Journal of Algebra 321 (2009) 1546–1554

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Gorenstein injective complexes of modules over Noetherian rings ☆

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ARTICLE INFO

Article history:
Received 17 July 2008
Available online 30 December 2008
Communicated by Luchezar L. Avramov

Keywords:
Gorenstein injective complex
Gorenstein injective module
Cover
Gorenstein injective dimension

ABSTRACT

A complex C is called Gorenstein injective if there exists an exact sequence of complexes $\cdots \to I_{-1} \to I_0 \to I_1 \to \cdots$ such that each I_i is injective, $C = \mathrm{Ker}(I_0 \to I_1)$ and the sequence remains exact when $\mathrm{Hom}(E, -)$ is applied to it for any injective complex E. We show that over a left Noetherian ring R, a complex C of left R-modules is Gorenstein injective if and only if C^m is Gorenstein injective in R-*Mod* for all $m \in \mathbb{Z}$. Also Gorenstein injective dimensions of complexes are considered.

1. Introduction and preliminaries

Throughout this paper, R denotes a ring with unity. A complex

$$\cdots \to C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \cdots$$

of left R-modules will be denoted (C, δ) or C.

It is an important question to establish relationships between a complex C and the modules C^m, $m \in \mathbb{Z}$. It is well known that a complex (C, δ) is injective (respectively projective) if and only if each left R-module $\mathrm{Ker}(\delta^m)$ is injective (respectively projective) in R-*Mod* and C is exact; and C is finitely generated if and only if C is bounded and C^m is finitely generated in R-*Mod* for all $m \in \mathbb{Z}$ [6, Lemma 2.2]. It is natural to consider the relationships of Gorenstein injectivity of a complex C and Gorenstein injectivity of all R-modules C^m, $m \in \mathbb{Z}$. If R is an n-Gorenstein ring (that is, R is left and right Noetherian and the injective dimensions of ${}_R R$ and R_R are at most n), then E.E. Enochs

☆ Supported by National Natural Science Foundation of China, TRAPOYT and the Cultivation Fund of the Key Scientific and Technical Innovation Project, Ministry of Education of China.

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.11.031

and J.R. Garcia Rozas in [5] (also, see [12]) showed that a complex C is Gorenstein injective if and only if C^m is Gorenstein injective in R-*Mod* for all $m \in \mathbb{Z}$. In this paper we will show that the same result holds if R is a left Noetherian ring. We also consider the Gorenstein injective dimensions of complexes by showing that if R is a left Noetherian ring and C a complex of left R-modules, then $\mathrm{Gid}(C) = \sup\{\mathrm{Gid}(C^m) \mid m \in \mathbb{Z}\}$ where $\mathrm{Gid}(-)$ denotes Gorenstein injective dimension.

In the following $\mathcal{C}$ will be the abelian category of complexes of left R-modules. This category has enough projectives and injectives. For complexes C and D, $\mathrm{Hom}(C, D)$ is the abelian group of morphisms from C to D in the category of complexes and $\mathrm{Ext}^i(C, D)$ for $i \geqslant 1$ will denote the groups we get from the right derived functor of Hom.

Let $\mathcal{B}$ be a class of objects in an abelian category $\mathcal{D}$. Let X be an object of $\mathcal{D}$. We recall the definition introduced in [4]. A homomorphism $\alpha : B \to X$, where B is in $\mathcal{B}$, is called a $\mathcal{B}$-precover of X if the diagram

$$\begin{array}{ccc} & & B' \\ & \overset{\gamma}{\swarrow} & \downarrow \beta \\ B & \xrightarrow{\alpha} & X \end{array}$$

can be completed for each homomorphism $\beta : B' \to X$ with B' in $\mathcal{B}$. If furthermore, when $B' = B$ and $\beta = \alpha$ the only such γ are automorphisms of B, then $\alpha : B \to X$ is called a $\mathcal{B}$-cover of X. Dually we have the concepts $\mathcal{B}$-preenvelope and $\mathcal{B}$-envelope. If $\mathcal{B}$ is the class of all injective objects of $\mathcal{D}$, then $\mathcal{B}$-precover and $\mathcal{B}$-cover are called injective precover and injective cover, respectively. There are a lot of results concerning covers and envelopes (see, for example, [6–9,14,1]).

Given a left R-module M, we will denote by $\overline{M}$ the complex

$$\cdots \to 0 \to 0 \to M \xrightarrow{\mathrm{id}} M \to 0 \to 0 \to \cdots$$

with the M in the -1 and 0th position. Given a complex C and an integer m, $C[m]$ denotes the complex such that $C[m]^n = C^{m+n}$ and whose boundary operators are $(-1)^m \delta^{m+n}$.

Throughout the paper we use both the subscript notation for complexes and the superscript notation. When we use superscripts for a complex we will use subscripts to distinguish complexes: for example, if $(K_i)_{i \in I}$ is a family of complexes, then K_i^n denotes the degree-n term of the complex K_i.

General background material can be found in [10,12,13].

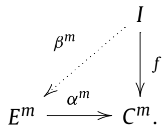
2. Gorenstein injective complexes

Lemma 1. *Let C be a complex and $\alpha : E \to C$ an injective precover of C. If C^m is Gorenstein injective for all $m \in \mathbb{Z}$, then α is surjective.*

Proof. Suppose that I is an injective left R-module and $f : I \to C^m$ an R-homomorphism. Consider the complex $\bar{I}[-m-1]$ and define a map of complexes $\gamma : \bar{I}[-m-1] \to C$ as following

$$\begin{array}{ccccccccc} \longrightarrow & 0 & \longrightarrow & I & \longrightarrow & I & \longrightarrow & 0 & \longrightarrow \\ & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \delta_C^m f} & & \downarrow{\scriptstyle 0} & \\ \longrightarrow & C^{m-1} & \longrightarrow & C^m & \longrightarrow & C^{m+1} & \longrightarrow & C^{m+2} & \longrightarrow . \end{array}$$

Since $\bar{I}[-m-1]$ is an injective complex and $\alpha : E \to C$ an injective precover of C, there exists a map of complexes $\beta : \bar{I}[-m-1] \to E$ such that $\alpha\beta = \gamma$. Thus we have a communicative diagram



This means that $\alpha^m : E^m \to C^m$ is an injective precover of C^m.

Since C^m is a Gorenstein injective left R-module, it is easy to see that there exists an epimorphism $I \to C^m$ with I injective. Hence α^m is an epimorphism. Therefore α is surjective. □

According to [12], a complex C is called Gorenstein injective if there exists an exact sequence of complexes

$$\cdots \to I_{-1} \to I_0 \to I_1 \to \cdots$$

such that

(1) each I_i is injective;
(2) $C = \mathrm{Ker}(I_0 \to I_1)$;
(3) the sequence remains exact when $\mathrm{Hom}(E, -)$ is applied to it for any injective complex E.

Lemma 2. *Let C be a complex. Then C is Gorenstein injective if and only if there exists an exact sequence of complexes*

$$\cdots \to I_{-1} \to I_0 \to I_1 \to \cdots$$

such that

(1) *each I_i is injective*;
(2) $C = \mathrm{Ker}(I_0 \to I_1)$;
(3) $\mathrm{Ext}^1(E, K_i) = 0$ *for all injective complexes E and all $K_i = \mathrm{Ker}(I_i \to I_{i+1})$, $i \in \mathbb{Z}$.*

Proof. It follows from the definition. □

Note that the similar result holds for left R-modules.

Lemma 3. *Let k be a positive integer. If a complex C satisfies $\mathrm{Ext}^k(E, C) = 0$ for all complexes E with finite injective dimension, then $\mathrm{Ext}^k(M, C^m) = 0$ for all $m \in \mathbb{Z}$ and all left R-modules M with finite injective dimension.*

Proof. Let M be a left R-module with finite injective dimension. Then there exists exact sequence

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_n \to 0$$

of left R-modules with each I_j injective. Thus we have the following exact sequence of complexes

$$0 \to \overline{M}[-m-1] \to \overline{I_0}[-m-1] \to \overline{I_1}[-m-1] \to \cdots \to \overline{I_n}[-m-1] \to 0.$$

Hence the injective dimension of $\overline{M}[-m-1]$ is finite since each $\overline{I_j}[-m-1]$ is injective. Consider exact sequence

$$0 \to Q \to P_{k-1} \to P_{k-2} \to \cdots \to P_0 \to M \to 0$$

with each P_j projective. Let $H = \operatorname{Im}(P_{k-1} \to P_{k-2})$ (if $k = 1$, then let $H = M$). Then $\operatorname{Ext}^k(M, C^m) \cong \operatorname{Ext}^1(H, C^m)$. Now from the exact sequence

$$0 \to Q \to P_{k-1} \to H \to 0$$

it follows that the sequence

$$0 \to \overline{Q}[-m-1] \to \overline{P_{k-1}}[-m-1] \to \overline{H}[-m-1] \to 0$$

is exact and $\overline{P_{k-1}}[-m-1]$ is projective. By the hypothesis, $\operatorname{Ext}^k(\overline{M}[-m-1], C) = 0$. Thus, from the exact sequence

$$0 \to \overline{H}[-m-1] \to \overline{P_{k-2}}[-m-1] \to \cdots \to \overline{P_0}[-m-1] \to \overline{M}[-m-1] \to 0$$

it follows that

$$\operatorname{Ext}^1\big(\overline{H}[-m-1], C\big) \cong \operatorname{Ext}^k\big(\overline{M}[-m-1], C\big) = 0$$

since each $\overline{P_j}[-m-1]$ is projective. Hence we have an exact sequence

$$\operatorname{Hom}\big(\overline{P_{k-1}}[-m-1], C\big) \to \operatorname{Hom}\big(\overline{Q}[-m-1], C\big) \to 0.$$

Let $f : Q \to C^m$ be an R-homomorphism. Define $\alpha^m = f$, $\alpha^{m+1} = \delta_C^m f$ and $\alpha^n = 0$ for any $n \neq m$, $m+1$. Then $\alpha : \overline{Q}[-m-1] \to C$ is a map of complexes. Thus there exists $\beta : \overline{P_{k-1}}[-m-1] \to C$ such that the diagram

$$\begin{array}{ccc} \overline{Q}[-m-1] & \longrightarrow & \overline{P_{k-1}}[-m-1] \\ \downarrow{\scriptstyle \alpha} & \swarrow{\scriptstyle \beta} & \\ C & & \end{array}$$

commutes. Hence, considering the degree-m term of the complexes yields that the sequence

$$\operatorname{Hom}\big(P_{k-1}, C^m\big) \to \operatorname{Hom}\big(Q, C^m\big) \to 0$$

is exact. On the other hand we have an exact sequence

$$\operatorname{Hom}\big(P_{k-1}, C^m\big) \to \operatorname{Hom}\big(Q, C^m\big) \to \operatorname{Ext}^1\big(H, C^m\big) \to 0.$$

Thus $\operatorname{Ext}^k(M, C^m) \cong \operatorname{Ext}^1(H, C^m) = 0$. □

Corollary 4. *If a complex C satisfies $\operatorname{Ext}^1(E, C) = 0$ for all injective complexes E, then $\operatorname{Ext}^1(I, C^n) = 0$ for all $n \in \mathbb{Z}$ and any injective left R-module I.*

Proof. It follows by analogy with the proof of Lemma 3. □

Corollary 5. *If C is a Gorenstein injective complex, then C^m is a Gorenstein injective left R-module for all $m \in \mathbb{Z}$.*

Proof. Suppose that C is a Gorenstein injective complex. We use the notation of Lemma 2. Then, for each $m \in \mathbb{Z}$, the following sequence

$$\cdots \to I^m_{-1} \to I^m_0 \to I^m_1 \to \cdots$$

is exact, each I^m_i is injective and $C^m = \mathrm{Ker}(I^m_0 \to I^m_1)$. Also by Lemma 2, $\mathrm{Ext}^1(E, K_i) = 0$ for all injective complexes E. Thus, by Corollary 4, $\mathrm{Ext}^1(I, K^m_i) = 0$ for any injective left R-module I and for all $m \in \mathbb{Z}$. Now, by the version for modules of Lemma 2, C^m is Gorenstein injective. □

A left R-module K is called an nth syzygy of a left R-module N, if there exists an exact sequence

$$0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to N \to 0$$

such that each P_j is projective.

Lemma 6. *Let M be a Gorenstein injective left R-module. Then for any syzygy K of an injective left R-module I, $\mathrm{Ext}^i(K, M) = 0$ for all $i \geqslant 1$.*

Proof. Let I be an injective left R-module and K an nth syzygy of I. Then There exists an exact sequence

$$0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to I \to 0$$

such that each P_j is projective. Thus for any $i \geqslant 1$,

$$\mathrm{Ext}^i(K, M) \cong \mathrm{Ext}^{i+n}(I, M) = 0$$

since M is Gorenstein injective and I is injective. □

Lemma 7 *(Dual version of [13, Corollary 2.11]). Let $0 \to M \to N \to L \to 0$ be a short exact sequence of left R-modules where N and L are Gorenstein injective. If $\mathrm{Ext}^1(I, M) = 0$ for all injective left R-modules I, then M is Gorenstein injective.*

Now we are in the position to give our main result. Note that the same result was shown in [5,12] if R is an n-Gorenstein ring.

Theorem 8. *Let R be a left Noetherian ring and G a complex of left R-modules. Then the following conditions are equivalent.*

(1) *G is a Gorenstein injective complex*;
(2) *G^m is a Gorenstein injective left R-module for all $m \in \mathbb{Z}$.*

Proof. (1) ⇒ (2). It follows from Corollary 5.

(2) ⇒ (1). Let I be an injective left R-module. Consider an exact sequence

$$0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to I \to 0$$

where each P_j is projective. Let $K_j = \operatorname{Ker}(P_{j-1} \to P_{j-2})$ for $j \geqslant 2$, $K_1 = \operatorname{Ker}(P_0 \to I)$ and $K_0 = I$. Then for any $m \in \mathbb{Z}$ the following sequence

$$0 \to \overline{K}[-m] \to \overline{P_{n-1}}[-m] \to \cdots \to \overline{P_1}[-m] \to \overline{P_0}[-m] \to \bar{I}[-m] \to 0$$

is exact and each $\overline{P_j}[-m]$ is a projective complex. Then

$$\operatorname{Ext}^i\big(\bar{I}[-m], G\big) \cong \operatorname{Ext}^1\big(\overline{K_{i-1}}[-m], G\big).$$

Let

$$0 \to G \to X \to \overline{K_{i-1}}[-m] \to 0$$

be an exact sequence. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^{m-2} & \longrightarrow & X^{m-2} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^{m-1} & \longrightarrow & X^{m-1} & \xrightarrow{\alpha^{m-1}} & K_{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^{m} & \longrightarrow & X^{m} & \xrightarrow{\alpha^{m}} & K_{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^{m+1} & \longrightarrow & X^{m+1} & \longrightarrow & 0 & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

Since G^{m-1} is Gorenstein injective and K_{i-1} is a syzygy of injective left R-module I, by Lemma 6, it follows that $\operatorname{Ext}^1(K_{i-1}, G^{m-1}) = 0$. Thus the exact sequence $0 \to G^{m-1} \to X^{m-1} \to K_{i-1} \to 0$ splits. Hence there exists $h : K_{i-1} \to X^{m-1}$ such that $\alpha^{m-1}h = 1$. Now define a map of complexes $\gamma : \overline{K_{i-1}}[-m] \to X$ via $\gamma^{m-1} = h$, $\gamma^m = \delta_X^{m-1}h$ and $\gamma^n = 0$ for $n \neq m-1, m$. Then $\alpha\gamma = 1$ and so the sequence $0 \to G \to X \to \overline{K_{i-1}}[-m] \to 0$ splits. Thus $\operatorname{Ext}^1(\overline{K_{i-1}}[-m], G) = 0$. Hence $\operatorname{Ext}^i(\bar{I}[-m], G) = 0$ for all $i \geqslant 1$, for all $m \in \mathbb{Z}$ and for all injective left R-modules I.

Now suppose that E is an injective complex. Then E is a direct sum of complexes in the form $\bar{I}[m]$ with I an injective left R-module. Thus $\operatorname{Ext}^i(E, G) = 0$ for all $i \geqslant 1$ and for all injective complexes E.

Since R is left Noetherian, by [11], there is a set $\mathcal{X}$ of injective left R-modules such that any injective left R-module is the direct sum of modules each isomorphic to an element of $\mathcal{X}$. Set

$$\mathcal{S} = \big\{\bar{I}[m] \bigm| I \in \mathcal{X},\ m \in \mathbb{Z}\big\}.$$

Then it is clearly that every injective complex is the direct sum of complexes each isomorphic to an element of $\mathcal{S}$. Thus, by [2, Theorem 3.2], any complex has an injective cover. (Or, by analogy with the proof of [4, Proposition 2.2], any complex has an injective precover. Now apply Zorn's Lemma for categories (see, [12]).) Suppose that $\alpha_{-1}: E_{-1} \to G$ is an injective cover of G. Then, by Lemma 1, we have an exact sequence

$$0 \to H_{-2} \to E_{-1} \to G \to 0$$

where $H_{-2} = \mathrm{Ker}(\alpha_{-1})$. A standard argument yields that $\mathrm{Ext}^i(E, H_{-2}) = 0$ for all $i \geqslant 1$ and for all injective complexes E.

Now consider exact sequence

$$0 \to H^m_{-2} \to E^m_{-1} \to G^m \to 0.$$

By the hypothesis, G^m is Gorenstein injective. Since E_{-1} is an injective complex, E^m_{-1} is injective and hence Gorenstein injective. By Corollary 4, $\mathrm{Ext}^1(I, H^m_{-2}) = 0$ for all injective left R-modules I since $\mathrm{Ext}^1(E, H_{-2}) = 0$ for all injective complexes E. Thus, from Lemma 7, it follows that H^m_{-2} is Gorenstein injective for all $m \in \mathbb{Z}$.

By analogy with above discussion, we have an exact sequence

$$0 \to H_{-3} \to E_{-2} \to H_{-2} \to 0$$

where $E_{-2} \to H_{-2}$ is an injective cover of H_{-2}, $H_{-3} = \mathrm{Ker}(E_{-2} \to H_{-2})$ and H^m_{-3} is Gorenstein injective for all $m \in \mathbb{Z}$.

Continuing this process yields the following exact sequence:

$$\cdots \to E_{-2} \to E_{-1} \to G \to 0$$

where E_{-1} is an injective cover of G and E_{-n} is an injective cover of $H_{-n} = \mathrm{Ker}(E_{-(n-1)} \to H_{-(n-1)})$ for all $n \geqslant 2$ (set $H_{-1} = G$). Note that this sequence remains exact when $\mathrm{Hom}(E, -)$ is applied to it for any injective complex E.

Taking injective envelopes yields the following exact sequence:

$$0 \to G \to E_0 \to E_1 \to \cdots.$$

Since $\mathrm{Ext}^i(E, G) = 0$ for all $i \geqslant 1$ and for all injective complexes E, it is easy to see that this sequence remains exact when $\mathrm{Hom}(E, -)$ is applied to it for any injective complex E.

Hence G is Gorenstein injective. □

3. Gorenstein injective dimensions

Let C be a complex of left R-modules. The Gorenstein injective dimension, $\mathrm{Gid}(C)$, of C is defined as $\mathrm{Gid}(C) = \inf\{n \mid$ there exists an exact sequence $0 \to C \to E_0 \to E_1 \to \cdots \to E_n \to 0$ with each E_i Gorenstein injective$\}$. If no such n exists, set $\mathrm{Gid}(C) = \infty$. Similarly, the Gorenstein injective dimension, $\mathrm{Gid}(M)$, of a left R-module M is defined. Details and results on Gorenstein injective dimension of modules appeared in [3,13].

Theorem 9. *Let R be a left Noetherian ring and G a complex of left R-modules. Then* $\mathrm{Gid}(G) = \sup\{\mathrm{Gid}(G^m) \mid m \in \mathbb{Z}\}$.

Proof. If $\sup\{\mathrm{Gid}(G^m) \mid m \in \mathbb{Z}\} = \infty$, then $\mathrm{Gid}(G) \leqslant \sup\{\mathrm{Gid}(G^m) \mid m \in \mathbb{Z}\}$. So naturally we may assume that $\sup\{\mathrm{Gid}(G^m) \mid m \in \mathbb{Z}\} = n$ is finite. Consider an injective resolution

$$0 \to G \to E_0 \to E_1 \to \cdots \to E_{n-1} \to K_n \to 0$$

of G, where each E_i is an injective complex. Then K_n^m is Gorenstein injective for all $m \in \mathbb{Z}$ by [13, Theorem 2.22]. Now, by Theorem 8, K_n is a Gorenstein injective complex. This shows that $\mathrm{Gid}(G) \leqslant n$ and so $\mathrm{Gid}(G) \leqslant \sup\{\mathrm{Gid}(G^m) \mid m \in \mathbb{Z}\}$.

Now it is enough to show that $\sup\{\mathrm{Gid}(G^m) \mid m \in \mathbb{Z}\} \leqslant \mathrm{Gid}(G)$. Naturally we may assume that $\mathrm{Gid}(G) = n$ is finite. Then there exists an exact sequence

$$0 \to G \to E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0$$

with each E_i Gorenstein injective. By Theorem 8, E_i^m is Gorenstein injective for all $m \in \mathbb{Z}$ and all $i = 0, 1, \cdots, n$. Thus $\mathrm{Gid}(G^m) \leqslant n$ and so $\sup\{\mathrm{Gid}(G^m) \mid m \in \mathbb{Z}\} \leqslant n = \mathrm{Gid}(G)$. □

For a general ring R, Propositions 10, 11 and Corollary 12 can be proved by applying the proofs of [13, Theorems 2.22, 2.15, 2.6 and (dual of) Corollary 2.11]. If R is left Noetherian, Propositions 10, 11 and Corollary 12 can be proved more easily by combining Theorem 9 with references to [13] given above.

Proposition 10. *Let G be a complex with finite Gorenstein injective dimension and n an integer. Then the following conditions are equivalent.*

(1) $\mathrm{Gid}(G) \leqslant n$;
(2) $\mathrm{Ext}^i(E, G) = 0$ *for all $i > n$ and all injective complexes E*;
(3) $\mathrm{Ext}^i(L, G) = 0$ *for all $i > n$ and all complexes L with finite injective dimension.*

Proposition 11. *Let C be a complex with finite Gorenstein injective dimension n. Then there exists an exact sequence $0 \to C \to K \to L \to 0$ with $C \to K$ a Gorenstein injective preenvelope and L a complex with finite injective dimension $n - 1$ (if C is Gorenstein injective, then this should be interpreted as $L = 0$).*

Corollary 12. *Let $0 \to L \to M \to N \to 0$ be an exact sequence of complexes.*

(1) *If L is Gorenstein injective, then M is Gorenstein injective if and only if N is Gorenstein injective.*
(2) *If M and N are Gorenstein injective, and if $\mathrm{Ext}^1(E, L) = 0$ for all injective complexes E, then L is Gorenstein injective.*

Acknowledgment

The authors wish to express their sincere thanks to the referee for his/her valuable suggestions which simplified the proof of the main results and led to the new version of Propositions 10, 11 and Corollary 12.

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