JOURNAL OF ALGEBRA 44, 152–159 (1977)

Coproducts of Topological Abelian Groups

P. J. HIGGINS

Department of Mathematics, King's College, Strand, London, WC2R 2LS, England

Communicated by A. Fröhlich

Received December 8, 1975

1. INTRODUCTION

In the category of topological Abelian groups the coproduct of any family $\{A_i\}_{i\in I}$ exists; it is the (restricted) direct sum $\sum_{i\in I} A_i$ with an appropriate topology. Attempts to characterize this topology by describing a fundamental system of neighborhoods of 0 in terms of neighborhoods of 0 in the given groups A_i have so far failed except in special cases. In this paper we narrow the search a little by showing that some reasonable conjectures are false.

For a countable coproduct of locally compact groups it was shown in [1] that the appropriate topology is the "rectangular" topology, which coincides (for all countable direct sums) with the "asterisk" topology of [4]. For an uncountable coproduct the situation is very different and we show that, in quite general circumstances, the coproduct topology, the asterisk topology, and the rectangular topology are all distinct.

We also show that, although $\sum_{i\in I} A_i$ is the union of the finite sums $\sum_{i\in F} A_i$ (for all finite $F \subset I$) and is algebraically their direct limit, the coproduct topology on $\sum_{i\in I} A_i$ is not, in general, the direct limit of the topologies on the finite sums. Results analogous to this for topological vector spaces can be found, for example, in [2, pp. 11–13; 3, pp. 98–99].

2. DEFINITIONS OF VARIOUS TOPOLOGIES ON $\sum A_i$

Let A_i $(i \in I)$ be topological Abelian groups and let $A = \sum_{i\in I} A_i$ be their direct sum as abstract groups. Then A is the coproduct in the category of Abelian groups of the A_i and we identify each A_i with its image in A. The existence of the coproduct in the category of topological Abelian groups can be seen by the following standard argument. Consider all pairs $(B, \mathscr{T})$ consisting of a quotient group $B = A/N$ and a group topology $\mathscr{T}$ on B (that is, a topology making B a topological group) such that the quotient map $A \to B$ induces a continuous map $A_i \to B$ for each $i \in I$. These pairs form a set S and we may form the product

$$P = \prod_{(B,\mathscr{T}) \in S} B$$

which is a topological group under the Tychonoff topology. The quotient maps $q(B, \mathscr{T}) : A \to B$ induce a canonical group morphism α: $A \to P$, and α is an injection because each of the A_i, with its given topology, occurs as one of the pairs $(B, \mathscr{T})$ (to within isomorphism). If we now give A the topology induced from P by this embedding, A becomes a topological group, and it is routine to verify that it is the coproduct of the topological Abelian groups A_i. We shall refer to this topology on $A = \sum A_i$ as the *coproduct topology*, $\mathscr{T}_c$. Clearly, $\mathscr{T}_c$ induces the given topology on each of the subgroups A_i and is the finest group topology on A with this property.

We now describe some other topologies on A all of which induce the given topologies on the A_i. First, as a subgroup of $\prod_{i \in I} A_i$, A can be given the topology induced by the Tychonoff topology. We denote this topology by $\mathscr{T}_\pi$; it is a group topology. Next, $\prod A_i$ can be made into a topological group by taking as a fundamental system of open neighborhoods of 0 all sets $\prod_{i \in I} V_i$, where V_i is an open neighborhood of 0 in A_i. This induces a topology $\mathscr{T}_r$ on $A = \sum A_i$ which is known as the *rectangular topology*.

The *asterisk topology* $\mathscr{T}_a$ on A was defined by Kaplan [4] as follows. In any topological Abelian group, if V is an open neighborhood of 0, the set $\{x; x \in V$ and $2x \in V\}$ is denoted by $(1/2)V$, and $(1/2^n)V$ is defined inductively as $(1/2)((1/2^{n-1})V)$. For $x \in V$, the number (x/V) is defined as the smallest $1/2^n$ for which $x \in (1/2^n)V$ (and is 0 if $x \in (1/2^n)V$ for all $n \geqslant 0$). Thus

$$(x/V) = (1/2^n) \Leftrightarrow x, 2x, 4x, \ldots, 2^n x \in V,\ 2^{n+1}x \notin V,$$

and $(x/V) = 0 \Leftrightarrow 2^n x \in V$ for all $n \geqslant 0$. Now if $\mathbf{V} = \{V_i\}_{i \in I}$, where V_i is an open neighborhood of 0 in A_i, for each i, we define

$$U(\mathbf{V}) = \left\{ \mathbf{x} \in A;\ x_i \in V_i \text{ for all } i, \text{ and } \sum_{i \in I} (x_i/V_i) < 1 \right\}.$$

Kaplan showed that, as the V_i vary, the $U(\mathbf{V})$ form a fundamental system of open neighborhoods of 0 for a group topology on A, called the asterisk topology. We generalize this by writing, for any positive real number p,

$$U^{(p)}(\mathbf{V}) = \left\{ \mathbf{x} \in A;\ x_i \in V_i \text{ for all } i, \text{ and } \sum_{i \in I} (x_i/V_i)^p < 1 \right\}.$$

As the V_i run through a fundamental system of open neighborhoods of 0 in A_i, for each i, the $U^{(p)}(\mathbf{V})$ (for fixed $p > 0$) form a fundamental system of open neighborhoods of 0 for a group topology $\mathscr{T}^{(p)}$ on A which we call the *p-topology*.

To verify this we first note the following consequences of the definition of (x/V).

(i) If $x \in W \subset V$, then $(x/V) \leqslant (x/W)$.

(ii) If $W + W \subset V$, and $x \in W$, then $(x/V) \leqslant \frac{1}{2}(x/W)$. Hence, if $W + W + \cdots + W \subset V$ (2^n copies of W) and $x \in W$, then $(x/V) \leqslant 2^{-n}(x/W)$.

(iii) If $W - W \subset V$ and $x, y \in W$, then

$$((x - y)/V) \leqslant \max((x/W), (y/W)).$$

Given fundamental neighborhoods V_i', V_i'' in A_i, let $V' = U^{(p)}(\mathbf{V}')$, $V'' = U^{(p)}(\mathbf{V}'')$. For each i, there is a fundamental neighborhood $V_i \subset V_i' \cap V_i''$ and, putting $V = U^{(p)}(\mathbf{V})$, we have $V \subset V' \cap V''$ by (i). Next, given $V = U^{(p)}(\mathbf{V})$, there exist fundamental neighborhoods W_i, X_i such that

$$W_i + W_i + \cdots + W_i \subset V_i \qquad (2^n \text{ copies, where } n > p^{-1})$$

and $X_i - X_i \subset W_i$. Put $X = U^{(p)}(\mathbf{X})$. Then $X - X \subset V$ since, if $\mathbf{x}$, $\mathbf{y} \in X$, we have

$$\sum (x_i/X_i)^p < 1 \qquad \text{and} \qquad \sum (y_i/X_i)^p < 1,$$

so

$$\begin{aligned} \sum ((x_i - y_i)/V_i)^p &\leqslant 2^{-pn} \sum ((x_i - y_i)/W_i)^p \qquad \text{by (ii)}, \\ &\leqslant \tfrac{1}{2} \sum \max((x_i/X_i)^p, (y_i/X_i)^p) \qquad \text{by (iii)}, \\ &\leqslant \tfrac{1}{2} \sum ((x_i/X_i)^p + (y_i/X_i)^p) \\ &< 1. \end{aligned}$$

It remains to show that if $V = U^{(p)}(\mathbf{V})$ and $\mathbf{x} \in V$ then there exists a fundamental neighborhood $W = U^{(p)}(\mathbf{W})$ such that $\mathbf{x} + W \subset V$. For fixed $x \in V$, the index set I can be partitioned as $I = F \cup G \cup H$, where

$$\begin{aligned} &i \in F \quad \text{if} \quad (x_i/V_i) \neq 0, \quad \text{say} \quad (x_i/V_i) = 2^{-n_i}, \\ &i \in G \quad \text{if} \quad (x_i/V_i) = 0 \quad \text{but} \quad x_i \neq 0, \\ &i \in H \quad \text{if} \quad x_i = 0. \end{aligned}$$

By definition of $\sum A_i$, the sets F and G are finite, and we have

$$s = \sum_{i \in F} (x_i/V_i)^p = \sum_{i \in I} (x_i/V_i)^p < 1.$$

For $i \in F$, we have $2^k x_i \in V_i$ for $0 \leqslant k \leqslant n_i$ and we choose W_i so that $2^k(x_i + y_i) \in V_i$ for $y_i \in W_i$, $0 \leqslant k \leqslant n_i$. Then

$$\sum_{i \in F} ((x_i + y_i)/V_i)^p \leqslant \sum_{i \in F} 2^{-pn_i} = \sum_{i \in F} (x_i/V_i)^p = s$$

whenever $y_i \in W_i$ for $i \in F$. For $i \in G$, we similarly choose W_i so that $2^k(x_i + y_i) \in V_i$ for $y_i \in W_i$, $0 \leqslant k \leqslant n$, where n is first chosen so that $2^{pn} \geqslant 2\,|G|\,(1-s)^{-1}$. Then

$$\sum_{i \in G} ((x_i + y_i)/V_i)^p \leqslant \sum_{i \in G} \frac{1-s}{2\,|G|} = \frac{1-s}{2}$$

whenever $y_i \in W_i$ for $i \in G$. For $i \in H$, we choose W_i so that $W_i + W_i + \cdots + W_i \subset V_i$ (2^m copies of W_i, where m is first chosen so that $2^{pm} \geqslant 2(1-s)^{-1}$). Then, by (ii)

$$\sum_{i \in H} ((x_i + y_i)/V_i)^p = \sum_{i \in H} (y_i/V_i)^p \leqslant \sum_{i \in H} 2^{-pm}(y_i/W_i)^p < \frac{1-s}{2}$$

whenever $y_i \in W_i$ and $\sum_{i \in H} (y_i/W_i)^p < 1$. Hence, for $\mathbf{y} \in W = U^{(p)}(\mathbf{W})$, we have

$$\sum_{i \in I} ((x_i + y_i)/V_i)^p < s + \frac{1-s}{2} + \frac{1-s}{2} = 1,$$

that is, $\mathbf{x} + W \subset V$. (We note that Kaplan's proof for the case $p = 1$ is deficient in that he assumes G to be empty.)

3. Comparison of Topologies on $\sum A_i$

We now have group topologies $\mathscr{T}_\pi$ (Tychonoff), $\mathscr{T}_r$ (rectangular), $\mathscr{T}_a = \mathscr{T}^{(1)}$ (asterisk), $\mathscr{T}^{(p)}$ ($p > 0$), and $\mathscr{T}_c$ (coproduct) on $A = \sum A_i$, and there are some obvious relations between them:

$$\mathscr{T}_\pi \prec \mathscr{T}_r \prec \mathscr{T}^{(q)} \prec \mathscr{T}_a \prec \mathscr{T}^{(p)} \prec \mathscr{T}_c \qquad \text{for} \quad 0 < p \leqslant 1 \leqslant q,$$
$$\mathscr{T}^{(q)} \prec \mathscr{T}^{(p)} \qquad \text{whenever} \quad 0 < p \leqslant q,$$

where $\mathscr{T} \prec \mathscr{T}'$ means that $\mathscr{T}'$ is finer than $\mathscr{T}$ (but not necessarily strictly so).

When I is finite, there is only one group topology on A consistent with the given topologies on the A_i, so all the above topologies coincide, and there is no problem.

When I is countable, $\mathscr{T}_r = \mathscr{T}_a$ in all cases (see [4, Theorem 1]), and $\mathscr{T}_r = \mathscr{T}_a = \mathscr{T}_c$ if all the A_i are locally compact (see [1, Proposition 1, Corollary]). It is not known whether the asterisk topology and the coproduct topology coincide on all countable direct sums.

When I is uncountable, there is very little information. Negrepontis claimed in [6] that $\mathscr{T}_a = \mathscr{T}_c$ always, but withdrew the claim in [7]. We show here that the claim is actually false, even for locally compact groups.

THEOREM 1. *Let A_i $(i \in I)$ be nondiscrete, Hausdorff, topological Abelian groups each possessing an open neighborhood U_i of 0 such that every neighborhood of 0 contains some $(1/2^{n(i)})U_i$. Let $0 < p < q$ and let I be uncountable. Then the p-topology on $A = \sum_{i \in I} A_i$ is strictly finer than the q-topology.*

Proof. Let U_i be as stated in the theorem and put $U = U^{(p)}(\mathbf{U})$, a fundamental neighborhood of 0 in the p-topology. We show that U contains no neighborhood of 0 in the q-topology. For suppose that V_i is, for each i, an open neighborhood of 0 in A_i and that $U \supset V = U^{(q)}(\mathbf{V})$, that is,

$$\left[x_i \in V_i \text{ and} \sum_{i \in I} (x_i/V_i)^q < 1\right] \Rightarrow \left[x_i \in U_i \text{ and} \sum_{i \in I} (x_i/U_i)^p < 1\right].$$

We can replace V_i by $U_i \cap V_i$ and so we may assume that $V_i \subset U_i$ for all i. Observe that all the neighborhoods $(1/2^k)V_i$ for $k = 0, 1, 2, \ldots$ are distinct since, if two were equal, we would have $(1/2^k)V_i = W_i$ for all large k and therefore $W_i \subset (1/2^k)U_i$ for all k; but this implies, by hypothesis, that every neighborhood of 0 in A_i contains the neighborhood W_i, which is impossible in a nondiscrete, Hausdorff group.

Now let F be any nonempty, finite subset of I, with $|F| = m$, say. Let N be the integer defined by $2^{N-1} \leqslant m^{1/q} < 2^N$. We can choose elements $x_i \in ((1/2^N)V_i)\backslash((1/2^{N+1})V_i)$ for $i \in F$ and then, putting $x_i = 0$ for $i \notin F$, we have

$$\sum_{i \in I} (x_i/V_i)^q = \sum_{i \in F} (x_i/V_i)^q = \sum_{i \in F} 2^{-Nq} < \sum_{i \in F} m^{-1} = 1,$$

that is, $\mathbf{x} = \{x_i\} \in V$. Hence $\mathbf{x} \in U$, which gives $\sum_{i \in F} (x_i/U_i)^p = \sum_{i \in I} (x_i/U_i)^p < 1$. By hypothesis, there exist integers $n(i)$ such that $V_i \supset (1/2^{n(i)})U_i$, and therefore $(1/2^k)V_i \supset (1/2^{k+n(i)})U_i$ for all $k \geqslant 0$. In particular since, for $i \in F$, $x_i \notin (1/2^{N+1})V_i$, we have $x_i \notin (1/2^{N+n(i)+1})U_i$, that is, $(x_i/U_i) \geqslant 2^{-(N+n(i))} \geqslant \delta_i m^{-1/q}$, where $\delta_i = 2^{-(n(i)+1)} > 0$. Hence

$$1 > \sum_{i \in F} (x_i/U_i)^p \geqslant m^{-p/q} \sum_{i \in F} \delta_i^p,$$

and since F is arbitrary, this gives

$$\sum_{i \in F} \delta_i^p < |F|^{p/q} \qquad \text{for all finite} \quad F \subset I.$$

It is easy to see that this is impossible. For let $I_m = \{i \in I;\ \delta_1 > m^{q^{-1}-p^{-1}}\}$ for $m = 1, 2, \ldots$. Then $I = \bigcup_{m=1}^{\infty} I_m$, since $\delta_i > 0$ for all i, and $q^{-1} - p^{-1} < 0$. But $|I_m| < m$, since otherwise I_m contains a subset F with $|F| = m$ and, for this F,

$$\sum_{i \in F} \delta_i^p > m(m^{q^{-1}-p^{-1}})^p = |F|^{p/q}.$$

Hence I is countable, contrary to hypothesis.

COROLLARY. *If the groups A_i are as stated in the theorem and I is uncountable, then the rectangular, asterisk and coproduct topologies on A are all different.*

It is worth noting the special case when each A_i is a copy $\mathbb{R}_i$ of the real line. We may take the standard neighborhood U_i to be the open interval $(-1, 1)$ in $\mathbb{R}_i$ and we may restrict attention to neighborhoods V_i of 0 of the form $\{x_i ; |x_i| < \delta_i\}$. Then (x_i/V_i) lies between $|x_i/\delta_i|$ and $2|x_i/\delta_i|$ and so the p-topology on $\sum \mathbb{R}_i$ can be described by the fundamental neighborhoods $U^{(p)}(\delta) = \{\mathbf{x}; \sum |x_i/\delta_i|^p < 1\}$. For $p \geqslant 1$ this topology is therefore determined by the family of vector space norms $\|\mathbf{x}\|_\delta^{(p)} = (\sum |x_i/\delta_i|^p)^{1/p}$, for varying $\delta = \{\delta_i\}$. For $p < 1$ these functions are not norms, but the topology is still defined by an obvious family of group norms in the sense of Markov [5], namely, the functions $N_\delta^{(p)}(\mathbf{x}) = \sum |x_i/\delta_i|^p$. The p-topologies all make $\sum \mathbb{R}_i$ a topological vector space and, according to the theorem, are all distinct when I is uncountable, although they all coincide when I is countable.

The union of the p-topologies, for $p > 0$, is itself a group topology, and one might conjecture that this is the coproduct topology on $A = \sum A_i$. In fact this is not the case in general, as can be shown by a similar argument. If f is any strictly increasing function on $[0, 1]$, continuous at the origin, with $f(0) = 0$, then one shows (as in the case $f(t) = t^p$) that the sets

$$U^{(f)}(\mathbf{V}, \epsilon) = \left\{\mathbf{x}; x_i \in V_i \text{ and } \sum f(x_i/V_i) < \epsilon\right\}$$

for $\epsilon > 0$ and V_i an open neighborhood of 0 in A_i, form a fundamental system of open neighborhoods of 0 for a group topology $\mathcal{T}^{(f)}$ on A. If $f(t) \leqslant g(t)$ for all sufficiently small positive t then $\mathcal{T}^{(f)} \prec \mathcal{T}^{(g)}$. So, for example, if we take $g(t) = -(\log_2 t)^{-1}$, we have $\mathcal{T}^{(p)} \prec \mathcal{T}^{(g)}$ for all $p > 0$. We claim that this $\mathcal{T}^{(g)}$ is strictly finer than the union of the p-topologies under the hypotheses of Theorem 1. For suppose not. Then $U^{(g)}(\mathbf{U}, 1)$ contains $U^{(q)}(\mathbf{V})$ for some $q > 0$ and some $\mathbf{V}$. The argument used for Theorem 1 gives, for any finite $F \subset I$ with $|F| = m$,

$$1 > \sum_{i \in F} g(x_i/U_i) \geqslant \sum_{i \in F} (N + n(i))^{-1},$$

where N, $n(i)$ and x_i have the same meaning as before. Since $2^{N-1} \leqslant m^{1/q}$, this gives

$$\sum_{i \in F} (1 + q^{-1} \log_2 m + n(i))^{-1} < 1$$

whenever $|F| = m$. Now put $I_m = \{i; 1 + q^{-1} \log_2 m + n(i) \geqslant m\}$. As before, $|I_m| \leqslant m$. But $I = \bigcup_{m=1}^{\infty} I_m$ since $m - q^{-1} \log_2 m \to \infty$ as $m \to \infty$. Hence I is countable, contrary to hypothesis. It seems likely that one can construct even finer group topologies by this method.

4. The Finite Topology on $\sum A_i$

Any group topology on $A = \sum_{i \in I} A_i$ compatible with the given topologies on the A_i must induce on each finite sum $A_F = \sum_{i \in F} A_i$ the product topology. We denote by $\mathscr{T}_f$ the *finite topology* on A which is the finest topology inducing the product topology on each A_F; that is, the open sets of $\mathscr{T}_f$ are all those subsets of A that meet each A_F in an open set of A_F (with its product topology). The finite topology is, of course, at least as fine as the coproduct topology $\mathscr{T}_c$, which is the finest *group topology* inducing the product topology on each A_F. It was shown [1, Proposition 1], that $\mathscr{T}_c = \mathscr{T}_f$ if I is countable and all the A_i are locally compact. In general, however, $\mathscr{T}_c \neq \mathscr{T}_f$ or, what is the same thing, $\mathscr{T}_f$ is not a group topology, as we now show.

Theorem 2. *Let A_i $(i \in I)$ be nondiscrete Hausdorff topological Abelian groups without small subgroups. If $|I| \geqslant 2^{\aleph_0}$ then the finite topology on $A = \sum_{i \in I} A_i$ is not a group topology.*

Proof. By hypothesis, there exists a surjection $\phi: I \to \mathbb{N}^{\mathbb{N}}$; we choose one and keep it fixed. Also, I contains a sequence $i_1, i_2, \ldots, i_r, \ldots$ of distinct elements; again we choose a fixed such sequence, and we define a function $g: I \times I \to \mathbb{N}$ as follows.

$$g(i_r, j) = \phi(j)(r) \qquad \text{for} \quad r \in \mathbb{N}, j \in I;$$
$$g(i, j) = 1 \qquad \text{if} \quad i \neq i_r \quad \text{for all } r \in \mathbb{N}.$$

Now let N_i be, for each i, a symmetric open neighborhood of 0 in A_i containing no nontrivial subgroup, and let $N = \sum N_i$ be the corresponding rectangular neighborhood of 0; it is open in the topology $\mathscr{T}_f$. We construct a smaller open set X of $\mathscr{T}_f$ by deleting from N the sets C_{ij} $(i, j \in I)$, where C_{ij} consists of all elements $a_i + a_j$ $(a_i \in A_i, a_j \in A_j)$ such that there exist positive integers $s, t \leqslant g(i, j)$ with $sa_i \notin N_i$ and $ta_j \notin N_j$. Clearly, C_{ij} is closed in $A_i + A_j \subset A$, whence $\bigcup_{i,j} C_{ij}$ is closed in the finite topology on A, and $X = N \backslash \bigcup C_{ij}$ is open.

Suppose that $(A, \mathscr{T}_f)$ is a topological group. Then there is an open neighborhood Y of 0 in A such that $Y + Y \subset X$. Since $Y \cap A_i$ is open in A_i, there exists, for each i, a symmetric neighborhood M_i of 0 in A_i such that $M_i \subset Y$ and hence $M_i + M_j \subset X$ for all $i, j \in I$. In particular $(M_i + M_j) \cap C_{ij} = \varnothing$ and so, for all $i, j \in I$, either

$$\begin{aligned} &sM_i \subset N_i \qquad \text{for} \quad s = 1, 2, \ldots, g(i, j), \\ \text{or} \qquad & \\ &tM_j \subset N_j \qquad \text{for} \quad t = 1, 2, \ldots, g(i, j). \end{aligned} \tag{1}$$

Since the neighborhoods N_i contain no nontrivial subgroups, we can choose integers $0 < n_1 < n_2 < n_3 < \cdots$ such that $n_r M_{i_r} \not\subset N_{i_r}$ for $r = 1, 2, \ldots$ (where

$\{i_r\}$ is the originally chosen sequence). Since ϕ is surjective, there is an element $j \in I$ such that $\phi(j)(r) = n_r$ for all r. For this fixed j we have $g(i_r, j) = n_r$ for all r. But $n_r M_{i_r} \not\subset N_{i_r}$, so the alternatives (1) give $tM_j \subset N_j$ for $t = 1, 2, \ldots, g(i_r, j) = n_r$. Since $n_r \to \infty$ as $r \to \infty$ and N_j is symmetric, this implies $tM_j \subset N_j$ for all integers t, which is impossible since N_j contains no nontrivial subgroup. This contradiction shows that $(A, \mathscr{T}_f)$ is not a topological group.

COROLLARY. *On a real vector space of dimension* $\geqslant 2^{\aleph_0}$ *the weak topology induced by the standard topology on the finite-dimensional subspaces is not a group topology.*

REFERENCES

1. R. BROWN, P. J. HIGGINS, AND S. A. MORRIS, Countable products and sums of lines and circles; their closed subgroups, quotients and duality properties, *Math. Proc. Cambridge Philos. Soc.* **78** (1975), 19–32.
2. D. J. H. GARLING, A generalized form of inductive limit topology for vector spaces, *Proc. London Math. Soc.* (3) **14** (1964), 1–28.
3. A. GROTHENDIECK, Sur les espaces (*F*) et (*DF*), *Summa Brasil. Math.* 3 (1954), 57–123.
4. S. KAPLAN, Extensions of Pontrjagin duality I: infinite products, *Duke Math. J.* **15** (1948), 649–658.
5. A. A. MARKOV, On free topological groups, *Izv. Akad. Nauk SSSR Ser. Mat.* **9** (1945), 3–64. English translation in *Amer. Math. Soc. Transl.* **30** (1950). Reprinted in *Amer. Math. Soc. Transl. Ser.* 1 **8** (1962), 195–272.
6. J. W. NEGREPONTIS, Duality in analysis from the point of view of triples, *J. Algebra* **19** (1971), 228–253.
7. J. W. NEGREPONTIS, Erratum to [6], *J. Algebra* **34** (1975), 188.