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Least Squares Finite-Element Solution of a Fractional Order Two-Point Boundary Value Problem

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Abstract—In this paper, a theoretical framework for the least squares finite-element approximation of a fractional order differential equation is presented. Mapping properties for fractional dimensional operators on suitable fractional dimensional spaces are established. Using these properties existence and uniqueness of the least squares approximation is proven. Optimal error estimates are proven for piecewise linear trial elements. Numerical results are included which confirm the theoretical results. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, we investigate the least squares approximation to the following two-point boundary value problem containing a fractional differential operator:

$$\begin{aligned} \frac{d^\alpha}{dx^\alpha}\left(k\frac{d\phi}{dx}\right) &= f, \qquad \text{in } [a,b], \\ \phi(a) &= \phi_0, \\ \phi(b) &= \phi_1, \end{aligned} \tag{1}$$

where $0 < \alpha \leq 1$, and $\frac{d^\alpha}{dx^\alpha}$ denotes a fractional order differential operator in the finite domain $[a, b]$. Physical models containing fractional differential operators have recently received renewed attention from scientists. The interest in such models is mainly due to the observation of power-law patterns in physical systems, in particular in contaminant transport of ground-water flow [1]. Using a probabilistic argument, one can show that a subdiffusive power law pattern leads to a partial differential equation containing fractional differential operators.

The diffusion equation may be derived by considering a continuous time random walk, governed by a jump probability density function whose first and second moments in space and time exist and are finite. However, the diffusion which occurs in complex systems may violate the assumption

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that the mean waiting time between two successive jump events is finite and/or the assumption that the variance of the jump length probability density is finite [2–5]. In this case, a fractional order diffusion equation results, i.e.,

$$\frac{\partial^{\gamma_1}\phi}{\partial t^{\gamma_1}} = K\frac{\partial^{\gamma_2}\phi}{\partial|x|^{\gamma_2}}, \tag{2}$$

where $0 < \gamma_1 \le 1$ and $1 < \gamma_2 \le 2$.

To date, all the analysis of fractional differential equations has been for the infinite domain, using the Fourier transform [1,6]. The inclusion of boundary conditions in fractional differential equations introduces additional difficulties. In this paper, we study the approximation of fractional differential equations on finite domains using variational methods. In [7,8], several definitions are given for a fractional differential operator on a finite domain. Herein, we consider the Riemann-Liouville fractional differential operator. Summarized in the Appendix are the definitions of the Riemann-Liouville operators, as well as several results used in this analysis.

The paper is organized as follows. First, we provide a derivation of the fractional advection-dispersion equation. Second, a fractional derivative space is introduced, and results are proven which aid in our analysis. Third, a least squares variational form for (1) is derived, and existence and uniqueness results are proven. Fourth, optimal error estimates are derived for the variational form for piecewise linear trial elements. Finally, numerical results are given which support the theoretical optimal error estimates.

2. PHYSICAL MODEL

Let $\phi(\mathbf{x}, t)$ represent the concentration of a solute at a point $\mathbf{x}$ at time t in an arbitrary bounded connected set $\Omega \subset \mathbb{R}^d$. The conservation of mass equation may be defined as

$$\frac{\partial\phi}{\partial t} = -\nabla\cdot\mathbf{F} + f, \qquad \text{in } \Omega, \tag{3}$$

where $\mathbf{F}$ is the mass flux and f denotes a source term. The mass flux term can be decomposed into

$$\mathbf{F} = \mathbf{F}_a + \mathbf{F}_d, \tag{4}$$

where $\mathbf{F}_a$ denotes the flux from advection, $\mathbf{F}_d$ from dispersion. An equation for advective flux is given by

$$\mathbf{F}_a = \phi\mathbf{v}, \tag{5}$$

where $\mathbf{v}$ is the (known) velocity of the fluid.

The advection-dispersion equation results when Fick's first law of diffusion is used to model the dispersive term, i.e.,

$$\mathbf{F}_d = -k\nabla\phi. \tag{6}$$

However, in the application of solute transport in highly heterogeneous porous media, dispersive flux does not appear to follow Fick's first law [1]. The argument for a fractional dispersive flux was given in [2] and interpreted in terms of a nonlocal superposition principle in [9], and in terms of a probabilistic argument in [10]. Similar to equation (20) in [2], the j^{th} component of the dispersive flux can be written as

$$(\mathbf{F}_d)_j = \left(p_{-\infty}D_{x_j}^{-\sigma} + q_{x_j}D_{\infty}^{-\sigma}\right)\left(-k\frac{\partial\phi}{\partial x_j}\right). \tag{7}$$

This equation may be interpreted as stating that the mass flux of a particle is related to the negative gradient via a combination of the left and right fractional integrals, where $p + q = 1$, and k is the diffusion coefficient. Equation (7) is physically interpreted as a Fick's law for concentrations of particles with a strong nonlocal interaction.

Combining (3) and (7), the d-dimensional fractional advection-dispersion equation (FADE) may be stated as

$$\frac{\partial \phi}{\partial t} = -\nabla \cdot (\mathbf{v}\phi) + \nabla^{\alpha} \cdot (k\nabla\phi) + f, \qquad \text{in } \Omega, \tag{8}$$

where the components of ∇^{α} are a linear combination of the left and right Riemann-Liouville fractional differential operators, and $0 < \alpha \leq 1$,

$$(\nabla^{\alpha})_j = p_{-\infty} D^{\alpha}_{x_j} - q_{x_j} D^{\alpha}_{\infty}. \tag{9}$$

In the case $\alpha = 1$, (8) reduces to the traditional advection dispersion equation. In a more general version of (8), k is replaced by a symmetric positive definite matrix.

The derivation of (8) in terms of a continuous time random walk results when representing the jump length probability density function as a product of d univariate Lévy densities. Another version of the FADE as presented in [11] relies on viewing the jump length p.d.f. as a multivariate Lévy density. The FADE has been studied in one dimension [1], and in three dimensions [6], over infinite domains using Fourier transform techniques.

3. FRACTIONAL DERIVATIVE SPACES

In order to perform a finite-element analysis for fractional differential equations, it is necessary to construct appropriate function spaces. In this section, we construct and analyze a set of spaces $J^s(\Omega)$, which depend on the square integrability of the Riemann-Liouville fractional derivative of a function. For the remainder of this paper, we let $\Omega = [0, b]$ and denote for α positive, negative, or zero,

$$\begin{aligned} \mathcal{D}^{\alpha} &:= {}_0D^{\alpha}_x, \\ \mathcal{D}^{\alpha *} &:= {}_xD^{\alpha}_b, \end{aligned} \tag{10}$$

where ${}_0D^{\alpha}_x$ and ${}_xD^{\alpha}_b$ represent the left and right fractional differential operators defined in (78) and (79), respectively.

DEFINITION 1. *For $s = m + \alpha$, where m is a nonnegative integer and $0 \leq \alpha < 1$, define*

$$J^s(\Omega) := \{u \in H^m(\Omega) \mid \mathcal{D}^{\alpha}u, \mathcal{D}^{\alpha *}u \in H^m(\Omega)\}. \tag{11}$$

We associate with $J^s(\Omega)$ the norm $\|\cdot\|_s$, and seminorm $|\cdot|_s$, given by

$$\|u\|_s := \left(\|u\|^2_m + \|\mathcal{D}^s u\|^2_0\right)^{1/2}, \qquad |u|_s := \|\mathcal{D}^s u\|_0 . \tag{12}$$

For s a nonnegative integer, the $J^s(\Omega)$ spaces coincide with the Sobolev spaces $H^s(\Omega)$ [12,13]. However, for s noninteger, the spaces $J^s(\Omega)$ and the fractional order Sobolev spaces $H^s(\Omega)$ are not equivalent. For example, for $\Omega = [0, b]$, constant functions are in $H^s(\Omega)$ for all s, yet constant functions are not in $J^s(\Omega)$ for $1/2 \leq s < 1$. We will prove equivalence of $J^s(\Omega)$ and $H^s(\Omega)$ for the case that $\Omega = \mathbb{R}$.

DEFINITION 2. *Define the space $J^s_0(\Omega)$ as the closure of $C^{\infty}_0(\Omega)$ under the J^s norm.*

LEMMA 1. *For $s \geq 0$, the spaces $H^s(\mathbb{R})$ and $J^s(\mathbb{R})$ are equivalent.*

PROOF. Let $u \in L^2(\mathbb{R})$, and $\hat{u}$ denote the Fourier transform of u. For $s = m+\alpha$ as in Definition 1, note that for $u \in H^s(\mathbb{R})$ then $u \in H^m(\mathbb{R})$. Similarly, for $u \in J^s(\mathbb{R})$, then $u \in H^m(\mathbb{R})$. From [13],

$$u \in H^s(\mathbb{R}) \Leftrightarrow \left(1 + |\omega|^2\right)^{s/2} \hat{u} \in L^2(\mathbb{R}).$$

Since $u \in L^2(\mathbb{R})$ we have

$$\left(1+|\omega|^2\right)^{s/2} \hat{u} \in L^2(\mathbb{R}) \Leftrightarrow (i\omega)^s \hat{u}, (-i\omega)^s \hat{u} \in L^2(\mathbb{R}) \Leftrightarrow {}_{-\infty}D_x^s u, {}_xD_\infty^s u \in L^2(\mathbb{R}),$$

using (88). From these observations, and the fact that $u \in H^m(\mathbb{R})$, we have the equivalence of $H^s(\mathbb{R})$ and $J^s(\mathbb{R})$.

Also, we have that the norms $\|\cdot\|_{H^s(\mathbb{R})}$ and $\|\cdot\|_{J^s(\mathbb{R})}$ are equivalent since

$$\|u\|_{H^s(\mathbb{R})} = \left\|\left(1+|\omega|^2\right)^{s/2} \hat{u}\right\|_{L^2(\mathbb{R})} \sim \|(1+|\omega|^s)\, \hat{u}\|_{L^2(\mathbb{R})} \sim \|u\|_{J^s(\mathbb{R})}.$$

We apply a variational formulation over the $J^s(\Omega)$ spaces, which are not in general equivalent to the $H^s(\Omega)$ spaces. Therefore, we develop properties of the $J^s(\Omega)$ spaces which will be useful in the later analysis in Sections 4 and 5.

LEMMA 2. *Let $s > 0$. Then $u \in L^p(\Omega)$ satisfies*

$$\left\|\mathcal{D}^{-s} u\right\|_{L^p(\Omega)} \leq \frac{b^s}{\Gamma(s+1)} \|u\|_{L^p(\Omega)}. \tag{13}$$

PROOF. Recall that for convolution integrals of functions in the L^p spaces [1], we have that

$$\|w * v\|_{L^p} \leq \|w\|_{L^1} \|v\|_{L^p}. \tag{14}$$

Noting that any fractional order integral operator can be written as

$$\mathcal{D}^{-s} u = \frac{x^{s-1}}{\Gamma(s)} * u(x), \tag{15}$$

substituting (15) into (14) yields (13). ∎

LEMMA 3. *Let $s > 0$. The following mapping properties hold:*

(i) *$\mathcal{D}^s : J^s(\Omega) \to L^2(\Omega)$ is a bounded linear operator;*

(ii) *$\mathcal{D}^{-s} : L^2(\Omega) \to J^s(\Omega)$ is a bounded linear operator.*

PROOF. Let $s = m + \alpha$ as in Definition 1. Property (i) follows directly from the definition of $J^s(\Omega)$, as

$$\|\mathcal{D}^s u\|_0 \leq \left(\|u\|_m^2 + \|\mathcal{D}^s u\|_0^2\right)^{1/2}.$$

(ii) From the definition of $J^s(\Omega)$,

$$\left\|\mathcal{D}^{-s} u\right\|_s = \left(\left\|\mathcal{D}^{-s} u\right\|_m^2 + \left\|\mathcal{D}^s \mathcal{D}^{-s} u\right\|_0^2\right)^{1/2}. \tag{16}$$

From (81) and the semigroup property (80), we have that, for $u \in L^2(\Omega)$, $\|\mathcal{D}^s \mathcal{D}^{-s} u\|_0^2 = \|u\|_0^2$, and

$$\begin{aligned} \left\|\mathcal{D}^{-s} u\right\|_m^2 &= \sum_{k=0}^m \left\|D^k \mathcal{D}^{-s} u\right\|_0^2 \\ &= \sum_{k=0}^m \left\|D^k \mathcal{D}^{-k} \mathcal{D}^{k-s} u\right\|_0^2 \\ &= \sum_{k=0}^m \left\|\mathcal{D}^{k-s} u\right\|_0^2. \end{aligned} \tag{17}$$

Using (13), there exist constants c_k, $k = 0, \dots, m$, such that

$$\left\|\mathcal{D}^{k-s} u\right\|_0^2 \leq c_k \|u\|_0^2. \tag{18}$$

Therefore, combining (18), (17), and (16) we obtain the bound

$$\left\|\mathcal{D}^{-s} u\right\|_s \leq \left(1 + \sum_{k=0}^m c_k\right)^{1/2} \|u\|_0.$$

∎

LEMMA 4. *For $u \in J_0^s(\Omega)$, we have*

$$\mathcal{D}^{-s}\mathcal{D}^s u = u. \tag{19}$$

PROOF. By definition of $J_0^s(\Omega)$, there exists a sequence $\{\phi_n\}_{n=1}^{\infty} \subset C_0^{\infty}(\Omega)$ such that

$$\lim_{n\to\infty} \|u - \phi_n\|_s = 0. \tag{20}$$

Applying the triangle inequality, we have

$$\left\|\mathcal{D}^{-s}\mathcal{D}^s u - u\right\|_s \le \left\|\mathcal{D}^{-s}\mathcal{D}^s(u-\phi_n)\right\|_s + \left\|\mathcal{D}^{-s}\mathcal{D}^s\phi_n - \phi_n\right\|_s + \|\phi_n - u\|_s. \tag{21}$$

Since $\phi_n \in C_0^{\infty}(\Omega)$, ϕ_n satisfies the hypotheses of A.7. Therefore, from (84),(82) we have $\|\mathcal{D}^{-s}\mathcal{D}^s\phi_n - \phi_n\|_0 = 0$. By the mapping properties in Lemma 3,

$$\left\|\mathcal{D}^{-s}\mathcal{D}^s(u-\phi_n)\right\|_s \le c_1\|u-\phi_n\|_s.$$

Thus,

$$\left\|\mathcal{D}^{-s}\mathcal{D}^s u - u\right\|_s \le c_2\|u-\phi_n\|_s.$$

Taking the limit as $n \to \infty$, (22) implies

$$\left\|\mathcal{D}^{-s}\mathcal{D}^s u - u\right\|_s = 0. \tag{22}$$ ∎

COROLLARY 1. *(Fractional Version of Poincaré-Friedrichs.) For all $u \in J_0^s(\Omega)$,*

$$\|u\|_{L^2(\Omega)} \le \frac{b^s}{\Gamma(s+1)}\|\mathcal{D}^s u\|_{L^2(\Omega)}. \tag{23}$$

PROOF. Combining Lemmas 2 and 4, we have

$$\|u\|_{L^2(\Omega)} = \left\|\mathcal{D}^{-s}\mathcal{D}^s u\right\|_{L^2(\Omega)} \le \frac{b^s}{\Gamma(s+1)}\|\mathcal{D}^s u\|_{L^2(\Omega)}.$$ ∎

We can now establish a semigroup property for fractional differential operators.

LEMMA 5. *For $s, t > 0$ and $u \in J_0^{s+t}(\Omega)$, we have*

$$\mathcal{D}^{s+t}u = \mathcal{D}^s\mathcal{D}^t u. \tag{24}$$

PROOF. Let $u \in J_0^{s+t}(\Omega)$. Then by Lemma 4, and the semigroup property for fractional integrals (80),

$$u = \mathcal{D}^{-s-t}\mathcal{D}^{s+t}u = \mathcal{D}^{-t}\mathcal{D}^{-s}\mathcal{D}^{s+t}u.$$

Applying $\mathcal{D}^s\mathcal{D}^t$ to both sides and using (81),

$$\mathcal{D}^s\mathcal{D}^t\mathcal{D}^{-t}\mathcal{D}^{-s}\mathcal{D}^{s+t}u = \mathcal{D}^s\mathcal{D}^t u,$$
$$\Rightarrow \mathcal{D}^{s+t}u = \mathcal{D}^s\mathcal{D}^t u.$$ ∎

COROLLARY 2. *For $u \in J_0^{s+t}(\Omega)$,*

$$\|\mathcal{D}^s u\|_{L^2(\Omega)} \le \frac{b^t}{\Gamma(t+1)}\left\|\mathcal{D}^{s+t}u\right\|_{L^2(\Omega)}. \tag{25}$$

PROOF. Replacing u in (23) with $\mathcal{D}^s u$ and using (24) and Lemma 4, the stated result follows. ∎

REMARK. All of the results in Lemmas 2–5, Corollaries 1 and 2 hold also for the right fractional differential operator $\mathcal{D}^{s*}$.

We will now provide a result which shows adjoint properties of the left and right Riemann-Liouville differential operators to be used later in the analysis. We begin by showing properties for the fractional order integral operators as defined in (76) and (77). First, we note the adjoint property of the left and right Riemann-Liouville fractional integral operator [8], which is

$$\left(\mathcal{D}^{-s}u, v\right)_{L^2(\Omega)} = \left(u, \mathcal{D}^{-s*}v\right)_{L^2(\Omega)}, \qquad \forall\, u, v \in L^2(\Omega). \tag{26}$$

An important property of the $J^\alpha(\Omega)$ spaces which generalizes from the fractional order Sobolev spaces $H^\alpha(\Omega)$ is that a function in J^α is continuous provided that $\alpha > 1/2$.

LEMMA 6. *Let $1/2 < \alpha < 1$, $u \in J^\alpha(\Omega)$. Then u is continuous on $\bar{\Omega}$.*

PROOF. From Theorem 3.6 in [8], we have that for all $f \in L^2(\Omega)$,

$$\left|\mathcal{D}^{-\alpha} f(x) - \mathcal{D}^{-\alpha} f(y)\right| \le c|x-y|^{\alpha-1/2}\|f\|_{L^2(\Omega)}. \tag{27}$$

As $\mathcal{D}^\alpha u \in L^2(\Omega)$, we apply (27) to obtain the continuity of the function $\mathcal{D}^{-\alpha}\mathcal{D}^\alpha u$ on $\bar{\Omega}$. Analogously, as $\mathcal{D}^{\alpha *} u \in L^2(\Omega)$, we obtain that the function $\mathcal{D}^{-\alpha *}\mathcal{D}^{\alpha *} u$ is also a continuous function on $\bar{\Omega}$.

We complete the argument using the following properties of the Riemann-Liouville fractional order operators [7]

$$\mathcal{D}^{-\alpha}\mathcal{D}^\alpha u(x) = u(x) - c_1 x^{\alpha-1}, \tag{28}$$

$$\mathcal{D}^{-\alpha *}\mathcal{D}^{\alpha *} u(x) = u(x) - c_2 (b-x)^{\alpha-1}. \tag{29}$$

Combining (28) and (29)

$$\mathcal{D}^{-\alpha}\mathcal{D}^\alpha u(x) - \mathcal{D}^{-\alpha *}\mathcal{D}^{\alpha *} u(x) = c_2(b-x)^{\alpha-1} - c_1 x^{\alpha-1}.$$

As $\mathcal{D}^{-\alpha}\mathcal{D}^\alpha u$ and $\mathcal{D}^{-\alpha *}\mathcal{D}^{\alpha *} u$ are both continuous on $\bar{\Omega}$, then $c_1 = c_2 = 0$. ∎

LEMMA 7. *Let $0 < \alpha < 1$, $u \in J^\alpha(\Omega)$. Then*

$$\mathcal{D}^{\alpha-1} u(x)\Big|_{x=0} = 0.$$

PROOF. We consider two cases for α.

CASE I. $0 < \alpha \le 1/2$.

Set $f(x) = \mathcal{D}^{\alpha-1}u(x)$. Note that $f \in H^1(\Omega)$, as $f \in L^2(\Omega)$ by Lemma 2 and $Df \in L^2(\Omega)$ by the assumption that $u \in J^\alpha(\Omega)$. Therefore, f is continuous on $\bar{\Omega}$.

As $u \in L^2(\Omega)$, there exists a sequence $\{\phi_n\}_{n=1}^\infty \subset C_0^\infty(\Omega)$ such that

$$\lim_{n\to\infty} \|u - \phi_n\|_{L^2(\Omega)} = 0. \tag{30}$$

Let $t > 0$. As $f(x)$ is continuous, applying the mean value theorem, we have that there exists η, $0 \le \eta \le t$ such that

$$f(\eta)t = \int_0^t f(x)\,dx.$$

From the definition of f and ϕ_n, we obtain

$$f(\eta)t = \int_0^t \mathcal{D}^{(\alpha-1)}\phi_n(x)\,dx + \int_0^t \mathcal{D}^{(\alpha-1)}(u(x) - \phi_n(x))\,dx. \tag{31}$$

Using the Cauchy-Schwarz inequality, we bound the third term in (31) as

$$\begin{aligned}\int_0^t \mathcal{D}^{(\alpha-1)}(u(x) - \phi_n(x))\,dx &\le \|1\|_{L^2[0,t]} \left\|\mathcal{D}^{(\alpha-1)}(u - \phi_n)\right\|_{L^2[0,t]} \\ &= t^{1/2}\left\|\mathcal{D}^{(\alpha-1)}(u - \phi_n)\right\|_{L^2[0,t]} \\ &\le c_1 t^{(3/2-\alpha)}\|u - \phi_n\|_{L^2(\Omega)}, \qquad \text{from (13).}\end{aligned} \tag{32}$$

To bound the first term in (31), we use the mean value theorem and the continuity of ϕ_n to obtain

$$\begin{aligned}
\int_0^t \mathcal{D}^{(\alpha-1)}\phi_n(x)\,dx &= \int_0^t \int_0^x \frac{(x-\xi)^{-\alpha}}{\Gamma(1-\alpha)}\phi_n(\xi)\,d\xi\,dx \\
&= \int_0^t \phi_n(\sigma(x)) \lim_{\delta\to 0}\int_0^{x-\delta} \frac{(x-\xi)^{-\alpha}}{\Gamma(1-\alpha)}\,d\xi\,dx, \qquad 0 \le \sigma(x) \le x, \\
&= \int_0^t \phi_n(\sigma(x)\frac{x^{(1-\alpha)}}{\Gamma(2-\alpha)}\,dx \\
&= \phi_n(\tau)\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}, \qquad 0 \le \tau \le t.
\end{aligned} \tag{33}$$

Combining (32) and (33), we obtain the bound

$$f(\eta)t \le \phi_n(\tau)\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + c_1 t^{(3/2-\alpha)}\|u-\phi_n\|_{L^2(\Omega)}. \tag{34}$$

Therefore, dividing through by t and taking the limit as $t \to 0$, we obtain that

$$|f(0)| \le c_1\|u-\phi_n\|_{L^2(\Omega)}. \tag{35}$$

In view of (30), it follows that $f(0) = \mathcal{D}^{\alpha-1}u(x)|_{x=0} = 0$.

CASE II. $1/2 < \alpha < 1$.

By Lemma 6, u is continuous on $\bar{\Omega}$. Therefore, we may apply a generalized mean value theorem to the integral $\mathcal{D}^{\alpha-1}u$ to obtain

$$\begin{aligned}
\mathcal{D}^{\alpha-1}u(x) &= u(x_0)\lim_{\delta\to 0}\int_0^{x-\delta}\frac{(x-\xi)^{-\alpha}}{\Gamma(1-\alpha)}\,d\xi, \qquad 0 \le x_0 \le x, \\
&= u(x_0)\lim_{\delta\to 0}\frac{x^{1-\alpha}-\delta^{1-\alpha}}{\Gamma(2-\alpha)} \\
&= u(x_0)\frac{x^{1-\alpha}}{\Gamma(2-\alpha)}.
\end{aligned}$$

Therefore, taking the limit as $x \to 0$, we obtain the stated result. ∎

LEMMA 8. *Let $0 \le \alpha \le 1$, $\phi \in H_0^1(\Omega)$, and $u \in J^\alpha(\Omega)$. Then*

$$(-D\phi, u)_{L^2(\Omega)} = \left(\mathcal{D}^{(1-\alpha)*}\phi, \mathcal{D}^\alpha u\right)_{L^2(\Omega)}. \tag{36}$$

PROOF. From the definition of the Riemann-Liouville fractional differential operators, (78),(79),

$$\left(\mathcal{D}^{(1-\alpha)*}\phi, \mathcal{D}^\alpha u\right) = \left(-D_x D_b^{-\alpha}\phi, D_0 D_x^{(1-\alpha)}u\right).$$

As $\phi \in H_0^1(\Omega)$, we can commute the differential operator and the fractional integral operator on the ϕ term. To see this, note that under the change of variable $\omega = \xi - x$, we obtain

$$\begin{aligned}
-D_x D_b^{-\alpha}\phi(x) &= -D\int_x^b \frac{(x-\xi)^{\alpha-1}}{\Gamma(\alpha)}\phi(\xi)\,d\xi \\
&= -D\int_0^{b-x}\frac{\omega^{\alpha-1}}{\Gamma(\alpha)}\phi(x+\omega)\,d\omega.
\end{aligned} \tag{37}$$

Using Leibniz's rule in (37), we obtain

$$-D_x D_b^{-\alpha}\phi(x) = \int_0^{b-x} \frac{\omega^{\alpha-1}}{\Gamma(\alpha)}(-D\phi(x+\omega))\,d\omega + \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)}\phi(b)$$
$$= {}_xD_b^{-\alpha}(-D)\phi(x).$$

Therefore, this result and (26) yields

$$\left(\mathcal{D}^{(1-\alpha)*}\phi, \mathcal{D}^{\alpha}u\right) = \left({}_xD_b^{-\alpha}(-D)\phi, D_0D_x^{(\alpha-1)}u\right)$$
$$= \left(-D\phi, {}_0D_x^{-\alpha}D_0D_x^{(\alpha-1)}u\right).$$

Similarly, commuting the operators ${}_0D_x^{-\alpha}$ and D on the u term, we obtain

$$\left(\mathcal{D}^{(1-\alpha)*}\phi, \mathcal{D}^{\alpha}u\right) = (-D\phi, D_0D_x^{-1}u) + \left(D\phi, \left[{}_0D_x^{\alpha-1}u(x)\right]_{x=0}\frac{x^{\alpha-1}}{\Gamma(\alpha)}\right).$$

Finally, using Lemma 7 and (81), we obtain (36). ∎

4. PROBLEM FORMULATION

We now apply the function spaces and operator properties developed in Section 3 to the solution of the following model problem.

PROBLEM 1. *For $0 < \alpha < 1$, $f \in L^2[0,1]$, find (ϕ, u) satisfying*

$$\begin{aligned} D\phi - u &= 0, &&\text{in } [0,1],\\ \mathcal{D}^{\alpha}u &= f, &&\text{in } [0,1],\\ \phi(0) &= \phi_0, &&\phi(1) = \phi_1. \end{aligned} \tag{38}$$

Note that this formulation corresponds to the steady-state FADE with no advection terms, and $p = 1$.

Separating out the nonhomogeneous boundary conditions, the solution (ϕ, u) satisfying Problem 1 can be decomposed into (ϕ_a, u_a) and (ϕ_b, u_b) satisfying Problems 1a and 1b, respectively.

PROBLEM 1A. *For $0 < \alpha < 1$, $f \in L^2[0,1]$, find (ϕ_a, u_a) satisfying*

$$\begin{aligned} D\phi_a - u_a &= 0, &&\text{in } [0,1],\\ \mathcal{D}^{\alpha}u_a &= f, &&\text{in } [0,1],\\ \phi_a(0) &= 0, &&\phi_a(1) = 0. \end{aligned} \tag{39}$$

PROBLEM 1B. *For $0 < \alpha < 1$, find (ϕ_b, u_b) satisfying*

$$\begin{aligned} D\phi_b - u_b &= 0, &&\text{in } [0,1],\\ \mathcal{D}^{\alpha}u_b &= 0, &&\text{in } [0,1],\\ \phi_b(0) &= \phi_0, &&\phi_b(1) = \phi_1. \end{aligned} \tag{40}$$

The solution to Problem 1 is given by $\phi = \phi_a + \phi_b$ where

$$\phi_b = \phi_0 + (\phi_1 - \phi_0)x^{\alpha}.$$

Therefore, we restrict our attention to the solution of Problem 1a. We apply a least-squares finite-element analysis similar to that in [14,15].

Let $\mathcal{L}$ and $\mathcal{V}$ denote

$$\mathcal{L} := \{\psi \mid \psi \in H_0^1(\Omega)\}, \tag{41}$$
$$\mathcal{V} := \{v \mid v \in J^\alpha(\Omega)\}. \tag{42}$$

Define the functional

$$L(\psi, v) := \frac{1}{2}\int_\Omega \left\{|D\psi - v|^2 + |\mathcal{D}^\alpha v - f|^2\right\} dx, \tag{43}$$

which is to be minimized over all $\psi \in \mathcal{L}$, $v \in \mathcal{V}$. Define the bilinear form $B([\cdot,\cdot],[\cdot,\cdot]) : [\mathcal{L}\times\mathcal{V}] \times [\mathcal{L}\times\mathcal{V}] \to \mathbb{R}$ as

$$B([\phi, u], [\psi, v]) := \int_\Omega \{(D\phi - u)(D\psi - v) + (\mathcal{D}^\alpha u \mathcal{D}^\alpha v)\}\, dx,$$

and the linear functional $F(\cdot,\cdot) : \mathcal{L}\times\mathcal{V} \to \mathbb{R}$ as

$$F(\psi, v) := \int_\Omega f\mathcal{D}^\alpha v\, dx.$$

Taking the first variation of $L(\psi, v)$, we have that the least squares solution (ϕ, u) of (43) satisfies the variational form

$$B([\phi, u], [\psi, v]) = F(\psi, v), \qquad \forall\, (\psi, v) \in \mathcal{L}\times\mathcal{V}. \tag{44}$$

Next we show that there exists a unique solution to (44).

THEOREM 1. *There exists a unique solution to (44) in the space $\mathcal{L}\times\mathcal{V}$.*

PROOF. We verify that the bilinear form $B([\cdot,\cdot],[\cdot,\cdot])$ is continuous and coercive on $(\mathcal{L}\times\mathcal{V}) \times (\mathcal{L}\times\mathcal{V})$ with respect to the norm

$$\|\phi, u\|_{\mathcal{L}\times\mathcal{V}} = \left(\|\phi\|_1^2 + \|u\|_\alpha^2\right)^{1/2}. \tag{45}$$

Continuity follows from an application of Cauchy-Schwarz and the triangle inequality.

$$\begin{aligned} B([\phi, u], [\psi, v]) &= (D\phi - u, D\psi - v) + (\mathcal{D}^\alpha u, \mathcal{D}^\alpha v) \\ &\le (\|D\phi\|_0 + \|u\|_0)(\|D\psi\|_0 + \|v\|_0) + \|\mathcal{D}^\alpha u\|_0 \|\mathcal{D}^\alpha v\|_0 \\ &\le 3\|\phi, u\|_{\mathcal{L}\times\mathcal{V}}\|\psi, v\|_{\mathcal{L}\times\mathcal{V}}. \end{aligned} \tag{46}$$

In order to prove coercivity, we first note that from the definition of the bilinear form B,

$$B([\phi, u], [\phi, u]) \ge \|D\phi - u\|_0^2, \tag{47}$$
$$B([\phi, u], [\phi, u]) \ge \|\mathcal{D}^\alpha u\|_0^2. \tag{48}$$

Using the definition of the adjoint (79) and Lemmas 3 and 5,

$$\begin{aligned} -D\phi &= \mathcal{D}^{\alpha *}\mathcal{D}^{(1-\alpha)*}\phi \\ \Rightarrow \|D\phi\|_0 &\ge c_1 \left\|\mathcal{D}^{(1-\alpha)*}\phi\right\|_0, \qquad \text{using (25)}. \end{aligned}$$

Thus,

$$\begin{aligned} \|D\phi - u\|_0^2 &= \|D\phi\|_0^2 + 2(-D\phi, u) + \|u\|_0^2 \\ &\ge c_1^2 \left\|\mathcal{D}^{(1-\alpha)*}\right\|_0^2 + 2(-D\phi, u) + \|u\|_0^2 \\ &= c_1^2 \left\|\mathcal{D}^{(1-\alpha)*}\phi\right\|_0^2 + 2\left(\mathcal{D}^{(1-\alpha)*}\phi, \mathcal{D}^\alpha u\right) + \|u\|_0^2, \end{aligned} \tag{49}$$

by Lemma 8. Combining (47)–(49), we obtain

$$\begin{aligned}\left(1+\frac{1}{c_1^2}\right) B([\phi,u],[\phi,u]) &\geq c_1^2 \left\|\mathcal{D}^{(1-\alpha)*}\phi\right\|_0^2 + 2\left(\mathcal{D}^{(1-\alpha)*}\phi, \mathcal{D}^\alpha u\right) + \frac{1}{c_1^2}\left\|\mathcal{D}^\alpha u\right\|_0^2 + \|u\|_0^2 \\ &= \left\|c_1 \mathcal{D}^{(1-\alpha)*}\phi + \frac{1}{c_1}\mathcal{D}^\alpha u\right\|_0^2 + \|u\|_0^2 \\ &\geq \|u\|_0^2. \\ \Leftrightarrow B([\phi,u],[\phi,u]) &\geq c_2^2 \|u\|_0^2.\end{aligned} \tag{50}$$

Then, from (47),(50), and the triangle inequality

$$\left(1+\frac{1}{c_2}\right)\sqrt{B([\phi,u],[\phi,u])} \geq \|D\phi - u\|_0 + \|u\|_0 \geq \|D\phi\|_0,$$

which, together with the Poincaré-Friedrichs lemma, implies

$$B([\phi,u],[\phi,u]) \geq c_3 \|\phi\|_1. \tag{51}$$

Combining (48), (50), (51) we obtain

$$B([\phi,u],[\phi,u]) \geq c_4 \|\phi,u\|_{\mathcal{L}\times\mathcal{V}}. \tag{52}$$

Finally, by (46) and (52), B satisfies the hypotheses for the Lax-Milgram theorem from which existence and uniqueness of a solution to (44) immediately follow. ∎

Note that $B([\cdot,\cdot],[\cdot,\cdot])$ defines an inner product on the space $\mathcal{L}\times\mathcal{V}$. Thus, we can define an energy norm with respect to this inner product as

$$|||\phi,u||| = (B([\phi,u],[\phi,u]))^{1/2}. \tag{53}$$

To compute an approximation (ϕ_h, u_h) to (ϕ, u) satisfying (44), we introduce finite dimensional subspaces $\mathcal{L}^h \subset \mathcal{L}$, $\mathcal{V}^h \subset \mathcal{V}$, and then seek $\phi_h \in \mathcal{L}^h$ and $u_h \in \mathcal{V}^h$ which satisfy the variational form

$$B([\phi_h,u_h],[\psi_h,v_h]) = F(\psi_h,v_h), \qquad \forall\,(\psi_h,v_h) \in \mathcal{L}_h \times \mathcal{V}_h. \tag{54}$$

COROLLARY 3. *There exists a unique solution $(\phi_h,u_h) \in \mathcal{L}^h \times \mathcal{V}^h$ satisfying (54).*

PROOF. As $\mathcal{L}^h \times \mathcal{V}^h$ are subspaces of $\mathcal{L}\times\mathcal{V}$ then the argument used to prove Theorem 1 can similarly be applied directly to $\mathcal{L}^h \times \mathcal{V}^h$. ∎

5. ERROR ANALYSIS

5.1. Approximation Properties

For this exposition, we consider piecewise polynomial approximations of ϕ and u. We shall first prove approximation properties for the spaces $J^s(\Omega)$ and $J_0^s(\Omega)$.

THEOREM 2. *Let $\{S_h\}$ denote a family of partitions of Ω, with grid parameter h. Associated with S_h we let $X_h(\Omega)$ represent the finite dimensional vector space of continuous piecewise linear polynomials. We denote by $\mathcal{I}^h\phi$ the continuous piecewise linear approximation to ϕ. Then the following approximation properties hold.*

(i) *For $\phi \in J^s(\Omega)$, there is a constant C_A such that for all $s-1 \leq t \leq s$,*

$$\left|\phi - \mathcal{I}^h\phi\right|_t \leq C_A h^{s-t}\left|\phi - \mathcal{I}^h\phi\right|_s. \tag{55}$$

(ii) *For $\phi \in J^k(\Omega)$, k integer, there is a constant C_A such that for all $k-1 \leq t \leq k$,*

$$\left|\phi - \mathcal{I}^h\phi\right|_t \leq C_A h^{k-t}|\phi|_k. \tag{56}$$

(iii) *For $\phi \in H^s(\Omega)$, there is a constant C_A such that for all $s-1 \leq t \leq s$,*

$$\left|\phi - \mathcal{I}^h\phi\right|_t \leq C_A h^{s-t}|\phi|_s. \tag{57}$$

PROOF. To prove Part (i), note that for each subinterval $I_j := [x_{j-1}, x_j]$ of S^h, $\phi - \mathcal{I}^h\phi \in H_0^s(I_j)$. (For functions $\phi \in J^s(\Omega)$ which are not continuous, we interpret $\mathcal{I}^h$ as the Clément interpolation operator [16].) Thus, from (23)

$$\begin{aligned}\left\|\mathcal{D}^t\left(\phi - \mathcal{I}^h\phi\right)\right\|_{L^2(I_j)} &\le c_1(x_j - x_{j-1})^{s-t}\left\|\mathcal{D}^s\left(\phi - \mathcal{I}^h\phi\right)\right\|_{L^2(I_j)}\\ &\le c_1 h^{s-t}\left\|\mathcal{D}^s\left(\phi - \mathcal{I}^h\phi\right)\right\|_{L^2(I_j)}.\end{aligned}$$

Summing over the intervals I_j yields the result.

To prove Part (ii), apply Part (i) together with the fact that for k integer [17],

$$\left\|\phi - \mathcal{I}^h\phi\right\|_k \le c_2 \inf_{v\in X_h} \|\phi - v\|_k.$$

The result follows when we substitute $v = 0$.

The proof of Part (iii) is found in [17]. ∎

5.2. Error Estimates in the Energy Norm

In this section we present an optimal error bound for the error in the energy norm $|||\cdot|||$. Denote

$$\epsilon = \phi - \phi_h, \qquad e = u - u_h, \tag{58}$$

where (ϕ, u) solves (44) and (ϕ_h, u_h) solves (54) with $\mathcal{L}^h = \mathcal{V}^h = X_h$.

LEMMA 9. *Let ϵ, e, ϕ, u be defined as in (58), then*

$$|||\epsilon, e||| \le 2|||\phi - \Phi, u - U|||, \qquad \forall\, \Phi \in X_h, \quad U \in X_h. \tag{59}$$

PROOF. Note that from (44) and (54), we have

$$B([\phi_h - \Phi, u_h - U], [\psi, v]) = B([\phi - \Phi, u - U], [\psi, v]), \qquad \forall\, [\psi, v], [\Phi, U] \in X_h \times X_h.$$

Substituting $\psi = \phi_h - \Phi$, $v = u_h - U$, we obtain

$$\begin{aligned}|||\phi_h - \Phi, u_h - U|||^2 &= B([\phi - \Phi, u - U], [\phi_h - \Phi, u_h - U])\\ &\le |||\phi - \Phi, u - U|||\,|||\phi_h - \Phi, u_h - U|||.\end{aligned} \tag{60}$$

Now, using (60),

$$\begin{aligned}|||\epsilon, e||| &= |||\phi - \phi_h, u - u_h|||\\ &\le |||\phi - \Phi, u - U||| + |||\phi_h - \Phi, u_h - U|||\\ &\le 2|||\phi - \Phi, u - U|||.\end{aligned} \tag{61}$$

∎

THEOREM 3. *Let ϵ, e, ϕ, u be defined as in (58). Then there exists a constant C_1 such that*

$$|||\epsilon, e||| \le C_1 h(|\phi|_2 + |u|_{1+\alpha}). \tag{62}$$

PROOF. From (53) and (46), we have

$$|||\phi - \Phi, u - U||| \le \sqrt{3}\left(|\phi - \Phi|_1 + |u - U|_\alpha\right), \qquad \forall\, \Phi \in X_h, \quad U \in X_h.$$

Since (Φ, U) is arbitrary, applying Lemma 9 yields

$$|||\epsilon, e||| \le 2\sqrt{3}\left(\inf_{\Phi\in X_h} |\phi - \Phi|_1 + \inf_{U\in X_h} |u - U|_\alpha\right). \tag{63}$$

From (56), we then obtain

$$|||\epsilon, e||| \le 2\sqrt{3}\left(C_A h|\phi|_2 + C_A h|u|_{1+\alpha}\right). \tag{64}$$

∎

5.3. Error Estimates for ϕ, u in L^2

Now that we have an error estimate in the energy norm, we apply a form of the Aubin-Nitsche lemma in order to pull the energy error estimate down to an estimate in the L^2 sense.

LEMMA 10. *Let ϵ, e be defined as in (58). Then there exists a constant C_2 such that*

$$\|\mathcal{D}^\alpha e\|_{-1} \le C_2 h |||\epsilon, e|||. \tag{65}$$

PROOF. We introduce the boundary value problem

$$\begin{aligned} Dw - z &= 0, && \text{in } \Omega, && (66)\\ \mathcal{D}^\alpha z &= \eta, && \text{in } \Omega, && (67)\\ w &= 0, && \text{on } \partial\Omega, &&\\ \eta &\in H_0^1(\Omega), && \text{with } \|\eta\|_1 \le 1. && \end{aligned}$$

Note that as η is continuous, equations (66) and (67) hold pointwise. First, we have the *a priori* bound

$$|w|_{2+\alpha} = |z|_{1+\alpha} \le C_E \left\|\mathcal{D}^{1+\alpha} z\right\|_0 = C_E \|\eta\|_1 \le C_E. \tag{68}$$

Multiplying (67) through by $\mathcal{D}^\alpha e$ and integrating over Ω yields

$$\begin{aligned} \int \eta \mathcal{D}^\alpha e\,dx &= \int \mathcal{D}^\alpha z \mathcal{D}^\alpha e\,dx \\ &= B([w,z],[\epsilon,e]) && \text{(using (66))}\\ &= B([w-w_h, z-z_h],[\epsilon,e]) && \text{(using Galerkin orthogonality)}\\ &\le |||w-w_h, z-z_h|||\,|||\epsilon,e||| \\ &\le \sqrt{3}(|w-w_h|_1 + |z-z_h|_\alpha)|||\epsilon,e||| \\ &\le \sqrt{3}\left(C_A h|w|_2 + C_A h|z|_{1+\alpha}\right)|||\epsilon,e||| \\ &\le \sqrt{3} C_A h\left(|w|_{2+\alpha} + |z|_{1+\alpha}\right)|||\epsilon,e||| \\ &\le 2\sqrt{3} C_A C_E h |||\epsilon,e|||. \end{aligned}$$

Finally, taking the supremum over all such η as defined in (67), we obtain (65). ∎

LEMMA 11. *Let ϵ, e be defined as in (58). Then there exist constants C_3, C_4 such that*

$$\|\epsilon\|_0 \le C_3 h |||\epsilon, e||| + C_4 \|\mathcal{D}^\alpha e\|_{-1}. \tag{69}$$

PROOF. Introduce the boundary value problem

$$\begin{aligned} -D^2 w &= \epsilon, && \text{in } \Omega,\\ w &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{70}$$

Again, as $\epsilon \in H_0^1(\Omega)$, (70) holds pointwise. First, note the *a priori* bound

$$\|w\|_2 \le C_E \|\epsilon\|_0. \tag{71}$$

Multiplying both sides of (70) by ϵ and integrating over Ω, we obtain

$$\begin{aligned} \|\epsilon\|_0^2 &= \int -D^2 w \epsilon\,dx \\ &= \int Dw D\epsilon\,dx. \end{aligned}$$

Now, adding and subtracting,

$$\begin{aligned}\|\epsilon\|_0^2 &= \int Dw(D\epsilon - e)\,dx + \int Dwe\,dx \\ &= B([w,0],[\epsilon,e]) - \int -Dwe\,dx \\ &= B([w-w_h,0],[\epsilon,e]) - \int \mathcal{D}^{(1-\alpha)*}w\mathcal{D}^{\alpha}e\,dx,\end{aligned}$$

where we have used Galerkin orthogonality in the first term and (36) in the second. Applying a form of the Cauchy-Schwarz inequality to each term, we have

$$\begin{aligned}\|\epsilon\|_0^2 &\le |||w-w_h,0|||\,|||\epsilon,e||| + \left\|\mathcal{D}^{(1-\alpha)*}w\right\|_1 \|\mathcal{D}^{\alpha}e\|_{-1} \\ &\le \sqrt{3}\,\|w-w_h\|_1|||\epsilon,e||| + \left(\left\|\mathcal{D}^{(1-\alpha)*}w\right\|_0^2 + \left\|\mathcal{D}^{(2-\alpha)*}w\right\|_0^2\right)^{1/2} \|\mathcal{D}^{\alpha}e\|_{-1} \\ &\le \sqrt{3}\,C_A h|w|_2|||\epsilon,e||| + \left(\|Dw\|_0^2 + \left\|D^2w\right\|_0^2\right)^{1/2} \|\mathcal{D}^{\alpha}e\|_{-1} \\ &\le \sqrt{3}\,C_A h\|w\|_2|||\epsilon,e||| + \|w\|_2\,\|\mathcal{D}^{\alpha}e\|_{-1} \\ &\le \sqrt{3}\,C_A C_E h\|\epsilon\|_0|||\epsilon,e||| + C_E\|\epsilon\|_0\,\|\mathcal{D}^{\alpha}e\|_{-1}.\end{aligned}$$

Finally, dividing through by $\|\epsilon\|_0$ we obtain (69). ∎

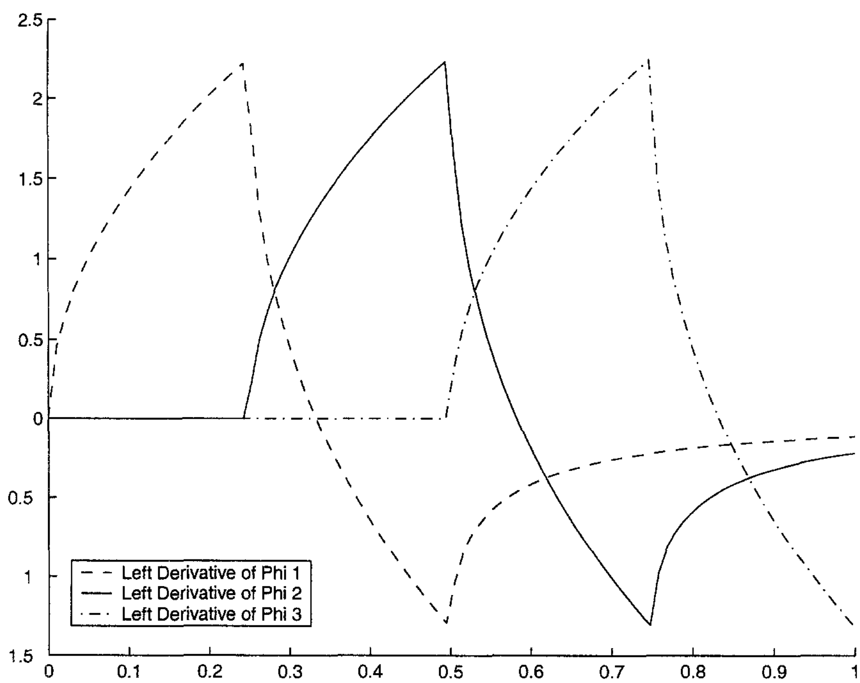
THEOREM 4. *Let ϵ, e, ϕ, u be defined as in (58). Then there exists a constant C_5 such that*

$$\|\epsilon\|_0 \le C_5 h^2\left(\|\phi\|_2 + \|u\|_{1+\alpha}\right), \qquad \textit{and} \tag{72}$$

$$\|e\|_0 \le C_1 h\left(\|\phi\|_2 + \|u\|_{1+\alpha}\right). \tag{73}$$

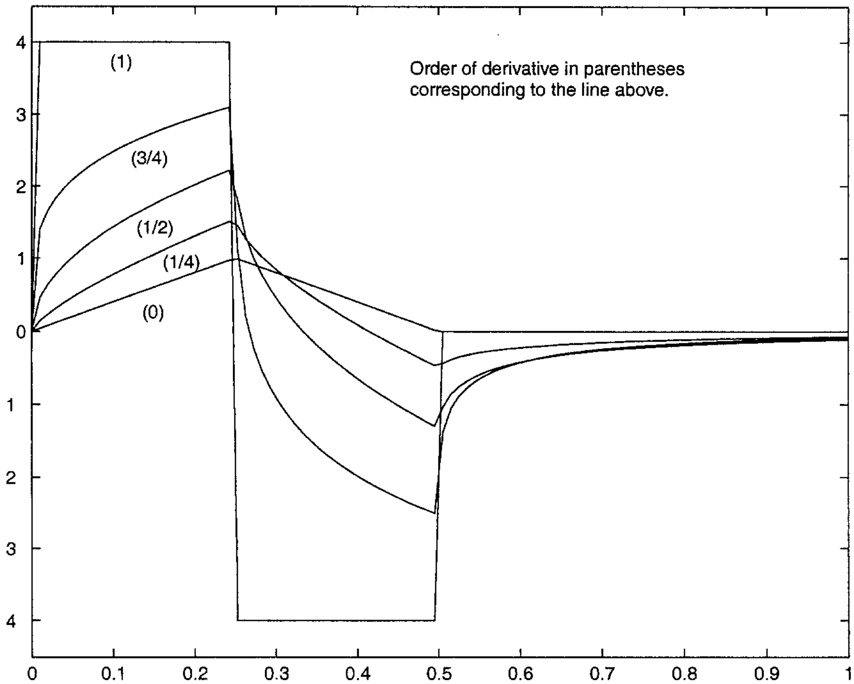
PROOF. The estimate in (72) follows from a combination of Lemmas 10, 11, and Theorem 3 with $C_5 = C_1(C_3 + C_2C_4)$. Estimate (73) follows from Theorem 3, as

$$\|e\|_0 \le |||\epsilon,e|||. \tag{74}$$ ∎



(a) Plots of $\mathcal{D}^{.5}\phi_i$, for $i = 1, 2, 3$ and $n = 1/h = 4$ on the interval $[0, 1]$.

Figure 1.



(b) Plots of $\mathcal{D}^\alpha \phi_1$ for $\alpha = 0, 0.25, 0.5, 0.75, 1$ on the interval $[0, 1]$.

Figure 1. (cont.)

Table 1. Experimental convergence results for Example 1.

h	$\left\|\phi - \hat{\phi}_h\right\|_0$	Cvgce. Rate	$\|u - \hat{u}_h\|_0$	Cvgce. Rate
$\frac{1}{4}$	$1.347085 \cdot 10^{-2}$		$1.451860 \cdot 10^{-1}$	
$\frac{1}{8}$	$3.292250 \cdot 10^{-3}$	2.03	$7.225961 \cdot 10^{-2}$	1.01
$\frac{1}{16}$	$8.070548 \cdot 10^{-4}$	2.03	$3.609218 \cdot 10^{-2}$	1.00
$\frac{1}{32}$	$1.988848 \cdot 10^{-4}$	2.02	$1.804189 \cdot 10^{-2}$	1.00
$\frac{1}{64}$	$4.922038 \cdot 10^{-5}$	2.01	$9.020557 \cdot 10^{-3}$	1.00
$\frac{1}{128}$	$1.221814 \cdot 10^{-5}$	2.01	$4.510428 \cdot 10^{-3}$	1.00

6. NUMERICAL RESULTS

In this section, we present numerical results for the least squares approximations which support the analysis in Section 5.

Let S^h denote a uniform partition on $[0, 1]$, and X^h the space of continuous piecewise linear functions on S^h. We associate with X^h the standard basis of hat functions on the uniform grid of size $h := 1/n$. For this choice of X^h the approximation property holds, and thus our error analysis is applicable.

For the "hat functions" $\phi_i(x)$, $i = 0 \cdots n$, there exists a closed form expression for $\mathcal{D}^\alpha \phi_i(x)$. In the left panel of Figure 1 are the left fractional derivatives of order one half for the hat functions ϕ_1, ϕ_2, ϕ_3, on the interval $[0, 1]$ for $n = 4$. Note that the fractional derivatives are nonlocal, with the support of ϕ_j being the unbounded interval $((j-1)/n, \infty)$. Illustrated in the right panel of Figure 1 are different order fractional derivatives for ϕ_1.

Table 2. Experimental convergence results for Example 2.

h	$\left\|\phi - \hat{\phi}_h\right\|_0$	Cvgce. Rate	$\|u - \hat{u}_h\|_0$	Cvgce. Rate
$\frac{1}{4}$	$6.682513 \cdot 10^{-3}$		$7.397892 \cdot 10^{-2}$	
$\frac{1}{8}$	$1.306160 \cdot 10^{-3}$	2.36	$3.713726 \cdot 10^{-2}$	0.99
$\frac{1}{16}$	$2.648486 \cdot 10^{-4}$	2.30	$1.841490 \cdot 10^{-2}$	1.01
$\frac{1}{32}$	$6.361133 \cdot 10^{-5}$	2.06	$8.9909481 \cdot 10^{-3}$	1.03

EXAMPLE 1. We consider $\phi(x) = x^2$ as the exact solution of the equation $\mathcal{D}^{1.5}\phi = f$. From the definition of the fractional derivative we compute

$$f(x) = \mathcal{D}^{1.5}\phi(x) = \frac{2\sqrt{x}}{\Gamma(1.5)}.$$

The associated boundary conditions are $\phi(0) = 0$, $\phi(1) = 1$. Because of the nonlocal supports of $\mathcal{D}^{\alpha}\phi_j$, the stiffness matrix is not sparse, but, as in [18], contains a fully occupied submatrix. Table 1 summarizes the numerical results for this example. Observe that experimental rates of convergence agree with the theoretical rates of 2 for ϕ and 1 for u as presented in Section 5.

EXAMPLE 2. For the second example, we set

$$\phi(x) := \frac{1}{\pi^{3/2}}\left(2\sqrt{x} - \sqrt{2}\cos(\pi x)\text{FresnelC}\left(\sqrt{2x}\right) - \sqrt{2}\sin(\pi x)\text{FresnelS}\left(\sqrt{2x}\right)\right), \tag{75}$$

where FresnelC(x) denotes the Fresnel cosine integral, and FresnelS(x) denotes the Fresnel sine integral. We then see that

$$\mathcal{D}^{1.5}\phi(x) = \sin(\pi x),$$

subject to the boundary conditions provided by (75). Presented in Table 2 are the numerical results for this example.

APPENDIX A

DEFINITIONS AND PROPERTIES OF THE RIEMANN-LIOUVILLE OPERATORS

We define the fractional integral and differential operators in terms of the Riemann-Liouville definition given in [7,8,19].

DEFINITION A.1. *(Left Riemann-Liouville Fractional Integral.) Let u be a function defined on $[a,b]$, and $\sigma > 0$. Then the left Riemann-Liouville fractional integral of order σ is defined to be*

$${}_aD_x^{-\sigma}u(x) := \frac{1}{\Gamma(\sigma)}\int_a^x (x-s)^{\sigma-1}u(s)\,ds. \tag{76}$$

DEFINITION A.2. *(Right Riemann-Liouville Fractional Integral.) Let u be a function defined on $[a,b]$, and $\sigma > 0$. Then the right Riemann-Liouville fractional integral of order σ is defined to be*

$${}_xD_b^{-\sigma}u(x) := \frac{1}{\Gamma(\sigma)}\int_x^b (s-x)^{\sigma-1}u(s)\,ds. \tag{77}$$

DEFINITION A.3. *(Left Riemann-Liouville Fractional Derivative.) Let u be a function defined on the interval $[a,b]$, $\mu > 0$, n be the smallest integer greater than μ $(n-1 \le \mu < n)$, and $\sigma = n - \mu$. Then the left Riemann-Liouville fractional derivative of order μ is defined to be*

$$ {}_aD_x^{\mu}u(x) = D^n\, {}_aD_x^{-\sigma}u(x) = \frac{d^n}{dx^n}\left(\frac{1}{\Gamma(\sigma)}\int_a^x (x-s)^{\sigma-1}u(s)\,ds\right). \tag{78} $$

DEFINITION A.4. *(Right Riemann-Liouville Fractional Derivative.) Let u be a function defined on the interval $[a,b]$, $\mu > 0$, n be the smallest integer greater than μ $(n-1 \le \mu < n)$, and $\sigma = n - \mu$. Then the right Riemann-Liouville fractional derivative of order μ is defined to be*

$$ {}_xD_b^{\mu}u(x) = (-D)^n\, {}_xD_b^{-\sigma}u(x) = (-1)^n\frac{d^n}{dx^n}\left(\frac{1}{\Gamma(\sigma)}\int_x^b (s-x)^{\sigma-1}u(s)\,ds\right). \tag{79} $$

With these definitions, we note some of the properties of the Riemann-Liouville fractional differential operators, as outlined in [7,8].

PROPERTY A.5. *(Semigroup Property.) The left and right Riemann-Liouville fractional integral operators follow the properties of a semigroup, i.e., for $u \in L^p[a,b]$ for $p \ge 1$,*

$$ \begin{aligned} {}_aD_x^{-\mu}\, {}_aD_x^{-\nu}u(x) &= {}_aD_x^{-\mu-\nu}u(x), & \forall\, x \in [a,b], \quad \forall\, \mu,\nu > 0, \\ {}_xD_b^{-\mu}\, {}_xD_b^{-\nu}u(x) &= {}_xD_b^{-\mu-\nu}u(x), & \forall\, x \in [a,b], \quad \forall\, \mu,\nu > 0. \end{aligned} \tag{80} $$

PROPERTY A.6. *(Fractional Fundamental Theorem of Calculus Part I.) The left (right) fractional derivative acts as a left inverse of the corresponding fractional integral, i.e., for $u \in L^p[a,b]$ for $p \ge 1$,*

$$ \begin{aligned} {}_aD_x^{\mu}\, {}_aD_x^{-\mu}u(x) &= u(x), & \forall\, x \in [a,b], \quad \forall\, \mu > 0, \\ {}_xD_b^{\mu}\, {}_xD_b^{-\mu}u(x) &= u(x), & \forall\, x \in [a,b], \quad \forall\, \mu > 0. \end{aligned} \tag{81} $$

PROPERTY A.7. *(Initial Conditions.) Let $\mu > 0$, and $n-1 \le \mu < n$, then the following statements are equivalent for $u \in C^{\infty}[a,b]$:*

$$ {}_aD_x^{-\mu}\, {}_aD_x^{\mu}u(x) = u(x), \qquad \forall\, x \in [a,b], \tag{82} $$

$$ \left[{}_aD_x^{\mu-j}u(x)\right]_{x=a} = 0, \qquad j = 1,2,\dots,n, \tag{83} $$

$$ u^{(j)}(a) = 0, \qquad j = 0,1,\dots,n-1. \tag{84} $$

PROPERTY A.8. *(Terminal Conditions.) Let $\mu > 0$, and $n-1 \le \mu < n$, then the following statements are equivalent for $u \in C^{\infty}[a,b]$:*

$$ {}_xD_b^{-\mu}\, {}_xD_b^{\mu}u(x) = u(x), \qquad \forall\, x \in [a,b], \tag{85} $$

$$ \left[{}_xD_b^{\mu-j}u(x)\right]_{x=b} = 0, \qquad j = 1,2,\dots,n, \tag{86} $$

$$ u^{(j)}(b) = 0, \qquad j = 0,1,\dots,n-1. \tag{87} $$

PROPERTY A.9. *(Fourier Transform.) The Fourier transform of a Riemann-Liouville operator satisfies the following:*

$$ \begin{aligned} \mathcal{F}\left({}_{-\infty}D_x^{\mu}u(x)\right) &= (i\omega)^{\mu}\hat{u}(\omega), \\ \mathcal{F}\left({}_xD_{\infty}^{\mu}u(x)\right) &= (-i\omega)^{\mu}\hat{u}(\omega), \end{aligned} \tag{88} $$

for all μ positive, negative, or zero, where $\hat{u}(\omega)$ denotes the Fourier transform of u.

REFERENCES

1. D.A. Benson, S.W. Wheatcraft and M.M. Meerschaert, The fractional order governing equations of Levy motion, *Water Resour. Res.* **36**, 1413–1423, (2000).
2. A.S. Chaves, A fractional diffusion equation to describe Levy flights, *Phys. Lett. A* **239**, 13–16, (1998).
3. A. Compte, Stochastic foundations of fractional dynamics, *Phys. Rev. E* **53** (4), 4191–4193, (1996).
4. R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, *Phys. Rep.* **339**, 1–77, (2000).
5. G.M. Zaslavsky, Chaos, fractional kinetics, and anomalous transport, *Phys. Rep.* **371**, 461–580, (2002).
6. S. Lu, F.J. Molz and G.J. Fix, Possible problems of scale dependency in applications of the three-dimensional fractional advection-dispersion equation to natural porous media, *Water Resour. Res.* **38** (9), 1165–1171, (2002).
7. I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, (1999).
8. S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, New York, (1993).
9. F.J. Molz, G.J. Fix and S. Lu, A physical interpretation for the fractional derivative in Lévy diffusion, *Appl. Math. Lett.* **15** (7), 907–911, (2002).
10. R. Schumer, D.A. Benson, M.M. Meerschaert and S.W. Wheatcraft, Eulerian derivation of the fractional advection-dispersion equation, *J. Cont. Hyd.* **48**, 69–88, (2001).
11. M.M. Meerschaert, D.A. Benson and B. Baeumer, Multidimensional advection and fractional dispersion, *Phys. Rev. E* **59** (5), 5026–5028, (1999).
12. R.A. Adams, *Sobolev Spaces*, Academic Press, New York, (1975).
13. P. Grisvard, *Singularities in Boundary Value Problems*, Springer-Verlag, New York, (1992).
14. G.J. Fix, M.D. Gunzburger and R.A. Nicolades, On finite element methods of the least squares type, *Computers Math. Applic.* **6** (2), 87–98, (1979).
15. B. Jiang, *The Least Squares Finite Element Method: Theory and Applications in Computational Fluid Dynamics and Electromagnetics*, Springer-Verlag, New York, (1998).
16. P. Clément, Approximation by finite element functions using local regularization, *RAIRO Anal. Numér.* **R-2**, 77–84, (1975).
17. S. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, New York, (1994).
18. D.M. Bedivan and G.J. Fix, Least squares methods for Volterra equations and generalizations, *Numer. Meth. P.D.E.* **14** (5), 679–693, (1998).
19. K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, (1993).