JOURNAL OF COMBINATORIAL THEORY 7, 116–121 (1969)

On Stirling Numbers of the Second Kind

B. C. RENNIE AND A. J. DOBSON

University College of Townsville, Queensland, Australia

Communicated by Gian-Carlo Rota

Received June 6, 1968

ABSTRACT

We first find inequalities between the Stirling numbers $S(n, r)$ for fixed n, then introduce functions L and U such that $L(n, r) \leqslant S(n, r) \leqslant U(n, r)$, and finally obtain the asymptotic value $n/\log n$ for the value of r for which $S(n, r)$ is maximal.

The Stirling number of the second kind $S(n, r)$ is the number of partitions of n things into r non-empty sets; it is positive if $1 \leqslant r \leqslant n$ and zero for other values of r. It satisfies the recurrence relation

$$S(n+1, r) = S(n, r-1) + rS(n, r). \tag{1}$$

Other properties are given by Riordan in [1].

It is known [2] that, for fixed n, $S(n, r)$ has a single maximum—more explicitly, that there is k_n such that

$$S(n, 1) < S(n, 2) < \cdots < S(n, k_n)$$

and

$$S(n, k_n) \geqslant S(n, k_n + 1) > \cdots > S(n, n).$$

LEMMA 1. *For all* $n \geqslant 1$, $S(n, 1) = S(n, n) = 1$,

$$S(n, 2) = 2^{n-1} - 1$$

and

$$S(n, n-1) = \tfrac{1}{2}n(n-1).$$

The first few Stirling numbers are given in the table below; $S(n, r)$ being the r-th element in row n:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&3&&1&& \\ &1&&7&&6&&1& \\ 1&&15&&25&&10&&1 \end{array}$$

PROOF: Each of the identities may be proved by induction on n using (1).

LEMMA 2. *If* $m > 2$ *and* $(m-2)\,S(2m-3, m-1) > mS(2m-3, m)$ *then*

$$S(2m-2, m-1) > S(2m-2, m).$$

PROOF: By the recursion (1),

$$\begin{aligned} &S(2m-2, m-1) - S(2m-2, m) \\ &\quad = S(2m-3, m-2) + (m-2)\,S(2m-3, m-1) - mS(2m-3, m); \end{aligned}$$

since the first term is positive, this is

$$\begin{aligned} &> (m-2)\,S(2m-3, m-1) - mS(2m-3, m) \\ &> 0 \qquad \text{by hypothesis.} \end{aligned}$$

THEOREM 1. *If* $n \geqslant 4$ *and* $\frac{1}{2}(n+1) \leqslant r \leqslant n-1$ *then*

$$(n-r)\,S(n, r) > (r+1)\,S(n, r+1) \tag{2}$$

and if $n \geqslant 2$ *then*

$$S(2n, n) > S(2n, n+1).$$

PROOF: The inequality (2) holds when $n = 4$. Also, for $n \geqslant 4$, (2) holds when $r = n-1$ since by Lemma 1

$$S(n, n-1) - nS(n, n) = \tfrac{1}{2}n(n-3) > 0.$$

Now we use induction on n, taking as our induction hypothesis:

$$(n-1-r)\,S(n-1, r) > (r+1)\,S(n-1, r+1)$$

if $\frac{1}{2}n \leqslant r \leqslant n-2$. Thus

$$S(n-1, r-1) > S(n-1, r)$$

for $\frac{1}{2}n + 1 \leqslant r \leqslant n-1$. By the recurrence relation (1) and the induction hypothesis:

$$\begin{aligned} \frac{S(n, r)}{S(n, r+1)} &= \frac{S(n-1, r-1) + rS(n-1, r)}{S(n-1, r) + (r+1)\,S(n-1, r+1)} \\ &> \frac{S(n-1, r-1) + rS(n-1, r)}{(n-r)\,S(n-1, r)}, \qquad \frac{1}{2}n \leqslant r \leqslant n-2 \\ &> \frac{r+1}{n-r} \qquad \text{for} \quad \frac{1}{2}n + 1 \leqslant r \leqslant n-2. \end{aligned}$$

If n is even,

$$\{r; \tfrac{1}{2}n + 1 \leqslant r \leqslant n - 2\} = \{r; \tfrac{1}{2}(n + 1) \leqslant r \leqslant n - 2\},$$

so that (2) has been established when $\frac{1}{2}(n + 1) \leqslant r \leqslant n - 2$.

If n is odd we still have to consider the case $r = \frac{1}{2}(n + 1)$. Let $n = 2m - 1$, then $r = m$. By the induction hypothesis

$$(m - 2)\, S(2m - 3, m - 1) > mS(2m - 3, m)$$

so that by Lemma 2 $S(2m - 2, m - 1) > S(2m - 2, m)$, that is, $S(n - 1, r - 1) > S(n - 1, r)$.

Thus whether n is even or odd the inequality (2) has been established for $\frac{1}{2}(n + 1) \leqslant r \leqslant n - 2$ but we also know that it holds when $r = n - 1$; therefore the proof of (2) by induction is complete.

The second part of the theorem follows from Lemma 2.

COROLLARY. $k_n \leqslant \frac{1}{2}(n + 1)$.

THEOREM 2. *If* $n \geqslant 2$ *and* $1 \leqslant r \leqslant \frac{1}{2}n$, *then*

$$S(n, r) \geqslant S(n, n - r + 1). \tag{3}$$

In fact if $n \geqslant 4$ *and* $2 \leqslant r \leqslant \frac{1}{2}n$, *the following stronger result is true*:

$$S(n, r) > S(n, n - r + 1). \tag{4}$$

PROOF: We begin by establishing (4) by induction on n. From the values given in Lemma 1 the result is clear when $n = 4$. Let us suppose that $S(n - 1, r) > S(n - 1, n - r)$ whenever $2 \leqslant r \leqslant \frac{1}{2}(n - 1)$. Using (1) we obtain

$$\begin{aligned} S(n, r) &= S(n - 1, r - 1) + rS(n - 1, r) \\ &> rS(n - 1, r) \\ &> rS(n - 1, n - r) \qquad \text{if} \quad 2 \leqslant r \leqslant \tfrac{1}{2}(n - 1). \end{aligned}$$

In Theorem 1 it was shown that

$$(r - 1)\, S(n - 1, n - r) > (n - r + 1)\, S(n - 1, n - r + 1)$$

if $\frac{1}{2}n \leqslant n - r \leqslant n - 2$, that is, if $2 \leqslant r \leqslant \frac{1}{2}n$. By (1)

$$\begin{aligned} S(n, n - r + 1) &= S(n - 1, n - r) + (n - r + 1)\, S(n - 1, n - r + 1) \\ &< S(n - 1, n - r) + (r - 1)\, S(n - 1, n - r) \\ &\qquad\qquad\qquad\qquad \text{if} \quad 2 \leqslant r \leqslant \tfrac{1}{2}n. \end{aligned}$$

Thus if $2 \leqslant r \leqslant \frac{1}{2}(n-1)$ we have

$$S(n, r) > rS(n-1, n-r) > S(n, n-r+1).$$

For $r = \frac{1}{2}n$, the inequality (4) is established in Theorem 1. Therefore if $2 \leqslant r \leqslant \frac{1}{2}n$ we have deduced that $S(n, r) > S(n, n-r+1)$ so that the proof by induction of the stronger result is complete.

The inequality (3) follows from (4) and the results of Lemma 1.

For $1 \leqslant r \leqslant n-1$, let

$$L(n, r) = \frac{1}{2}(r^2 + r + 2)\, r^{n-r-1} - 1$$

and

$$U(n, r) = \frac{1}{2}\binom{n}{r} r^{n-r}.$$

THEOREM 3. *If* $n \geqslant 2$ *and* $1 \leqslant r \leqslant n-1$, *then*

$$L(n, r) \leqslant S(n, r) \leqslant U(n, r).$$

PROOF: The inequalities are proved by induction on n. From Lemma 1 it can be seen that they are valid for $n = 2$ and valid for any n when $r = 1$ or when $r = n - 1$. In fact, there is equality when $r = n - 1$.

First, suppose that $L(n-1, r) \leqslant S(n-1, r)$ if $1 \leqslant r \leqslant n-2$. Using (1) we have, if $2 \leqslant r \leqslant n-2$,

$$\begin{aligned} S(n, r) &= S(n-1, r-1) + rS(n-1, r) \\ &\geqslant \tfrac{1}{2}(r^2 - r + 2)(r-1)^{n-r-1} - 1 + \tfrac{1}{2}(r^2 + r + 2)\, r^{n-r-1} - r \\ &\geqslant \tfrac{1}{2}(r^2 + r + 2)\, r^{n-r-1} - 1 = L(n, r), \end{aligned}$$

and since the inequality also holds for $r = 1$ and $n - 1$ it holds for $1 \leqslant r \leqslant n-1$, which completes the proof by induction.

Second, suppose that $S(n-1, r) \leqslant U(n-1, r)$ if $1 \leqslant r \leqslant n-2$. Again using (1) we find if $2 \leqslant r \leqslant n-2$ that

$$\begin{aligned} S(n, r) &\leqslant \frac{1}{2}\binom{n-1}{r-1}(r-1)^{n-r} + \frac{1}{2}\binom{n-1}{r} r^{n-r} \\ &\leqslant \frac{1}{2} r^{n-r}\left(\binom{n-1}{r-1} + \binom{n-1}{r}\right) \\ &= \frac{1}{2} r^{n-r}\binom{n}{r} \\ &= U(n, r) \end{aligned}$$

by the recurrence relation for binomial coefficients. Thus this inequality holds for $1 \leqslant r \leqslant n-1$, because for the cases $r = 1$ and $r = n-1$ it is already proved. This completes the proof by induction, so that we have established the theorem.

Finally we derive an asymptotic value for k_n.

THEOREM 4. *When n is large*

$$k_n = \frac{n}{\log n} + 0(n(\log\log n)^{1/2}(\log n)^{-3/2}).$$

PROOF: For all sufficiently large n define integers r_1, r_2, and r_3 as follows:

$$r_1 = \frac{n}{\log n}(1-\epsilon) + 0(1),$$

$$r_2 = \frac{n}{\log n} + 0(1),$$

and

$$r_3 = \frac{n}{\log n}(1+\epsilon) + 0(1),$$

where $\epsilon = 2(\log\log n)^{1/2}(\log n)^{-1/2}$ and the 0(1) terms are included to ensure that r_1, r_2, and r_3 are integers.

$$\begin{aligned}\log L(n, r_2) &= (n - r_2)\log r_2 + 0(\log n)\\ &= n\log n - n\log\log n - n + \frac{n\log\log n}{\log n} + 0(\log n).\end{aligned}$$

Stirling's formula for factorials gives

$$\log U(n, r) = n\log n + (n - 2r)\log r - (n - r)\log(n - r) + 0(\log n)$$

so that both $\log U(n, r_1)$ and $\log U(n, r_3)$ are

$$n\log n - n\log\log n - n + 2n\frac{\log\log n}{\log n} - \frac{1}{2}n\epsilon^2 + 0\left(\frac{n}{\log n}\right) < \log L(n, r_2)$$

Therefore by Theorem 3, for all sufficiently large n,

$$S(n, r_2) \geqslant L(n, r_2) > U(n, r_1) \geqslant S(n, r_1)$$

and

$$S(n, r_2) \geqslant L(n, r_2) > U(n, r_3) \geqslant S(n, r_3).$$

Since $r_1 < r_2 < r_3$ it follows that $r_1 < k_n < r_3$.

COROLLARY.

$$\max_{1 \leqslant r \leqslant n-1} \log S(n, r) = n \log n - n \log \log n - n + 0\left(n \frac{\log \log n}{\log n}\right).$$

REFERENCES

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