 7 we obtain the branching law in the case where one factor in  $G'$  is compact, and in particular when one factor is just  $O(1)$ . In this case we have Corollary 7.2.1, stating that  $\varpi^{p,q}$  restricted to  $O(p, q-1)$  is the direct sum of two representations, realized in even, respectively, odd functions on the hyperboloid for  $O(p, q-1)$ . Note here the analogy with the metaplectic representation. Also note here Theorem 7.2.2, which gives a mysterious extension of  $\varpi^{p,q}$  by  $\varpi^{p+1, q-1}$ —both inside the space of solutions to the Yamabe equation on the hyperboloid  $X(p, q-1) = O(p, q-1)/O(p-1, q-1)$ . We also point out that the representations  $\pi_{+, \lambda}^{p,q}$  for  $\lambda = 0, -\frac{1}{2}$  are rather exceptional; they are unitary, but outside the usual “fair range” for derived functor modules, see the remarks in Section 8.4. Section 8 contains a proof of Theorem 3.9.1 of [15] on the spectra of the Knapp–Stein intertwining operators and gives the explicit Parseval–Plancherel formulas for the branching laws. Finally, in Section 9 we use certain Sobolev estimates to construct infinitely many discrete spectra when both factors in  $G'$  are non-compact. We also conjecture the form of the full discrete spectrum (true in the case of a compact factor). It should be interesting to calculate the full Parseval–Plancherel formula in the case of both discrete and continuous spectrum.

#### 4. Criterion for discrete decomposable branching laws

**4.1.** Our object of study is the discrete spectra of the branching law of the restriction  $\varpi^{p,q}$  with respect to a symmetric pair  $(G, G') = (O(p, q), O(p', q') \times O(p'', q''))$ . The aim of this section is to give a necessary and sufficient condition on  $p', q', p''$  and  $q''$  for the branching law to be discretely decomposable.

We start with general notation. Let  $G$  be a linear reductive Lie group, and  $G'$  its subgroup which is reductive in  $G$ . We take a maximal compact subgroup  $K$  of  $G$  such that  $K' := K \cap G'$  is also a maximal compact subgroup. Let  $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$  be a Cartan decomposition, and  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  its complexification. Accordingly, we have a direct decomposition  $\mathfrak{g}^* = \mathfrak{k}^* + \mathfrak{p}^*$  of the dual spaces.

Let  $\pi \in \hat{G}$ . We say that the restriction  $\pi|_{G'}$  is  $G'$ -admissible if  $\pi|_{G'}$  splits into a direct Hilbert sum of irreducible unitary representations of  $G'$  with each multiplicity finite (see [7]). As an algebraic analogue of this notion, we say the underlying  $(\mathfrak{g}, K)$ -module  $\pi_K$  is discretely decomposable as a  $(\mathfrak{g}', K')$ -module, if  $\pi_K$  is decomposed into an algebraic direct sum of irreducible  $(\mathfrak{g}', K')$ -modules (see [10]). We note that if the restriction  $\pi|_{K'}$  is  $K'$ -admissible, then the restriction  $\pi|_{G'}$  is also  $G'$ -admissible

[7, Theorem 1.2] and the underlying  $(\mathfrak{g}, K)$ -module  $\pi_K$  is discretely decomposable (see [10, Proposition 1.6]). Here are criteria for  $K'$ -admissibility and discrete decomposability:

**Fact 4.1** (see Kobayashi [9, Theorem 2.9] for (1); [10, Corollary 3.4] for (2)).

- (1) If  $\text{AS}_K(\pi) \cap \text{Ad}^*(K)(\mathfrak{f}')^\perp = 0$ , then  $\pi$  is  $K'$ -admissible and also  $G'$ -admissible.
- (2) If  $\pi_K$  is discretely decomposable as a  $(\mathfrak{g}', K')$ -module, then  $\text{pr}_{\mathfrak{g} \rightarrow \mathfrak{g}'}(\mathcal{V}_{\mathfrak{g}}(\pi_K)) \subset \mathcal{N}_{\mathfrak{g}'^*}^*$ .

Here,  $\text{AS}_K(\pi)$  is the asymptotic cone of

$$\text{Supp}_K(\pi) := \{\text{highest weight of } \tau \in \widehat{K_0}: [\pi|_{K_0}: \tau] \neq 0\},$$

where  $K_0$  is the identity component of  $K$ , and  $(\mathfrak{f}')^\perp \subset \mathfrak{f}^*$  is the annihilator of  $\mathfrak{f}'$ . Let  $\mathcal{N}_{\mathfrak{p}}^*$  ( $\subset \mathfrak{p}^*$ ) be the nilpotent cone for  $\mathfrak{p}$ .  $\mathcal{V}_{\mathfrak{g}}(\pi_K)$  denotes the associated variety of  $\pi_K$ , which is an  $\text{Ad}^*(K_{\mathbb{C}})$ -invariant closed subset of  $\mathcal{N}_{\mathfrak{p}}^*$ . We write the projection  $\text{pr}_{\mathfrak{p} \rightarrow \mathfrak{p}'} : \mathfrak{p}^* \rightarrow \mathfrak{p}'^*$  dual to the inclusion  $\mathfrak{p}' \hookrightarrow \mathfrak{p}$ .

**4.2.** Let us consider our setting where  $\pi = \varpi^{p,q}$  and  $(G, G') = (\text{O}(p, q), \text{O}(p', q') \times \text{O}(p'', q''))$ .

**Theorem 4.2.** Suppose  $p' + p'' = p (\geq 2)$ ,  $q' + q'' = q (\geq 2)$  and  $p + q \in 2\mathbb{N}$ . Then the following three conditions on  $p', q', p'', q''$  are equivalent:

- (i)  $\varpi^{p,q}$  is  $K'$ -admissible;
- (ii)  $\varpi_K^{p,q}$  is discretely decomposable as a  $(\mathfrak{g}', K')$ -module;
- (iii)  $\min(p', q', p'', q'') = 0$ .

Implication (i)  $\Rightarrow$  (ii) holds by a general theory as we explained [10, Proposition 1.6]; (ii)  $\Rightarrow$  (iii) will be proved in Section 4.4, and (iii)  $\Rightarrow$  (i) in Section 4.5, by an explicit computation of the asymptotic cone  $\text{AS}_K(\varpi^{p,q})$  and the associated variety  $\mathcal{V}_{\mathfrak{g}}(\varpi_K^{p,q})$  which are used in Fact 4.1.

**Remark.** Analogous results to the equivalence (i)  $\Leftrightarrow$  (ii) in Theorem 4.2 were first proved in [10], Theorem 4.2 in the setting where  $(G, G')$  is any reductive symmetric pair and the representation is any  $A_q(\lambda)$  module in the sense of Zuckerman–Vogan, which may be regarded as “representations attached to elliptic orbits”. We note that our representations  $\varpi^{p,q}$  are supposed to be attached to nilpotent orbits. We refer [13], Conjecture A to relevant topics.

**4.3.** The following corollary is a direct consequence of Theorem 4.2, which will be an algebraic background for the proof of the explicit branching law (Theorem 7.1).

**Corollary 4.3.** *Suppose that  $\min(p', q', p'', q'') = 0$ .*

- (1) *The restriction of the unitary representation  $\varpi^{p,q}|_G$  is also  $G'$ -admissible.*
- (2) *The space of  $K'$ -finite vectors  $\varpi_K^{p,q}$  coincides with that of  $K$ -finite vectors  $\varpi_K^{p,q}$ .*

**Proof.** See [7, Theorem 1.2] for (1) and [10, Proposition 1.6] for (2).  $\square$

A geometric counterpart of Corollary 4.3(2) is reflected as the removal of singularities of matrix coefficients for the discrete spectra in the analysis that we study in Section 6; namely, any analytic function defined on an open subset  $M_+$  (see Section 6 for notation) of  $M$  which is a  $K'$ -finite vector of a discrete spectrum, extends analytically on  $M$  if  $p'' = 0$ . The reason for this is not only the decay of matrix coefficients but a matching condition of the leading terms for  $t \rightarrow \pm \infty$ . This is not the case for  $\min(p', q', p'', q'') > 0$  (see Section 9).

**4.4.** Proof of (ii)  $\Rightarrow$  (iii) in Theorem 4.2.

We identify  $\mathfrak{p}^*$  with  $\mathfrak{p}$  via the Killing form, which is in turn identified with  $M(p, q; \mathbb{C})$  by

$$M(p, q; \mathbb{C}) \xrightarrow{\sim} \mathfrak{p}, \quad X \mapsto \begin{pmatrix} O & X \\ {}^tX & O \end{pmatrix}.$$

Then the nilpotent cone  $\mathcal{N}_{\mathfrak{p}}^*$  corresponds to the following variety:

$$\{X \in M(p, q; \mathbb{C}) : \text{both } X^tX \text{ and } {}^tXX \text{ are nilpotent matrices}\}. \quad (4.4.1)$$

We put

$$M_{0,0}(p, q; \mathbb{C}) := \{X \in M(p, q; \mathbb{C}) : X^tX = O, {}^tXX = O\}.$$

Then  $M_{0,0}(p, q; \mathbb{C}) \setminus \{O\}$  is the unique  $K_{\mathbb{C}} \simeq O(p, \mathbb{C}) \times O(q, \mathbb{C})$ -orbit of dimension  $p + q - 3$ . The associated variety  $\mathcal{V}_{\mathfrak{g}}(\varpi_K^{p,q})$  of  $\varpi^{p,q}$  is of dimension  $p + q - 3$ , which follows easily from the  $K$ -type formula of  $\varpi^{p,q}$  (see [15, Theorem 3.6.1]). Thus, we have proved:

**Lemma 4.4.** *The associated variety  $\mathcal{V}_{\mathfrak{g}}(\varpi_K^{p,q})$  equals  $M_{0,0}(p, q; \mathbb{C})$ .*

The projection  $\text{pr}_{\mathfrak{p} \rightarrow \mathfrak{p}'} : \mathfrak{p}^* \rightarrow \mathfrak{p}'^*$  is identified with the map

$$\text{pr}_{\mathfrak{p} \rightarrow \mathfrak{p}'} : M(p, q; \mathbb{C}) \rightarrow M(p', q'; \mathbb{C}) \oplus M(p'', q''; \mathbb{C}), \quad \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \mapsto (X_1, X_4).$$

Suppose  $p'p''q'q'' \neq 0$ . If we take

$$X := E_{1,1} - E_{p'+1,q'+1} + \sqrt{-1}E_{p'+1,1} + \sqrt{-1}E_{1,q'+1} \in M_{0,0}(p, q; \mathbb{C}),$$

then  $\text{pr}_{\mathfrak{p} \rightarrow \mathfrak{p}'}(X) = (E_{1,1}, -E_{p'+1, q'+1})$ . But  $E_{1,1} \notin \mathcal{N}_{\mathfrak{o}(p', q')}$  and  $-E_{p'+1, q'+1} \notin \mathcal{N}_{\mathfrak{o}(p'', q'')}$ . Thus,  $\text{pr}_{\mathfrak{g} \rightarrow \mathfrak{g}'}(X) \notin \mathcal{N}_{\mathfrak{g}'}$ . It follows from Fact 4.1(2) that  $\varpi_K^{p, q}$  is not discrete decomposable as a  $(\mathfrak{g}', K')$ -module. Hence (ii)  $\Rightarrow$  (iii) in Theorem 4.2 is proved.  $\square$

**4.5. Proof of (iii)  $\Rightarrow$  (i) in Theorem 4.2.**

We take an orthogonal complementary subspace  $\mathfrak{k}_0''$  of  $\mathfrak{k}_0'$  in  $\mathfrak{k}_0 \simeq \mathfrak{o}(p) + \mathfrak{o}(q)$ . Let  $\mathfrak{t}_0^c$  be a Cartan subalgebra of  $\mathfrak{k}_0$  such that  $\mathfrak{t}_0' := \mathfrak{t}_0^c \cap \mathfrak{k}_0''$  is a maximal abelian subspace in  $\mathfrak{k}_0''$ . We choose a positive system  $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$  which is compatible with a positive system of the restricted root system  $\Sigma(\mathfrak{k}, \mathfrak{t}'')$ . Then we can find a basis  $\{f_i: 1 \leq i \leq \lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor\}$  on  $\sqrt{-1}\mathfrak{t}_0^*$  such that a positive root system of  $\mathfrak{k}$  is given by

$$\begin{aligned} \Delta^+(\mathfrak{k}, \mathfrak{t}^c) &= \left\{ f_i \pm f_j: 1 \leq i < j \leq \left\lfloor \frac{p}{2} \right\rfloor \right\} \\ &\cup \left\{ f_i \pm f_j: \left\lfloor \frac{p}{2} \right\rfloor + 1 \leq i < j \leq \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \right\} \\ &\cup \left( \left\{ f_l: 1 \leq l \leq \left\lfloor \frac{p}{2} \right\rfloor \right\} (p: \text{odd}) \right) \\ &\cup \left( \left\{ f_l: \left\lfloor \frac{p}{2} \right\rfloor + 1 \leq l \leq \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \right\} (q: \text{odd}) \right), \end{aligned}$$

and such that

$$\sqrt{-1}(\mathfrak{t}_0'')^* = \sum_{j=1}^{\min(p', p'')} \mathbb{R}f_j + \sum_{j=1}^{\min(q', q'')} \mathbb{R}f_{\lfloor \frac{p}{2} \rfloor + j} \tag{4.5.1}$$

if we regard  $(\mathfrak{t}_0'')^*$  as a subspace of  $(\mathfrak{t}_0^c)^*$  by the Killing form.

Suppose  $p'q'p''q'' = 0$ . Without loss of generality we may and do assume  $p'' = 0$ , namely,  $G' = O(p, q') \times O(q'')$  with  $q' + q'' = q$ .

Let us first consider the case  $p \neq 2$ . Then the irreducible  $O(p)$ -representation  $\mathcal{H}^a(\mathbb{R}^p)$  remains irreducible when restricted to  $SO(p)$ . The corresponding highest weight is given by  $af_1$ . It follows from the  $K$ -type formula of  $\varpi^{p, q}$  (Theorem 3.6.1) that

$$\text{Supp}_K(\varpi^{p, q}) = \left\{ af_1 + bf_{\lfloor \frac{p}{2} \rfloor + 1}: a, b \in \mathbb{N}, a + \frac{p}{2} = b + \frac{q}{2} \right\}.$$

Therefore, we have proved

$$\text{AS}_K(\varpi^{p, q}) = \mathbb{R}_+(f_1 + f_{\lfloor p/2 \rfloor + 1}). \tag{4.5.2}$$

Then  $\text{AS}_K(\varpi^{p, q}) \cap \sqrt{-1}(\mathfrak{t}_0'')^* = \{0\}$  from (4.5.1) and (4.5.2), which implies

$$\text{AS}_K(\varpi^{p, q}) \cap \sqrt{-1}\text{Ad}^*(K)(\mathfrak{k}_0')^\perp = \{0\}$$



because  $(\mathfrak{t}_0'')^*$  meets any  $\text{Ad}^*(K)$ -orbit through  $(\mathfrak{t}_0')^\perp$ . Therefore, the restriction  $\varpi^{p,q}|_{K'}$  is  $K'$ -admissible by Fact 4.1(1).

If  $p = 2$ , then  $\varpi^{p,q}$  splits into two representations (see Remark 3.7.3), say  $\varpi_+^{2,q}$  and  $\varpi_-^{2,q}$ , when restricted to the connected component  $\text{SO}_0(2, q)$ . Likewise,  $\mathcal{H}^a(\mathbb{R}^p)$  is a direct sum of two one-dimensional representations when restricted to  $\text{SO}(2)$  if  $a \geq 1$ . Then we have

$$\text{AS}_K(\varpi_\pm^{2,q}) = \mathbb{R}_+(\pm f_1 + f_{p+1}).$$

Applying Fact 4.1(1) to the identity components  $(G_0, G_0')$  of groups  $(G, G')$ , we conclude that the restriction  $\varpi_\pm^{p,q}|_{K_0}$  is  $K_0'$ -admissible. Hence the restriction  $\varpi^{p,q}|_{K'}$  is also  $K'$ -admissible. Thus, (iii)  $\Rightarrow$  (i) in Theorem 4.2 is proved.

Now the proof of Theorem 4.2 is completed.  $\square$

### 5. Minimal elliptic representations of $\mathbf{O}(p, q)$

**5.1.** In this section, we introduce a family of irreducible representations of  $G = \mathbf{O}(p, q)$ , denoted by  $\pi_{+\lambda}^{p,q}, \pi_{-\lambda}^{p,q}$ , for  $\lambda \in A_0(p, q)$ , in three different realizations. These representations are supposed to be attached to minimal elliptic orbits, for  $\lambda > 0$  in the sense of the Kirillov–Kostant orbit method. Here, we set

$$A_0(p, q) := \begin{cases} \{\lambda \in \mathbb{Z} + \frac{p+q}{2}: \lambda > -1\} & (p > 1, q \neq 0), \\ \{\lambda \in \mathbb{Z} + \frac{p+q}{2}: \lambda \geq \frac{p}{2} - 1\} & (p > 1, q = 0), \\ \emptyset & (p = 1, q \neq 0) \text{ or } (p = 0), \\ \{-\frac{1}{2}, \frac{1}{2}\} & (p = 1, q = 0). \end{cases} \tag{5.1.1}$$

It seems natural to include the parameter  $\lambda = 0, -\frac{1}{2}$  in the definition of  $A_0(p, q)$  as above, although  $\lambda = -\frac{1}{2}$  is outside the weakly fair range parameter in the sense of Vogan [22]. Cohomologically induced representations for  $\lambda = -\frac{1}{2}$  and  $-1$  will be discussed in details in a subsequent paper. In particular, the case  $\lambda = -1$  is of importance in another geometric construction of the minimal representation via Dolbeault cohomology groups (see Part I, Introduction, Theorem B(4)).

**5.2.** Let  $\mathbb{R}^{p,q}$  be the Euclidean space  $\mathbb{R}^{p+q}$  equipped with the flat pseudo-Riemannian metric:

$$g_{\mathbb{R}^{p,q}} = dv_0^2 + \cdots + dv_{p-1}^2 - dv_p^2 - \cdots - dv_{p+q-1}^2.$$

We define a hyperboloid by

$$X(p, q) := \{(x, y) \in \mathbb{R}^{p,q}: |x|^2 - |y|^2 = 1\}.$$

We note  $X(p, 0) \simeq S^{p-1}$  and  $X(0, q) = \emptyset$ . If  $p = 1$ , then  $X(p, q)$  has two connected components. The group  $G$  acts transitively on  $X(p, q)$  with isotropy subgroup  $O(p - 1, q)$  at

$$x^o := {}^t(1, 0, \dots, 0). \tag{5.2.1}$$

Thus  $X(p, q)$  is realized as a homogeneous manifold:

$$X(p, q) \simeq O(p, q)/O(p - 1, q).$$

We induce a pseudo-Riemannian metric  $g_{X(p,q)}$  on  $X(p, q)$  from  $\mathbb{R}^{p,q}$  (see [15, Section 3.2]), and write  $\Delta_{X(p,q)}$  for the Laplace–Beltrami operator on  $X(p, q)$ . As in [15, Example 2.2], the Yamabe operator is given by

$$\tilde{\Delta}_{X(p,q)} = \Delta_{X(p,q)} - \frac{1}{4}(p + q - 1)(p + q - 3). \tag{5.2.2}$$

For  $\lambda \in \mathbb{C}$ , we set

$$\begin{aligned} C_\lambda^\infty(X(p, q)) &:= \left\{ f \in C^\infty(X(p, q)) : \Delta_{X(p,q)} f = \left(-\lambda^2 + \frac{1}{4}(p + q - 2)^2\right) f \right\} \\ &= \left\{ f \in C^\infty(X(p, q)) : \tilde{\Delta}_{X(p,q)} f = \left(-\lambda^2 + \frac{1}{4}\right) f \right\}. \end{aligned} \tag{5.2.3}$$

Furthermore, for  $\varepsilon = \pm$ , we write

$$C_{\lambda,\varepsilon}^\infty(X(p, q)) := \{ f \in C_\lambda^\infty(X(p, q)) : f(-z) = \varepsilon f(z), z \in X(p, q) \}.$$

Then we have a direct sum decomposition

$$C_\lambda^\infty(X(p, q)) = C_{\lambda,+}^\infty(X(p, q)) + C_{\lambda,-}^\infty(X(p, q)) \tag{5.2.4}$$

and each space is invariant under left translations of the isometry group  $G$  because  $G$  commutes with  $\Delta_{X(p,q)}$ . With the notation in Section 3.5, we note if  $q = 0$ , then  $C_{\lambda,\text{sgn}(-1)^k}^\infty(X(p, 0))$  is finite-dimensional and isomorphic to the space of spherical harmonics:

$$\mathcal{H}^k(\mathbb{R}^p) \simeq C_\lambda^\infty(X(p, 0)) = C_{\lambda,\text{sgn}(-1)^k}^\infty(X(p, 0)) \quad \left( k := \lambda + \frac{p-2}{2} \right).$$

**5.3.** Let  $G = O(p, q)$  where  $p, q \geq 1$  and let  $\theta$  be the Cartan involution corresponding to  $K = O(p) \times O(q)$ . We extend a Cartan subalgebra  $\mathfrak{t}_0^c$  of  $\mathfrak{k}_0$  (given in Section 4.5) to that of  $\mathfrak{g}_0$ , denoted by  $\mathfrak{h}_0^c$ . If both  $p$  and  $q$  are odd, then  $\dim \mathfrak{h}_0^c = \dim \mathfrak{t}_0^c + 1$ ; otherwise  $\mathfrak{h}_0^c = \mathfrak{t}_0^c$ . The complexification of  $\mathfrak{h}_0^c$  is denoted by  $\mathfrak{h}^c$ .

We can take a basis  $\{f_i : 1 \leq i \leq \lfloor \frac{p+q}{2} \rfloor\}$  of  $(\mathfrak{h}^c)^*$  (see Section 4.5; by a little abuse of notation if both  $p$  and  $q$  are odd) such that the root system of  $\mathfrak{g}$  is

given by

$$\begin{aligned} \Delta(\mathfrak{g}, \mathfrak{h}^c) &= \left\{ \pm(f_i \pm f_j) : 1 \leq i < j \leq \left\lfloor \frac{p+q}{2} \right\rfloor \right\} \\ &\cup \left( \left\{ \pm f_l : 1 \leq l \leq \left\lfloor \frac{p+q}{2} \right\rfloor \right\} \cup \{f_1\} \right) \quad (p+q: \text{ odd}). \end{aligned}$$

Let  $\{H_i\} \subset \mathfrak{h}^c$  be the dual basis for  $\{f_i\} \subset (\mathfrak{h}^c)^*$ . Set  $\mathfrak{t} := \mathbb{C}H_1$  ( $\subset \mathfrak{t}^c \subset \mathfrak{h}^c$ ). Then the centralizer  $L$  of  $\mathfrak{t}$  in  $G$  is isomorphic to  $\text{SO}(2) \times \text{O}(p-2, q)$ . Let  $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$  be a  $\theta$ -stable parabolic subalgebra of  $\mathfrak{g}$  with nilpotent radical  $\mathfrak{u}$  given by

$$\Delta(\mathfrak{u}, \mathfrak{h}^c) := \left\{ f_1 \pm f_j : 2 \leq j \leq \left\lfloor \frac{p+q}{2} \right\rfloor \right\} \cup \{f_1\} \quad (p+q: \text{ odd}),$$

and with a Levi part  $\mathfrak{l} = \mathfrak{l}_0 \otimes \mathbb{C}$  given by

$$\mathfrak{l}_0 \cong \text{Lie}(L) \simeq \mathfrak{o}(2) + \mathfrak{o}(p-2, q).$$

Any character of the Lie algebra  $\mathfrak{l}_0$  (or any complex character of  $\mathfrak{l}$ ) is determined by its restriction to  $\mathfrak{h}_0^c$ . So, we shall write  $\mathbb{C}_\nu$  for the character of the Lie algebra  $\mathfrak{l}_0$  whose restriction to  $\mathfrak{h}^c$  is  $\nu \in (\mathfrak{h}^c)^*$ . With this notation, the character of  $L$  acting on  $\wedge^{\dim \mathfrak{u}} \mathfrak{u}$  is written as  $\mathbb{C}_{2\rho(\mathfrak{u})}$  where

$$\rho(\mathfrak{u}) := \left( \frac{p+q}{2} - 1 \right) f_1. \tag{5.3.1}$$

The homogeneous manifold  $G/L$  carries a  $G$ -invariant complex structure with canonical bundle  $\wedge^{\text{top}} T^*G/L \simeq G \times_L \mathbb{C}_{2\rho(\mathfrak{u})}$ . As an algebraic analogue of a Dolbeault cohomology of a  $G$ -equivariant holomorphic vector bundle over a complex manifold  $G/L$ , Zuckerman introduced the cohomological parabolic induction  $\mathcal{R}_q^j \equiv (\mathcal{R}_q^{\mathfrak{q}})^j$  ( $j \in \mathbb{N}$ ), which is a covariant functor from the category of metaplectic  $(\mathfrak{l}, (L \cap K)^\sim)$ -modules to that of  $(\mathfrak{g}, K)$ -modules. Here,  $\tilde{L}$  is a metaplectic covering of  $L$  defined by the character of  $L$  acting on  $\wedge^{\dim \mathfrak{u}} \mathfrak{u} \simeq \mathbb{C}_{2\rho(\mathfrak{u})}$ . In this paper, we follow the normalization in [21, Definition 6.20] which is different from the one in [19] by a ‘ $\rho$ -shift’.

The character  $\mathbb{C}_{\lambda f_1}$  of  $\mathfrak{l}_0$  lifts to a metaplectic  $(\mathfrak{l}, (L \cap K)^\sim)$ -module if and only if  $\lambda \in \mathbb{Z} + \frac{p+q}{2}$ . In particular, we can define  $(\mathfrak{g}, K)$ -modules  $\mathcal{R}_q^j(\mathbb{C}_{\lambda f_1})$  for  $\lambda \in A_0(p, q)$ . The  $\mathcal{Z}(\mathfrak{g})$ -infinitesimal character of  $\mathcal{R}_q^j(\mathbb{C}_{\lambda f_1})$  is given by

$$\left( \lambda, \frac{p+q}{2} - 2, \frac{p+q}{2} - 3, \dots, \frac{p+q}{2} - \left\lfloor \frac{p+q}{2} \right\rfloor \right) \in (\mathfrak{h}^c)^*$$

in the Harish-Chandra parametrization if it is non-zero. In the sense of Vogan [22], we have

$$\mathbb{C}_{\lambda f_1} \text{ is in the good range} \Leftrightarrow \lambda > \frac{p+q}{2} - 2,$$

$$\mathbb{C}_{\lambda f_1} \text{ is in the weakly fair range} \Leftrightarrow \lambda \geq 0.$$

We note that  $\mathcal{R}_q^j(\mathbb{C}_{\lambda_{f_1}}) = 0$  if  $j \neq p - 2$  and if  $\lambda \in A_0(p, q)$ . This follows from a general result in [20] for  $\lambda \geq 0$ ; and [6] for  $\lambda = -\frac{1}{2}$ .

**5.4.** For  $b \in \mathbb{Z}$ , we define an algebraic direct sum of  $K = O(p) \times O(q)$ -modules by

$$\Xi(K:b) \equiv \Xi(O(p) \times O(q):b) := \bigoplus_{\substack{m,n \in \mathbb{N} \\ m-n \geq b \\ m-n \equiv b \pmod{2}}} \mathcal{H}^m(\mathbb{R}^p) \boxtimes \mathcal{H}^n(\mathbb{R}^q). \tag{5.4.1}$$

For  $\lambda \in A_0(p, q)$ , we put

$$b \equiv b(\lambda, p, q) := \lambda - \frac{p}{2} + \frac{q}{2} + 1 \in \mathbb{Z}, \tag{5.4.2}$$

$$\varepsilon \equiv \varepsilon(\lambda, p, q) := (-1)^b. \tag{5.4.3}$$

We define the line bundle  $\mathcal{L}_n$  over  $G/L$  by the character  $n\mathfrak{f}_1$  of  $L$  (see Section 5.3).

Here is a summary for different realizations of the representation  $\pi_{+,\lambda}^{p,q}$ :

**Fact 5.4.** *Let  $p, q \in \mathbb{N}$  ( $p > 1$ ).*

(1) *For any  $\lambda \in A_0(p, q)$ , each of the following 5 conditions defines uniquely a  $(\mathfrak{g}, K)$ -module, which are mutually isomorphic. We shall denote it by  $(\pi_{+,\lambda}^{p,q})_K$ . The  $(\mathfrak{g}, K)$ -module  $(\pi_{+,\lambda}^{p,q})_K$  is non-zero and irreducible.*

- (i) *A subrepresentation of the degenerate principal representation  $\text{Ind}_{\mathfrak{p}_{\max}}^G(\varepsilon \otimes \mathbb{C}_\lambda)$  (see Section 3.7) with  $K$ -type  $\Xi(K:b)$ .*
- (i)' *A quotient of  $\text{Ind}_{\mathfrak{p}_{\max}}^G(\varepsilon \otimes \mathbb{C}_{-\lambda})$  with  $K$ -type  $\Xi(K:b)$ .*
- (ii) *A subrepresentation of  $C_\lambda^\infty(X(p, q))_K$  with  $K$ -type  $\Xi(K:b)$ .*
- (iii) *The underlying  $(\mathfrak{g}, K)$ -module of the Dolbeault cohomology group  $H_{\bar{\partial}}^{p-2}(G/L, \mathcal{L}_{(\lambda+\frac{p+q-2}{2})})_K$ .*
- (iii)' *The Zuckerman–Vogan derived functor module  $\mathcal{R}_q^{p-2}(\mathbb{C}_{\lambda_{f_1}})$ .*

(2) *In the realization of (ii), if  $f \in (\pi_{+,\lambda}^{p,q})_K$ , then there exists an analytic function  $a \in C^\infty(S^{p-1} \times S^{q-1})$  such that*

$$f(\omega \cosh t, \eta \sinh t) = a(\omega, \eta)e^{-(\lambda+\rho)t}(1 + te^{-2t}O(1)) \quad \text{as } t \rightarrow \infty.$$

Here, we put  $\rho = \frac{p+q-2}{2}$ .

For details, we refer, for example, to [5] for (i) and (i)'; to [18] for (ii) and also for a relation with (i) (under some parity assumption on eigenspaces); to [7, Section 6] (see

also [6]) for (iii)'  $\Leftrightarrow$  (ii); and to [23] for (iii)  $\Leftrightarrow$  (iii)'. The second statement follows from a general theory of the boundary value problem with regular singularities; or also follows from a classical asymptotic formula of hypergeometric functions (see (8.3.1)) in our specific setting.

**Remark.** (1) By definition, (i) and (i)' make sense for  $p > 1$  and  $q > 0$ ; and others for  $p > 1$  and  $q \geq 0$ .

(2) Each of realization (i), (i)', (ii), and (iii) also gives a globalization of  $\pi_{+,\lambda}^{p,q}$ , namely, a continuous representation of  $G$  on a topological vector space. Because all of  $(\pi_{+,\lambda}^{p,q})_K$  ( $\lambda \in A_0(p, q)$ ) are unitarizable we may and do take the globalization  $\pi_{+,\lambda}^{p,q}$  to be the unitary representation of  $G$ .

(3) If  $\lambda > 0$  and  $\lambda \in A_0(p, q)$ , then the realization (ii) of  $\pi_{+,\lambda}^{p,q}$  gives a discrete series representation for  $X(p, q)$ . Conversely,

$$\{\pi_{+,\lambda}^{p,q} : \lambda \in A_0(p, q), \lambda > 0\}$$

exhausts the set of discrete series representations for  $X(p, q)$ .

If  $(p, q) = (1, 0)$ , then  $O(p, q) \simeq O(1)$  and it is convenient to define representations of  $O(1)$  by

$$\pi_{+,\lambda}^{1,0} = \begin{cases} \mathbf{1} & (\lambda = -\frac{1}{2}), \\ \text{sgn} & (\lambda = \frac{1}{2}), \\ 0 & (\text{otherwise}). \end{cases}$$

As we defined  $\pi_{+,\lambda}^{p,q}$  in Fact 5.4, we can also define an irreducible unitary representation, denoted by  $\pi_{-,\lambda}^{p,q}$ , for  $\lambda \in A_0(q, p)$  such that the underlying  $(\mathfrak{g}, K)$ -module has the following  $K$ -type:

$$\bigoplus_{\substack{m,n \in \mathbb{N} \\ m-n \leq -\lambda + \frac{q}{2} - \frac{p}{2} - 1 \\ m-n \equiv -\lambda + \frac{q}{2} - \frac{p}{2} - 1 \pmod{2}}} \mathcal{H}^m(\mathbb{R}^p) \boxtimes \mathcal{H}^n(\mathbb{R}^q).$$

Similarly to  $\pi_{+,\lambda}^{p,q}$ , the representations  $\pi_{-,\lambda}^{p,q}$  are realized in function spaces on another hyperboloid  $O(p, q)/O(p, q - 1)$ .

In order to understand the notation here, we remark:

- (i)  $\pi_{-,\lambda}^{p,q} \in \widehat{O(p, q)}$  corresponds to the representation  $\pi_{+,\lambda}^{q,p} \in \widehat{O(p, q)}$  if we identify  $O(p, q)$  with  $O(q, p)$ .
- (ii)  $\pi_{+,\lambda}^{p,0} \simeq \mathcal{H}^k(\mathbb{R}^p)$ , where  $k = \lambda - \frac{p-2}{2}$  and  $p \geq 1, k \in \mathbb{N}$ .

**5.5.** The case  $\lambda = \pm\frac{1}{2}$  is delicate, which happens when  $p + q \in 2\mathbb{N} + 1$ .

First, we assume  $p + q \in 2\mathbb{N} + 1$ . By using the equivalent realizations of  $\pi_{+, \lambda}^{p, q}$  in Fact 5.4 and by the classification of the composition series of the most degenerate principal series representation  $\text{Ind}_{\rho_{\max}}^G(\varepsilon \otimes \mathbb{C}_\lambda)$  (see [5]), we have non-splitting short exact sequences of  $(\mathfrak{g}, K)$ -modules:

$$0 \rightarrow (\pi_{-, -\frac{1}{2}}^{p, q})_K \rightarrow \text{Ind}_{\rho_{\max}}^G \left( (-1)^{\frac{p-q+1}{2}} \otimes \mathbb{C}_{-\frac{1}{2}} \right) \rightarrow (\pi_{+, \frac{1}{2}}^{p, q})_K \rightarrow 0, \tag{5.5.1}$$

$$0 \rightarrow (\pi_{+, -\frac{1}{2}}^{p, q})_K \rightarrow \text{Ind}_{\rho_{\max}}^G \left( (-1)^{\frac{p-q-1}{2}} \otimes \mathbb{C}_{-\frac{1}{2}} \right) \rightarrow (\pi_{-, \frac{1}{2}}^{p, q})_K \rightarrow 0. \tag{5.5.2}$$

Because  $\pi_{+, \lambda}^{p, q}$  ( $\lambda \in A_0(p, q)$ ) is self-dual, the dual  $(\mathfrak{g}, K)$ -modules of (5.5.1) and (5.5.2) give the following non-splitting short exact sequences of  $(\mathfrak{g}, K)$ -modules:

$$0 \rightarrow (\pi_{+, \frac{1}{2}}^{p, q})_K \rightarrow \text{Ind}_{\rho_{\max}}^G \left( (-1)^{\frac{p-q+1}{2}} \otimes \mathbb{C}_{\frac{1}{2}} \right) \rightarrow (\pi_{-, -\frac{1}{2}}^{p, q})_K \rightarrow 0, \tag{5.5.3}$$

$$0 \rightarrow (\pi_{-, \frac{1}{2}}^{p, q})_K \rightarrow \text{Ind}_{\rho_{\max}}^G \left( (-1)^{\frac{p-q-1}{2}} \otimes \mathbb{C}_{\frac{1}{2}} \right) \rightarrow (\pi_{+, -\frac{1}{2}}^{p, q})_K \rightarrow 0. \tag{5.5.4}$$

Next, we assume  $p + q \in 2\mathbb{N}$ . Then,  $\varpi^{p, q}$  is realized as a subrepresentation of some degenerate principal series (see [15, Lemma 3.7.2]). More precisely, we have non-splitting short exact sequences of  $(\mathfrak{g}, K)$ -modules

$$0 \rightarrow \varpi_K^{p, q} \rightarrow \text{Ind}_{\rho_{\max}}^G \left( (-1)^{\frac{p-q}{2}} \otimes \mathbb{C}_{-1} \right) \rightarrow ((\pi_{-, 1}^{p, q})_K \oplus (\pi_{+, 1}^{p, q})_K) \rightarrow 0, \tag{5.5.5}$$

$$0 \rightarrow ((\pi_{-, 1}^{p, q})_K \oplus (\pi_{+, 1}^{p, q})_K) \rightarrow \text{Ind}_{\rho_{\max}}^G \left( (-1)^{\frac{p-q}{2}} \otimes \mathbb{C}_1 \right) \rightarrow \varpi_K^{p, q} \rightarrow 0, \tag{5.5.6}$$

and an isomorphism of  $(\mathfrak{g}, K)$ -modules:

$$\text{Ind}_{\rho_{\max}}^G \left( (-1)^{\frac{p-q+2}{2}} \otimes \mathbb{C}_0 \right) \simeq (\pi_{-, 0}^{p, q})_K \oplus (\pi_{+, 0}^{p, q})_K. \tag{5.5.7}$$

These results will be used in another realization of the unipotent representation  $\varpi^{p, q}$ , namely, as a submodule of the Dolbeault cohomology group in a subsequent paper (cf. Part I, Introduction, Theorem B(4)).

### 6. Conformal embedding of the hyperboloid

This section prepares the geometric setup which will be used in Sections 7 and 9 for the branching problem of  $\varpi^{p, q}|_G$ . Throughout this section, we shall use the following

notation:

$$|x|^2 := |x'|^2 + |x''|^2 = \sum_{i=1}^{p'} (x'_i)^2 + \sum_{j=1}^{p''} (x''_j)^2, \quad \text{for } x := (x', x'') \in \mathbb{R}^{p'+p''} = \mathbb{R}^p,$$

$$|y|^2 := |y'|^2 + |y''|^2 = \sum_{i=1}^{q'} (y'_i)^2 + \sum_{j=1}^{q''} (y''_j)^2, \quad \text{for } y := (y', y'') \in \mathbb{R}^{q'+q''} = \mathbb{R}^q.$$

**6.1.** We define two open subsets of  $\mathbb{R}^{p+q}$  by

$$\mathbb{R}_+^{p'+p'', q'+q''} := \{(x, y) = ((x', x''), (y', y'')) \in \mathbb{R}^{p'+p'', q'+q''} : |x'| > |y'|\},$$

$$\mathbb{R}_-^{p'+p'', q'+q''} := \{(x, y) = ((x', x''), (y', y'')) \in \mathbb{R}^{p'+p'', q'+q''} : |x'| < |y'|\}.$$

Then the disjoint union  $\mathbb{R}_+^{p'+p'', q'+q''} \cup \mathbb{R}_-^{p'+p'', q'+q''}$  is open dense in  $\mathbb{R}^{p+q}$ . Let us consider the intersection of  $\mathbb{R}_\pm^{p'+p'', q'+q''}$  with the submanifolds  $M$  and  $\Xi$  given in Section 3.2:

$$M \subset \Xi \subset \mathbb{R}^{p,q}.$$

Then, we define two open subsets of  $M \simeq S^{p-1} \times S^{q-1}$  by

$$M_\pm := M \cap \mathbb{R}_\pm^{p'+p'', q'+q''}. \tag{6.1.1}$$

Likewise, we define two open subsets of the cone  $\Xi$  by

$$\Xi_\pm := \Xi \cap \mathbb{R}_\pm^{p'+p'', q'+q''}. \tag{6.1.2}$$

We notice that if  $(x, y) = ((x', x''), (y', y'')) \in \Xi$  then

$$|x'| > |y'| \Leftrightarrow |x''| < |y''|$$

because  $|x'|^2 + |x''|^2 = |y'|^2 + |y''|^2$ . The following statement is immediate from definition:

$$\Xi_+ = \emptyset \Leftrightarrow M_+ = \emptyset \Leftrightarrow p'q'' = 0, \tag{6.1.3}$$

$$\Xi_- = \emptyset \Leftrightarrow M_- = \emptyset \Leftrightarrow p''q' = 0. \tag{6.1.4}$$

**6.2.** We embed the direct product of hyperboloids

$$X(p', q') \times X(q'', p'') = \{((x', y'), (y'', x'')) : |x'|^2 - |y'|^2 = |y''|^2 - |x''|^2 = 1\}$$

into  $\Xi_+$  ( $\subset \mathbb{R}^{p,q}$ ) by the map

$$X(p', q') \times X(q'', p'') \hookrightarrow \Xi_+, ((x', y'), (y'', x'')) \mapsto (x', x'', y', y''). \tag{6.2.1}$$

The image is transversal to rays (see [15, Section 3.3] for definition) and the induced pseudo-Riemannian metric  $g_{X(p',q') \times X(q'',p'')}$  on  $X(p', q') \times X(q'', p'')$  has signature  $(p' - 1, q') + (p'', q'' - 1) = (p - 1, q - 1)$ . With the notation in Section 5.2, we have

$$g_{X(p',q') \times X(q'',p'')} = g_{X(p',q')} \oplus (-g_{X(q'',p'')}).$$

We note that if  $p'' = q' = 0$ , then  $X(p', q') \times X(q'', p'')$  is diffeomorphic to  $S^{p-1} \times S^{q-1}$ , and  $g_{X(p',0) \times X(q'',0)}$  is nothing but the pseudo-Riemannian metric  $g_{S^{p-1} \times S^{q-1}}$  of signature  $(p - 1, q - 1)$  (see [15, Section 3.3]).

By the same computation as in (3.4.1), we have the relationship among the Yamabe operators on hyperboloids (see also (5.2.2)) by

$$\tilde{\Delta}_{X(p',q') \times X(q'',p'')} = \tilde{\Delta}_{X(p',q')} - \tilde{\Delta}_{X(q'',p'')}. \tag{6.2.2}$$

We denote by  $\Phi_1$  the composition of (6.2.1) and the projection  $\Phi: \Xi \rightarrow M$  (see [15, (3.2.4)]), namely,

$$\Phi_1: X(p', q') \times X(q'', p'') \hookrightarrow M, ((x', y'), (y'', x'')) \mapsto \left( \frac{(x', x'')}{|x|}, \frac{(y', y'')}{|y|} \right). \tag{6.2.3}$$

**Lemma 6.2.** (1) *The map  $\Phi_1: X(p', q') \times X(q'', p'') \rightarrow M$  is a diffeomorphism onto  $M_+$ . The inverse map  $\Phi_1^{-1}: M_+ \rightarrow X(p', q') \times X(q'', p'')$  is given by the formula:*

$$((u', u''), (v', v'')) \mapsto \left( \frac{(u', v')}{\sqrt{|u'|^2 - |v'|^2}}, \frac{(v'', u'')}{\sqrt{|v''|^2 - |u''|^2}} \right). \tag{6.2.4}$$

(2)  $\Phi_1$  is a conformal map with conformal factor  $|x|^{-1} = |y|^{-1}$ , where  $x = (x', x'') \in \mathbb{R}^{p'+p''}$  and  $y = (y', y'') \in \mathbb{R}^{q'+q''}$ . Namely, we have

$$\Phi_1^*(g_{S^{p-1} \times S^{q-1}}) = \frac{1}{|x|^2} g_{X(p',q') \times X(q'',p'')}.$$

**Proof.** The first statement is straightforward in light of the formula

$$|u'|^2 - |v'|^2 = |v''|^2 - |u''|^2 > 0$$

for  $(u, v) = ((u', u''), (v', v'')) \in M_+ \subset S^{p-1} \times S^{q-1}$ .

The second statement is a special case of Lemma 3.3.  $\square$



**6.3.** Now, the conformal diffeomorphism  $\Phi_1 : X(p', q') \times X(q'', p'') \xrightarrow{\sim} M_+$  establishes a bijection of the kernels of the Yamabe operators owing to Proposition 2.6:

**Lemma 6.3.**  $\widetilde{\Phi}_1^*$  gives a bijection from  $\text{Ker } \widetilde{\Delta}_{M_+}$  onto  $\text{Ker } \widetilde{\Delta}_{X(p', q') \times X(q'', p'')}$ .

Here, the twisted pull-backs  $\widetilde{\Phi}_1^*$  and  $(\widetilde{\Phi}_1^{-1})^*$  (see Definition 2.3), namely,

$$\widetilde{\Phi}_1^* : C^\infty(M_+) \rightarrow C^\infty(X(p', q') \times X(q'', p'')) \tag{6.3.1}$$

$$(\widetilde{\Phi}_1^{-1})^* : C^\infty(X(p', q') \times X(q'', p'')) \rightarrow C^\infty(M_+) \tag{6.3.2}$$

are given by the formulae

$$(\widetilde{\Phi}_1^* F)(x', y', y'', x'') := (|x'|^2 + |x''|^2)^{-\frac{p+q-4}{4}} F \left( \frac{(x', x'')}{\sqrt{|x'|^2 + |x''|^2}}, \frac{(y', y'')}{\sqrt{|y'|^2 + |y''|^2}} \right),$$

$$((\widetilde{\Phi}_1^{-1})^* f)(u', u'', v', v'') := (|u'|^2 - |v'|^2)^{-\frac{p+q-4}{4}} f \left( \frac{(u', v')}{\sqrt{|u'|^2 - |v'|^2}}, \frac{(u'', v'')}{\sqrt{|u''|^2 - |v''|^2}} \right),$$

respectively. We remark that  $(\widetilde{\Phi}_1^{-1})^* = (\widetilde{\Phi}_1^*)^{-1}$ .

**6.4.** Similarly to Section 6.2, we consider another embedding

$$X(q', p') \times X(p'', q'') \hookrightarrow \Xi_-, \quad ((y', x'), (x'', y'')) \mapsto (x', x'', y', y''). \tag{6.4.1}$$

The composition of (6.4.1) and the projection  $\Phi : \Xi \rightarrow M$  is denoted by

$$\Phi_2 : X(q', p') \times X(p'', q'') \hookrightarrow M, \quad ((y', x'), (x'', y'')) \mapsto \left( \frac{(x', x'')}{|x|}, \frac{(y', y'')}{|y|} \right). \tag{6.4.2}$$

Obviously, results analogous to Lemmas 6.2 and 6.3 hold for  $\Phi_2$ . For example, here is a lemma parallel to Lemma 6.2:

**Lemma 6.4.** *The map  $\Phi_2 : X(q', p') \times X(p'', q'') \rightarrow M_-$  is a conformal diffeomorphism onto  $M_-$ . The inverse map  $\Phi_2^{-1} : M_- \rightarrow X(q', p') \times X(p'', q'')$  is given by*

$$((u', u''), (v', v'')) \mapsto \left( \frac{(v', u')}{\sqrt{|v'|^2 - |u'|^2}}, \frac{(u'', v'')}{\sqrt{|u''|^2 - |v''|^2}} \right).$$

**7. Explicit branching formula (discrete decomposable case)**

**7.1.** If one of  $p', q', p''$  or  $q''$  is zero, then the restriction  $\varpi^{p,q}|_{G'}$  is decomposed discretely into irreducible representations of  $G' = O(p', q') \times O(p'', q'')$  as we saw in Section 4. In this case, we can determine the branching laws of  $\varpi^{p,q}|_{G'}$  as follows:

**Theorem 7.1.** *Let  $p + q \in 2\mathbb{N}$ . If  $q'' \geq 1$  and  $q' + q'' = q$ , then we have an irreducible decomposition of the unitary representation  $\varpi^{p,q}$  when restricted to  $O(p, q') \times O(q'')$ :*

$$\begin{aligned} \varpi^{p,q}|_{O(p,q') \times O(q'')} &\simeq \sum_{l=0}^{\infty} \oplus \pi_{+,l+\frac{q''}{2}-1}^{p,q'} \boxtimes \pi_{-,l+\frac{q''}{2}-1}^{0,q''} \\ &\simeq \sum_{l=0}^{\infty} \oplus \pi_{+,l+\frac{q''}{2}-1}^{p,q'} \boxtimes \mathcal{H}^l(\mathbb{R}^{q''}). \end{aligned} \tag{7.1.1}$$

We shall prove Theorem 7.1 in Section 7.5 after we prepare an algebraic lemma in Section 7.3 and a geometric lemma in Section 7.4.

**Remark.** The formula in Theorem 7.1 is nothing but a  $K$ -type formula (see Theorem 3.6.1) when  $q' = 0$ .

**7.2.** The branching law (7.1.1) is an infinite direct sum for  $q'' > 1$ . This subsection treats the case  $q'' = 1$ , which is particularly interesting, because the branching formula consists of only two irreducible representations (we recall  $\mathcal{H}^l(\mathbb{R}^1) \neq 0$  if and only if  $l = 0, 1$ ). For simplicity, we shall assume  $q \geq 3$  in Section 7.2.

It follows from Theorem 7.1 with  $q' = 1$  that

$$\begin{aligned} \varpi^{p,q}|_{O(p,q-1) \times O(1)} &\simeq \left( \pi_{+,-\frac{1}{2}}^{p,q-1} \boxtimes \pi_{-,-\frac{1}{2}}^{0,1} \right) \oplus \left( \pi_{+,\frac{1}{2}}^{p,q-1} \boxtimes \pi_{-,\frac{1}{2}}^{0,1} \right) \\ &\simeq \left( \pi_{+,-\frac{1}{2}}^{p,q-1} \boxtimes \mathbf{1} \right) \oplus \left( \pi_{+,\frac{1}{2}}^{p,q-1} \boxtimes \text{sgn} \right). \end{aligned} \tag{7.2.1}$$

This means that  $\varpi_K^{p,q}$  can be realized in a subspace of  $C_{\frac{1}{2}}^{\infty}(X(p, q - 1))$ , namely, the kernel of the Yamabe operator  $\tilde{\Delta}_{X(p,q-1)}$  (see (5.2.3)).

More precisely, according to the direct sum decomposition (see (5.2.4)), we have

$$\text{Ker } \tilde{\Delta}_{X(p,q-1)} \equiv C_{\frac{1}{2}}^{\infty}(X(p, q - 1)) = C_{\frac{1}{2},+}^{\infty}(X(p, q - 1)) + C_{\frac{1}{2},-}^{\infty}(X(p, q - 1)). \tag{7.2.2}$$

We recall that the central element  $-I_{p+q} \in G$  acts on  $\varpi^{p,q}$  with scalar  $\delta$ , where

$$\delta := (-1)^{\frac{p-q}{2}}. \tag{7.2.3}$$

In view of the composition series of eigenspaces on the hyperboloid (see [18] for the case  $\delta = +$ ; similarly for  $\delta = -$ ), we have non-splitting exact sequences of Harish-Chandra modules of  $O(p, q - 1)$ :

$$0 \rightarrow (\pi_{+, -\frac{1}{2}}^{p, q-1})_K \rightarrow C_{\frac{1}{2}, \delta}^\infty(X(p, q - 1))_K \rightarrow (\pi_{-, \frac{1}{2}}^{p, q-1})_K \rightarrow 0, \tag{7.2.4}$$

$$0 \rightarrow (\pi_{+, \frac{1}{2}}^{p, q-1})_K \rightarrow C_{\frac{1}{2}, -\delta}^\infty(X(p, q - 1))_K \rightarrow (\pi_{-, -\frac{1}{2}}^{p, q-1})_K \rightarrow 0. \tag{7.2.5}$$

Here is a realization of  $\varpi^{p, q}$  in a subspace of the kernel of the Yamabe operator on the hyperboloid  $X(p, q - 1)$ , on which  $O(p, q)$  acts as meromorphic conformal transformations.

**Corollary 7.2.1.** *Let  $W_\pm$  be the unique non-trivial subrepresentation of  $O(p, q - 1)$  in  $(\text{Ker } \tilde{\Delta}_{X(p, q-1)})_\pm \equiv C_{\frac{1}{2}, \pm}^\infty(X(p, q - 1))$ . Each of the underlying  $(\mathfrak{g}, K)$ -modules is infinitesimally unitarizable, and we denote the resulting unitary representation by  $\overline{W}_\pm$ . Then, the irreducible unitary representation  $\varpi^{p, q}$  of  $O(p, q)$  is realized on the direct sum  $\overline{W}_+ + \overline{W}_-$ .*

We note that  $\overline{W}_{-\delta} \subset L^2(X(p, q - 1))$  and  $\overline{W}_\delta \not\subset L^2(X(p, q - 1))$  where  $\delta = (-1)^{\frac{p-q}{2}}$ .

It is interesting to note that the Laplacian  $\Delta_{X(p, q-1)}$  acts on a discrete series  $\pi_{+, \lambda}^{p, q-1}$  for the hyperboloid  $X(p, q - 1)$  as a scalar  $-\lambda^2 + \frac{1}{4}(p + q - 3)^2$ , which attains the maximum when  $\lambda = \frac{1}{2}$  if  $p + (q - 1) \in 2\mathbb{N} + 1$ .

Taking the direct sum of two exact sequences (7.2.4) and (7.2.5), we have the following:

**Theorem 7.2.2.** *There is a non-split exact sequence of Harish-Chandra modules for  $O(p - 1, q)$ .*

$$0 \rightarrow (\varpi^{p, q})_K \rightarrow (\text{Ker } \tilde{\Delta}_{X(p, q-1)})_K \rightarrow (\varpi^{p+1, q-1})_K \rightarrow 0. \tag{7.2.6}$$

It is a mysterious phenomenon in (7.2.6) that  $\varpi^{p, q}$  extends to a representation of  $O(p, q)$  and  $(\varpi^{p+1, q-1})_K$  to that of  $O(p + 1, q - 1)$ . So, different real forms of  $O(p + q, \mathbb{C})$  act on subquotients of the kernel of the Yamabe operator on the hyperboloid  $X(p, q - 1) = O(p, q - 1)/O(p - 1, q - 1)$ !

Here, we remark that  $\text{Ker } \tilde{\Delta}_{X(p, q-1)} \cap L^2(X(p, q - 1)) \neq \{0\}$  if and only if  $p + q \in 2\mathbb{Z}$ , by the classification of discrete series for the hyperboloid  $X(p, q - 1)$  for  $p > 1$ .

**7.3.** By Theorem 4.2, the restriction  $\varpi^{p, q}|_{K'}$  is  $K'$ -admissible, where  $K' \simeq O(p) \times O(q')$ . Let us first find the  $K'$ -structure of  $\varpi^{p, q}$ . We recall a classical

branching law with respect to  $(O(q), O(q') \times O(q''))$  ( $q = q' + q'', q' \geq 1, q'' \geq 1$ ):

$$\mathcal{H}^n(\mathbb{R}^q) \simeq \bigoplus_{\substack{k, l \in \mathbb{N} \\ k+l \leq n \\ k+l \equiv n \pmod{2}}} \mathcal{H}^k(\mathbb{R}^{q'}) \boxtimes \mathcal{H}^l(\mathbb{R}^{q''}). \tag{7.3.1}$$

We define  $b = \frac{1}{2}(q - p)$ . Then we have isomorphisms as  $K'$ -modules:

$$\begin{aligned} & \bigoplus_{\substack{m, n \in \mathbb{N} \\ m-n=b}} \mathcal{H}^m(\mathbb{R}^p) \boxtimes \mathcal{H}^n(\mathbb{R}^q) \Big|_{O(p) \times O(q') \times O(q'')} \\ & \simeq \bigoplus_{\substack{m, n \in \mathbb{N} \\ m-n=b}} \bigoplus_{\substack{k, l \in \mathbb{N} \\ k+l \leq n \\ k+l \equiv n \pmod{2}}} \mathcal{H}^m(\mathbb{R}^p) \boxtimes \mathcal{H}^k(\mathbb{R}^{q'}) \boxtimes \mathcal{H}^l(\mathbb{R}^{q''}) \\ & \simeq \bigoplus_{l \in \mathbb{N}} \bigoplus_{\substack{m, k \in \mathbb{N} \\ m-b \geq k+l \\ m-b \equiv k+l \pmod{2}}} \mathcal{H}^m(\mathbb{R}^p) \boxtimes \mathcal{H}^k(\mathbb{R}^{q'}) \boxtimes \mathcal{H}^l(\mathbb{R}^{q''}) \\ & \simeq \bigoplus_{l \in \mathbb{N}} \Xi(O(p) \times O(q'): b+l) \boxtimes \mathcal{H}^l(\mathbb{R}^{q''}). \end{aligned}$$

In view of Theorem 3.6.1, we have proved:

**Lemma 7.3.** *We have an isomorphism of  $K'$ -modules:*

$$\varpi_K^{p,q} \simeq \bigoplus_{l \in \mathbb{N}} \Xi(O(p) \times O(q'): \frac{q-p}{2} + l) \boxtimes \mathcal{H}^l(\mathbb{R}^{q''}). \tag{7.3.2}$$

**7.4.** By Theorem 4.2 and [10, Lemma 1.3], the underlying  $(\mathfrak{g}, K)$ -module  $\varpi_K^{p,q}$  is decomposed into an algebraic direct sum of irreducible  $(\mathfrak{g}', K')$ -modules:

$$\varpi_K^{p,q} \simeq \bigoplus_{\tau} m_{\tau} \tau \simeq \bigoplus_{\tau_1, \tau_2} m_{\tau_1, \tau_2} \tau_1 \boxtimes \tau_2, \tag{7.4.1}$$

where  $m_{\tau} \in \mathbb{N}$  and  $\tau$  runs over irreducible  $(\mathfrak{g}', K')$ -modules or equivalently,  $\tau_1$  runs over irreducible  $(\mathfrak{g}'_1, K'_1)$ -modules (with obvious notation for  $G'_1 := O(p, q')$ ) and  $\tau_2$  runs over irreducible  $O(q'')$ -modules. It follows from Lemma 7.3 that for each  $l$  there exists a  $(\mathfrak{g}'_1, K'_1)$ -module  $W_l$  which is a direct sum of irreducible  $(\mathfrak{g}'_1, K'_1)$ -module such that  $W_l$  is isomorphic to  $\Xi(O(p) \times O(q'): \frac{q-p}{2} + l)$  as  $K'_1$ -modules. Let us prove that  $W_l$  is in fact irreducible as a  $(\mathfrak{g}'_1, K'_1)$ -module.

**Lemma 7.4.**  *$W_l$  is realized in a subspace of  $C_{\lambda}^{\infty}(X(p, q'))$  with  $\lambda = l + \frac{q''}{2} - 1$ .*

**Proof.** In our conformal construction of  $\varpi^{p,q}$  in Section 3, we recall that each  $K$ -finite vector of  $\varpi^{p,q}$  is an analytic function satisfying the Yamabe equation on  $M \simeq S^{p-1} \times S^{q-1}$ . By using the conformal diffeomorphism  $\Phi_1 : X(p, q') \times$

$S^{q''-1} \rightarrow M_+$ , an open dense subset of  $M$  (see (6.1.4) with  $p'' = 0$ ), we can realize  $W_l \boxtimes \mathcal{H}^l(\mathbb{R}^{q''})$  in the space of smooth functions on  $X(p, q') \times S^{q''-1}$  satisfying the Yamabe equation by the following diagram:

$$\begin{array}{ccccccc}
 \mathcal{A}(M) & \subset & C^\infty(M) & \hookrightarrow & C^\infty(M_+) & \xrightarrow{\sim} & C^\infty(X(p, q') \times S^{q''-1}) \\
 & \text{dense} & & \text{restriction} & & & \\
 \cup & & \cup & & \cup & & \cup \\
 W_l \boxtimes \mathcal{H}^l(\mathbb{R}^{q''}) & \subset & \varpi_K^{p, q'} & \subset & \text{Ker } \tilde{\Delta}_M & \hookrightarrow & \text{Ker } \tilde{\Delta}_{M_+} \xrightarrow{\sim} \text{Ker } \tilde{\Delta}_{X(p, q') \times S^{q''-1}} \\
 & & \text{dense} & \text{restriction} & & & 
 \end{array}$$

Because  $\tilde{\Delta}_{X(p, q') \times S^{q''-1}} = \tilde{\Delta}_{X(p', q')} - \tilde{\Delta}_{S^{q''-1}}$  (see (6.2.2)) acts on  $W_l \boxtimes \mathcal{H}^l(\mathbb{R}^{q''})$  as 0, and because  $\tilde{\Delta}_{S^{q''-1}}$  acts on  $\mathcal{H}^l(\mathbb{R}^{q''})$  as a scalar  $\frac{1}{4} - (l + \frac{q''}{2} - 1)^2$  (see (3.5.1)), we conclude that  $\tilde{\Delta}_{X(p', q')}$  acts on  $W_l$  as the same scalar. Hence, Lemma is proved.  $\square$

**7.5.** Let us complete the proof of Theorem 7.1. It follows from Lemma 7.4 together with the  $K'_1$ -structure of  $W_l$  in Section 7.3 that  $W_l$  is irreducible and isomorphic to  $(\pi_{+, l + \frac{q''-2}{2}}^{p, q'})_{K'_1}$  as  $(\mathfrak{g}'_1, K'_1)$ -module (see Fact 5.4). Therefore, we have an isomorphism of  $(\mathfrak{g}', K')$ -modules

$$\varpi_K^{p, q'} \simeq \bigoplus_{l=0}^{\infty} (\pi_{+, l + \frac{q''-1}{2}}^{p, q'})_{K'_1} \boxtimes \mathcal{H}^l(\mathbb{R}^{q''}) \quad (\text{algebraic direct sum}).$$

Taking the closure in the Hilbert space, we have (7.1.1). Hence Theorem 7.1 is proved.  $\square$

**7.6.** So far, we have not used the irreducibility of  $\varpi^{p, q}$  in the branching law. Although the irreducibility of  $\varpi^{p, q}$  ( $p, q \neq (2, 2)$ ) is known [1], we can give a new and simple proof for it, as an application of the branching formulae in Theorem 7.1.

**Theorem 7.6.** *Let  $p, q \geq 2$ ,  $p + q \in 2\mathbb{Z}$  and  $(p, q) \neq (2, 2)$ . Then,  $\varpi^{p, q}$  is an irreducible representation of  $O(p, q)$ .*

**Proof.** Suppose  $W \neq \{0\}$  is a closed invariant subspace of the unitary representation  $(\varpi^{p, q}, \overline{V^{p, q}})$ . We want to prove  $W = \overline{V^{p, q}}$ .

Without loss of generality ( $p$  and  $q$  play a symmetric role), we may assume  $q \geq 3$ , and fix  $q' \geq 1, q'' \geq 2$  such that  $q' + q'' = q$ . We write  $\overline{V^{p, q}} = \sum_{l=0}^{\infty} \oplus V_l$  according to the irreducible decomposition (7.1.1) of  $G' = G'_1 \times G'_2 := O(p, q') \times O(q'')$ .

Because  $W$  is non-zero,  $W$  contains a  $K$ -type of the form  $\mathcal{H}^a(\mathbb{R}^p) \boxtimes \mathcal{H}^b(\mathbb{R}^q)$ , which we fix once for all. In view of the branching formula (7.3.1),  $\mathcal{H}^b(\mathbb{R}^q)$  contains a non-zero  $O(q'')$ -fixed vector. That is,

$$V_0 := \overline{V^{p, q}}^{G'_2} \supset W^{G'_2} \supset \mathcal{H}^a(\mathbb{R}^p) \boxtimes (\mathcal{H}^b(\mathbb{R}^q))^{G'_2} \neq \{0\}.$$

Because  $G'_1 = O(p, q')$  acts on  $V_0$  by  $\pi_{+, \frac{q''}{2}-1}^{p, q'}$  (see Theorem 7.1) which is irreducible, and because  $V_0 \supset W^{G_2}$  is stable under the action of  $G'_1$ , we conclude that  $V_0 = W^{G_2}$ . Hence, we have proved

$$W \supset V_0. \tag{7.6.1}$$

If  $W \neq \overline{V^{p, q}}$ , the orthogonal complement  $W^\perp \supset V_0$  by the same argument. This would contradict to  $W \cap W^\perp = \{0\}$ . Therefore,  $W$  must coincides with  $\overline{V^{p, q}}$ . Hence,  $\varpi^{p, q}$  is irreducible.  $\square$

**8. Inner product on  $\varpi^{p, q}$  and the Parseval–Plancherel formula**

In this section, we prove the Parseval–Plancherel-type formula for our discrete decomposable branching law given in Theorem 7.1. The proof of Theorem 3.9.1 of Part I [15] is also given. Our main result in this section is Theorem 8.6.

**8.1.** In Sections 8.1–8.3, we give some explicit formulas on the Jacobi functions, which are key to the proof of the Parseval–Plancherel-type formula of branching laws of representations attached to minimal nilpotent orbits (Theorem 8.6) and also to minimal elliptic orbits [14].

For readers’ convenience, we include here some of the proofs.

We begin with a brief summary of some known fact on the Jacobi function (see [16]). Let us consider the differential operator

$$L := \frac{d^2}{dt^2} + ((2\lambda' + 1) \tanh t + (2\lambda'' + 1) \coth t) \frac{d}{dt}.$$

We recall that for  $\lambda, \lambda', \lambda'' \in \mathbb{C}$ ,  $\lambda'' \neq -1, -2, \dots$ , the Jacobi function  $\varphi_{i\lambda}^{(\lambda'', \lambda')}(t)$  is the unique even solution to the following differential equation

$$(L + ((\lambda' + \lambda'' + 1)^2 - \lambda^2))\varphi = 0 \tag{8.1.1}$$

such that  $\varphi(0) = 0$ .

Under the change of variables  $y = e^{-2t}$ ,  $L$  has also a regular singularity at  $y = 0$ . Then (8.1.1) has characteristic exponents  $\frac{1}{2}(\pm\lambda - (\lambda' + \lambda'' + 1))$  at  $y = 0$ . Thus, for each  $\lambda \in \mathbb{C}$ , there exists a unique analytic solution  $\Psi_\lambda^{(\lambda'', \lambda')}(t)$  to (8.1.1) on  $t > 0$  such that  $\lim_{t \rightarrow \infty} e^{-(\lambda - \lambda' - \lambda'' - 1)t} \Psi_\lambda^{(\lambda'', \lambda')}(t) = 1$ . If  $\lambda \neq 0$  then  $\Psi_\lambda^{(\lambda'', \lambda')}(t)$  and  $\Psi_{-\lambda}^{(\lambda'', \lambda')}(t)$  are linearly independent, and  $\varphi_{i\lambda}^{(\lambda'', \lambda')}(t)$  is a linear combination of both. Since  $\varphi_{i\lambda}^{(\lambda'', \lambda')}(t) = \varphi_{-i\lambda}^{(\lambda'', \lambda')}(t)$ , we can write as

$$\varphi_{i\lambda}^{(\lambda'', \lambda')}(t) = c^{(\lambda'', \lambda')}(\lambda) \Psi_\lambda^{(\lambda'', \lambda')}(t) + c^{(\lambda'', \lambda')}(-\lambda) \Psi_{-\lambda}^{(\lambda'', \lambda')}(t). \tag{8.1.2}$$

We have from [16, (2.18)] (or from [2, 2.10 (2)]) that

$$c^{(\lambda'', \lambda')}(\lambda) \equiv c^{(\lambda'', -\lambda')}(\lambda) := \frac{2^{-\lambda+\lambda'+\lambda''+1} \Gamma(1 + \lambda'') \Gamma(\lambda)}{\Gamma(\frac{\lambda''-\lambda'+1+\lambda}{2}) \Gamma(\frac{\lambda''+\lambda'+1+\lambda}{2})}. \tag{8.1.3}$$

For  $\lambda', \lambda'' > 0$  and  $\text{Re } \lambda \geq 0$ , we note

$$c^{(\lambda'', \lambda')}(\lambda) = 0 \quad \text{if and only if} \quad \lambda > 0 \quad \text{and} \quad -\lambda + \lambda' - \lambda'' - 1 \in 2\mathbb{N}. \tag{8.1.4}$$

In terms of the hypergeometric function, we have the following expressions:

$$\varphi_{i\lambda}^{(\lambda'', \lambda')}(t) = {}_2F_1\left(\frac{\lambda' + \lambda'' + 1 - \lambda}{2}, \frac{\lambda' + \lambda'' + 1 + \lambda}{2}; \lambda'' + 1; -\sinh^2 t\right),$$

$$\Psi_{\lambda}^{(\lambda'', \lambda')}(t) = (2 \sinh t)^{\lambda-\lambda'-\lambda''-1} {}_2F_1\left(\frac{\lambda' + \lambda'' + 1 - \lambda}{2}, \frac{\lambda' - \lambda'' + 1 - \lambda}{2}; 1 - \lambda; -\frac{1}{\sinh^2 t}\right).$$

We introduce a meromorphic function of three variables  $\lambda, \lambda'$  and  $\lambda''$  by

$$M \equiv M(\lambda, \lambda', \lambda'') := (-1)^{\frac{\lambda'-\lambda''-\lambda-1}{2}} \frac{\Gamma(\frac{\lambda'+\lambda''-\lambda+1}{2})\Gamma(\lambda+1)}{\Gamma(\frac{\lambda'-\lambda''+\lambda+1}{2})\Gamma(\lambda''+1)}. \tag{8.1.5}$$

**Lemma 8.1** (Triangular relation of the Jacobi function, cf. [14]). *Assume that  $\lambda, \lambda', \lambda'' > 0$  and  $\lambda' - \lambda'' - \lambda - 1 \in 2\mathbb{N}$ . If  $0 < \theta < \frac{\pi}{2}$  and  $0 < t$  satisfy  $\sin \theta \cosh t = 1$ , then*

$$\varphi_{i\lambda'}^{(\lambda, \lambda'')}(i\theta) = M(\cosh t)^{\lambda+\lambda'+\lambda''+1} \varphi_{i\lambda}^{(\lambda'', \lambda')}(t).$$

For the sake of completeness, we give a proof here. The coordinate change between  $\theta$  and  $t$  appears in a conformal embedding that we shall use later in Section 8.5.

**Proof.** We recall one of Kummer’s relations among hypergeometric functions (see [2, 2.9 (33)])

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} u_6 + \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} u_2$$

where  $u_6 := z^{b-c}(1 - z)^{c-a-b} {}_2F_1(c - b, 1 - b; c + 1 - a - b; 1 - z^{-1})$  and  $u_2 := z^{-b} {}_2F_1(b + 1 - c, b; a + b + 1 - c; 1 - z^{-1})$ . The substitution

$$a = \frac{-\lambda' + \lambda'' + 1 + \lambda}{2}, \quad b = \frac{\lambda' + \lambda'' + 1 + \lambda}{2}, \quad c = 1 + \lambda, \quad z = \sin^2 \theta = \cosh^{-2} t$$

gives the desired equation because the first term vanishes if  $-a \in \mathbb{N}$ .  $\square$

**8.2.** We recall the definition of  $A_0(p, q)$  from (5.1.1), and set

$$A_+(p, q) := A_0(p, q) \cap \{\lambda \in \mathbb{R} : \lambda > 0\},$$

$$A_{+-}(\lambda) := \{(\lambda', \lambda'') \in A_+(p', q') \times A_+(q'', p'') : \lambda' - \lambda'' - \lambda - 1 \in 2\mathbb{N}\},$$

$$A_{++}(\lambda) := \{(\lambda', \lambda'') \in A_+(p', q') \times A_+(p'', q'') : \lambda - \lambda' - \lambda'' - 1 \in 2\mathbb{N}\}.$$

We define meromorphic functions of three variables  $\lambda, \lambda'$  and  $\lambda''$  by

$$V_{+-,\lambda}^{(\lambda', \lambda'')} := \frac{(\Gamma(\lambda'' + 1))^2 \Gamma(\frac{\lambda' - \lambda'' + \lambda + 1}{2}) \Gamma(\frac{\lambda' - \lambda'' - \lambda + 1}{2})}{2\lambda \Gamma(\frac{\lambda' + \lambda'' + \lambda + 1}{2}) \Gamma(\frac{\lambda' + \lambda'' - \lambda + 1}{2})}, \tag{8.2.1}$$

$$V_{++,\lambda}^{(\lambda', \lambda'')} := \frac{(\Gamma(\lambda'' + 1))^2 \Gamma(\frac{-\lambda' - \lambda'' + \lambda + 1}{2}) \Gamma(\frac{\lambda' - \lambda'' + \lambda + 1}{2})}{2\lambda \Gamma(\frac{-\lambda' + \lambda'' + \lambda + 1}{2}) \Gamma(\frac{\lambda' + \lambda'' + \lambda + 1}{2})}. \tag{8.2.2}$$

Then we have readily

$$\lambda' V_{+-,\lambda}^{(\lambda', \lambda'')} = M(\lambda, \lambda', \lambda'')^2 \lambda V_{+-,\lambda}^{(\lambda', \lambda'')}. \tag{8.2.3}$$

Here are the meanings of the functions  $V_{+-,\lambda}^{(\lambda', \lambda'')}$  and  $V_{++,\lambda}^{(\lambda', \lambda'')}$ .

**Lemma 8.2.** (1) For  $\lambda > 0$  and  $(\lambda', \lambda'') \in A_{+-}(\lambda)$ , we have

$$\int_0^\infty |\varphi_{i\lambda}^{(\lambda'', \lambda')} (t)|^2 (\cosh t)^{2\lambda'+1} (\sinh t)^{2\lambda''+1} dt = V_{+-,\lambda}^{(\lambda', \lambda'')}. \tag{8.2.4}$$

(2) For  $\lambda > 0$  and  $(\lambda', \lambda'') \in A_{++}(\lambda)$ , we have

$$\int_0^{\frac{\pi}{2}} |\varphi_{i\lambda}^{(\lambda'', \lambda')} (i\theta)|^2 (\cos \theta)^{2\lambda'+1} (\sin \theta)^{2\lambda''+1} d\theta = V_{++,\lambda}^{(\lambda', \lambda'')}. \tag{8.2.5}$$

**Proof.** (1) The condition  $(\lambda', \lambda'') \in A_{+-}(\lambda)$  implies that  $c^{(\lambda'', \lambda')}(\lambda) = 0$  (see (8.1.4)) and that  $\varphi_{i\lambda}^{(\lambda'', \lambda')}$  appears in the discrete spectrum of the Plancherel–Parseval formula for the Jacobi transform [16, Theorem 2.4], from which the  $L^2$ -norm (8.2.4) is obtained by the following residue computation:

$$\text{Res}_{v=\lambda} \frac{2^{2(\lambda'+\lambda''+1)}}{c^{(\lambda'', \lambda')}(v) c^{(\lambda'', \lambda')}(-v)} = \frac{1}{V_{+-,\lambda}^{(\lambda', \lambda'')}}. \tag{8.2.6}$$



(2) The Jacobi function reduces to the (classical) Jacobi polynomial, denoted by  $P_m^{(\alpha,\beta)}(x)$  as usual, up to a scalar multiple if  $(\lambda', \lambda'') \in A_{++}(\lambda)$ , and then we have

$$\varphi_{i\lambda}^{(\lambda'', \lambda')}(i\theta) = \frac{\Gamma(\lambda'' + 1) m!}{\Gamma(\frac{-\lambda' + \lambda'' + \lambda + 1}{2})} P_m^{(\lambda'', \lambda')}(\cos 2\theta), \tag{8.2.7}$$

where we put  $m := \frac{1}{2}(\lambda - \lambda' - \lambda'' - 1) \in \mathbb{N}$ . We now recall a classical integral formula of the Jacobi polynomial  $P_m^{(\lambda'', \lambda')}$  ( $\lambda', \lambda'' > -1$ ) (see [3, 16.4 (5)]):

$$\begin{aligned} & \int_{-1}^1 (P_m^{(\lambda'', \lambda')}(x))^2 (1-x)^{\lambda''} (1+x)^{\lambda'} dx \\ &= \frac{2^{\lambda' + \lambda'' + 1}}{2m + \lambda' + \lambda'' + 1} \frac{\Gamma(m + \lambda'' + 1) \Gamma(m + \lambda' + 1)}{m! \Gamma(m + \lambda' + \lambda'' + 1)} \end{aligned}$$

The substitution  $(m, x) := (\frac{1}{2}(\lambda - \lambda' - \lambda'' - 1), \cos 2\theta)$  leads to (8.2.5).  $\square$

**8.3.** In this subsection, we prove Theorem 3.9.1 of [15], which gives explicit eigenvalues of the Knapp–Stein integral operator (see (3.9.3)).

Analogously to the Knapp–Stein integral operator, we define the Poisson transform for an affine symmetric space  $X(p, q)$  by

$$\mathcal{P}_{-\lambda, \varepsilon} : \text{Ind}_{\mathfrak{p}_{\max}}^G(\varepsilon \otimes \mathbb{C}_{\lambda}) \rightarrow C_{\lambda, \varepsilon}^{\infty}(X(p, q)), \tag{8.3.1}$$

$$(\mathcal{P}_{-\lambda, \varepsilon} f)(x) := \int_M \psi_{\lambda - \rho, \varepsilon}([x, b]) f(b) db \quad (x \in X(p, q)).$$

Here,  $\rho = \frac{p+q-2}{2}$  and  $db$  is the Riemannian measure on  $M \simeq S^{p-1} \times S^{q-1}$ , a double cover of  $G/P^{\max}$ .

For  $\text{Re } \lambda > 0$ , we can define the boundary value map

$$\beta_{\lambda} : C_{\lambda, \varepsilon}^{\infty}(X(p, q))_K \rightarrow \text{Ind}_{\mathfrak{p}_{\max}}^G(\varepsilon \otimes \mathbb{C}_{-\lambda}) \tag{8.3.2}$$

such that for  $f \in C_{\lambda, \varepsilon}^{\infty}(X(p, q))_K$  we have

$$(\beta_{\lambda} f)(g \zeta^o) := 2^{\lambda - \rho} \lim_{t \rightarrow \infty} e^{(\rho - \lambda)t} f(g \exp(tE)x^o).$$

Here, we recall from Sections 3.7 and 5.2 that  $\exp(tE)x^o = {}^t(\cosh t, 0, \dots, 0, \sinh t) \in X(p, q)$  and  $\zeta^o = {}^t(1, 0, \dots, 0, 1) \in \Xi$ . Again as in Section 3.7, we have regarded an element of  $\text{Ind}_{\mathfrak{p}_{\max}}^G(\varepsilon \otimes \mathbb{C}_{-\lambda})$  as a function over  $\Xi$ .

We recall the barrier functions from (3.9.5):

$$B_{\lambda}^{\varepsilon_1, \varepsilon_2} \equiv B_{\lambda}^{\varepsilon_1, \varepsilon_2}(m, n) = \lambda - 1 - \varepsilon_1 \left( m + \frac{p}{2} - 1 \right) - \varepsilon_2 \left( n + \frac{q}{2} - 1 \right),$$

and define

$$\alpha(\lambda, m, n) := \frac{c^{(n+\frac{q}{2}-1, m+\frac{p}{2}-1)}(\lambda)}{2^{m+n-\lambda+\rho}} = \frac{\Gamma(n + \frac{q}{2}) \Gamma(\lambda)}{\Gamma(1 + B_\lambda^{-}) \Gamma(1 + B_\lambda^{+-})}, \tag{8.3.3}$$

$$\beta(\lambda, m, n) := \frac{4\pi^{\frac{p+q-1}{2}} (-1)^{\lfloor \frac{m-n}{2} \rfloor} \Gamma(1 + B_\lambda^{-})}{\Gamma(n + \frac{q}{2}) \Gamma(\frac{2\lambda+p+q-1-\varepsilon}{4}) \Gamma(-B_\lambda^{+-})}. \tag{8.3.4}$$

**Lemma 8.3.** *Suppose  $h_{m,n} \in \mathcal{H}^m(\mathbb{R}^p) \otimes \mathcal{H}^n(\mathbb{R}^q)$ .*

$$\mathcal{P}_{\lambda,\varepsilon} h_{m,n} = \beta(\lambda, m, n) \tilde{h}_{m,n}, \tag{8.3.5}$$

$$\beta_\lambda \tilde{h}_{m,n} = \alpha(\lambda, m, n) h_{m,n}. \tag{8.3.6}$$

Here,  $\tilde{h}_{m,n} \in C^\infty(X(p, q))$  is given by

$$\tilde{h}_{m,n}(x \cosh t, y \sinh t) := h_{m,n}(x, y) (\cosh t)^m (\sinh t)^n \varphi_{i\tilde{\lambda}}^{(n+\frac{q}{2}-1, m+\frac{p}{2}-1)}(t). \tag{8.3.7}$$

**Proof.** (1) The case  $\varepsilon = 1$  was proved in [18, Lemma 7.2] which is based on the integral formula of [4, Appendix B]. The case  $\varepsilon = -1$  is similar. Note that we have normalized  $\mathcal{P}_{\lambda,\varepsilon}$  slightly different from [18].

(2) This is a direct consequence of the asymptotic behavior of the Jacobi function (see (8.1.2)).  $\square$

We are now ready to prove Theorem 3.9.1 of [15]:

**Proof of Theorem 3.9.1.** For  $\text{Re } \lambda > 0$ , we have

$$A_{\lambda,\varepsilon} = \beta_\lambda \circ \mathcal{P}_{-\lambda,\varepsilon} \tag{8.3.8}$$

because the boundary value of the Poisson kernel is essentially the Knapp–Stein kernel:

$$2^{\lambda-\rho} \lim_{t \rightarrow \infty} e^{(\rho-\lambda)t} \psi_{\rho-\lambda,\varepsilon}([g \exp(tE)x^o, b]) = \psi_{\rho-\lambda,\varepsilon}([g\xi^o, b]).$$

In particular, it follows from (8.3.3) and (8.3.4) that  $A_{\lambda,\varepsilon}$  acts as a scalar

$$\alpha(\lambda, m, n) \beta(-\lambda, m, n) = \frac{4\pi^{\frac{p+q-1}{2}} (-1)^{\lfloor \frac{m-n}{2} \rfloor} \Gamma(\lambda) \Gamma(-B_\lambda^{++})}{\Gamma(\frac{-2\lambda+p+q-1-\varepsilon}{4}) \Gamma(1 + B_\lambda^{-}) \Gamma(1 + B_\lambda^{+-}) \Gamma(1 + B_\lambda^{++})}$$

on the  $K$ -type  $\mathcal{H}^m(\mathbb{R}^p) \otimes \mathcal{H}^n(\mathbb{R}^q)$ . Now, Theorem 3.9.1 follows by the analytic continuation of this formula on  $\lambda$ .  $\square$

**8.4.** Unitarization of  $\pi_{+, \lambda}^{p, q}$  ( $\lambda \in A_0(p, q)$ ).

In this subsection, we give an explicit unitary inner product on  $\pi_{+, \lambda}^{p, q}$  for  $\lambda \in A_0(p, q)$ . In view of Fact 5.4, the  $(\mathfrak{g}, K)$ -module  $(\pi_{+, \lambda}^{p, q})_K$  is realized in  $C_{\lambda}^{\infty}(X(p, q))_K$  with  $K$ -types  $\Xi(K:b)$  where  $b = \lambda - \frac{p}{2} + \frac{q}{2} + 1$ . Suppose  $f \in C_{\lambda}^{\infty}(X(p, q))_K$  belongs to the  $K$ -type  $\mathcal{H}^m(\mathbb{R}^p) \boxtimes \mathcal{H}^n(\mathbb{R}^q) \in \Xi(K:b)$ . Then  $f$  is of the form

$$f(\omega \cosh t, \eta \sinh t) = h_m(\omega)h_n(\eta)(\cosh t)^m(\sinh t)^n \varphi_{i\lambda}^{(n+\frac{q}{2}-1, m+\frac{p}{2}-1)}(t), \tag{8.4.1}$$

where  $h_m \in \mathcal{H}^m(\mathbb{R}^p)$ ,  $h_n \in \mathcal{H}^n(\mathbb{R}^q)$ ,  $\omega \in S^{p-1}$ ,  $\eta \in S^{q-1}$ , and  $t > 0$ . We put

$$\|f\|_{\pi_{+, \lambda}^{p, q}}^2 := \|h_m\|_{L^2(S^{p-1})}^2 \|h_n\|_{L^2(S^{q-1})}^2 \lambda V_{+, \lambda}^{(m+\frac{p-2}{2}, n+\frac{q-2}{2})}. \tag{8.4.2}$$

**Proposition 8.4.** *For  $\lambda \in A_0(p, q)$ ,  $\|\cdot\|_{\pi_{+, \lambda}^{p, q}}$  defines an inner product on  $(\pi_{+, \lambda}^{p, q})_K$  and one can define an irreducible unitary representation of  $G = O(p, q)$  on its Hilbert completion.*

**Proof.** Proposition 8.4 follows from the Parseval–Plancherel formula of the branching law of the minimal representations (see Theorem 8.6) (for this, we need to replace  $(q, q', q'')$  in Theorem 8.6 by  $(q + c, q, c)$  for some  $c > 0$ ). See also Remark (4).  $\square$

There is an obvious inner product for  $\lambda > 0$ , because  $(\pi_{+, \lambda}^{p, q})_K \subset L^2(X(p, q))$ . The relation between our norm  $\|\cdot\|_{\pi_{+, \lambda}^{p, q}}$  and the  $L^2$ -norm  $\|\cdot\|_{L^2(X(p, q))}$  is given by

$$\|f\|_{\pi_{+, \lambda}^{p, q}}^2 = \lambda \|f\|_{L^2(X(p, q))}^2 \quad \text{for any } f \in (\pi_{+, \lambda}^{p, q})_K, \tag{8.4.3}$$

if  $\lambda > 0$  owing to Lemma 8.2(1). This observation gives an alternative proof of Proposition 8.4 for  $\lambda > 0$ . We should note that even for  $\lambda \in A_0(p, q)$  such that  $\lambda \leq 0$  (this can happen if  $\lambda = 0, -\frac{1}{2}$ ),  $\pi_{+, \lambda}^{p, q}$  is still unitarizable by the inner product  $(\cdot, \cdot)_{\pi_{+, \lambda}^{p, q}}$  given in (8.4.2), which we have proved to be positive definite.

**Remark (Unitarity).** As we explained, all of  $(\pi_{+, \lambda}^{p, q})_K$  are unitarizable for  $\lambda \in A_0(p, q)$ . We summarize four different approaches to the proof of unitarizability:

- (1) If  $\lambda > 0$ , then  $(\pi_{+, \lambda}^{p, q})_K$  is unitarizable because of the realization in  $L^2(X(p, q))$ .
- (2) If  $\lambda \geq 0$ , then  $(\pi_{+, \lambda}^{p, q})_K$  is unitarizable because of the realization of Zuckerman–Vogan’s derived functor module  $\mathcal{R}_q^{p-2}(\mathbb{C}_{\lambda})$  with the parameter  $\lambda$  in the weakly fair range [20].

We note that the case  $\lambda = -\frac{1}{2}$  is not treated in the above two methods. However, the following two methods cover all  $\lambda \in A_0(p, q)$  including the singular cases  $\lambda = 0$  and  $-\frac{1}{2}$ .

- (3) Use the classification of unitarizable subquotients of  $\text{Ind}_{p_{\max}}^G(\varepsilon \otimes \mathbb{C}_\lambda)$  given in [5, Section 3].

At last, here is a new proof of the unitarizability of  $(\pi_{+, \lambda}^{p, q})_K$  for all  $\lambda \in A_0(p, q)$ . The idea is to use our branching formula Theorem 7.1, for which the proof does not use the unitarizability of  $(\pi_{+, \lambda}^{p, q})_K$ .

- (4) All of  $(\pi_{+, \lambda}^{p, q})_K$  are unitarizable because they appear as discrete spectra in the branching law of a unitary representation  $\varpi^{p, q+c}$  of a larger group  $O(p, q+c)$  to  $O(p, q) \times O(c)$  for some  $c > 0$  (see Theorems 7.1 and 8.6). For this purpose,  $c \leq 3$  will do.

**8.5.** We notice that the map  $\Phi_1: X(p, q') \times S^{q''-1} \rightarrow M_+$  (6.2.3) (in the case  $p'' = 0$ ) is given by

$$((\omega \cosh t, \eta' \sinh t), \eta'') \mapsto (\omega, (\eta' \cos \theta, \eta'' \sin \theta)),$$

where  $\theta$  and  $t$  satisfy  $\sin \theta \cosh t = 1$ . Suppose  $f \in C^\infty(X(p, q') \times S^{q''-1})$  belongs to

$$\pi_{+, l+\frac{q''}{2}-1}^{p, q'} \boxtimes \mathcal{H}^l(\mathbb{R}^{q''})$$

as an  $O(p, q') \times O(q'')$ -module, and furthermore to  $\mathcal{H}^m(\mathbb{R}^p) \boxtimes \mathcal{H}^n(\mathbb{R}^{q'})$  as an  $O(p) \times O(q')$ -module in the first factor. Then  $f$  is of the form:

$$f((\omega \cosh t, \eta' \sinh t), \eta'') = h_m(\omega) h_k(\eta') h_l(\eta'') (\cosh t)^m (\sinh t)^k \varphi_{i(l+\frac{q''}{2}-1)}^{(k+\frac{q'}{2}-1, m+\frac{p}{2}-1)}(t),$$

where  $h_m \in \mathcal{H}^m(\mathbb{R}^p)$ ,  $h_k \in \mathcal{H}^k(\mathbb{R}^{q'})$ ,  $h_l \in \mathcal{H}^l(\mathbb{R}^{q''})$ ,  $\omega \in S^{p-1}$ ,  $\eta' \in S^{q'-1}$ ,  $\eta'' \in S^{q''-1}$ , and  $t > 0$ .

**Lemma 8.5.** *The twisted pull-back  $\widetilde{(\Phi_1^{-1})^*}: C^\infty(X(p, q') \times S^{q''-1}) \rightarrow C^\infty(M_+)$  (see (6.3.2) for definition) is given by the formula:*

$$\begin{aligned} & \widetilde{(\Phi_1^{-1})^*} f(\omega, (\eta' \cos \theta, \eta'' \sin \theta)) \\ &= M^{-1} h_m(\omega) h_k(\eta') h_l(\eta'') (\cos \theta)^k (\sin \theta)^l \varphi_{i(n+\frac{q}{2}-1)}^{(l+\frac{q''}{2}-1, k+\frac{q'}{2}-1)}(i\theta). \end{aligned} \tag{8.5.1}$$

**Proof.** In view of the definition, we have

$$((\widetilde{\Phi}_1^{-1})^* f)(\omega, (\eta' \cos \theta, \eta'' \sin \theta)) = (\cosh t)^{\frac{p+q-4}{2}} f((\omega \cosh t, \eta' \sinh t), \eta'').$$

Then (8.5.1) follows from the triangular relation of the Jacobi-function (Lemma 8.1) and from  $m + \frac{p}{2} = n + \frac{q}{2}$ .  $\square$

**8.6.** Let us consider the restriction from  $G = O(p, q)$  to  $G' = O(p, q') \times O(q'')$ . Because we use an explicit map to prove the branching law, the generalized Parseval–Plancherel formula makes sense.

**Theorem 8.6.** (1) *If we develop  $F \in \text{Ker } \tilde{\Delta}_M$  as  $F = \sum_l^\infty F_l^{(1)} F_l^{(2)}$  according to the irreducible decomposition (see Theorem 7.1)*

$$(\widetilde{\Phi}_1)^* : \varpi^{p,q} |_{O(p,q') \times O(q'')} \xrightarrow{\sim} \sum_{l=0}^\infty \oplus_{+, l + \frac{q''}{2} - 1} \pi^{p,q'} \boxtimes \mathcal{H}^l(\mathbb{R}^{q''}) \tag{8.6.1}$$

then we have

$$\|F\|_{\varpi^{p,q}}^2 = \sum_{l=0}^\infty \|F_l^{(1)}\|_{\pi^{p,q'}}^2 \|F_l^{(2)}\|_{L^2(S^{q''-1})}^2. \tag{8.6.2}$$

(2) *In particular, if  $q'' \geq 3$ , then all of  $\pi^{p,q'}_{+, l + \frac{q''}{2} - 1}$  are discrete series for the hyperboloid*

$X(p, q')$  and

$$\|F\|_{\varpi^{p,q}}^2 = \sum_{l=0}^\infty (l + \frac{q''}{2} - 1) \|F_l^{(1)}\|_{L^2(X(p,q'))}^2 \|F_l^{(2)}\|_{L^2(S^{q''-1})}^2. \tag{8.6.3}$$

**Remark.** Formula (8.6.3) coincides with the Kostant–Binegar–Zierau formula (see Section 3) in the special case where  $q' = 0$  (namely, where  $G'$  is a compact subgroup).

**Proof.** We write  $\lambda := l + \frac{q''}{2} - 1$ ,  $\lambda' := m + \frac{p}{2} - 1 = n + \frac{q}{2} - 1$  and  $\lambda'' := k + \frac{q'}{2} - 1$ . If  $F$  is of the form of the right side of (8.5.1), then

$$\frac{\|(\widetilde{\Phi}_1)^* F\|_{\pi^{p,q'}_{+, \lambda} \boxtimes \mathcal{H}^l(\mathbb{R}^{q''})}^2}{\|F\|_{\varpi^{p,q}}^2} = \frac{M^2 \|h_m\|_{L^2(S^{p-1})}^2 \|h_k\|_{L^2(S^{q'-1})}^2 \|h_l\|_{L^2(S^{q''-1})}^2 \lambda V_{+, \lambda}^{(\lambda', \lambda'')}}{\|h_m\|_{L^2(S^{p-1})}^2 \|h_k\|_{L^2(S^{q'-1})}^2 \|h_l\|_{L^2(S^{q''-1})}^2 \lambda' V_{+, \lambda'}^{(\lambda'', \lambda)}} = 1$$

Here, the first equality follows from definition (8.4.2), Lemmas 8.5 and 8.2(2). The second equality is given by (8.2.3). Hence the first statement is proved. The second statement follows from (8.4.3).  $\square$

**9. Construction of discrete spectra in the branching laws**

**9.1.** In Section 7, we determined explicitly the branching law  $\varpi^{p,q}|_{G'}$  of the minimal unipotent representation  $\varpi^{p,q} \in \hat{G}$  where  $G' = O(p, q') \times O(q'')$ . The resulting branching law has no continuous spectrum (see Theorems 4.2 and 7.1). In this section, we treat a more general case where continuous spectrum may appear, namely, the branching law with respect to the semisimple symmetric pair

$$(G, G') = (O(p, q), O(p', q') \times O(p'', q'')),$$

where  $p' + p'' = p (\geq 2)$ ,  $q' + q'' = q (\geq 2)$ ,  $p + q \in 2\mathbb{N}$ , and  $(p, q) \neq (2, 2)$ .

We shall construct explicitly discrete spectra by using the conformal geometry.

Retain the notation in Section 5.1. We set

$$A'(p, q) := A_0(p, q) \cap \{\lambda \in \mathbb{R} : \lambda > 1\}. \tag{9.1.1}$$

**Theorem 9.1.** *The restriction of the unitary representation  $\varpi^{p,q}|_{G'}$  contains*

$$\sum_{\lambda \in A'(p', q') \cap A'(q'', p'')}^{\oplus} \pi_{+, \lambda}^{p', q'} \boxtimes \pi_{-, \lambda}^{p'', q''} \oplus \sum_{\lambda \in A'(q', p') \cap A'(p'', q'')}^{\oplus} \pi_{-, \lambda}^{p', q'} \boxtimes \pi_{+, \lambda}^{p'', q''}$$

as a discrete summand. Here,  $\pi_{+, \lambda}^{p', q'} \boxtimes \pi_{-, \lambda}^{p'', q''} \in \widehat{G'}$  is the outer tensor product of  $\pi_{+, \lambda}^{p', q'} \in \widehat{O(p', q')}$  and  $\pi_{-, \lambda}^{p'', q''} \in \widehat{O(p'', q'')}$ .

We have already established the full branching law  $\varpi^{p,q}|_{G'}$  if  $p'q'p''q'' = 0$ . Thus, the main part of this section will be devoted to the proof of Theorem 9.1 when  $p'q'p''q'' \neq 0$ . We shall give some remarks on Theorem 9.1 at the end of this subsection.

**9.2.** Let  $K' = O(p') \times O(q') \times O(p'') \times O(q'')$ . We realize the unitary representation  $\varpi^{p,q}$  on the Hilbert space  $\overline{V^{p,q}}$ . Here we recall the notation of Section 3.9, briefly as follows:

$$\overline{V^{p,q}} = \text{Ker}(D_p - D_q) \underset{\text{closed}}{\subset} \mathcal{V} = \text{Dom}(D_p) \cap \text{Dom}(D_q) \underset{\text{dense}}{(\subset} L^2(M))$$

$$M = S^{p-1} \times S^{q-1} \underset{\text{dense}}{\supset} M_+ \cup M_-.$$

Different from Corollary 4.3 in the discretely decomposable case ( $p'q'p''q'' = 0$ ), a  $K'$ -finite vector of a  $G'$ -irreducible summand (i.e. a discrete spectrum) in  $\varpi^{p,q}|_{G'}$  is not necessarily a real analytic function on  $M$  if  $p'q'p''q'' \neq 0$  in our conformal construction of  $\varpi^{p,q}$ . With this in mind, we extend  $(\Phi_1^{-1})^* f \in C^\infty(M_+)$  (see Section 6

for notation) to a function, denoted by  $T_+f$ , on  $S^{p-1} \times S^{q-1}$  as

$$T_+f := \begin{cases} (\widetilde{\Phi_1^{-1}})^*f & \text{on } M_+, \\ 0 & \text{on } M \setminus M_+. \end{cases} \tag{9.2.1}$$

Here is a key lemma:

**Lemma 9.2.** *Suppose  $\lambda' \in A_0(p', q')$  and  $\lambda'' \in A_0(q'', p'')$ . Let  $f \in C^\infty(X(p', q') \times X(q'', p''))$  be a  $K'$ -finite function which belongs to  $\pi_{+\lambda'}^{p', q'} \boxtimes \pi_{-\lambda''}^{p'', q''}$  (see Fact 5.4(1) (ii)).*

- (1) *If  $\lambda' \geq \frac{1}{2}$  and  $\lambda'' \geq \frac{1}{2}$ , then  $T_+f \in \mathcal{V}$ .*
- (2) *If  $\lambda' > 1$  and  $\lambda'' > 1$ , then  $Y(T_+f) \in L^2(M)$  and  $YY'(T_+f) \in L^1(M)$  for any smooth vector fields  $Y, Y'$  on  $M$ .*
- (3) *If  $\lambda' = \lambda'' > 1$  then  $T_+f \in \overline{V^{p,q}}$ .*

Before proving Lemma 9.2, we first show that Lemma 9.2 implies Theorem 9.1. In fact, Lemma 9.2(3) constructs a non-zero  $(\mathfrak{g}', K')$ -homomorphism

$$T_+ : (\pi_{+\lambda'}^{p', q'})_{K'_1} \boxtimes (\pi_{-\lambda''}^{p'', q''})_{K'_2} \rightarrow \varpi^{p,q},$$

for  $\lambda \in A'(p', q') \cap A'(q'', p'')$ . Then  $T_+$  is injective because  $(\pi_{+\lambda'}^{p', q'})_{K'_1} \boxtimes (\pi_{-\lambda''}^{p'', q''})_{K'_2}$  is irreducible. Since  $\varpi^{p,q}$  is a unitary representation of  $G$ ,  $T_+$  extends to an isometry of unitary representations of  $G'$ :

$$\pi_{+\lambda'}^{p', q'} \boxtimes \pi_{-\lambda''}^{p'', q''} \rightarrow \varpi^{p,q},$$

by taking the closure with respect to the inner product induced from  $\varpi^{p,q}$ . This proves Theorem 9.1 for the irreducible representation  $\pi_{+\lambda'}^{p', q'} \boxtimes \pi_{-\lambda''}^{p'', q''}$  that appears in the first summand. The second summand is constructed similarly by using the conformal diffeomorphism (see Lemma 6.4)

$$\Phi_2 : X(q', p') \times X(p'', q'') \xrightarrow{\sim} M_- \ (\subset M).$$

Hence, the proof of Theorem 9.1 is completed by assuming Lemma 9.2.

**9.3.**

**Remark 9.3.** There exist  $\lambda' \in A_0(p', q')$  and  $\lambda'' \in A_0(q'', p'')$  satisfying  $\lambda' = \lambda'' > 1$  if and only if  $p' + q' \equiv p'' + q'' \pmod{2}$  and  $p' \geq 2, q'' \geq 2$ . This implies  $p + q \in 2\mathbb{N}$ ,  $p \geq 2$  and  $q \geq 2$ , and  $V^{p,q}$  is non-zero (see Section 3). Of course, Lemma 9.2(3) also implies  $V^{p,q} \neq \{0\}$ .

**9.4.** It follows from Lemma 3.8.1 that the first statement of Lemma 9.2 is proved if  $Y(T_+f) \in L^{2-\varepsilon}$  for any  $\varepsilon > 0$  and for any smooth vector field  $Y$  on  $M$ . Therefore, both of (1) and (2) of Lemma 9.2 are proved by studying the asymptotic behavior of  $T_+f$  near the boundary of  $M_+$  in  $M$ . This asymptotic estimate is studied below.

Any  $K'$ -finite vector  $f$  is a finite linear combination of the form  $f_1f_2$ , where

$$\begin{cases} f_1 \in C^\infty(X(p', q')) \text{ is an } O(p') \times O(q')\text{-finite vector that belongs to } \pi_{+, \lambda'}^{p', q'}, \\ f_2 \in C^\infty(X(q'', p'')) \text{ is an } O(p'') \times O(q'')\text{-finite vector that belongs to } \pi_{-, \lambda''}^{p'', q''}. \end{cases}$$

In order to prove Lemma 9.2, we may and do assume  $f$  is of the form  $f_1f_2$ . Then we have

$$\begin{aligned} & (T_+(f_1f_2))(u, v) \\ &= (|u'|^2 - |v'|^2)_+^{\frac{p+q-4}{4}} f_1 \left( \frac{(u', v')}{\sqrt{|u'|^2 - |v'|^2}} \right) f_2 \left( \frac{(u'', v'')}{\sqrt{|u''|^2 - |v''|^2}} \right). \end{aligned} \tag{9.4.1}$$

Here, we have used the following notation:

$$r_+^v := \begin{cases} r^v & (r > 0), \\ 0 & (r \leq 0). \end{cases}$$

**9.5.** In order to analyze the asymptotic behavior of  $T_+(f_1f_2)$  (see (9.4.1)) near the boundary of  $M_+$ , we consider a change of variables on  $S^{p-1} \times S^{q-1}$  by the surjective map

$$(S^1 \times S^{p-1} \times S^{p-1}) \times (S^1 \times S^{q-1} \times S^{q-1}) \rightarrow S^{p-1} \times S^{q-1},$$

$$(e^{i\theta}, \omega', \omega''), (e^{i\varphi}, \eta', \eta'') \mapsto (u, v)$$

defined by

$$(u, v) \equiv (u', u'', v', v'') := (\omega' \cos \theta, \omega'' \sin \theta, \eta' \cos \varphi, \eta'' \sin \varphi). \tag{9.5.1}$$

Because  $|u'|^2 - |v'|^2 = |\omega' \cos \theta|^2 - |\eta' \cos \varphi|^2 = |\cos \theta|^2 - |\cos \varphi|^2$ ,  $M_\pm$  defined in (6.1.1) is rewritten as

$$M_+ = \{(\omega' \cos \theta, \omega'' \sin \theta, \eta' \cos \varphi, \eta'' \sin \varphi) : |\cos \theta| > |\cos \varphi|\},$$

$$M_- = \{(\omega' \cos \theta, \omega'' \sin \theta, \eta' \cos \varphi, \eta'' \sin \varphi) : |\cos \theta| < |\cos \varphi|\}.$$



Here is an elementary computation corresponding to the change of variables (9.5.1):

**Lemma 9.5.** (1) *The volume element  $du dv$  on  $S^{p-1} \times S^{q-1}$  is given by*

$$du dv = |\cos \theta|^{p'-1} |\sin \theta|^{p''-1} |\cos \varphi|^{q'-1} |\sin \varphi|^{q''-1} d\theta d\varphi d\omega' d\omega'' d\eta' d\eta'',$$

where  $d\omega'$  is the volume element on  $S^{p'-1}$  and so on.

(2) *Any smooth vector field on  $S^{p-1} \times S^{q-1}$  is a linear combination of*

$$\frac{1}{\cos \theta} X', \frac{1}{\sin \theta} X'', \frac{1}{\cos \varphi} Y', \frac{1}{\sin \varphi} Y'', \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \theta},$$

whose coefficients are smooth functions of  $(\omega', \omega'', \eta', \eta'', \theta, \varphi)$ . Here,  $X', X'', Y'$  and  $Y''$  are smooth vector fields on  $S^{p'-1}, S^{p''-1}, S^{q'-1}$  and  $S^{q''-1}$ , respectively.

Fact 5.4(2) describes the asymptotic behavior of  $K$ -finite functions that belong to discrete series representations for a hyperboloid. Applying it to  $f_1$  with respect to coordinate (9.5.1), we find  $a_1 \in C^\infty(S^{p'-1} \times S^{q'-1})$  and  $h_1 \in C^\infty(\mathbb{R})$  such that

$$f_1 \left( \frac{(u', v')}{\sqrt{|u'|^2 - |v'|^2}} \right) = a_1(\omega', \eta') \left( \frac{\cos^2 \theta + \cos^2 \varphi}{\cos^2 \theta - \cos^2 \varphi} \right)_+^{-\frac{2\lambda' + p' + q' - 2}{4}} h_1 \left( \sqrt{\frac{\cos^2 \theta - \cos^2 \varphi}{\cos^2 \theta + \cos^2 \varphi}} \right).$$

Likewise, there exist  $a_2 \in C^\infty(S^{q''-1} \times S^{p''-1})$  and  $h_2 \in C^\infty(\mathbb{R})$  such that

$$f_2 \left( \frac{(v'', u'')}{\sqrt{|v''|^2 - |u''|^2}} \right) = a_2(\eta'', \omega'') \left( \frac{\sin^2 \theta + \sin^2 \varphi}{\cos^2 \theta - \cos^2 \varphi} \right)_+^{-\frac{2\lambda'' + p'' + q'' - 2}{4}} h_2 \left( \sqrt{\frac{\cos^2 \theta - \cos^2 \varphi}{\sin^2 \theta + \sin^2 \varphi}} \right).$$

We treat the boundary  $\partial M_+$  of  $M_+ \subset M$  locally in the following three cases:

Case (1)  $\cos^2 \theta - \cos^2 \varphi = 0$ ,  $(\cos \theta, \cos \varphi) \neq (0, 0)$ ,  $(\sin \theta, \sin \varphi) \neq (0, 0)$ .

Case (2)  $\cos \theta = \cos \varphi = 0$ .

Case (3)  $\sin \theta = \sin \varphi = 0$ .

We note that Case (2) or (3) happens only when  $p'q'p''q'' \neq 0$ .

**9.6. Case 1:** In this subsection, we consider a generic part of the boundary  $\partial M_+$  corresponding to Case 1. In a local coordinate  $(\omega', \omega'', \eta', \eta'', \theta, \varphi) \in S^{p'-1} \times S^{p''-1} \times$

$S^{q'-1} \times S^{q''-1} \times \mathbb{R}^2$ ,  $T_+(f_1 f_2)(u, v)$  is written as

$$A(\cos^2 \theta - \cos^2 \varphi)_+^{\frac{p+q-4}{4} + \frac{2\lambda'+p'+q'-2}{4} + \frac{2\lambda''+p''+q''-2}{4}} = A(\cos^2 \theta - \cos^2 \varphi)_+^{\frac{\lambda'+\lambda''}{2}}.$$

Here,  $A$  is a smooth function of variables  $\omega', \omega'', \eta', \eta'', (\cos^2 \theta - \cos^2 \varphi)_+^{\frac{1}{2}}$ . Therefore, by using Lemma 9.5, we have:

$$\lambda' + \lambda'' > -1 \Rightarrow T_+f \in L_{\text{loc}}^2,$$

$$\lambda' + \lambda'' \geq 1 \Rightarrow Y(T_+f) \in L_{\text{loc}}^{2-\varepsilon} \text{ for any } \varepsilon > 0, Y \in \mathfrak{X}(M),$$

$$\lambda' + \lambda'' > 2 \Rightarrow Y_1 Y_2(T_+f) \in L_{\text{loc}}^1 \text{ for any } Y_1, Y_2 \in \mathfrak{X}(M),$$

in a neighborhood of the boundary point of  $M_+$  for Case (1).

**9.7. Case 2:** In this subsection we consider a neighborhood of  $(\omega', \omega'', \eta', \eta'', \theta, \varphi)$  satisfying the condition of Case 2. We take a polar coordinate for  $(\cos \theta, \cos \varphi) \doteq (0, 0)$  as

$$\begin{cases} \cos \theta = r \cos \psi, \\ \cos \varphi = r \sin \psi. \end{cases}$$

The composition to (9.5.1) yields a new coordinate on  $S^{p-1} \times S^{q-1}$  given by

$$(u, v) = (u', u'', v', v'') = (\omega' r \cos \psi, \omega'' \sqrt{1 - r^2 \cos^2 \psi}, \eta' r \sin \psi, \eta'' \sqrt{1 - r^2 \sin^2 \psi}),$$

where  $\omega' \in S^{p-1}$ ,  $\omega'' \in S^{p''-1}$ ,  $\eta' \in S^{q-1}$ ,  $\eta'' \in S^{q''-1}$ ,  $r \geq 0$ , and  $\psi \in \mathbb{R}$ . Then our interest is in a neighborhood of  $r = 0$ . In this coordinate, we have

$$M_+ = \{r \neq 0, \cos 2\psi > 0\}.$$

The Jacobian matrix of the transform  $(\theta, \varphi) \rightarrow (r, \psi)$  is given by

$$\begin{pmatrix} \frac{\partial r}{\partial \theta} & \frac{\partial r}{\partial \varphi} \\ \frac{\partial \psi}{\partial \theta} & \frac{\partial \psi}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} -\cos \psi \sin \theta & -\sin \psi \sin \varphi \\ \frac{1}{r} \sin \psi \sin \theta & \frac{-1}{r} \cos \psi \sin \varphi \end{pmatrix}.$$

**Lemma 9.7.** (1) *The standard measure on  $S^{p-1} \times S^{q-1}$  is locally represented as smooth function of  $(r, \psi, \omega', \eta', \omega'', \eta'') \times r^{p+q-1} dr d\psi d\omega' d\eta' d\omega'' d\eta''$ .*

(2) Any smooth vector field on  $S^{p-1} \times S^{q-1}$  near  $r = 0$  is a linear combination of

$$\frac{1}{r \cos \psi} X', X'', \quad \frac{1}{r \sin \psi} Y', Y'', \quad \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \psi},$$

whose coefficients are smooth functions of  $(\omega', \omega'', \eta', \eta'', r, \psi)$ . Here,  $X', X'', Y'$  and  $Y''$  are smooth vector fields on  $S^{p-1}, S^{p-1}, S^{q-1}$  and  $S^{q-1}$ , respectively.

By noting the relations

$$\frac{\cos^2 \theta + \cos^2 \varphi}{\cos^2 \theta - \cos^2 \varphi} = \frac{1}{\cos 2\psi}, \quad \frac{\sin^2 \varphi + \sin^2 \theta}{\sin^2 \varphi - \sin^2 \theta} = \frac{2 - r^2}{r^2 \cos 2\psi},$$

$T_+(f_1 f_2)(u, v)$  is locally written as

$$\begin{aligned} B (r^2 \cos 2\psi)^{-\frac{p+q-4}{4}} (\cos 2\psi)_+^{\frac{2\lambda'+p'+q'-2}{4}} (r^2 \cos 2\psi)_+^{\frac{2\lambda''+p''+q''-2}{4}} \\ = B r_+^{\frac{2\lambda''-p'-q'+2}{2}} (\cos 2\psi)_+^{\frac{\lambda'+\lambda''}{2}}, \end{aligned}$$

where  $B$  is a smooth function of variables  $\omega', \omega'', \eta', \eta'', r_+, (\cos 2\psi)_+^{\frac{1}{2}}$ . Therefore, by using Lemma 9.7, we have:

$$2\lambda'' > -1, \quad \lambda' + \lambda'' > -1 \Rightarrow T_+ f \in L_{\text{loc}}^2,$$

$$2\lambda'' \geq 0, \quad \lambda' + \lambda'' \geq 1 \Rightarrow Y(T_+ f) \in L_{\text{loc}}^{2-\varepsilon} \quad \text{for any } \varepsilon > 0, Y \in \mathfrak{X}(M),$$

$$2\lambda'' > 2 - p' - q', \quad \lambda' + \lambda'' > 2 \Rightarrow Y_1 Y_2(T_+ f) \in L_{\text{loc}}^1 \quad \text{for any } Y_1, Y_2 \in \mathfrak{X}(M),$$

in a neighborhood of the boundary  $\partial M_+$  for Case (2).

The asymptotic estimate for Case (3) is parallel to that of Case (2).

**Remark 9.7.** Assume  $\lambda' = \lambda'' \in \mathbb{Z} + \frac{p'+q'}{2}$ . Then  $r_+^{\frac{2\lambda''-p'-q'+2}{2}}$  is bounded near  $r = 0$  if and only if  $\lambda' \geq \frac{p'+q'}{2} - 1$ , equivalently,  $\lambda' > \frac{p'+q'}{2} - 2$ , which means that  $\mathbb{C}_{\lambda'}$  is in the good range with respect to the  $\theta$ -stable parabolic subalgebra defined by  $\mathbb{C}_{\lambda'}$  in the sense of Vogan.

**9.8.** We end with some remarks and conjectures, primarily concerning the precise form of the discrete spectrum and also the continuous spectrum (where it would be very interesting to develop the complete Plancherel formula, given our explicit intertwining operator).

(1) *Multiplicity free property.* Each irreducible component in Theorem 9.1 occurs as multiplicity free. If  $p = 2$ , then  $\varpi^{p,q}$  is the direct sum of an irreducible highest weight module and a lowest one. It was proved in [8,12] that the multiplicity in the full Plancherel formula (both discrete and continuous spectrum) is at most one, in the branching law of any highest weight module of scalar type with respect to any symmetric pair.

(2) *At most finitely many discrete spectrum, and full discrete spectrum.* If  $p', q', p''$  and  $q''$  satisfy

$$\min(p', q'') \leq 1 \quad \text{and} \quad \min(q', p'') \leq 1, \tag{9.8.1}$$

then the parameter set in Theorem 9.1 is empty, namely,

$$A'(p', q') \cap A'(q'', p'') = A'(q', p') \cap A'(p'', q'') = \emptyset.$$

We conjecture that there are at most finitely many discrete spectra in the branching law  $\varpi^{p,q}|_G$  if (9.8.1) holds. We further conjecture that the full discrete spectrum is as in Theorem 9.1 with  $A'(p, q)$  replaced by  $A_0(p, q)$  everywhere.

(3) *No discrete spectrum.* Furthermore, if we exclude the case such as  $G' = O(p, q - 1) \times O(1)$  (see Section 7.2), namely, if

$$\min(p', q'') \leq 1, \quad \min(q', p'') \leq 1, \quad p' + q' > 1, \quad \text{and} \quad p'' + q'' > 1, \tag{9.8.2}$$

then

$$A_0(p', q') \cap A_0(q'', p'') = A_0(q', p') \cap A_0(p'', q'') = \emptyset.$$

It is likely that there is no discrete spectra in the branching law  $\varpi^{p,q}|_G$  if (9.8.2) is satisfied.

We note that condition (9.8.2) is equivalent to that at least one of  $X(p', q')$  or  $X(q'', p'')$  is a non-compact Riemannian symmetric space and at least one of  $X(q', p')$  or  $X(p'', q'')$  is a non-compact Riemannian symmetric space.

(4) *Discretely decomposable case.* The opposite extremal case is when

$$\min(p', p'', q', q'') = 0. \tag{9.8.3}$$

As we have proved in Theorem 4.2, the restriction  $\varpi^{p,q}|_G$  is discretely decomposable without any continuous spectrum. We have obtained the full branching formula in Theorem 7.1 by using Theorem 4.2 and the  $K$ -type formula of  $\varpi^{p,q}$ .

If we employ only the method in this section to the special case (9.8.3), then we do not have to consider Cases (2) and (3) in Section 9.7. Then, Theorem 9.1 exhausts all discrete spectra in Theorem 7.1 in most cases, but there are a few exceptions. To be precise, we consider the case  $p'' = 0$  without loss of generality. Then, in view of Theorem 7.1, the right side of Theorem 9.1 exhausts all discrete spectra if  $q'' \geq 5$ ; while at most two of  $(\mathfrak{g}', K')$ -modules  $\pi_{+, \lambda}^{p, q'}$  and  $\pi_{-, \lambda}^{0, q''}$  are missing in Theorem 9.1 if  $q' \leq 4$ . The precise missing parameters in the case  $p'' = 0$  and  $q' = 0$  are:  $\lambda = \pm \frac{1}{2}$  ( $q'' = 1$ );

$\lambda = 0, 1$  ( $q'' = 2$ );  $\lambda = \frac{1}{2}$  ( $q'' = 3$ ); and  $\lambda = 1$  ( $q'' = 4$ ). In order to cover all missing parameter by the purely geometric method of this section, we should note a specific feature in the case  $p'' = 0$ :

- (i)  $M_+$  is a dense subspace of  $M$ .
- (ii) Any real analytic functions on  $M_+$  satisfying the Yamabe equation corresponding to  $K'$ -finite vectors of the  $(\mathfrak{g}', K')$ -module with the above missing parameter extend to real analytic functions on  $M$  (see Corollary 4.3(2)).

(5) *Explicit continuous spectrum.*

We conjecture that

$$L^2\text{-Ind}_{p_1^{\max}}^{G_1'}(\varepsilon \otimes \mathbb{C}_{\sqrt{-1}\lambda}) \boxtimes L^2\text{-Ind}_{p_2^{\max}}^{G_2'}(\varepsilon \otimes \mathbb{C}_{\sqrt{-1}\lambda}) \quad (\lambda \in \mathbb{R})$$

is a continuous spectrum with multiplicity free if  $\min(p', p'', q', q'') > 0$ .

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