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# Localization at Epimorphisms and Quasi-injectives*

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## INTRODUCTION

The general theory of localization, developped by Rattray and the author [19] from a method of Fakir [7], yields not only the localization of Bourbaki–Gabriel [2], but also the localization associated with any epimorphism of rings, such as that of Cohn [3, 4]. (Incidentally, ring epimorphisms are characterized in terms of systems of equations whose solutions are uniquely determined.) After a discussion of a slightly generalized version of Jacobson's density theorem [13], the density theorem of the author [17, 18] is shown to be extendable to this general situation. Quasi-injectives do not, in general, appear to permit localization in Mod $R$, but they do in Cont $R$, the category of "continuous" $R$-modules. In this category, localization alone accomplishes what requires localization and completion under favorable circumstances in Mod $R$. Crucial to this result is a lemma of Harada [12]. Some observations about Mod $R$ may be deduced by specializing the topology to be discrete.

## 1. LOCALIZATION IN COMPLETE CATEGORIES

We begin by giving a brief exposition of the theory developed in [19]. Let $\mathcal{A}$ be a complete category, $I$ a given object of $\mathcal{A}$. We introduce the functor

$$T = T_I = I^{\mathrm{Hom}(-,I)}: \mathcal{A} \to \mathcal{A}.$$

It comes equipped with a natural transformation $\eta = \eta_I$: id $\to T$ from the identity functor, which satisfies $\pi_f \eta(A) = f$ for each $f \in \mathrm{Hom}(A, I)$, where $\pi_f$: $T(A) \to I$ is the canonical projection associated with $f$.

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Following the idea of [7], for each object $A$ of $\mathfrak{A}$, we let $\kappa(A) = \kappa_I(A): Q(A) \to T(A)$ be the equalizer of the two morphisms $\eta T(A)$, $T\eta(A)$: $TA \rightrightarrows T^2(A)$. As was shown in [19], it is also the intersection of the equalizers of all pairs of morphisms $\varphi, \psi$: $T(A) \rightrightarrows I$ such that $\varphi\eta(A) = \psi\eta(A)$. One obtains a natural transformation $\lambda = \lambda_I$: $\mathrm{id} \to Q$ such that $\kappa \circ \lambda = \eta$. The following is known [19].

THEOREM 1.1. *The following conditions on $I$ are equivalent:*

(1) $\lambda Q$ *is an isomorphism;*

(2) *for each* $f: Q(A) \to I$, *there exists an* $f'$: $T(A) \to I$ *such that* $f'\kappa(A) = f$;

(3) *the full subcategory* $\mathrm{Fix}(Q, \lambda)$ *of* $\mathfrak{A}$, *which consists of all objects* $A$ *of* $\mathfrak{A}$ *for which* $\lambda(A)$ *is an isomorphism, is the smallest full subcategory of* $\mathfrak{A}$ *that contains* $I$ *and is closed under limits.*

Property (1) assures that $(Q, \lambda)$ determines an idempotent triple [7].

In view of property (2), we call the object $I$ *$\kappa$-injective.*

Property (3) assures that $\mathrm{Fix}(Q, \lambda)$ is a full reflective subcategory of $\mathfrak{A}$, the reflector being given by the assignment $A \mapsto Q(A)$.

Many examples of this situation have been discussed in [19], among them the following: $\mathfrak{A}$ is the category of topological spaces, the category of uniform spaces, the category of presheaves on a topological space, the category $(\mathrm{Mod}\ R)^{\mathrm{op}}$, and the category $\mathrm{Mod}\ R$.

## 2. LOCALIZATION IN Mod $R$

We shall briefly review some known facts about Mod $R$ and add some new ones. Which are the $\kappa$-injectives of Mod $R$?

EXAMPLE 2.1. Let $I$ be an injective $R$-module and $\mathscr{D} = \mathscr{D}_I$ the set of all those right ideals $D$ of $R$ for which $\mathrm{Hom}_R(R/D, I) = 0$. Then, $\mathscr{D}$ is an "idempotent filter" in the sense of Bourbaki–Gabriel [2] and

$$Q(A) = \varinjlim_{D \in \mathscr{D}} \mathrm{Hom}(D, A/A_0),$$

where

$$A_0 = \{a \in A \mid \exists_{D \in \mathscr{D}}\, aD = 0\}.$$

Conversely, any idempotent filter of right ideals of $R$ gives rise to an injective

$$I = \prod_{C \in \mathscr{C}} E(R/C),$$

where $\mathscr{C}$ is the set of all those right ideals $C$ of $R$ such that

$$\forall_{r \notin C}\, r^{-1}C \notin \mathscr{D},$$

and $E(-)$ is the injective envelope. One calls $Q(A)$ "module of quotients" of $A$ and $Q$ "localization functor."

EXAMPLE 2.2. Let $e\colon R \to R'$ be a given epimorphism in the category of rings. Then, every injective in the full subcategory Mod $R'$ is $\kappa$-injective in Mod $R$. In particular, one may take $I$ to be an injective cogenerator of Mod $R'$, for instance, $I = \mathrm{Hom}_{\mathbb{Z}}(R', \mathbb{Q}/\mathbb{Z})$. Then, $\mathrm{Fix}(Q, \lambda) \cong \mathrm{Mod}\, R'$ and $Q(A) \cong A \otimes_R R'$.

Note that the intersection of Examples 2.1 and 2.2 is the localization theory determined by an epimorphism $R \to R'$, such that the left module ${}_R R'$ is flat, which theory has been discussed by Silver [24]. This vague observation will be made precise in Corollary 2.7 below.

The localization of Cohn [3, 4] is a special case of Example 2.2 not included in Example 2.1. He obtained an epimorphism $e\colon R \to R'$ by forcing a set of matrices with entries from $R$ to become invertible in $R'$.

To find a two-sided inverse of an $n \times n$ matrix, one has to satisfy $2n^2$ equations in $n^2$ unknowns; but we know that the solutions, if they exist, are unique. There are other such systems of equations with uniquely determined solutions, for example,

$$rxr = r, \qquad xrx = x, \qquad rx = xr.$$

Consider a ring extension $h\colon R \to R[X]$ obtained from $R$ by adjoining a set $X$ of indeterminates.[1] A set of polynomials

$$p(h(r_1), \dots, h(r_m), x_1, \dots, x_n) \in R[X]$$

with "coefficients" $r_i \in R$ and "unknowns" $x_j \in X$ will be said to have a *solution* $\varphi\colon X \to R'$ in an extension ($=$ ring homomorphism) $f\colon R \to R'$ if

$$p(f(r_1), \dots, f(r_m), \varphi(x_1), \dots, \varphi(x_m)) = 0$$

for each polynomial $p$ in the set.

A set of polynomials always has a solution in some extension, for example in $\pi h\colon R \to R[X]/K$, where $K$ is the ideal generated by the given polynomials and $\pi\colon R[X] \to R[X]/K$ is the canonical projection: the solution is of course $x \mapsto \pi(x)$.

We say that a set of polynomials in $R[X]$ is *uniquely solvable* for $X$ if it has, at most, one solution in each extension of $R$.

[1] They do not commute with elements of $R$ or with each other.

THEOREM 2.3. *Let $K$ be an ideal of $R[X]$, then $K$ is uniquely solvable for $X$ if and only if $R \to R[X]/K$ is an epimorphism of rings. Moreover, every epimorphism $R \to S$ is of this form, up to isomorphism over $R$.*

*Proof.* Assume $K$ is uniquely solvable for $X$. We claim that $\pi h$ is an epimorphism. Indeed, suppose $u, v: R[X]/K \rightrightarrows R'$ are such that $u\pi h = v\pi h = f$, say. Then, $x \mapsto u\pi(x)$ and $x \mapsto v\pi(x)$ are solutions in $f: R \to R'$, hence, $u\pi = v\pi$ and therefore, $u = v$.

Conversely, suppose $\pi h$ is an epimorphism of rings and suppose $\varphi$: $X \to R'$ is a solution of $K$ in the extension $f: R \to R'$. Then,

$$p(f(r_1),..., f(r_m), \varphi(x_1),..., \varphi(x_n)) = 0$$

for each polynomial $p$ in $K$. Now, there exists a unique ring homomorphism $\varphi'$: $R[X] \to R'$, such that $\varphi' h = f$ and $\varphi'(x) = \varphi(x)$ for all $x \in X$. Clearly,

$$\varphi'(p(h(r_1),..., h(r_m), x_1 ,..., x_n)) = 0$$

for each $p$ in $K$, hence, there exists a unique $\varphi^*$: $R[X]/K \to R'$ such that $\varphi^*\pi = \varphi'$. It follows that $\varphi^*\pi h = f$ and $\varphi^*\pi(x) = \varphi(x)$ for all $x \in X$. If $\psi$: $X \to R'$ is another solution of $K$ in $f: R \to R'$, then likewise, we obtain $\psi^*$, such that $\psi^*\pi h = f$ and $\psi^*\pi(x) = \psi(x)$ for all $x \in X$. Since $\pi h$ is epi, we have $\varphi^* = \psi^*$, hence, $\varphi = \psi$.

Finally, let $e$: $R \to S$ be any epimorphism of rings. To each element $s \in S$, assign the indeterminate $x_s$ and let $X = \{x_s \mid s \in S\}$. Let $K$ be the ideal of $R[X]$ generated by the following polynomials:

$$x_0 , \qquad x_1 - 1, \qquad x_s + x_{-s} , \qquad x_{s+s'} - x_s - x_{s'} , \qquad x_{ss'} - x_s x_{s'} ,$$
$$x_{e(r)} - h(r),$$

for all $r \in R$ and $s, s' \in S$. Let $g$: $R[X] \to S$ be the unique ring homomorphism such that $gh = e$ and $g(x_s) = s$ for all $s \in S$. Then, $g(x_0) = 0$, $g(x_1 - 1) = 1 - g(1) = 0$, etc., hence, $g(K) = 0$. Therefore, there exists a unique ring homomorphism $g'$: $R[X]/K \to S$ such that $g'\pi = g$. We claim that $g'$ is an isomorphism. Since $g'\pi h = gh = e$, this will complete the proof.

Clearly, $g'\pi(x_s) = g(x_s) = s$, hence, $g'$ is a surjection. It remains to show that it is one-to-one. Suppose

$$g(p(h(r_1),..., h(r_m), x_{s_1} ,..., x_{s_n})) = 0,$$

that is to say,

$$p(e(r_1),..., e(r_m), s_1 ,..., s_n) = 0.$$

Then, modulo $K$, we have

$$\begin{aligned} p(h(r_1),\ldots, h(r_m), x_{s_1},\ldots, x_{s_n}) &\equiv p(x_{e(r_1)},\ldots, x_{e(r_m)}, x_{s_1},\ldots, x_{s_n}) \\ &\equiv x_{p(e(r_1),\ldots,e(r_m),s_1,\ldots,s_n)} \\ &\equiv x_0 \equiv 0. \end{aligned}$$

Thus, $g'\pi(p) = 0$ implies that $\pi(p) = 0$, and our proof is complete.

*Remark* 2.4. It also follows from considerations similar to the above that $\pi h\colon R \to R[X]/K$ has the following universal property: If the polynomials in $K$ are also solvable in $f\colon R \to R'$, then there exists a unique ring homomorphism $g\colon R[X]/K \to R'$ such that $g\pi h = f$.

Finally, let it be mentioned that these observations about epimorphisms are not special to the category of rings. They are valid for all algebraic categories, provided one notes that, in general, an equation has the form $p = q$ and not just $p = 0$. Thus, one should speak of solutions for pairs of polynomials and one should replace the ideal $K$ by a suitable congruence relation.

The following well-known result depends only on the fact that $\mathrm{Fix}(Q, \lambda)$ is a reflective subcategory of Mod $R$.

LEMMA 2.5. *Let $I$ be a $\kappa$-injective right $R$-module; then*

(a) *$Q(R)$ is a ring,*

(b) *$\lambda(R)$ is a ring homomorphism,*

(c) *every object of $\mathrm{Fix}(Q, \lambda)$ is a right $Q(R)$-module,*

(d) *every $R$-homomorphism between objects of $\mathrm{Fix}(Q, \lambda)$ is a $Q(R)$-homomorphism.*

For a proof, see [16]. As explained there, Mod $Q(R)$ is, in a certain technical sense, the "best coapproximation" of $\mathrm{Fix}(Q, \lambda)$ by a module category.

The following result is known in case $I$ is injective.

PROPOSITION 2.6. *Let $I$ be a $\kappa$-injective right $R$-module. Then, the following statements are equivalent:*

(0) *$\eta(B)$ is mono for every $Q(R)$-module $B$,*

(1) $\mathrm{Fix}(Q, \lambda) = \mathrm{Mod}\, Q(R)$,

(2) $Q \cong (-) \otimes_R Q(R)$,

(3) *$Q$ preserves colimits,*

(4) *$\mathrm{Fix}(Q, \lambda)$ is closed under colimits.*

*Proof.* We show (0) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (0).

(0) $\Rightarrow$ (1). Let $B \in \operatorname{Mod} Q(R)$, then, also, $T(B)/\bar{B} \in \operatorname{Mod} Q(R)$. By (0), $\eta(B)$ and $\eta(T(B)/B)$ are both mono.[2] The first fact implies that $B \cong \operatorname{im} \eta(B)$ and the second fact implies that $\operatorname{im} \eta(B) = Q(B)$. Thus, $\operatorname{Mod} Q(R) \subseteq \operatorname{Fix}(Q, \lambda)$ and the converse inclusion follows from Lemma 2.5.

(1) $\Rightarrow$ (2). The reflectors $A \mapsto Q(A)$ and $A \mapsto A \otimes_R Q(R)$ must be isomorphic.

(2) $\Rightarrow$ (3). Since $(-) \otimes_R Q(R)$ preserves colimits, so does $Q$.

(3) $\Rightarrow$ (4). This is obvious.

(4) $\Rightarrow$ (0). Let $B \in \operatorname{Mod} Q(R)$. Then, there is an exact sequence

$$\sum^{\oplus} Q(R) \rightarrow \sum^{\ominus} Q(R) \rightarrow B \rightarrow 0$$

in $\operatorname{Mod} R$. By (4), $B \in \operatorname{Fix}(Q, \lambda)$, hence, $\eta(B)$ is mono.

Two $\kappa$-injectives $I$ and $I'$ are called *similar* if they have the same limit closures, that is, if they give rise to isomorphic localization functors.

COROLLARY 2.7. *There is a one-to-one correspondence between*

(a) *similarity classes of $\kappa$-injectives in* Mod $R$ *satisfying the equivalent conditions of Proposition* 2.6,

(b) *isomorphism classes of epimorphisms $R \rightarrow R'$ in the category of rings.*

*Moreover, ${}_RR'$ is a flat left $R$-module if and only if the associated similarity class contains an injective.*

*Proof.* To the $\kappa$-injective $I$, we associate the ring homomorphism $\lambda_I(R)\colon R \rightarrow Q_I(R)$, as usual. Under the equivalent conditions of Proposition 2.6, this is an epi, for then, $\operatorname{Mod} Q(R) = \operatorname{Fix}(Q, \lambda)$ is a full subcategory of $\operatorname{Mod} R$. (We note that $I$ is a cogenerator of $\operatorname{Mod} Q(R)$.)

To the ring epimorphism $e\colon R \rightarrow R'$, we associate the injective cogenerator $I_e$ of $\operatorname{Mod} R'$, as in Example 2.2. But then, $Q_{I_e} \cong (-) \otimes_R R'$, so that $Q_{I_e}(R) \cong R'$, hence, the equivalent conditions of Proposition 2.6 are satisfied.

If $I' = I_{\lambda_I(R)}$, then $Q_{I'} \cong (-) \otimes_R Q_I(R) \cong Q_I$, hence, $I'$ is similar to $I$.

Also, $Q_{I_e}$ induces the reflector $\operatorname{Mod} R \rightarrow \operatorname{Mod} R'$, hence, $Q_{I_e} \cong (-) \otimes_R R'$, hence, $Q_{I_e}(R) \cong R \otimes_R R' \cong R'$. One may check that $\lambda_{I_e}(R) \cong e$.

If $I$ is injective, $Q_I \cong (-) \otimes_R R'$ preserves monomorphisms, hence, ${}_RR'$ is flat.

If ${}_RR'$ is flat, the injective cogenerator $I = \operatorname{Hom}_{\mathbb{Z}}(R', \mathbb{Q}/\mathbb{Z})$ of $\operatorname{Mod} R'$ is injective as a right $R$-module.

[2] Put $\bar{B} = \operatorname{im} \eta(B)$ and observe that, in view of Section 1, $Q(B)/\bar{B} = \ker \eta(T(B)/\bar{B})$.

## 3. Density Theorems

Given a module $I \in \operatorname{Mod} R$, let $E$ be its ring of endomorphisms, so that $I$ may be viewed as a bimodule ${}_E I_R$. For any $A \in \operatorname{Mod} R$, we define its *double dual*

$$S(A) = S_I(A) = \operatorname{Hom}_E(\operatorname{Hom}_R(A, I), I).$$

Clearly, it is a submodule of $T(A)$ and the canonical morphism $\eta(A)\colon A \to T(A)$ factors through $S(A)$. In particular, $S(R)$ is a ring, the *bicommutator* of $I$ and $R \to S(R)$ is a ring homomorphism.

The following is an obvious generalization of Jacobson's Density Theorem [13] from rings to modules.

Proposition 3.1. *Let $I$ be a completely reducible right $R$-module. Then, for any $A \in \operatorname{Mod} R$, $A \to S(A)$ is "dense" in the following sense: for all $f_1, \ldots, f_n\colon A \to I$ and all $s \in S(A)$, there exists $a \in A$ such that*

$$(f_1)\, s = f_1(a), \ldots, (f_n)\, s = f_n(a).$$

*Proof* (inspired by [20]). First consider the case $n = 1$. Let $f\colon A \to I$ and $s \in S(A)$ be given. Since $I$ is completely reducible, $f(A) = eE$ for some $e^2 = e \in E$, hence, $f = ef$ and

$$(f)\, s = (ef)\, s = e((f)\, s) = f(a)$$

for some $a \in A$.

In general, suppose $f_1, \ldots, f_n : A \to I$. Then, we obtain $f = \langle f_1, \ldots, f_n \rangle\colon A \to I^n$, where

$$f(a) = (f_1(a), \ldots, f_n(a)).$$

Moreover, any $s \in S_I(A)$ induces $s_n \in S_{I^n}(A)$, where

$$(f)\, s_n = ((f_1)\, s, \ldots, (f_n)\, s).$$

Since $I^n$ is completely reducible, we may apply the special case of the theorem already proved and obtain $a \in A$, such that $(f)\, s_n = f(a)$, that is,

$$((f_1)\, s, \ldots, (f_n)\, s) = (f_1(a), \ldots, f_n(a)).$$

This result may be interpreted by saying that the image of $A \to S(A)$ is actually dense in $S(A)$ for the *finite* topology, that is, the topology induced by the product topology of $T(A)$, a power of the discrete module $I$.

*Remark* 3.2. Suppose $A$ is endowed with the $I$-adic topology, that is to say, fundamental open neighbourhoods of zero have the form $\operatorname{Ker} f$, where $f: A \to I^n$. Then, $S(A)$ is the Hausdorff completion of $A$. If $A$ is already Hausdorff and complete, it follows that the canonical homomorphism $A \to S(A)$ is an isomorphism.

Goldman [11] has recently shown that a ring $R$ is isomorphic to a direct product of endomorphism rings of vector spaces over division rings if and only if it is Hausdorff and complete in what he calls the *intrinsic* topology: fundamental open neighborhoods of zero have the form

$$e_1{}^r \cap \cdots \cap e_n{}^r,$$

where the $e_i$ are idempotent elements such that the $Re_i$ are minimal left ideals and $e_i{}^r = \{r \in R \mid e_i r = 0\}$. (I have taken the liberty of interchanging right and left in Goldman's definition.)

One part of Goldman's result may be deduced from the above considerations, if one checks that the intrinsic topology agrees with the $I$-adic topology for suitable $I$. In fact, let us suppose that $R$ is semiprime and that $I$ is its socle. Then, a fundamental neighborhood of 0 in the $I$-adic topology on $R$ is a finite intersection of $i^r = \{r \in R \mid ir = 0\}$, where $i \in I = Re_1 \oplus \cdots \oplus Re_n$. Now, $i^r \supseteq e_1{}^r \cap \cdots \cap e_n{}^r$, hence, the two topologies coincide.

We shall make frequent use of a lemma that, among other things, will yield another proof of Proposition 3.1.

LEMMA 3.3. *Let $I$ be any right $R$-module. For any $f: A \to I^n$, define $f^*: T(A) \to I^n$ by*

$$f^*(t) = ((f_1)\,t, \ldots, (f_n)\,t),$$

*for all $t \in T(A)$. Then, for any $h: I^n \to I$, if $hf(A) = 0$, then $hf^*S(A) = 0$.*

*Proof.* Suppose $h: I^n \to I$ is such that $hf(A) = 0$. Take any $s \in S(A)$, then

$$hf^*(s) = \sum_{i=1}^{n} h\kappa_i((f_i)\,s),$$

where the $\kappa_i: I \to I^n$ are the canonical injections. Now, $h\kappa_i \in E$ and $s$ is an $E$-homomorphism, hence,

$$hf^*(s) = \sum_{i=1}^{n} (h\kappa_i f_i)\,s = (hf)\,s = (0)\,s = 0.$$

This simple argument is essentially the same as that given by Müller [22] in an earlier result, which may be obtained as an application of the lemma, while the present author used an argument involving triples in [18].

APPLICATION 3.4 (Müller).[3] If $I$ is a cogenerator of Mod $R$, $\eta(A)\colon A \to S(A)$ is dense for all $A \in \text{Mod } R$.

*Proof.* Let $f\colon A \to I^n$. Since $I$ is a cogenerator, there is a set $X$ and a monomorphism $I^n/f(A) \to I^X$. In view of Lemma 3.3, $f^*S(A) = f(A)$, that is, for any $s \in S(A)$, there exists $a \in A$ such that $f^*(s) = f(a)$.

APPLICATION 3.5 (Proposition 3.1). If $I$ is completely reducible, $\eta(A)\colon A \to S(A)$ is dense for all $A \in \text{Mod } R$.

*Proof.* Let $f\colon A \to I^n$. Since $I$ is completely reducible, $f(A) = eI^n$ for some idempotent endomorphism $e$ of $I^n$. Hence, $I^n/f(A) \simeq (1 - e) I^n$, and we proceed as above.

Frequently, it is not the image of $A \to S(A)$ that is dense in $S(A)$, but $Q(A)$. For completeness, we include the following known observation.

LEMMA 3.6. $Q(A) \subseteq S(A)$.

*Proof.* Given $q \in Q(A)$, one wants to show that

$$(f + f')\, g = (f)\, q + (f')\, q, \qquad (ef)\, q = e((f)\, q)$$

for all $f, f'\colon A \to I$ and all $e \in E$. For example, let us verify the second equation, the first being shown similarly.

Define $\varphi, \psi\colon T(A) \to I$ by

$$\varphi(t) = (ef)\, t, \qquad \psi(t) = e((f)\, t),$$

for all $t \in T(A)$. Recall that $\eta(A)\colon A \to T(A)$ was defined by $\pi_f \eta(A) = f$. Put $\eta(A)\,(a) = \hat{a}$, then $(f)\, \hat{a} = f(a)$, hence,

$$\varphi\eta(A)\,(a) = (ef)\, \hat{a} = (ef)\,(a) = e(f(a)) = e((f)\, \hat{a}) = \psi\eta(A)\,(a).$$

Therfore, $\varphi\eta(A) = \psi\eta(A)$, whence, $\varphi(q) = \psi(q)$, by the definition of $Q(A)$, that is to say, $(ef)\, q = e((f)\, q)$.

THEOREM 3.7. *Suppose $R \to R'$ is an epimorphism and $I$ is an injective in Mod $R'$ such that the associated localization functor $Q$ is right exact. Then, for all $A$ in Mod $R$, $Q(A)$ is dense in $S(A)$ in the finite topology and $S(A)$ is the completion of $Q(A)$ in the $I$-adic topology.*

*Proof.* Let $f\colon A \to I^n$ and consider the following diagram in which the two squares commute and the top and bottom rows are exact:

$$\begin{array}{ccccccc} Q(A) & \to & I^n & \to & I^n/f^*Q(A) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ Q^2(A) & \to & Q(I^n) & \to & Q(I^n/f^*Q(A)) & \to & 0 \end{array}$$

[3] Applications 3.4 and 3.5 are comprised by Sandomierski's Lemma 3.4 (see [27]).

Since $Q(A)$ and $I^n$ are in $\mathrm{Fix}(Q, \lambda)$, in view of Theorem 1.1 and Example 2.2, the first two vertical maps are isomorphisms, hence, so is the last.

In particular, there is a monomorphism of $I^n/f^*Q(A)$ into some power of $I$. Thus, for any element $j$ of $I^n$ not in $f^*Q(A)$, one can find $h: I^n \to I$ such that $hf^*Q(A) = 0$ and $h(j) \neq 0$. But, by Lemma 3.3, $hf^*S(A) = 0$, hence, $j$ is not in $f^*S(A)$. Therefore, $f^*S(A) \subseteq f^*Q(A)$, that is, for any $s \in S(A)$, there exists $q \in Q(A)$ such that $f^*(s) = f^*(q)$.

This shows that $Q(A)$ is a dense submodule of $S(A)$ in the finite topology. On the other hand, it is easily seen that $S(A)$ induces the $I$-adic topology on $Q(A)$ and that $S(A)$ is complete in the finite topology, being a closed subspace of $T(A)$.

The above result has been shown for $R = R'$ in [18] and many examples were given. Let us now add one new example.

EXAMPLE 3.8. Let $R \to R'$ be an epimorphism of rings and $I$ an injective cogenerator of Mod $R'$. Then, $Q \cong (-) \otimes_R R'$ is right exact, hence, Theorem 3.7 applies.

A density theorem for quasi-injectives was also proved by Johnson and Wong [14] in case $A = R$. However, they stipulate that the elements $i_1, \ldots, i_n \in I$ are "$E$-linearly independent," which is not assumed when Theorem 3.7 is specialized to the case $A = R$ (so that $f_1, \ldots, f_n$ become $i_1, \ldots, i_n$). An interesting density theorem in their sense was obtained by Thierrin [25] (see also [15, p. 151]).

## 4. Quasi-injective Modules

An object $I$ of Mod $R$ is called *quasi-injective* if every partial endomorphism of $I$ can be extended to an endomorphism. Clearly, injective as well as completely reducible modules are quasi-injective. (For discussions of quasi-injective modules see [5, 9, 14], for example.)

To obtain a $\kappa$-injective object, in the sense of Section 1, one requires a slightly stronger property: for every set $X$ and for each submodule $B$ of $I^X$, every homomorphism $f: B \to I$ can be extended to a homomorphism $f': I^X \to I$. Evidently, this is equivalent to saying that every power of $I$ is quasi-injective. As was shown by Fuller [10] and Tisseron [26], it is also equivalent to saying that $I$ is injective as an $R/N$-module, where $N = \{r \in R \mid Ir = 0\}$. Since $R \to R/N$ is a ring epimorphism, we have here a special case of Example 2.2. (This stronger property is discussed in [23]. For a short proof of the above equivalence, see Proposition 6.4 below.)

Unfortunately, not every quasi-injective has this nice property. For example [10], if $R = \mathbb{Z}$ and $I$ is the direct sum of all $\mathbb{Z}/p\mathbb{Z}$, $p$ ranging over the prime numbers, $N = 0$, but $I$ is not injective. A weak form of this property is, however, contained in the following lemma of Harada [12]. As we shall use this lemma quite heavily and as Harada's original proof is not very direct, we present a direct proof here.

LEMMA 4.1 (Harada). *Let $I$ be a any quasi-injective right $R$-module, $n$ a natural number, and $B$ a submodule of $I^n$. Then, every homomorphism $f: B \to I$ can be extended to a homomorphism $f': I^n \to I$. That is to say, every finit power of $I$ is quasi-injective.*

*Proof.* Let $B$ be a submodule of $I^n = I^{n-1} \oplus I$ and $f: B \to I$. The restriction of $f$ to $I^{n-1} \cap B$ may be extended to $g: I^{n-1} \to I$ by inductional assumption. Define $h: I^{n-1} + B \to I$ by

$$h(x + b) = g(x) + f(b),$$

for any $x \in I^{n-1}$ and $b \in B$. (First check that $x + b = 0$ implies $g(x) + f(b) = 0$.) Let

$$K = \{k \in I \mid \exists_{x \in I^{n-1}} x + k \in B\}.$$

Then, $K$ is a submodule of $I$, $I^{n-1} + B = I^{n-1} + K$ and the latter sum is direct. Now, $h$ is determined by two homomorphisms: $I^{n-1} \to I$ and $K \to I$. Extending the latter to $I \to I$, we can extend $h$ to $I^{n-1} + I \to I$.

We shall see in the next section that the stronger property mentioned above is valid provided we replace Mod $R$ by a larger category.

With the help of a quasi-injective module $I$ (or any module $I$ for that matter), one can define the *$I$-adic* topology on another module $A$, by declaring as fundamental open neighbourhoods of zero all kernels of homomorphisms $A \to I^n$, $n$ being any natural number. When $I$ is injective, this topology has been studied extensively, the interesting fact being that the induced topology on any submodule $B$ of $A$ is also the $I$-adic topology of $B$. Actually, this fact may be used to characterize the injectives among the quasi-injectives.

PROPOSITION 4.2. *Let $I$ be a quasi-injective right $R$-module. Then, the following properties of $I$ are equivalent:*

(1) *$I$ is injective.*

(2) *The $I$-adic topology of each submodule $B$ of any module $A$ is induced by the $I$-adic topology of $A$.*

(3) *The $I$-adic topology on each (essential) right ideal $D$ of $R$ is induced by the $I$-adic topology of $R$.*

*Proof.* The implication (1) $\Rightarrow$ (2) is known (e.g., [18]) and straightforward. The implication (2) $\Rightarrow$ (3) is evident. It remains to show that (3) $\Rightarrow$ (1). In view of Baer's criterion, it suffices to show that any $f: D \to I$ can be extended to $R \to I$, where $D$ is any right ideal of $R$, or even any essential right ideal. (For, we can always extend to $D + D' \to I$, where $D'$ is maximal among right ideals such that $D \cap D' = 0$ and then, $D + D'$ is an essential right ideal.)

Now, $\operatorname{Ker} f$ is an open neighborhood of 0 in $D$. Therefore, by (3), there exists $g: R \to I^n$, such that $\operatorname{Ker} f \supseteq \operatorname{Ker} g$. Thus, we have a mapping $g(d) \mapsto f(d)$, $d \in D$. By Harada's lemma, we may extend this to $h: I^n \to I$. Hence, $h(g(d)) = f(d)$ for all $d \in D$ and so, $hg: R \to I$ extends $f$.

Corollary 4.3. *A right $R$-module $I$ is injective if and only if it is quasi-injective and the $I$-adic topology of any (essential) right ideal $D$ of $R$ is induced by that of $R$.*

## 5. Localization of Continuous Modules

Instead of endowing modules with various topologies to suit the purpose of the moment, we find it convenient to work in the category $\operatorname{Cont} R$ of "continuous" $R$-modules. The objects of this category are topological abelian groups on which the ring $R$ acts continuously. Thus, a *continuous* $R$-module is an $R$-module $A$ that is also a topological group such that, for each $r \in R$, the mapping $a \mapsto ar$ from $A$ to $A$ is continuous. The morphisms of $\operatorname{Cont} R$ are continuous $R$-homomorphisms. We write $f \in \operatorname{Cont}_R(A, B)$ if $f \in \operatorname{Hom}_R(A, B)$ and $f$ is continuous. There are, of course, the forgetful functor $\operatorname{Cont} R \to \operatorname{Mod} R$ and its left adjoint, which endows each ordinary module with the discrete topology.

A subobject (monomorphism) $m: A \to B$ is called *regular* in a category $\mathcal{A}$ if it is the equalizer of a pair of maps $u, v: B \rightrightarrows C$ in $\mathcal{A}$.

Lemma 5.1. *In the category* $\operatorname{Cont} R$, *every regular subobject has the induced topology. Conversely, any submodule with the induced topology is a regular subobject in* $\operatorname{Cont} R$.

*Proof.* Given two continuous $R$-homomorphisms $u, v: B \rightrightarrows C$, put $A = \{b \in B \mid u(b) = v(b)\}$ with the induced topology and let $m: A \to B$ be the inclusion. We claim that $m: A \to B$ is the equalizer of $(u, v)$ in $\operatorname{Cont} R$.

Indeed, suppose $g: D \to B$ equalizes $(u, v)$. Then, for each $d \in D$, $g(d) \in A$, hence, $g$ factors through $A$ in $\operatorname{Mod} R$, that is, there exists an $R$-homomorphism $h: D \to A$ such that $g = mh$. To see that $h$ is continuous, take any open set in $A$, it must be of the form $A \cap V = m^{-1}V$, where $V$ is open in $B$. Then, $h^{-1}(A \cap V) = (mh)^{-1} V = g^{-1}V$ is open in $D$, since $g$ is continuous, hence, $h$ is continuous.

Since any equalizer of $(u, v)$ is isomorphic to the above $m: A \to B$, every regular subobject has the induced topology.

Conversely, suppose $m: A \to B$ is mono in Mod $R$ and $A$ has the induced topology. We claim that $A$ is a regular subobject, in fact, that $m: A \to B$ is the equalizer of the two canonical homomorphisms $u, v: B \rightrightarrows C$, where $C = (B \times B)/K$ and $K = \{(m(a), -m(a)) \mid a \in A\}$. We may endow $C$ with any topology that renders $u$ and $v$ continuous, for instance, the indiscrete topology.

Indeed, $u(b)$ is the equivalence class of $(b, 0)$ modulo $K$ and $v(b)$ if the equivalence class of $(0, b)$ modulo $K$. Hence, $u(b) = v(b)$ if and only if $(b, 0) \equiv (0, b)$ modulo $K$, that is, $(b, -b) \in K$. But this means that $b = m(a)$ for some $a \in A$. Take $m$ to be the inclusion. We see that $A = \{b \in B \mid u(b) = v(b)\}$. Then, it follows from the assertion at the beginning of this proof that $m: A \to B$ is the equalizer of $(u, v)$, as was to be shown.

PROPOSITION 5.2. *Let $I$ be a quasi-injective right $R$-module endowed with the discrete topology. Then, for any set $X$ and any regular subobject $B$ of $I^X$, every morphism $g: B \to I$ of* Cont $R$ *can be extended to $I^X \to I$.*

*Proof.* By Lemma 5.1, $B$ has the induced topology of a subspace of $I^X$, which is a product of copies of the discrete module $I$. A fundamental open neighborhood of zero in $I^X$ has the form Ker $x^*$, where $x^*: I^X \to I^n$ is associated with $x \in X^n$ by the formula $x^*(t) = ((x_1)\, t, \dots, (x_n)\, t)$, for each $t \in I^X$.

Since $g$ is continuous and $I$ is discrete, $g^{-1}(0)$ contains a fundamental neighborhood of zero in $B$, that is,

$$\operatorname{Ker} g \supseteq \operatorname{Ker} x^* \cap B,$$

for some element $x \in X^n$. Thus, for each $b \in B$, $x^*(b) = 0$ implies $g(b) = 0$, hence, there exists an $R$-homomorphism $h: x^*(B) \to I$ such that $hx^*(b) = g(b)$. By Harada's lemma, $h$ may be extended to $h': I^n \to I$ and $h'$ is continuous, since $I^n$ is discrete.

Moreover, $x^*$ is continuous, by definition of the product topology, hence, so is $h'x^*: I^X \to I$. Since $h'x^*(b) = hx^*(b) = g(b)$, $h'x^*$ extends $g$, as required.

Let $I$ be a given object of Cont $R$. As in other categories, we write

$$T(A) = T_I(A) = I^{\mathrm{Cont}_I(A,I)}$$

and let $\kappa(A): Q(A) \to T(A)$ be the intersection of the equalizers of all pairs $\varphi, \psi \in \mathrm{Cont}_R(T(A), I)$ such that $\varphi\eta(A) = \psi\eta(A)$, where $\eta(A): A \to T(A)$ is the canonical morphism. We recall that $I$ is $\kappa$-injective in Cont $R$ if every continuous $R$-homomorphism $Q(A) \to I$ can be extended to a continuous $R$-homomorphism $T(A) \to I$. We also write

$$S(A) = S_I(A) = \mathrm{Hom}_E(\mathrm{Cont}_R(A, I), I),$$

where $E = \mathrm{Cont}_R(I, I)$. When $A$ and $I$ are discrete, this agrees with our earlier definition for Mod $R$.

PROPOSITION 5.3. *Let I be a quasi-injective right R-module endowed with the discrete topology. Then, I is $\kappa$-injective in* Cont $R$ *and* $S(A) = Q(A)$ *for any object* $A$ *of* Cont $R$.

*Proof.* That $I$ is $\kappa$-injective follows immediately from Proposition 5.2. The proof that $Q(A) \subseteq S(A)$ is the same as that for Mod $R$ in Lemma 3.6. To show the converse inclusion, we let $s \in S(A)$ and consider any $\varphi, \psi \in \mathrm{Cont}_R(T(A), I)$ such that $\varphi\eta(A) = \psi\eta(A)$. To assure that $s \in Q(A)$, we want to check that $\varphi(S) = \psi(s)$. Since Cont $R$ is an additive category, we may take $\psi = 0$.

Since $\varphi$ is continuous, there exists $f: A \to I^n$, such that $\operatorname{Ker} \varphi \supseteq \operatorname{Ker} f^*$, where $f^*: T(A) \to I^n$ is defined as in the proof of Proposition 5.2 or in Lemma 3.3 by

$$f^*(t) = ((f_1)\, t, \ldots, (f_n)\, t),$$

for all $t \in T(A)$. Thus, $f^*(t) = 0$ implies $\varphi(t) = 0$, hence, there exists an $R$-homomorphism $g: f^*(T(A)) \to I$ such that $gf^*(t) = \varphi(t)$. By Harada's lemma, one may extend $g$ to $g': I^n \to I$, hence,

$$g'f = g'f^*\eta(A) = gf^*\eta(A) = \varphi\eta(A) = 0.$$

Therefore, for any $s \in S(A)$,

$$\varphi(s) = gf^*(s) = g'f^*(s) = 0,$$

by Lemma 3.3.

Upon examining the above argument more closely, we find we can prove a slightly stronger result.

THEOREM 5.4. *Let $R \to R'$ be an epimorphism of rings, I a quasi-injective $R'$-module equipped with the discrete topology. Then, I is $\kappa$-injective in* Cont $R$ *and* $S(A) = Q(A)$ *for any* $A \in$ Cont $R$.

*Proof.* Any continuous $R$-homomorphism $Q(A) \to I$ is a continuous

$R'$-homomorphism, as Mod $R'$ is a full subcategory of Mod $R$ (see [24], hence, it may be extended to $S(A) \to I$, by Proposition 5.2. Thus, $I$ is $\kappa$-injective.

The proof that $Q(A) \subseteq S(A)$, as in Lemma 3.6 above, works for any $R$-module $I$, hence, for an $R'$-module. In the proof that $S(A) \subseteq Q(A)$, we observe that $f^*: T(A) \to I^n$ is an $R'$-homomorphism and so, Harada's lemma may still be applied in Mod $R'$.

It may well happen that $S(A)$ has the discrete topology, as we shall now see in a result adapted from [18].

PROPOSITION 5.5. *Let $I$ be a quasi-injective right $R$-module endowed with the discrete topology. Then, the following conditions are equivalent for any* $A \in \operatorname{Cont} R$.

(1) *There exists* $f \in \operatorname{Cont}_R(A, I^n)$, *such that* $\operatorname{Ker} \eta(A) = \operatorname{Ker} f$.

(2) $\operatorname{Cont}_R(A, I)$ *is a finitely generated left $E$-module.*

(3) $S(A)$ *has the discrete topology.*

*Proof.* We shall show that (1) ⇒ (2) ⇒ (3) ⇒ (1).

Suppose (1) and let $g \in \operatorname{Cont}_R(A, I)$. Then $\operatorname{Ker} g \supseteq \operatorname{Ker} f$, hence, there exists an $R$-homomorphism $h: f(a) \mapsto g(a)$ and this may be extended to $h': I^n \to I$ by Harada's lemma. Thus,

$$g(a) = h'f(a) = \sum_{i=1}^{n} h'\kappa_i f_i(a),$$

where $\kappa_i: I \to I^n$ canonically. Since $h'\kappa_i \in E$, we see that $g$ is generated by the $f_i$, $i = 1,..., n$.

Suppose (2) and let $f_1, ..., f_n$ be a basis of $\operatorname{Cont}_R(A, I)$. Put $f = \langle f_1, ..., f_n \rangle: A \to I^n$ and consider an element $s \in \operatorname{Ker} f^* \cap S(A)$. Then, $(f_1) s = 0, ..., (f_n) s = 0$, hence, $(\sum_{i=1}^{n} e_i f_i) s = 0$ for all $e_1, ..., e_n \in E$. It follows that the fundamental neighbourhood of zero $\operatorname{Ker} f^* \cap S(A) = 0$, hence, $S(A)$ has the discrete topology.

Suppose (3), that is, $\operatorname{Ker} f^* \cap S(A) = 0$, for some $f \in \operatorname{Cont}_R(A, I^n)$. Let $a \in \operatorname{Ker} f$, then, $\eta(A)(a) \in \operatorname{Ker} f^* \cap S(A) = 0$, hence, $a \in \operatorname{Ker} \eta(A)$. It follows that $\operatorname{Ker} f \subseteq \operatorname{Ker} \eta(A)$ and the converse inclusion holds in any case.

*Remark* 5.6. Condition (1) is easily verified when the underlying $R$-module of $A$ is Artinian, or even when the descending chain condition holds for closed submodules of $A$. On the other hand, it is easy to check condition (2) if $A$ is discrete and $I = E(A)$ is the injective envelope of $A$. Indeed, the inclusion $f_0: A \to I$ then generates $\operatorname{Hom}_R(A, I)$ as a left $E$-module, since each $f: A \to I$ can be extended to some $e: I \to I$, hence, $f = ef_0$.

PROPOSITION 5.7. *Let $I$ be a quasi-injective right $R$-module endowed with the discrete topology. Then, for any $A \in \operatorname{Cont} R$, $\operatorname{Cont}_R(A, I)$ is principal as a left $E$-module if and only if* $\operatorname{Ker} \eta(A) \supseteq \operatorname{Ker} g$ *for some* $g \in \operatorname{Cont}_R(A, I)$.

*Proof.* Assume $\operatorname{Cont}_R(A, I) = Ef$ and let $a \in \operatorname{Ker} f$, $g \in \operatorname{Cont}_R(A, I)$, then, $g = ef$ for some $e \in E$, hence, $g(a) = ef(a) = 0$, thus, $\operatorname{Ker} g \subseteq \operatorname{Ker} \eta(A)$.

Conversely, assume $\operatorname{Ker} g \subseteq \operatorname{Ker} \eta(A)$ and let $f \in \operatorname{Cont}_R(A, I)$. The assignment $g(a) \mapsto f(a)$ is an $R$-homomorphism $g(A) \to I$, hence, it may be extended to $e \in E$. It follows that $f = eg$.

## 6. APPLICATIONS TO MOD $R$

It may be of interest that sometimes, results about ordinary modules may be deduced from results about continuous modules. The following proposition is known when $I$ is injective [16, 21]. It makes sense even when $I$ is only quasi-injective, although $\operatorname{Fix}(Q, \lambda)$ need not then be a reflective subcategory.

PROPOSITION 6.1. *Suppose $I$ is a quasi-injective right $R$-module, $A \in \operatorname{Mod} R$ and $\operatorname{Hom}_R(A, I)$ is a finitely generated left $E$-module, where $E = \operatorname{End}_R(I)$. Then, $S(A) = Q(A)$, already in* $\operatorname{Mod} R$.

*Proof.* If $A$ and $I$ are endowed with the discrete topology, we need not distinguish between $T(A)$ in $\operatorname{Mod} R$ and $T(A)$ in $\operatorname{Cont} R$ and similarly, for $S(A)$. For the moment, let us write $Q(A)$ for the equalizer of $T(A) \rightrightarrows T^2(A)$ in $\operatorname{Mod} R$, whereas, the same equalizer in $\operatorname{Cont} R$ is $S(A)$, by Proposition 5.3. In view of Lemma 3.6, we need only show $S(A) \subseteq Q(A)$.

Let $s \in S(A)$ and suppose $\varphi, \psi\colon T(A) \rightrightarrows I$ are such that $\varphi\eta(A) = \psi\eta(A)$. The homomorphisms $\varphi$ and $\psi$ need not be continuous, but their restrictions to $S(A)$ are, since $S(A)$ has the discrete topology, by Proposition 5.5. Evidently, $S(A)$ is a regular subobject of $T(A)$ in $\operatorname{Cont} R$, as it is the intersection of equalizers of pairs of mappings $t \mapsto e(f)\, t + e'(f')\, t$ and $t \mapsto (ef + e'f')\, t$. Therefore, by Proposition 5.2, $\varphi \mid S(A)$ and $\psi \mid S(A)$ may be extended to continuous $R$-homomorphisms $\varphi', \psi'\colon T(A) \rightrightarrows I$. Since the image of $\eta(A)$ is contained in $S(A)$, one has $\varphi'\eta(A) = \varphi\eta(A) = \psi\eta(A) = \psi'\eta(A)$. Therefore, by Proposition 5.3, $\varphi' \mid S(A) = \psi' \mid S(A)$, that is, $\varphi \mid S(A) = \psi \mid S(A)$. It follows that $S(A) \subseteq Q(A)$.

However, it is only fair to point out that one just as easily can give a direct proof of the inclusion $S(A) \subseteq Q(A)$, as was done in [18]. If $f_1, \ldots, f_n$ are the generators of $\operatorname{Hom}_R(A, I)$, one puts $f = \langle f_1, \ldots, f_n \rangle$ and shows that $f^* \mid S(A)$ is one-to-one, then uses Lemma 3.3.

EXAMPLE 6.2. If $I = E(A)$ is the injective hull of $A$, one calls $Q(A)$ the *rational completion* of $A$ [8]. Now, it follows from Remark 5.6 that the rational completion of $A$ is given by $S(A)$.

In the considerations that follow, we write $S_I(A)$ for $S(A)$ to indicate its dependence on $I$. Even when $Q_I(A) \neq S_I(A)$, one may ask if $Q_I(A) = S_J(A)$ for some $J \neq I$. The answer is "yes" when $I$ is injective. In the special case of rings, this is known [16, Proposition 2.8]. Here, we shall give a proof in the general case of modules, which is a bit simpler.

First, let us recall some facts from [16]. Let $I$ be a given injective. One calls $A$ an *I-torsion* module if $\mathrm{Hom}_R(A, I) = 0$ and one says that $A$ is *I-torsionfree* if $\eta_I(A)$ is a monomorphism, that is, if there is a monomorphism from $A$ to a power of $I$. Necessary and sufficient conditions for $h: A \to B$ to be isomorphic to $\lambda_I(A): A \to Q_I(A)$, the module of quotients of $A$ with respect to $I$, are the following:

(a) Ker $h$ and Cok $h$ are $I$-torsion modules,

(b) $B$ and $E(B)/B$ are $I$-torsionfree.

PROPOSITION 6.3. *Suppose $I$ is an injective right $R$-module. Then, for any $A \in \mathrm{Mod}\, R$, $Q_I(A) \simeq S_J(A)$, where*

$$J = E(Q_I(A)) \oplus E(E(Q_I(A))/Q_I(A)).$$

*Proof.* We want to show that $h = \lambda_I(A): A \to Q_I(A) = B$ is isomorphic to $\eta_J(A): A \to S_J(A)$, where $J = E(B) \oplus E(E(B)/B)$.

Clearly, $B$ and $E(B)/B$ are $J$-torsionfree. Moreover, $J$ is $I$-torsionfree, hence, the $I$-torsion modules Ker $h$ and Cok $h$ are also $J$-torsion modules. Therefore, $h: A \to B$ is isomorphic to $\lambda_J(A): A \to Q_J(A)$.

Since Ker $h$ is a $J$-torsion module and since $h: A \to B \subseteq J$, it follows from Proposition 5.7 that $\mathrm{Hom}_R(A, J)$ is principal as a left $\mathrm{End}_R(J)$-module. Therefore, by Proposition 6.1, $Q_J(A) = S_J(A)$ and this entails $\lambda_J(A) = \eta_J(A)$.

For completeness, we shall also include the following known results [9, 10, 6].

PROPOSITION 6.4. *Let $I$ be a quasi-injective right $R$-module, $E$ its ring of endomorphisms, $N = \{r \in R \mid Ir = 0\} = \mathrm{Ker}\, \eta(R)$. Then, $I$ is injective as a right $R/N$-module if any one of the following conditions is satisfied:*

(1) *(Fuchs) $I$ is principal as a left $E$-module.*

(2) *(Fuller) $I^X$ is a quasi-injective right $R$-module for some set $X$ of generators of $I$ as a left $E$-module.*

(3) *(Faith) $I$ is finitely generated as a left $E$-module.*

*Proof.* (1) Suppose $I = Ei$, for some $i \in I$. For any right ideal $D$ of $R$ containing $N$, consider $f \in \mathrm{Hom}_{R/N}(D/N, I)$. Put $g(id) = f([d])$, where $[d]$

is the equivalence class of $d \in D$ modulo $N$, and extend $g \in \mathrm{Hom}_R(iD, I)$ to $e \in E$, then $f([d]) = e(id) = (ei)\,[d]$. Thus, $I_{R/N}$ is injective, by Baer's criterion.

(2) Let $g \in I^X$ correspond to the inclusion $g: X \to I$. Then, $gr = 0$ implies $Xr = 0$, hence, $Ir = 0$, that is, $r \in N$. Therefore, $\mathrm{Ker}\, g \in N$; but $N$ is the kernel of $\eta_{I^X}(R): R \to T_{I^X}(R)$, hence, we may apply Proposition 5.7 with $A = R$ and deduce that $I^X$ is principal over its ring of endomorphisms. In view of the result of Fuchs, already proved, $I^X$ is injective as an $R/N$-module, hence, so is $I$.

(3) Suppose $I$ is generated by a finite set $X$ of candinality $n$. Then, $I^X = I^n$ is quasi-injective by Harada's lemma, hence, we may invoke the result of Fuller already proved.

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