Global minimum and orthogonality in $C_1$-classes

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Abstract

In this paper we characterize the global minimum of an arbitrary function defined on a Banach space, in terms of a new concept of derivatives adapted for our case from a recent work due to D.J. Keckic (J. Operator Theory, submitted for publication). Using these results we establish several new characterizations of the global minimum of the map $F_{\phi}: U \rightarrow \mathbb{R}^+$ defined by $F_{\phi}(X) = \|\psi(X)\|_1$, where $\psi: U \rightarrow C_1$ is a map defined by $\psi(X) = S + \phi(X)$ and $\phi: B(H) \rightarrow B(H)$ is a linear map, $S \in C_1$, and $U = \{X \in B(H): \phi(X) \in C_1\}$. Further, we apply these results to characterize the operators which are orthogonal to the range of elementary operators.

Keywords: Elementary operators; Schatten $p$-classes; Orthogonality; $\phi$-directional derivative

1. Introduction

Let $E$ be a complex Banach space. We first define orthogonality in $E$. We say that $b \in E$ is orthogonal to $a \in E$ if for all complex $\lambda$ there holds

$$\|a + \lambda b\| \geq \|a\|.$$  \hfill (1.1)

This definition has a natural geometric interpretation. Namely, $b \perp a$ if and only if the complex line $\{a + \lambda b | \lambda \in \mathbb{C}\}$ is disjoint with the open ball $K(0, \|a\|)$, i.e., iff this complex line is a tangent one. Note that if $b$ is orthogonal to $a$, then $a$ need not be orthogonal to $b$. If $E$ is a Hilbert space, then from (1.1) follows $\langle a, b \rangle = 0$, i.e., orthogonality in the usual...
sense. Next we define the von Neumann–Schatten classes $C_p$ ($1 \leq p < \infty$). Let $B(H)$ denote the algebra of all bounded linear operators on a complex separable and infinite-dimensional Hilbert space $H$ and let $T \in B(H)$ be compact, and let $s_1(X) \geq s_2(X) \geq \cdots \geq 0$ denote the singular values of $T$, i.e., the eigenvalues of $|T| = (T^*T)^{1/2}$ arranged in their decreasing order. The operator $T$ is said to be belong to the Schatten $p$-classes $C_p$ if

$$\|T\|_p = \left[ \sum_{i=1}^{\infty} s_i(T)^p \right]^{1/p} = \left[ \text{tr}(T^p) \right]^{1/p}, \quad 1 \leq p < \infty,$$

where $\text{tr}$ denotes the trace functional. Hence $C_1$ is the trace class, $C_2$ is the Hilbert–Schmidt class, and $C_\infty$ corresponds to the class of compact operators with

$$\|T\|_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\|$$

denoting the usual operator norm. For the general theory of the Schatten $p$-classes the reader is referred to [14]. Recall (see [14]) that the norm $\| \cdot \|$ of the $B$-space $V$ is said to be Gâteaux differentiable at non-zero elements $x \in V$ if

$$\lim_{R \ni t \to 0} \frac{\|x + ty\| - \|x\|}{t} = \text{Re} \, D_x(y)$$

for all $y \in V$. Here $R$ denotes the set of all reals, $\text{Re}$ denotes the real part, and $D_x$ is the unique support functional (in the dual space $V^*$) such that $\|D_x\| = 1$ and $D_x(x) = \|x\|$. The Gâteaux differentiability of the norm at $x$ implies that $x$ is a smooth point of the sphere of radius $\|x\|$. It is well known (see [8] and the references therein) that for $1 < p < \infty$, $C_p$ is a uniformly convex Banach space. Therefore every non-zero $T \in C_p$ is a smooth point and in this case the support functional of $T$ is given by

$$D_T(X) = \text{tr} \left[ \frac{|T|^{p-1}UX^*}{\|T\|_p^{p-1}} \right] \quad (1.2)$$

for all $X \in C_p$, where $T = U|T|$ is the polar decomposition of $T$. The first result concerning the orthogonality in a Banach space was given by Anderson [1] showing that if $A$ is a normal operator on a Hilbert space $H$, then $AS = SA$ implies that for any bounded linear operator $X$ there holds

$$\|S + AX - XA\| \geq \|S\|. \quad (1.3)$$

This means that the range of the derivation $\delta_A : B(H) \to B(H)$ defined by $\delta_A(X) = AX - XA$ is orthogonal to its kernel. This result has been generalized in two directions: by extending the class of elementary mappings

$$E : B(H) \to B(H), \quad E(X) = \sum_{i=1}^{n} A_i XB_i$$

and

$$\tilde{E} : B(H) \to B(H), \quad \tilde{E}(X) = \sum_{i=1}^{n} A_i XB_i - X,$$
where \( (A_1, A_2, \ldots, A_n) \) and \( (B_1, B_2, \ldots, B_n) \) are \( n \)-tuples of bounded operators on \( H \), and by extending the inequality (1.3) to \( C_p \)-classes with \( 1 < p < \infty \) see [4,9,12]. The Gâteaux derivative concept was used in [3,6,9–11], in order to characterize those operators which are orthogonal to the range of a derivation. In these papers, the attention was directed to \( C_p \)-classes for some \( p > 1 \). The main purpose of this note is to characterize the global minimum of the map

\[
X \mapsto \| S + \phi(X) \|_{C_1}, \quad \phi \text{ a linear map in } B(H),
\]

in \( C_1 \) at points which are not necessarily smooth by using the \( \phi \)-directional derivative. These results are then applied to characterize the operators \( S \in C_1 \) which are orthogonal to the range of elementary operators, where \( S \) is not necessarily a smooth point. It is very interesting to point out that this result has been done in \( C_p \)-classes with \( 1 < p < \infty \) but, at least to our knowledge, it was not given, till now, for \( C_1 \)-classes. Recall that the operator \( S \) is a smooth point of the corresponding sphere in \( C_1 \) if and only if either \( S \) or \( S^* \) is injective.

2. Preliminaries

**Definition 2.1.** Let \( (X, \| \cdot \|) \) be an arbitrary Banach space and \( F: X \to \mathbb{R} \). We define the \( \phi \)-directional derivative of \( F \) at a point \( x \in X \) in direction \( y \in X \) by

\[
D_{\phi}F(x; y) = \lim_{t \to 0^+} \frac{F(x + te^{\phi}y) - F(x)}{t}.
\]

Note that when \( \phi = 0 \) the \( \phi \)-directional derivative of \( F \) at \( x \) in direction \( y \) coincides with the usual directional derivative of \( F \) at \( x \) in a direction \( y \) given by

\[
DF(x; y) = \lim_{t \to 0^+} \frac{F(x + ty) - F(x)}{t}. \tag{2.1}
\]

According to the notation given in [7] we will denote \( D_{\phi}F(x; y) \) for \( F(x) = \| x \| \) by \( D_{\phi,x}(y) \) and for the same function we write \( D_{x}(y) \) for \( DF(x; y) \).

**Remark 2.1.** In [7] the author used the term \( \phi \)-Gâteaux derivative instead of the term “\( \phi \)-directional derivative” that we use here. It seems to us that the most appropriate term is the “\( \phi \)-directional derivative,” because in the classical case when we do not have \( \phi \), as in (2.1) the existence of this limit corresponds to the directional differentiability of \( F \) at \( x \) in the direction \( y \), while the Gâteaux differentiability of \( F \) at \( x \) corresponds to the existence of the same limit in any direction \( y \in E \) and moreover the function \( y \mapsto DF(x; y) \) is linear and continuous. We note that the existence of \( DF(x; y) \) for any \( y \in E \) does not imply the Gâteaux differentiability of \( F \) at \( x \). As a simple example of what precedes we take the function \( F(x) = \| x \| \). We can easily check that \( DF(x; y) = \| y \| \) for any \( y \in E \) but the function \( y \mapsto DF(x; y) \) is not linear and so the Gâteaux derivative does not exist.

We recall (see [7, Proposition 6]) that the function \( y \mapsto D_{\phi,x}(y) \) is subadditive and

\[
|D_{\phi,x}(y)| \leq \| y \|. \tag{2.2}
\]
We end this section by establishing a necessary optimality condition in terms of $\varphi$-directional derivative for a minimization problem.

**Theorem 2.1.** Let $(X, \| \cdot \|)$ be an arbitrary Banach space and $F : X \to \mathbb{R}$. If $F$ has a global minimum at $v \in X$, then

$$\inf_{\varphi} D_{\varphi} F(v; y) \geq 0$$

(2.3)

for all $y \in X$.

**Proof.** Assume that $F$ has a global minimum at $v$, i.e.,

$$F(x) \geq F(v)$$

(2.4)

for all $v \in X$. Let $t > 0$, $\varphi$, and $y \in X$ be taken arbitrarily. Then (2.4) with $x := v + te^{i\varphi}y$ yields

$$F(v + te^{i\varphi}y) - F(v) \geq 0,$$

which implies

$$\frac{F(v + te^{i\varphi}y) - F(v)}{t} \geq 0$$

for all $t > 0$. Letting $t \to 0^+$ we obtain

$$\lim_{t \to 0^+} \frac{F(v + te^{i\varphi}y) - F(v)}{t} \geq 0, \quad \forall \varphi, y.$$

Thus

$$D_{\varphi} F(v; y) \geq 0, \quad \forall \varphi, y,$$

and hence

$$\inf_{\varphi} D_{\varphi} F(v; y) \geq 0, \quad \forall y \in X.$$

This completes the proof. \(\blacksquare\)

**3. Main results**

Let $\phi : B(H) \to B(H)$ be a linear map, that is, $\phi(\alpha X + \beta Y) = \alpha \phi(X) + \beta \phi(Y)$ for all $\alpha, \beta, X, Y$, and let $S \in C_1$. Put

$$\mathcal{U} = \{ X \in B(H) : \phi(X) \in C_1 \}.$$

Let $\psi : \mathcal{U} \to C_1$ be defined by

$$\psi(X) = S + \phi(X).$$

(3.1)

Define the function $F_\psi : \mathcal{U} \to \mathbb{R}^+$ by $F_\psi(X) = \| \psi(X) \|_{C_1}$. Now we are ready to prove our first result in $C_1$-classes. It gives a necessary and sufficient optimality condition for minimizing $F_\psi$. 
Theorem 3.1. The map $F_\phi$ has a global minimum at $V \in U$ if and only if
\[
\inf_{\psi} D_{\psi,\phi(V)}(\psi(Y)) \geq 0, \quad \forall Y \in U.
\] (3.2)

Before proving this theorem we need the following lemma.

Lemma 3.1. The following equalities hold for all $V,Y \in U$,
\[
D_{\phi} F_\psi(V,Y) = D_{\phi} \| \cdot \|_{C_1}(\psi(V); \phi(Y)) - D_{\phi,\psi(V)}(\phi(Y)).
\]

Proof. We have
\[
D_{\phi} F_\psi(V,Y) = \lim_{t \to 0^+} \frac{F(V + t e^{i\phi}Y) - F(Y)}{t} = \lim_{t \to 0^+} \frac{\|\psi(V + t e^{i\phi}Y)\|_{C_1} - \|\psi(V)\|_{C_1}}{t} = \lim_{t \to 0^+} \frac{\|S + \phi(V) + t e^{i\phi} \phi(Y)\|_{C_1} - \|\psi(V)\|_{C_1}}{t} = \lim_{t \to 0^+} \frac{\|\psi(V) + t e^{i\phi} \phi(Y)\|_{C_1} - \|\psi(V)\|_{C_1}}{t} = D_{\phi} \| \cdot \|_{C_1}(\psi(V); \phi(Y)).
\]

Proof of Theorem 3.1. For the necessity we have just to combine Theorem 2.1 and Lemma 3.1.

Conversely, assume that (3.2) is satisfied. First, observe that
\[
\|\psi(V)\|_{C_1} = -D_{\phi,\psi(V)}(e^{i(\pi-\phi)} \psi(V)).
\]

From this, we have
\[
\|\psi(V)\|_{C_1} = -D_{\phi,\psi(V)}(e^{i(\pi-\phi)} \psi(V)).
\]

Let $Y \in U$ be arbitrary and put $\tilde{Y} = Y + e^{i(\pi-\phi)} V + \phi^{-1}(S + e^{i(\pi-\phi)} S)$. It is easy to see that $\tilde{Y} \in U$. Then by (3.2) we have $D_{\phi,\psi(V)}(\phi(\tilde{Y})) \geq 0$ and hence by the subadditivity of $D_{\phi,\psi(V)}(\cdot)$ and the linearity of $\phi$ we get
\[
\|\psi(V)\|_{C_1} \leq -D_{\phi,\psi(V)}(e^{i(\pi-\phi)} \psi(V)) + D_{\phi,\psi(V)}(\phi(\tilde{Y})) = D_{\phi,\psi(V)}(\phi(\tilde{Y}) - e^{i(\pi-\phi)} \psi(V)) = D_{\phi,\psi(V)}(\phi(Y) + e^{i(\pi-\phi)} \phi(V) + S + e^{i(\pi-\phi)} S - e^{i(\pi-\phi)} \psi(V)) = D_{\phi,\psi(V)}(\psi(Y)).
\]
By using (2.2) we obtain
\[ \|\psi(V)\|_{C_1} \leq D_{\psi,\phi}(Y) \leq \|\dot{\psi}(Y)\|_{C_1}. \]

Finally as \( Y \) is arbitrary in \( \mathcal{U} \), then \( F_\psi \) has a global minimum at \( V \) on \( \mathcal{U} \). \( \square \)

Note that in our proofs of Theorem 3.1 and Lemma 3.1 we do not use the form of the norm in \( C_1 \)-classes and we can check that they still hold in any \( C_p \)-classes with \( 1 \leq p \leq \infty \). Now, we restrict our attention on \( C_1 \)-classes. First, let us recall the following result proved in \([7, \text{Theorem 2}]\) for \( C_1 \)-classes.

**Theorem 3.2.** Let \( X,Y \in C_1 \). Then, there holds
\[ D_X(Y) = \Re\{\tr(U^*Y)\} + \|QP\|_{C_1}, \]
where \( X = U|X| \) is the polar decomposition of \( X \), \( P = P_{\ker X}, Q = Q_{\ker X^*} \) are projections.

The following corollary establishes a characterization of the \( \varphi \)-directional derivative of the norm in \( C_1 \)-classes.

**Corollary 3.1.** Let \( X,Y \in C_1 \). Then, there holds
\[ D_{\varphi,X}(Y) = \Re\{e^{i\varphi}\tr(U^*Y)\} + \|QP\|_{C_1} \]
for all \( \varphi \), where \( X = U|X| \) is the polar decomposition of \( X \), \( P = P_{\ker X}, Q = Q_{\ker X^*} \) are projections.

**Proof.** Let \( X,Y \in C_1 \). Put \( \tilde{Y} = e^{i\varphi}Y \). Applying Theorem 3.2 with \( \varphi \), \( X \), and \( \tilde{Y} \) we get
\[ D_{\varphi,X}(Y) = \lim_{t \to 0^+} \|X + te^{i\varphi}Y\|_{C_1} - \|X\|_{C_1} = \lim_{t \to 0^+} \|X + t\tilde{Y}\|_{C_1} - \|X\|_{C_1} = D_X(\tilde{Y}) \]
\[ = \Re\{\tr(U^*\tilde{Y})\} + \|Q\tilde{Y}P\|_{C_1} = \Re\{\tr(U^*e^{i\varphi}Y)\} + \|Qe^{i\varphi}YP\|_{C_1} \]
\[ = \Re\{e^{i\varphi}\tr(U^*Y)\} + \|QP\|_{C_1}. \]
This completes the proof. \( \square \)

In the following theorem we use Theorem 3.1 to give another characterization of the global minimum of \( F_\psi \) as global minimum of the function \( L_{V,\psi} : \mathcal{U} \to \mathbb{R} \) defined by
\[ L_{V,\psi}(Y) := \|Q\psi(Y)P\|_{C_1} - |\tr(U^*\psi(Y))|, \]
where \( \psi(V) = U|\psi(V)| \).

**Theorem 3.3.**

1. \( F_\psi \) has a global minimum on \( \mathcal{U} \) at \( V \) if and only if
   \[ L_{V,\psi}(Y) \geq 0, \quad \forall Y \in \mathcal{U}. \quad (3.3) \]
If $V \in \ker \phi$, then $F_\psi$ has a global minimum on $\mathcal{U}$ at $V$ if and only if $L_{V, \phi}$ has a global minimum on $\mathcal{U}$ at $V$.

**Proof.** (1) We prove the necessity of part (1). Assume that $F_\psi$ has a global minimum on $\mathcal{U}$ at $V$. Then by Theorem 3.1 we have

$$\inf_{\psi} D_{\psi, \phi}(Y) \geq 0, \quad \forall \psi, Y \in \mathcal{U},$$

which ensures by Corollary 3.1 that

$$\inf_{\psi} \Re \left[ e^{i\psi} \text{tr}(U^* \phi(Y)) \right] + \left\| Q \phi(Y) P \right\|_{C_1} \geq 0$$

with $\psi(V) = U|\psi(V)|$ is the polar decomposition of $\psi(V)$ and $P = P_{\ker \psi(V)}$, $Q = Q_{\ker \psi(V)^*}$ or equivalently

$$\left\| Q \phi(Y) P \right\|_{C_1} \geq -\inf_{\psi} \Re \left[ e^{i\psi} \text{tr}(U^* \phi(Y)) \right].$$

By choosing the most suitable $\psi$ we get

$$\left\| Q \phi(Y) P \right\|_{C_1} \geq -\text{tr}(U^* \phi(Y))$$

and so $L_{V, \phi}(Y) \geq 0$ for all $Y \in \mathcal{U}$.

Conversely, assume that (3.3) is satisfied. Let $\psi$ be arbitrary and $Y \in \mathcal{U}$. By (3.3) we have

$$\left\| Q \phi(Y) P \right\|_{C_1} \geq -\Re \left( \text{tr}(U^* \phi(Y)) \right)$$

with $\tilde{Y} = e^{i\psi} Y \in \mathcal{U}$. Hence, by the linearity of $\phi$ we obtain

$$\left\| Q \phi(Y) P \right\|_{C_1} \geq -\Re \left( e^{i\psi} \text{tr}(U^* \phi(Y)) \right)$$

for $Y \in \mathcal{U}$ and all $\psi$ and so

$$\inf_{\psi} \left[ \left\| Q \phi(Y) P \right\|_{C_1} + \Re \left( e^{i\psi} \text{tr}(U^* \phi(Y)) \right) \right] \geq 0$$

for $Y \in \mathcal{U}$ and all $\psi$. Thus Theorem 3.1 and Lemma 3.1 complete the proof of part (1).

(2) Assume that $V \in \ker \phi$, that is, $\phi(V) = 0$; then $L_{V, \phi}(Y) = 0$ and so (3.3) is equivalent to

$$L_{V, \phi}(Y) \geq L_{V, \phi}(V), \quad \forall Y \in \mathcal{U}.$$

This means that $L_{V, \phi}$ has a global minimum at $V$. Therefore part (1) ends the proof. 

Now we characterize the global minimum of $F_\psi$ on $\mathcal{U}$, when $\phi$ is a linear map satisfying the following useful condition:

$$\text{tr}(X \phi(Y)) = \text{tr}(\phi^*(X) Y), \quad \forall X, Y \in \mathcal{C}$$

where $\phi^*$ is an appropriate conjugate of the linear map $\phi$. We state some examples of $\phi$ and $\phi^*$ which satisfy condition (3.5).

(1) The elementary operator $E : \mathcal{I} \to \mathcal{I}$ defined by

$$E(X) = \sum_{i=1}^{n} A_i X B_i,$$
where \((A_1, A_2, \ldots, A_n)\) and \((B_1, B_2, \ldots, B_n)\) are \(n\)-tuples of bounded Hilbert space operators and \(I\) is a separable ideal of compact operators associated with some unitarily invariant norm. In [7, Proposition 8] the author showed that the conjugate operator \(E^* : I^* \rightarrow I^*\) of \(E\) has the form

\[
E^*(X) = \sum_{i=1}^{n} B_i X A_i,
\]

and that the operators \(E\) and \(E^*\) satisfy condition (3.5).

(2) The elementary operator \(\tilde{E} : I \rightarrow I\) defined by

\[
\tilde{E}(X) = \sum_{i=1}^{n} A_i X B_i - X,
\]

where \((A_1, A_2, \ldots, A_n)\) and \((B_1, B_2, \ldots, B_n)\) are \(n\)-tuples of bounded Hilbert space operators and \(I\) is a separable ideal of compact operators associated with some unitarily invariant norm. Using the same ideas of the proof of [7, Proposition 8] we can check that the conjugate operator \(\tilde{E}^* : I^* \rightarrow I^*\) of \(\tilde{E}\) has the form

\[
\tilde{E}^*(X) = \sum_{i=1}^{n} B_i X A_i - X,
\]

and that the operators \(\tilde{E}\) and \(\tilde{E}^*\) satisfy condition (3.5).

Now, we are in position to prove the following theorem.

**Theorem 3.4.** Let \(V \in C_1\), and let \(\psi(V)\) have the polar decomposition \(\psi(V) = U|\psi(V)|\). Then \(F_{\psi}\) has a global minimum on \(C_1\) at \(V\) if and only if \(U^* \in \ker \phi^*\).

**Proof.** Assume that \(F_{\psi}\) has a global minimum on \(C_1\) at \(V\). Then

\[
\inf_{\psi} D_{\psi,\psi(V)}(\phi(Y)) \geq 0 \quad (3.6)
\]

for all \(Y \in C_1\). That is,

\[
\inf_{\psi} \text{Re} \left\{ e^{i \phi(Y)} \right\} + \left\| Q\phi(Y) P \right\|_{C_1} \geq 0, \quad \forall Y \in C_1.
\]

Take \(\phi\) so that

\[
\text{Re}\left\{ \text{tr}(U^* \phi(Y)) \right\} \geq 0. \quad (3.7)
\]

Let \(f \otimes g\) be the rank one operator defined by \(x \mapsto \langle x, f \rangle g\), where \(f, g\) are arbitrary vectors in the Hilbert space \(H\). Take \(Y = f \otimes g\), since the map \(\phi\) satisfies (3.5) one has

\[
\text{tr}(U^* \phi(Y)) = \text{tr}(\phi^*(U^*)Y).
\]

Then (3.7) is equivalent to \(\text{Re}\{\text{tr}(\phi^*(U^*)Y)\} \geq 0\) for all \(Y \in C_1\), or equivalently

\[
\text{Re}\left\{ \phi^*(U^*)g, f \right\} \geq 0, \quad \forall f, g \in H.
\]

As \(f, g\) are arbitrary we can easily check that
\[ \text{Re}\left\{ \langle \phi^*(U^*)g, f \rangle \right\} = 0, \quad \forall f, g \in H. \]

Thus \( \phi^*(U^*) = 0 \), i.e., \( U^* \in \ker \phi^* \).

Conversely, let \( \psi \) be arbitrary. If \( U^* \in \ker \phi^* \), then \( e^{i\psi} U^* \in \ker \phi^* \). It is easily seen (using the same arguments above) that

\[ \text{Re}\left\{ e^{i\psi} \operatorname{tr}(U^* \phi(Y)) \right\} + \| Q\phi(Y) P \|_{C_1} \geq 0, \quad \forall Y \in C_1. \]

Now as \( \psi \) is taken arbitrary, we get (3.6).

We state our first corollary of Theorem 3.4. Let \( \phi = \delta_{A,B} \), where \( \delta_{A,B} : B(H) \to B(H) \) is the generalized derivation defined by \( \delta_{A,B}(X) = AX - XB \).

**Corollary 3.2.** Let \( V \in C_1 \), and let \( \psi(V) \) have the polar decomposition \( \psi(V) = U|\psi(V)| \). Then \( F_\psi \) has a global minimum on \( C_1 \) at \( V \), if and only if \( U^* \in \ker \delta_{B,A} = \ker \delta_{A,B} \).

**Proof.** It is a direct consequence of Theorem 3.4. \( \square \)

This result may be reformulated in the following form where the global minimum \( V \) does not appear. It characterizes the operators \( S \) in \( C_1 \) which are orthogonal to the range of a derivation.

**Theorem 3.5.** Let \( S \in C_1 \), and let \( \psi(S) \) have the polar decomposition \( \psi(S) = U|\psi(S)| \). Then

\[ \| S + (AX - XB) \|_{C_1} \geq \| \psi(S) \|_{C_1} \]

for all \( X \in C_1 \) if and only if \( U^* \in \ker \delta_{B,A} \).

As a corollary of this theorem we have

**Corollary 3.3.** Let \( S \in C_1 \), and let \( \psi(S) \) have the polar decomposition \( \psi(S) = U|\psi(S)| \). Then the two following assertions are equivalent:

1. \( \| S + (AX - XB) \|_{C_1} \geq \| S \|_{C_1} \) for all \( X \in C_1 \);
2. \( U^* \in \ker \delta_{B,A} \).

**Remark 3.1.**
1. Note that results similar to Corollary 3.3 but only when \( S \) is a smooth point have been already considered by Kittaneh [8] and Duggal [5].
2. Note that the orthogonality notion considered in this paper is in the sense of Birkhoff [2].
3. We point out that, thanks to our general results given previously with more general linear maps \( \phi \), Theorem 3.5 and its Corollary 3.3 still true for more general classes of operators than \( \delta_{A,B} \) like the elementary operators \( E(X) \) and \( \bar{E}(X) \).
4. The case of \( C_\infty \)-classes and other investigations on the \( \phi \)-directional derivatives are studied by the authors in [13]. Many applications of our results presented in this paper and in [13] will be given in a series of works by the authors.
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