Existence of Solutions for P-laplace Equation with Critical Exponent

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Abstract

This paper discussed the Dirichlet problem for a p-Laplace equation with critical exponent. We prove existence of solutions using the fibering method introduced by Pohozaev and concentration compactness principle.

1. Introduction

This paper discussed the Dirichlet problem for a p-Laplace equation with critical exponent.

\[-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i}) = |u|^{p^*-2} u + \lambda |u|^{p-2} u \]

(1.1)

Let \( \Omega \subset \mathbb{R}^n \) be a smooth bounded domain, \( p^* = \frac{Np}{N-p} \), \( 0 < \lambda < \lambda_1 \). We denote by \( \lambda_1 \) the first eigenvalue with zero Dirichlet condition on \( \Omega \).

Equation involving critical exponent have been studied by various authors. Among other results, existence and non-existence theorems were obtained. They used the critical point theory in general. (see[1][2]). Now in this paper, we transform the method and still get the existence of the above equation. Our main tool here is the so-called fibering method introduced and developed by Pohozaev. (see[3]-[5]).

Our main result is the following:
Theorem 1.1. Assume $N \geq p^2$. Then for every $\lambda \in (0, \lambda_0)$, there exists a solution of (1.1).

2. Some Notations And Lemmas

We define the Sobolev spaces $L^s(\Omega)$ and $W_0^{1,p}(\Omega)$ equipped with the norms

$$
\|v\|_s = \left( \int_\Omega |v|^s \, dx \right)^{1/s},
$$

$$
\|v\| = \left( \|\nabla v\|_p^p + \|v\|_p^p \right)^{1/p}
$$

"→" denotes the weak convergence in the corresponding function space. We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1.1) if

$$
\int_\Omega |\nabla u|^{p-2} \nabla u \varphi dx = \int_\Omega |u|^{p-2} u \varphi dx + \int_\Omega \lambda |\varphi|^{p-2} \varphi dx
$$

For any $\varphi \in C_0^\infty (\mathbb{R}^n)$.

Lemma 2.1. The weak solutions of (1.1) correspond to critical points of the functional

$$
E_\lambda = \frac{1}{p} \int_\Omega (|\nabla u|^p - \lambda |u|^p)dx - \frac{1}{p} \int_\Omega |u|^p dx
$$

Lemma 2.2. Let $X$ be a real Banach space with norm differentiable on $X \setminus \{0\}$, let $f$ be a functional on $X$ of class $C^1 (X \setminus \{0\})$. We associate with $f$ a functional $\tilde{f}$ defined by

$$
\tilde{f}(t,v) = f(tv)
$$

where $(t,v) \in J \times X, tv \neq 0, J \subset R$. Let $(t,v)$ be a conditionally stationary point of the functional $\tilde{f}(t,v)$ under condition

$$
\{H_v, v\} \neq tH_v \quad \text{for} \quad H(t,v) = c
$$

Then the point $u = tv$ is a nonzero critical point of the original functional $f$, i.e. $f'(u) = 0$ and $u \neq 0$. (Proof see [3])

Lemma 2.3. Assume $u_m \to u$ on $W_0^{1,p}(\Omega)$, then

$$
\lim_{n \to \infty} \int_\Omega |\nabla u_m|^p dx - \int_\Omega |\nabla u|^p dx - \int_\Omega |(\nabla u_m - u)|^p dx = 0
$$

$$
\lim_{n \to \infty} \int_\Omega |u_m|^p dx - \int_\Omega |u|^p dx - \int_\Omega |(u_m - u)|^p dx = 0
$$

(Proof see [2])

3. Main Results

Now we prove the existence of equation (1.1.) used fibering method introduced by Pohozaev.

Let $E_\lambda$ be defined by

$$
E_\lambda = \frac{1}{p} \int_\Omega (|\nabla u|^p - \lambda |u|^p)dx - \frac{1}{p} \int_\Omega |u|^p dx \quad (3.1)
$$

Through Lemma 2.1., the critical points of $E_\lambda$ are the weak solutions of problem (1.1).

We express $u$ in the form $u = tv, t \neq 0$. Substituting it into (3.1) we obtain

$$
E_\lambda(tv) = \frac{t^p}{p} \int_\Omega (|\nabla v|^p - \lambda |v|^p)dx - \frac{t^p}{p} \int_\Omega |v|^p dx \quad (3.2)
$$
If $u$ is a critical point of $E_{\lambda}$ then $\frac{\partial}{\partial t}E_{\lambda}(tv)=0$. We have

$$t^{n-1}(H(v)-t^{p'-p}\|v\|_{p'}^{p'}) = 0$$

From the equation, we find the the real nonzero solutions

$$t(v) = \pm \left( \frac{H(v)}{\|v\|_{p'}^{p'}} \right)^{\frac{1}{p'-p}}$$

Where $H(v) = \int_{\Omega} (|\nabla v|^p - \lambda |v|^p) dx$. Then

$$\tilde{E}_{\lambda}(v) = E_{\lambda}(t(v)v) = \frac{1}{n} \left( \frac{H(v)}{\|v\|_{p'}^{p'}} \right)^{\frac{p'}{p'-p}}$$

(3.3)

Return to the initial question, we take $H(v) = 1$ as the fibering functional and we can prove the nondegeneracy condition, i.e.

$$\left\langle H'_v, v \right\rangle \neq tH'_v \quad \text{for} \quad H(v) = 1$$

Through Lemma 2.2., the problem of finding critical points for $\tilde{E}_{\lambda}(v)$ reduces to the constrained variational problem

$$P_{\lambda} : 0 < M_{\lambda} = \sup_{0 \neq v \in W_0^{1,p}(\Omega)} \left\{ \|v\|_{p'}^p \right\} \left\{ H(v) = 1 \right\}$$

(3.4)

**Lemma 3.1.** Assume $N \geq p^2$. Then for every $\lambda \in (0, \lambda_0)$, $M_{\lambda} > S^{-1}$, $S$ corresponds to the best constant for the Sobolev.

Proof: $M_{\lambda} = \sup_{0 \neq v \in W_0^{1,p}(\Omega)} Q_{\lambda}(v)$ We shall estimate the ratio $Q_{\lambda}(v) = \frac{\|v\|_{p'}^p}{\|\nabla v\|_p^p - \lambda \|v\|_p^p}$ with

$$V_\lambda(x) = \frac{\eta(x)}{(\varepsilon + |x|^{p'}\}}$$

Where $\phi \in C^\infty_0(\Omega)$ is a fixed function such that $\phi(x) = 1$ for $x$ in some neighborhood of $0$. We claim that, as $\varepsilon \to 0$, we have

$$\|v_\varepsilon\|_{p'}^p = \|v\|_p^p + o(\varepsilon)$$

$$\|v_\varepsilon\|_p^p = K_1 \varepsilon^{-\frac{N-p}{p}} + o(1), N > p^2$$

$$\|\nabla v_\varepsilon\|_p^p = K_1 |\ln \varepsilon| + o(1), N = p^2$$

$$\|\nabla v_\varepsilon\|_p^p = |\nabla v\|_p^p + o(1)$$
\[ K_1 = \int_{\mathbb{R}^N} \frac{1}{(1 + |y|^p)^{N-p}} \, dy, \quad v(x) = \frac{1}{(1 + |x|^p)^{N-p}} \]

When \( N > p^2 \)

\[
Q_\lambda(v) = \frac{\|v\|_p^p \epsilon^{N-p} + o(\epsilon)}{\|\nabla v\|_p^p \epsilon^{N-p} + o(1) - \lambda (K_1 \epsilon^{N-p} + o(1))} = \frac{1 + o(\epsilon^p)}{S + o(\epsilon^p) - \lambda (K_1 \epsilon^{N-p} + o(\epsilon^p))} > S^{-1}.
\]

When \( N = p^2 \)

\[
Q_\lambda(v) = \frac{\|v\|_p^p \epsilon^{N-p} + o(\epsilon)}{\|\nabla v\|_p^p \epsilon^{N-p} + o(1) - \lambda (K_1 \ln \epsilon + o(1))} = \frac{1 + o(\epsilon^p)}{S + o(\epsilon^p) - \lambda (K_1 \ln \epsilon + o(1))} > S^{-1}
\]

So \( N \geq p^2, M_\lambda > S^{-1} \). This completes the proof of Lemma 3.1.

**Lemma 3.2.** If \( M_\lambda > S^{-1} \), the supremum in (3.4) is achieved, the achieved function \( v \) is the solution of \( P_\lambda \).

**Proof:** Let \( W_\lambda = \{v \in W_0^{1,p}(\Omega) \mid H_\lambda(v) = 1\} \).

\( v_1 \) is the first eigenvalue function of \( \lambda_1 < \lambda < \lambda_1 \), then \( H(v_1) > 0 \). So \( W_\lambda \) is a nonempty set.

For any \( v \in W_0^{1,p}(\Omega) \)

\[
\int_{\Omega} |\nabla v|^p \, dx = 1 + \lambda \int_{\Omega} |v|^p \, dx \leq 1 + \frac{\lambda}{\lambda_1} \int_{\Omega} |\nabla v|^p \, dx
\]

\[
\|v\|_p^p = \int_{\Omega} |\nabla v|^p \, dx \leq \frac{\lambda_1}{\lambda_1 - \lambda}
\]

\( W_\lambda \) is bounded in \( W_0^{1,p}(\Omega) \).

Thus, for any maximizing sequence \( \{v_m\}_{m=1}^\infty \) in \( W_0^{1,p}(\Omega) \) must be bounded. And \( W_0^{1,p}(\Omega) \) is a reflexive Banach space, so there exists convergent subsequence \( \{v_{m_k}\}_{k=1}^\infty \) of \( \{v_m\}_{m=1}^\infty \) in \( W_0^{1,p}(\Omega) \). We assume

\[
v_m \rightharpoonup v \quad \text{weakly in} \quad W_0^{1,p}(\Omega)
\]

\[
v_m \to v \quad \text{a.e. in} \quad \Omega
\]

And \( M_\lambda + o(1) = \|v_m\|_p^p = \left( \int_{\Omega} |v_m|^p \, dx \right)^p \), then
We obtain a inequality
\[ M_{\alpha}^p + o(1) = \| v_m \|^p_* = \left\| \| v_m \|_p - \lambda v_m \right\|_p^p. \]

In order to get the above inequality, we used
\[
M_{\alpha} = \sup_{0 \neq \nu \in \mathcal{H}^1_0(\Omega)} \frac{\| \nu \|^p_p}{\| \nabla \nu \|^p_p},
\]
\[
S = \inf_{0 \neq \nu \in \mathcal{H}^1_0(\Omega)} \frac{\| \nabla \nu \|^p_p}{\| \nu \|^p_p},
\]
\[
H_\lambda(v_m) = \int_\Omega \left( |\nabla v_m|^p - \lambda |v_m|^p \right) dx = 1.
\]
Since \( M_\alpha > S^{-1} \), we can obtain (3.5) if \( \int_\Omega |\nabla (v_m - v)|^p dx = 0 \) i.e. \( v_m \to v \) in \( W^{1,p}_0(\Omega) \).

Thus \( H_\lambda = \| \nabla v \|^p_p - \lambda \| v \|^p_p = \| \nabla v_m \|^p_p - \lambda \| v_m \|^p_p = 1 \)
i.e. \( v \in M_\lambda \), so \( M_\lambda \) can be achieved. The achieved function is a solution of (1.1). This completes the proof of Lemma 3.1.

By Lemma 3.1 and Lemma 3.2, we completed the proof of Theorem 1.1.

**Reference**


