Linear Algebra and its Applications 430 (2009) 1849–1866

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

The Lyapunov order for real matrices

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A R T I C L E I N F O

Article history:
Received 11 November 2007
Accepted 10 October 2008
Available online 17 December 2008

Submitted by V. Mehrmann

Dedicated to Tom Laffey, an appreciated colleague.

Keywords:
Lyapunov matrix equation
Convex cone
Convex invertible cone
Controllability
Pick matrix

A B S T R A C T

The real Lyapunov order in the set of real $n \times n$ matrices is a relation defined as follows: $A \leqslant B$ if, for every real symmetric matrix S, $SB + B^tS$ is positive semidefinite whenever $SA + A^tS$ is positive semidefinite. We describe the main properties of the Lyapunov order in terms of linear systems theory, Nevenlinna–Pick interpolation and convexity.

1. Introduction

The set $GL(n, \mathbb{R})$ of real $n \times n$ non-singular matrices is divided into $n + 1$ classes of regular inertia $(\nu, \delta, \pi) = (k, 0, n - k), k = 0, \ldots, n$. In this paper, we study an ordering relation between matrices in each of these inertia classes, called the Lyapunov order, whose maximal elements are the scaled involutions.

The Lyapunov order is defined as follows. Let $\mathbb{S}$ (resp. $\overline{\mathbb{P}}$) denote the sets of symmetric (resp. symmetric positive semidefinite) real $n \times n$ matrices. Given $A, B \in \mathbb{R}^{n \times n}$ we say that $A \leqslant B$ if

$$S \in \mathbb{S}, \quad SA + A^tS \in \overline{\mathbb{P}} \;\Rightarrow\; SB + B^tS \in \overline{\mathbb{P}}. \tag{1}$$

The Lyapunov order is a preorder (a reflexive and transitive relation) in $\mathbb{R}^{n \times n}$.

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[1] Supported by the Conselho Nacional de Pesquisa (CNPq), Brazil.

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doi:10.1016/j.laa.2008.10.020

Although definition (1) appears to be new, the situation where $A \leqslant B$ and $B \leqslant A$ occur simultaneously was studied by Loewy in the 1970s in the context of the algebraic Lyapunov equation [26]. Our interest in the subject arose in the context of interpolation theory and the theory of convex cones. The Lyapunov order seems to be a unifying link between these three areas.

Below we summarize our main results.

1.1. Basic properties

We call $A \in \mathbb{R}^{n\times n}$ *Lyapunov regular* if its eigenvalues λ_i satisfy the standard condition [15, Section 15.5]

$$\lambda_i + \lambda_j \neq 0, \quad 1 \leqslant i,j \leqslant n. \tag{2}$$

In what follows we assume that $A, B, C \in \mathbb{R}^{n\times n}$; A is Lyapunov regular; and $A \leqslant B, C$. With these assumptions, the following holds:

(I) $A^t \leqslant B^t$.

(II) $A^{\pm 1} \leqslant B^{\pm 1}$, $\quad A \leqslant B + C$, $\quad aA \leqslant bB$ $(ab > 0)$ [8, Section 3]. Loewy showed that if A is Lyapunov regular then $B \leqslant A$ and $A \leqslant B$ hold simultaneously if and only if $A = aB$ or $A = aB^{-1}$, $a > 0$ [26], making the Lyapunov relation essentially a partial order.

(III) *Trivially*, $TAT^{-1} \leqslant TBT^{-1}$ *for all* $T \in GL(n, \mathbb{R})$.

(IV) *Every real A-invariant subspace is B-invariant*. (In general, complex A-invariant subspaces need not be B-invariant.) More precisely, if A is block upper triangular then, conformably, so is B, and the diagonal blocks satisfy $A_{ii} \leqslant B_{ii}$. This has strong implications in terms of putting the triple (A, B, S) in a common real Jordan or Schur type canonical form (see [4]).

(V) *A and B have the same stable–antistable dichotomy*. Let $\mathscr{V}_s(X), \mathscr{V}_{as}(X) \subset \mathbb{R}^n$ denote the stable and antistable invariant subspaces of a real matrix X. Then $\mathscr{V}_s(A) = \mathscr{V}_s(B)$ and $\mathscr{V}_{as}(A) = \mathscr{V}_{as}(B)$. In particular, A and B have the same inertia, and the stable and antistable parts of A and B satisfy $A_s \leqslant B_s$ and $A_{as} \leqslant B_{as}$. In particular, if A is Hurwitz stable, so is B.

(VI) *Eigenvalues have diminished argument*. We use the polar representation $z = \pm\rho e^{i\phi}$, $\pi/2 > |\phi|$ in the left and right half planes. It turns out that each eigenvalue of B has lower argument $|\phi|$ than its A counterpart. In particular, B has at least as many real eigenvalues as A.

(VII) *The jointly real parts commute*. More precisely, let $\mathscr{V}_r$ be the spectral subspace of A with respect to the entire real axis. Then $\mathscr{V}_r$ is a common reducing subspace for A and B. Moreover, if A_r, B_r are the corresponding restrictions of A, B to this subspace then B_r has real eigenvalues and $B_r \in \{A_r\}''$. (An analogous Lyapunov order between *complex* matrices may be defined by comparing the Hermitian positive definiteness of $HA + A^*H$ and $HB + B^*H$ for H Hermitian (see [7, Proposition 3.9], [12, 27]). The complex Lyapunov relation $A \leqslant B$ is stronger and implies that $B \in \{A\}''$.)

(VIII) *If A is Hurwitz stable then B has more quadratic Lyapunov functions than A*. Indeed, if $SA + A^tS \in \mathbb{P}$, then, by Lyapunov's stability theorem (e.g. [30, Theorem 7.11]) $-S \in \mathbb{P}$ and so the function $\nu_S(x) = -x^tSx$ is a quadratic Lyapunov function for the dynamical system $\frac{dx}{dt} = Ax$. If moreover $A \leqslant B$ then we have $SB + B^tS \in \mathbb{P}$, hence $\nu_S(x)$ is a quadratic Lyapunov function also for the system $\frac{dx}{dt} = Bx$.

(IX) *The Lyapunov order respects the positive real lemma*. Assume that $A \in \mathbb{R}^{n\times n}$ is Hurwitz stable and $(A, b, c; S)$ satisfy the conditions of the Positive Real Lemma [1, Chapter 5], [11, Section 4.3], [29]. According to this lemma, the strictly proper rational function $f_A(s) := c^t(sI - A)^{-1}b$ is positive real. It is easy to check that $(B, b, c; S)$ satisfy the same conditions if $A \leqslant B$, hence $f_B(s) = c^t(sI - B)^{-1}b$ is also positive real.

(X) *The Lyapunov order respects controllability*. For every real $n \times m$ matrix V, (A, V) is controllable if (B, V) is controllable.

Involutions play a unique role in the Lyapunov order: in each of the $n + 1$ regular inertia classes in $GL(n, \mathbb{R})$, the maximal elements are the scaled involutions. If A has regular inertia the cone

$$\overline{\mathscr{C}}_{\mathfrak{L}}(A) := \{B \in \mathbb{R}^{n\times n} : A \leqslant B\} \tag{3}$$

contains, up to scaling, a single maximal element, namely, the involution $E := \mathrm{Sign}(A)$ (see [7, Proposition 2.5], [17, Chapter 5], [22]). Increase in the Lyapunov order is well correlated with increased involution-type behavior of the matrix: in particular more real-invariant subspaces in item IV, better eigenvalue clustering in item VI, and in the stable case, increase in the set of Lyapunov functions in item VIII.

1.2. Convexity issues

An inner product space formalism in the space $\mathbb{S}$ of $n \times n$ symmetric matrices provides a geometric interpretation of the Lyapunov order. Following [26] we define the Lyapunov operator:

$$\mathfrak{L}_A : \mathbb{S} \to \mathbb{S}, \quad \mathfrak{L}_A(S) = SA + A^t S \tag{4}$$

and the forward and backward Lyapunov cones in $\mathbb{S}$:

$$\overline{\mathbb{Q}}(A) = \mathfrak{L}_A(\overline{\mathbb{P}}), \quad \overline{\mathbb{S}}(A) = \mathfrak{L}_A^{-1}(\overline{\mathbb{P}}).$$

The closed cones $\overline{\mathbb{S}}(A)$ and $\overline{\mathbb{Q}}(A^t)$ are dual cones within $\mathbb{S}$ in the Frobenius inner product [26]. Assuming that A is Lyapunov regular we show that the relation $A \leqslant B$ is equivalent to any of the following cone inclusions[2]:

$$\overline{\mathbb{S}}(A) \subset \overline{\mathbb{S}}(B), \quad \mathfrak{L}_B\mathfrak{L}_A^{-1}(\overline{\mathbb{P}}) \subset \overline{\mathbb{P}}, \quad \overline{\mathbb{Q}}(B) \subset \overline{\mathbb{Q}}(A),$$
$$\mathbb{S}(A) \subset \mathbb{S}(B), \quad \mathfrak{L}_B\mathfrak{L}_A^{-1}(\mathbb{P}) \subset \mathbb{P}, \quad \mathbb{Q}(B) \subset \mathbb{Q}(A).$$

This generalizes Loewy's result [26] stating the equivalence of the following statements:

$$\overline{\mathbb{S}}(A) = \overline{\mathbb{S}}(B), \quad \overline{\mathbb{Q}}(A^t) = \overline{\mathbb{Q}}(B^t), \quad B = aA^{\pm 1} \ (a > 0).$$

The transpose in his result may be dropped due to Property 1.1.I.

1.3. Convex invertible cones

A **cic** (or convex invertible cone) in $\mathbb{R}^{n\times n}$ is a convex cone of matrices which is closed under matrix inversion. The smallest **cic** which contains a given matrix A is a commutative set denoted by $\mathscr{C}(A)$ (see [7–12,25]); other **cic**s of interest are the cones $\overline{\mathscr{C}}_{\mathfrak{L}}(A)$ and $\mathscr{C}_{\mathfrak{L}}(A)$, where $\overline{\mathscr{C}}_{\mathfrak{L}}(A)$ is defined in (3) and $\mathscr{C}_{\mathfrak{L}}(A)$ is the set of matrices of regular inertia in $\overline{\mathscr{C}}_{\mathfrak{L}}(A)$ (here, the subscript $\mathfrak{L}$ refers to the Lyapunov order). The following inclusions hold:

$$\begin{array}{ll} \mathscr{C}(A) \subset \overline{\mathscr{C}}_{\mathfrak{L}}(A) \cap \{A\}'' & (A \text{ general}); \\ \mathscr{C}(A) \subset \mathscr{C}_{\mathfrak{L}}(A) \cap \{A\}'' & (A \text{ with regular inertia}); \\ \mathscr{C}(A) = \mathscr{C}_{\mathfrak{L}}(A) \cap \{A\}'' & (A \text{ Lyapunov regular}). \end{array} \tag{5}$$

We study cases where $\mathscr{C}(A) = \mathscr{C}_{\mathfrak{L}}(A)$. We also study the equalities $\mathscr{C}(A) = \mathscr{C}(B)$ and $\mathscr{C}_{\mathfrak{L}}(A) = \mathscr{C}_{\mathfrak{L}}(B)$ for a pair A, B, which are related to Loewy's work in [26].

1.4. Extreme points and the Pick test

The Lyapunov order involves inclusion between Lyapunov cones. In general, a cone inclusion $K \subset K'$ is established by showing that any extreme ray of K belongs to K'. Thus, we need to be able to characterize the set of extreme rays of a forward or backward Lyapunov cone. In the final analysis, this involves the description of extreme rays of $\overline{\mathbb{P}}$, which is well known. Take for example the backward Lyapunov cone $\overline{\mathbb{S}}(A)$.

Assume that A is Lyapunov regular. Extreme rays in $\overline{\mathbb{S}}(A)$ consist of matrices $S = S(u) \in \mathbb{S}$ such that $SA + A^tS = uu^t$ for some non-zero $u \in \mathbb{R}^{n\times n}$. These will be called *A-extreme matrices*.

[2] Throughout this work the inclusion sign "$\subset$" will always be interpreted as "$\subseteq$".

Extreme matrices $S(u)$ can be ordered by r, the dimension of the controllable subspace associated with the pair (A^t, u). Those matrices $S(u)$ for which r is maximized will be called *A-maximal matrices*. They form a relatively open and dense subset of A-extreme matrices, implying the following:

The weak Pick test. *Let A be Lyapunov regular. A necessary and sufficient condition for* $A \leqslant B$ *is that* $S(u)B + B^tS(u) \in \overline{\mathbb{P}}$ *for every A-maximal matrix* $S(u)$.

We add that the A-maximal matrices form a single orbit under the action of the group $GL(n,\mathbb{R}) \cap \{A\}'$. Moreover, when $B \in \{A\}''$ (this case occurs in interpolation) the corresponding Pick matrices can be shown to be pairwise congruent, and so checking just one of them is sufficient, leading to the following result:

The strong Pick test. *Let A be Lyapunov regular and* $B \in \{A\}''$. *Then a necessary and sufficient condition for* $A \leqslant B$ *is that the Pick matrix* $\Pi := S(u)B + B^tS(u)$ *is positive semidefinite for some A-maximal matrix* $S(u)$.

It is in this context that Pick matrices appear in interpolation and moment problems. Indeed, in [11, Corollary 5.2.2] it is shown that $B \in \mathscr{C}(A)$ if and only if $B = f(A)$ for some positive real odd function $f(s)$ on the right half plane; and, if A is Lyapunov regular, we know that $\mathscr{C}(A) = \mathscr{C}_{\mathfrak{L}}(A) \cap \{A\}''$ according to (5). So the strong Pick test is a necessary and sufficient condition for the interpolation $B = f(A)$. This topic is described in detail in [12].

In the last section of the paper, we mention several open research problems associated with the Lyapunov order.

2. Matrix preliminaries

2.1. Commutation

The set of real matrices, $\mathbb{R}^{n\times n}$, is a non-commutative algebra. We associate with each $A \in \mathbb{R}^{n\times n}$ two subalgebras of $\mathbb{R}^{n\times n}$, the commutant and bicommutant of A:

$$\{A\}' = \{B \in \mathbb{R}^{n\times n} : AB = BA\},$$
$$\{A\}'' = \{C \in \mathbb{R}^{n\times n} : BC = CB \text{ for all } B \in \{A\}'\}.$$

$\{A\}''$ is the set of real polynomials in A. The following statements are equivalent: (i) $\{A\}' = \{A\}''$; (ii) the minimal and characteristic polynomials of A are equal; (iii) different blocks in the Jordan form of A have distinct eigenvalues. Such a matrix is called non-derogatory, see [19, Theorem 3.2.4.2, Problem 3.2.1], [21 p. 93], [24, p. 419], [31, p. 23].

2.2. Lattices and invariant subspaces

A lattice is a partially ordered set with a maximum and a minimum for each subset [14]. The sets $Lat(A)$ (resp. $Lat_r(A)$) of complex (resp. real) invariant subspaces of a matrix A, ordered by inclusion, are classical examples. Consider the following subalgebras of $\mathbb{R}^{n\times n}$:

$$\{A\}_c = \{B \in \mathbb{R}^{n\times n} : Lat(A) \subset Lat(B)\},$$
$$\{A\}_r = \{B \in \mathbb{R}^{n\times n} : Lat_r(A) \subset Lat_r(B)\}. \tag{6}$$

We have the obvious inclusion relations $\{A\}'' \subset \{A\}'$ and $\{A\}'' \subset \{A\}_c \subset \{A\}_r$.

2.3. Inertia

As usual we denote the inertia of A by $In(A) = (\nu, \delta, \pi)$ [20, Definition 2.1.1]. We call the inertia regular, (Hurwitz) stable or antistable if $\delta = 0$, $\nu = n$ or $\pi = n$, respectively. If $A \in \mathbb{R}^{n\times n}$ has regular inertia, we denote by $\mathrm{Sign}(A)$ the unique matrix E which satisfies

$$E^2 = I, \quad E \in \{A\}', \quad AE \text{ is antistable.} \tag{7}$$

Actually $E \in \{A\}''$. There are several equivalent definitions: in terms of the Jordan form, or in terms of the sign function algorithm. According to the latter, $\mathrm{Sign}(A)$ is the limit of the sequence $\{A_k\}$ defined by

$$A_o = A, \quad A_{k+1} = \frac{1}{2}(A_k + A_k^{-1}), \ k = 0, 1, \ldots, \tag{8}$$

see e.g. [22,17, Chapter 5]. In particular, we get $\mathrm{Sign}(A) = I$ if and only if A is antistable.

3. The Lyapunov order

3.1. Definitions

We consider the space $\mathbb{S}$ of real symmetric $n \times n$ matrices as a real inner product space in the trace inner product and Frobenius norm defined by

$$\langle S_1, S_2 \rangle = \mathrm{tr}(S_1 S_2), \quad \|S\|_F^2 = \mathrm{tr}(S^2).$$

In it, we consider the two convex cones $\mathbb{P}, \overline{\mathbb{P}}$ of symmetric positive definite and semidefinite matrices, respectively. Given $A \in \mathbb{R}^{n\times n}$, the Lyapunov operator $\mathfrak{L}_A$ (4) is invertible exactly when A is Lyapunov regular.[3] The preimages of $\mathbb{P}, \overline{\mathbb{P}}$ under the Lyapunov operator are the convex cones[4] in $\mathbb{S}$:

$$\mathbb{S}(A) = \{S \in \mathbb{S} : \ SA + A^*S \in \mathbb{P}\}, \quad \overline{\mathbb{S}}(A) = \{S \in \mathbb{S} : SA + A^*S \in \overline{\mathbb{P}}\}$$

called the *inverse Lyapunov cones* for A. Their members are sometimes called strong/weak Lyapunov factors for A. Matrices in $\mathbb{S}(A)$ are invertible, and it is not difficult to see that $\mathbb{S}(A) \neq \emptyset$ if and only if A has regular inertia. On the other hand, $\overline{\mathbb{S}}(A)$ is never trivial: when A is Lyapunov regular, it is the closure of $\mathbb{S}(A)$; otherwise, it contains the kernel of the Lyapunov operator. In the sequel we take $\overline{\mathbb{S}}(A)$ as a basis for comparing matrices.

Definition 1. Let $A, B \in \mathbb{R}^{n\times n}$. We say that $A \leqslant B$ if $\overline{\mathbb{S}}(A) \subset \overline{\mathbb{S}}(B)$.

Theorem 15(vi) implies that whenever $A \leqslant B$ and A has regular inertia, so does B. However, if A is Lyapunov regular B need not be Lyapunov regular, as shown by the example $A = \mathrm{diag}\{1, -2\}$, $B = \mathrm{Sign}(A) = \mathrm{diag}\{1, -1\}$.

3.2. Cone duality and transpose

It is well known that $\overline{\mathbb{P}} \subset \mathbb{S}$ is a self-dual cone (in the trace inner product). This has the following implication. Define the convex cones

$$\mathbb{Q}(A) = \mathfrak{L}_A(\mathbb{P}), \quad \overline{\mathbb{Q}}(A) = \overline{\mathfrak{L}}_A(\mathbb{P}).$$

Lemma 2 [26]. *For $A \in \mathbb{R}^{n\times n}$, $\overline{\mathbb{S}}(A)$ is the dual cone of $\overline{\mathbb{Q}}(A^t)$.*

The proof was omitted in [26] and is included here for completeness. Assume that $S, K' \in \mathbb{S}$. If $SA + A^tS = K$ and $S' = K'A^t + AK'$ then also $S', K \in \mathbb{S}$. Using basic properties of the trace function we get

$$\begin{aligned}\langle S, S' \rangle &= \mathrm{tr}(SS') = \mathrm{tr}(S(AK' + K'A^t)) = \mathrm{tr}(SAK') + \mathrm{tr}(SK'A^t) \\ &= \mathrm{tr}(SAK') + \mathrm{tr}(A^tSK') = \mathrm{tr}((SA + A^tS)K') = \mathrm{tr}(KK') = \langle K, K' \rangle.\end{aligned}$$

[3] The same condition also guarantees invertibility of the Kronecker product operator $A \otimes I + I \otimes A$, see e.g. [20, Corollary 4.4.7]. Lyapunov regularity, defined by (2), implies regular inertia, but not conversely. Our terminology is not consistent with [16], where a matrix with regular inertia is called Lyapunov regular.

[4] Topologically, $\overline{\mathbb{P}}$ is the closure of $\mathbb{P}$ but $\overline{\mathbb{S}}(A)$ need not be the closure of $\mathbb{S}(A)$, and the same applies to $\overline{\mathbb{Q}}(A)$ in Lemma 2. Nevertheless, we maintain the bar notation for convenience.

If $S \in \overline{\mathbb{S}}(A)$ and $S' \in \overline{\mathbb{Q}}(A^t)$ then $K, K' \in \overline{\mathbb{P}}$, hence by self-duality $\langle S, S'\rangle = \langle K, K'\rangle \geqslant 0$. Conversely, if $S' \in \mathbb{S}$ and $\langle S, S'\rangle \geqslant 0$ for all $S \in \overline{\mathbb{S}}(A)$ then $K' \in \mathbb{S}$ and $\langle K, K'\rangle \geqslant 0$, hence by self-duality $K' \in \overline{\mathbb{P}}$, namely, $S' \in \overline{\mathbb{Q}}(A^t)$. □

The sets $\mathbb{S}(A)$ and $\mathbb{S}(A^t)$ consist of non-singular matrices and are connected by the map $S \to S^{-1}$, immediately implying the following:

Lemma 3. *For all $A, B \in \mathbb{R}^{n\times n}$, $\mathbb{S}(A) \subset \mathbb{S}(B)$ if and only if $\mathbb{S}(A^t) \subset \mathbb{S}(B^t)$.*

3.3. Comparison of several cone inclusions

A theory of Lyapunov order could, in principle, be based on any of the following cone inclusions:

$$\begin{aligned} &\text{(i)}\quad \overline{\mathbb{S}}(A) \subset \overline{\mathbb{S}}(B); \qquad \text{(ii)}\quad \mathbb{S}(A) \subset \mathbb{S}(B);\\ &\text{(iii)}\quad \overline{\mathbb{Q}}(B) \subset \overline{\mathbb{Q}}(A); \qquad \text{(iv)}\quad \mathbb{Q}(B) \subset \mathbb{Q}(A). \end{aligned} \tag{9}$$

However, these conditions are not necessarily equivalent. For example, when $A = 0$ and B is arbitrary one gets $\emptyset = \mathbb{S}(A) \subset \mathbb{S}(B) \subset \overline{\mathbb{S}}(B) \subset \overline{\mathbb{S}}(A) = \mathbb{S}$, showing that (i) and (ii) can be incompatible. On the positive side, conditions (i) and (iii) are always equivalent by Lemma 2, and the implication (iv) $\Rightarrow$ (iii) holds since in general $\overline{\mathbb{Q}}(A)$ is the closure of $\mathbb{Q}(A)$.

Lyapunov regularity guarantees complete equivalence in (9), based on a simple topological observation.

Definition 4. An ordered pair $(Y, \overline{Y})$ of subsets of a topological space $\mathscr{X}$ is called proper if $Y \neq \emptyset$, $\overline{Y}$ is the closure of Y, and Y is the interior of $\overline{Y}$.

Given two proper pairs $(Y, \overline{Y})$ and $(Z, \overline{Z})$ of X, the inclusions $Y \subset Z$ and $\overline{Y} \subset \overline{Z}$ are clearly equivalent.

Proposition 5. *Assume that A, B are Lyapunov regular. Then*:

(I) *The four items in (9) are equivalent as definitions for $A \leqslant B$*;
(II) *$A \leqslant B$ if and only if $A^t \leqslant B^t$.*

Proof. (I) Concerning the inverse Lyapunov cones, we start with the proper pair $(\mathbb{P}, \overline{\mathbb{P}})$ in $\mathbb{S}$. If A is Lyapunov regular, invertibility of the Lyapunov operator implies that the pair $(\mathbb{S}(A), \overline{\mathbb{S}}(A))$ is proper; similarly $(\mathbb{S}(B), \overline{\mathbb{S}}(B))$ is proper, implying the equivalence of (i) and (ii). For the forward Lyapunov cones, a similar argument shows the equivalence of (iii) and (iv).

(II) follows from the equivalence of (i) and (ii) together with Lemma 3. □

For completeness we point out that a discussion relevant to (9) appeared in [16].

We do not know whether Proposition 5 extends to matrices A, B with regular inertia. For $A = \operatorname{diag}\{1, -1\}$ the pair $(\mathbb{S}(A), \overline{\mathbb{S}}(A))$ is proper while the pair $(\mathbb{Q}(A), \overline{\mathbb{Q}}(A))$ is not. Indeed, the interior of $\overline{\mathbb{Q}}(A)$ is empty.

4. The sets $\mathscr{C}(A), \overline{\mathscr{C}}_{\mathfrak{L}}(A), \mathscr{C}_{\mathfrak{L}}(A)$

The following definitions and properties can be found in [8, Section II]. A convex cone $\mathscr{C} \in \mathbb{R}^{n\times n}$ is called a **cic** (convex invertible cone) if it is closed under inversion in the following sense: $A^{-1} \in \mathscr{C}$ whenever $A \in \mathscr{C} \cap GL(n, \mathbb{R})$. We call $\mathscr{C}$ *non-singular* if it consists entirely of non-singular matrices. It is easy to see that the intersection of (non-singular) **cic**s is a (non-singular) **cic**. A non-singular **cic** $\mathscr{C}$ contains a single involution E and for every matrix $A \in \mathscr{C}$ we have $\operatorname{Sign}(A) = E$. In particular, all the matrices in a non-singular **cic** have the same regular inertia.

Given $A \in \mathbb{R}^{n\times n}$, the subalgebras $\{A\}'$, $\{A\}''$, $\{A\}_c$, $\{A\}_r$ defined in Section 2 are examples of matrix **cic**s. In the context of the Lyapunov order, though, we shall study the following two **cic**s: $\mathscr{C}(A)$, the smallest **cic** which contains A, and the cone $\overline{\mathscr{C}}_{\mathfrak{L}}(A)$ defined in (3). We shall refer to $\mathscr{C}(A)$ as the **cic** generated by A. It is always a strict subset of $\{A\}''$. Assuming A non-singular, both A and A^{-1} lie on extreme rays in $\mathscr{C}(A)$, and we define the set $\mathfrak{conf}(A)$ as the union of these two rays[5]:

$$\mathfrak{conf}(A) = \{aA, aA^{-1} : a > 0\}. \tag{10}$$

The closed convex set $\overline{\mathscr{C}}_{\mathfrak{L}}(A)$ is also a **cic**. This can be proved directly from the definition using Properties 1.1.II. An alternative argument is as follows. For all $S \in \mathbb{S}$ define the convex cone $\overline{\mathbb{A}}(S) = \{A \in \mathbb{R}^{n\times n} : SA + A^tS \in \overline{\mathbb{P}}\}$ which is a closed **cic** [8, Section 3, 16]. By a routine set-theoretic argument we have

$$\overline{\mathscr{C}}_{\mathfrak{L}}(A) = \bigcap_{S\in\overline{\mathbb{S}}(A)} \overline{\mathbb{A}}(S). \tag{11}$$

We conclude that $\overline{\mathscr{C}}_{\mathfrak{L}}(A) \subset \mathbb{R}^{n\times n}$ is a closed **cic**, as the intersection of closed **cic**s.

Due to Property 1.1.II it is clear that $\mathscr{C}(A) \subset \overline{\mathscr{C}}_{\mathfrak{L}}(A)$. This inclusion is strict for several reasons: (a) while $\mathscr{C}(A)$ is always commutative, $\overline{\mathscr{C}}_{\mathfrak{L}}(A)$ is not (see case (ii) in the proof of Lemma 12); (b) $\overline{\mathscr{C}}_{\mathfrak{L}}(A)$ but not $\mathscr{C}(A)$, is always closed (see Example 7); (c) if A has regular inertia then $\mathscr{C}(A)$ consists of matrices with regular inertia while $\overline{\mathscr{C}}_{\mathfrak{L}}(A)$ contains the zero matrix and often other singular matrices. Inspired by item (c) we define $\mathscr{C}_{\mathfrak{L}}(A)$ as the set of matrices with regular inertia in $\overline{\mathscr{C}}_{\mathfrak{L}}(A)$.

Theorem 6

(i) *For all $A \in \mathbb{R}^{n\times n}$ we have $\mathscr{C}(A) \subset \overline{\mathscr{C}}_{\mathfrak{L}}(A) \cap \{A\}''$.*
(ii) *If $A \in \mathbb{R}^{n\times n}$ has regular inertia then $\mathscr{C}(A) \subset \mathscr{C}_{\mathfrak{L}}(A) \cap \{A\}''$.*
(iii) *If A is Lyapunov regular then $\mathscr{C}(A) = \mathscr{C}_{\mathfrak{L}}(A) \cap \{A\}''$.*

Proof (i) both $\{A\}''$ and $\overline{\mathscr{C}}_{\mathfrak{L}}(A)$ are **cic**s which contain A, hence contain $\mathscr{C}(A)$. So, their intersection also contains $\mathscr{C}(A)$. (ii) If A has regular inertia, $\mathscr{C}(A)$ is a non-singular **cic**, hence by (i) we have $\mathscr{C}(A) \subset \mathscr{C}_{\mathfrak{L}}(A) \cap \{A\}''$. The equality in (iii) is not trivial and is proven in [12] using the Pick test for interpolation. □

When A has imaginary eigenvalues, $\mathscr{C}(A)$ is not necessarily closed (even when the zero matrix is included). Here is one example.

Example 7. Consider the matrices of type

$$A(b,c) = \mathrm{diag}\{B(b), C(c)\}, \quad B(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad C(c) = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$$

and the sets

$$\mathscr{C}' = \{rA(b,c) : -1 < b < 1,\ r > 0,\ c \in \mathbb{R}\}, \quad \mathscr{C}'' = \{rA(1,1), rA(-1,-1) :\ r > 0\}.$$

Then $\mathscr{C}' \cup \mathscr{C}''$ is the **cic** generated by $A{:=}A(1,1)$ and is not closed in $\mathbb{R}^{4\times 4}$.

Indeed, it is easy to see that $\mathscr{C}'$ is a convex cone, $\mathscr{C}', \mathscr{C}''$ are closed under inversions, and any non-trivial convex combination in $\mathscr{C}' \cup \mathscr{C}''$ belongs to $\mathscr{C}'$. This shows that $\mathscr{C}' \cup \mathscr{C}''$ is a **cic**. It is not closed since, for example, $A(1,2)$ is on its boundary.

Now we show that this set is generated by $A(1,1)$. First, $\mathscr{C}''$ consists of (positively scaled) A and its inverse, and its convex hull contains the matrices $A(x,x)$ $(-1 < x < 1)$. By inversion we get also

[5] In [8] (see also [11, Section 2]) we describe an inductive process of constructing a **cic** in terms of its generators. Considering $\mathscr{C}(A)$ with the single generator A, the first inductive step gives the set $\mathfrak{conf}(A)$. The notation "$\mathfrak{conf}$" is suggestive since $\mathfrak{conf}(s) = \{as, a/s :\ a > 0\}$ consists of the real-conformal maps on the right and left half planes $\mathbb{C}_+, \mathbb{C}_-$, respectively.

$A(-x, -1/x)$ $(-1 < x < 1, x \neq 0)$. By considering $\frac{1}{2}(A(-x, -1/x) + A(x, x))$ we obtain all matrices of the form $A(0, c)$ $(c \in \mathbb{R})$. Positive combinations of $A(x, x)$ and $A(0, c)$ fill all of $\mathscr{C}'$.

5. Matrices which generate the same cic

Consider the following four equivalence relations in $\mathbb{R}^{n\times n}$:

$$\text{(i)}\quad B \in \mathfrak{conf}(A), \qquad \text{(ii)}\quad \mathscr{C}(A) = \mathscr{C}(B),$$
$$\text{(iii)}\quad \overline{\mathscr{C}}_{\mathfrak{L}}(A) = \overline{\mathscr{C}}_{\mathfrak{L}}(B), \qquad \text{(iv)}\quad \overline{\mathbb{Q}}(A) = \overline{\mathbb{Q}}(B). \tag{12}$$

Condition (i) means that B is a positive multiple of A or A^{-1}, and so is easily decidable. Condition (iii) may be viewed as the natural equivalence relation in $\mathbb{R}^{n\times n}$, denoted here by $A \sim B$, defined by the simultaneous inequalities $A \leqslant B$ and $B \leqslant A$. It was characterized by Loewy in [26] in terms of condition (i), and we show that the same characterization holds for condition (ii) (see also [10, Theorem 4.1]). The following result is analogous to Proposition 5.

Theorem 8. *Let $A, B \in \mathbb{R}^{n\times n}$ be Lyapunov regular. Then the four conditions in (12) are equivalent definitions for $A \sim B$.*

Proof. Statements (iii) and (iv) are equivalent unconditionally, in view of duality (Lemma 2; a similar argument can be found in [26]). The equivalence of (i) and (iii) is shown in [26], assuming the Lyapunov regularity of A only. Clearly, (i) implies (ii); also, (ii) implies (iii) as follows: if $\mathscr{C}(A) = \mathscr{C}(B)$ then $B \in \mathscr{C}(A) \subset \overline{\mathscr{C}}_{\mathfrak{L}}(A)$ and, similarly, $A \in \mathscr{C}(B) \subset \overline{\mathscr{C}}_{\mathfrak{L}}(B)$. So $B \leqslant A$ and $A \leqslant B$. In other words, $\overline{\mathscr{C}}_{\mathfrak{L}}(A) = \overline{\mathscr{C}}_{\mathfrak{L}}(B)$. $\square$

In the remaining part of this section we study the exact relation between conditions (i) and (ii) in (12) for general complex matrices. The definition of $\mathfrak{conf}(A)$ for non-singular matrices, given in (10), is extended to singular matrices as $\mathfrak{conf}(A) = \{rA : r > 0\}$.

Definition 9. We say that $A \in \mathbb{C}^{n\times n}$ is uniquely generating if the equality $\mathscr{C}(A) = \mathscr{C}(B)$ for $B \in \mathbb{C}^{n\times n}$ implies $B \in \mathfrak{conf}(A)$.

The unique generating property depends critically on the existence of simple imaginary eigenvalues. In the following result we exclude the case $n = 1$ which is always uniquely generating.

Lemma 10. *Assume that $A \in \mathbb{C}^{n\times n}$ with $n \geqslant 2$.*

(I) *The following are equivalent*:
  (i) *A is not uniquely generating*;
  (ii) *A is similar to* $\operatorname{diag}\{E, D\}$ *where $D \in \mathbb{C}^{m\times m}$ is non-singular skew-Hermitian, $m \geqslant 1$, $E \in \mathbb{C}^{(n-m)\times(n-m)}$ is a scaled involution, and in case $n = m$ then iD is not a scaled involution*.
(II) *In particular, A is uniquely generating if it has regular inertia, or it is singular, or non-diagonalizable*.

Proof. Up to similarity we may assume that A is in complex upper Jordan form $A = \operatorname{diag}\{J_{k_1}(\lambda_1), J_{k_2}(\lambda_2), \ldots\}$ where $J_{k_j}(\lambda_j)$ is a Jordan block of size k_j associated with an eigenvalue λ_j. Every matrix $B \in \mathscr{C}(A)$ has similar block structure $B = \operatorname{diag}\{B_1, B_2, \ldots\}$ where each B_j is a $k_j \times k_j$ Toeplitz upper triangular. We denote by $\mu_{1j}, \mu_{2j}, \ldots, \mu_{k_j,j}$ the first row of B_j. A is clearly uniquely generating if it contains a block with the following characteristic:

(1) $\lambda_j, i\lambda_j \notin \mathbb{R}$ (since $|\arg(\mu_{1j})| \leqslant |\arg(\lambda_j)|$ with equality only when $B \in \mathfrak{conf}(A)$);
(2) $i\lambda_j \notin \mathbb{R}$, $k_j > 1$ (since $|\mu_{2j}/\mathrm{Re}[\mu_{1j}]| \leqslant 1/|\mathrm{Re}[\lambda_j]|$ with equality only when $B \in \mathfrak{conf}(A)$);

(3) $i\lambda_j \in \mathbb{R}$ and $k_j > 1$ (since $|\mu_{1j}/\mu_{2j}| \leqslant |\lambda|$ with equality only when $B \in \mathfrak{conf}(A)$);
(4) $\lambda_j, \lambda_i \in \mathbb{R}$ and $|\lambda_j| \neq |\lambda_i|$ (since $|\log|\mu_{1j}/\mu_{1i}|| \leqslant |\log|\lambda_j/\lambda_i||$ with equality only when $B \in \mathfrak{conf}(A)$).

This argument is based on the steady decrease of certain functionals along the given **cic**, so that matrices with lower value of the given functional cannot generate matrices with a higher value. In each of the four cases, the functional in question remains the same upon positive scaling and inversion, but strictly decreases on sums.

Now we prove item (I). If we assume (i), cases 1–4 are discarded, and up to re-ordering of diagonal elements we conclude that $A = \mathrm{diag}\{E, D\}$; E is $(n-m) \times (n-m)$ diagonal and real; D is $m \times m$ diagonal and imaginary, hence skew-Hermitian. Due to case 4, E is a scaled involution. We cannot have $m = 0$, otherwise A itself is a scaled involution, hence is uniquely generating. A similar argument discards the case where $m = n$ and $iA = iD$ is a scaled involution, hence uniquely generating. Thus (ii) holds.

Conversely, assume (ii). Up to similarity we may write $A = \mathrm{diag}\{E, D\}$. There are two cases:

In the first case, $m = n$. Let $d_1 < \cdots < d_t$ be the distinct moduli of the (imaginary) eigenvalues of $A = D$. Up to permutation of the diagonal terms we have $A = \mathrm{diag}\{id_1E_1, \ldots, id_tE_t\}$ with E_i diagonal involutions. It can be seen with some calculation that $\mathscr{C}(A)$ consists of all matrices of the form $B = \mathrm{diag}\{ir_1E_1, \ldots, ir_tE_t\}$ with $r_i \in \mathbb{R}$, and as long as r_i are non-zero and distinct, B is also a generator. Since iA is not a scaled involution, $t > 1$, hence there exist generators B which are not proportional to either A or A^{-1}, implying (i).

In the second case, $n \geqslant m + 1$. Here we may write $A = \mathrm{diag}\{E, id_1E_1, \ldots, id_tE_t\}$. Since $m \geqslant 1$, we have $t > 0$. It can easily be seen that $\mathscr{C}(A)$ consists of all matrices of the form $B = \mathrm{diag}\{qE, ir_1E_1, \ldots, ir_tE_t\}$ with $r_i \in \mathbb{R}$ and $q > 0$. As long as r_i are non-zero and distinct, B is also a generator. Again, there exist generators B which are not proportional to either A or A^{-1}, implying (i).

Item (II) in the Lemma follows directly from (I). $\square$

6. The equality $\mathscr{C}(A) = \mathscr{C}_{\mathfrak{L}}(A)$

Assume that A has regular inertia. According to Theorem 6, $\mathscr{C}(A) \subset \mathscr{C}_{\mathfrak{L}}(A)$ with equality only if the latter set is commutative. Here we bring evidence that this happens only when A has real eigenvalues.

The sets $\overline{\mathbb{S}}(A), \mathscr{C}(A), \overline{\mathscr{C}}_{\mathfrak{L}}(A)$ (and $\mathscr{C}_{\mathfrak{L}}(A)$) respect similarity in the sense that for all $T \in GL(n, \mathbb{R})$

$$\overline{\mathbb{S}}(T^{-1}AT) = T^t\overline{\mathbb{S}}(A)T, \quad \mathscr{C}(T^{-1}AT) = T^{-1}\mathscr{C}(A)T, \quad \overline{\mathscr{C}}_{\mathfrak{L}}(T^{-1}AT) = T^{-1}\overline{\mathscr{C}}_{\mathfrak{L}}(A)T.$$

Thus we can always assume A to be in real Jordan form; but positive scaling remains an additional degree of freedom.

Lemma 11. *Assume that $A \in \mathbb{R}^{n\times n}$ has regular inertia. Then*:

(i) *If A is real diagonalizable then $\mathscr{C}(A) = \mathscr{C}_{\mathfrak{L}}(A)$*;
(ii) *If $\mathscr{C}(A) = \mathscr{C}_{\mathfrak{L}}(A)$ then A has only real eigenvalues*.

Proof of (i). Up to real similarity we may assume that A is diagonal, say $A = \mathrm{diag}\{a_iE_i\}$ where $0 < a_1 < a_2 < \cdots$ and E_i are diagonal involutions of size k_i. Theorem 14 (items (i) plus (iii)) implies that $\mathscr{C}_{\mathfrak{L}}(A)$ consists of real diagonal matrices. By restriction to each block it can be seen that each B in $\mathscr{C}_{\mathfrak{L}}(A)$ is of the form $B = \mathrm{diag}\{b_iE_i\}$ with $b_i > 0$. Thus, $\mathscr{C}_{\mathfrak{L}}(A) \subset \{A\}''$. By a different restriction, this time to a single slot in each block, plus a sign change if necessary, we may assume that $A = \mathrm{diag}\{a_i\}$, which is Lyapunov regular. Then by item (ii) in Theorem 6 $\mathscr{C}(A) = \mathscr{C}_{\mathfrak{L}}(A)$. $\square$

We shall prove item (ii) at the end of the section. Meanwhile, let us examine the general 2×2 case with regular inertia.

Lemma 12. *For $A \in \mathbb{R}^{2\times 2}$ Lyapunov regular, $\mathscr{C}(A) = \mathscr{C}_{\mathfrak{L}}(A)$ if and only if A has real spectrum*.

Proof. We may apply positive scaling and similarity on A. Cases involving two distinct real eigenvaluesare treated by Lemma 11. Thus A is stable or antistable, and to avoid redundancy we only consider

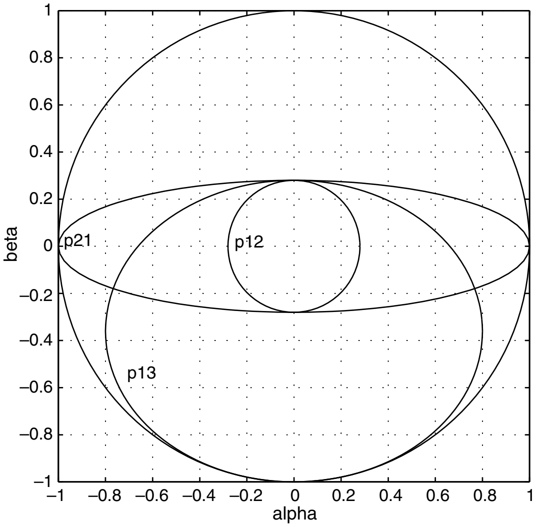

Fig. 1. Ellipses for A, B_0, B_1.

the antistable case. There are two possibilities, either A is non-diagonalizable or it has a pair of complex eigenvalues.

All matrices will be rescaled to have trace equal to 2. Rescaled matrices $S \in \mathbb{S}(A) \subset \mathbb{P}$ can be parameterized by a single complex number $z = \alpha + i\beta$ ($|z| \leqslant 1$) via

$$S = S(\alpha, \beta) = \begin{pmatrix} 1+\alpha & \beta \\ \beta & 1-\alpha \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}, \quad \alpha^2 + \beta^2 \leqslant 1.$$

These matrices may be identified with an open domain in the unit disk, in fact, the interior of an ellipse contained in the closed unit disk (see Fig. 1). We shall denote the ellipse itself by $\mathscr{E}ll(A)$. The Lyapunov relation $A \leqslant B$ means that $\mathscr{E}ll(A)$ is inscribed inside $\mathscr{E}ll(B)$:

(i) One (positive) real eigenvalue. Assume that A has a single eigenvalue λ. If A is diagonalizable we are back in Lemma 11, so we may assume that $A = I + 2X$ where $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It is easy to show that $\mathscr{C}(A) = \mathrm{conv}(A, A^{-1})$, hence rescaled matrices in $\mathscr{C}(A)$ have the form $I + \gamma X$ with $\gamma \in [-2, 2]$. Also, direct calculation gives for $\mathscr{E}ll(A)$ the ellipse $2\alpha(\alpha+1) + \beta^2 = 0$. Now assume that $0 \neq B \in \mathscr{C}_{\mathfrak{Q}}(A)$. By Theorem 15(ii), B is upper triangular.
We show that $B_{11} = B_{22}$. Indeed, assuming the contrary, up to a similarity of the form $T = I + tX$, which leaves A invariant, we may assume that B is diagonal. Rescaling, we may assume that $B = \mathrm{diag}\{1+c, 1-c\}$, where $-1 < c < 1$, and $c \neq 0$. Direct calculation gives for $\mathscr{E}ll(B)$ the ellipse $\alpha^2 + \beta^2/(1-c^2) = 1$. We compare the behavior of $\mathscr{E}ll(A)$ and $\mathscr{E}ll(B)$ at the point $z = -1$ by calculating a tangent parabola in each case. Setting $\alpha = \epsilon - 1$ with $\epsilon \searrow 0$, a first order approximation of the ellipse equation gives $\beta^2 \sim 2\epsilon$ for $\mathscr{E}ll(A)$ and $\beta^2 \sim 2\epsilon(1-c^2)$ for $\mathscr{E}ll(B)$. This shows that $\mathscr{E}ll(A)$ is not inscribed inside $\mathscr{E}ll(B)$ for any c non-zero, contradicting the assumption $A \leqslant B$.
So $B_{11} = B_{22}$. Since A, B have the same stability type, up to positive scaling we have $B = I + rX \in \{A\}''$ with $r \in \mathbb{R}$. Choosing $S = -I \in \overline{\mathbb{S}}(A)$ we see that $|r| < 2$, hence $B \in \mathscr{C}(A)$.

(ii) Two complex antistable eigenvalues. We may take $A = I + bZ$ with $0 \neq b \in \mathbb{R}$ and $Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Rescaled non-zero matrices $B \in \mathscr{C}_{\mathfrak{Q}}(A)$ will be written in the form:

$$B(c, d, e) = \begin{pmatrix} 1+c & d+e \\ d-e & 1-c \end{pmatrix}, \quad c, d, e \in \mathbb{R}.$$

We now show that:

(iia) Rescaled matrices in $\mathscr{C}(A)$ have the form $I + eZ = B(0, 0, e)$ with $|e| \leqslant |b|$;
(iib) Rescaled matrices in $\mathscr{C}_{\mathfrak{L}}(A)$ have the form $B(c, d, e)$ with

$$|e| + \sqrt{(1 + b^2)(c^2 + d^2)} \leqslant |b|. \tag{13}$$

This would show that in the case of non-real eigenvalues $\mathscr{C}(A)$ is indeed a proper subset of $\mathscr{C}_{\mathfrak{L}}(A)$. First we prove item (iia): $c = d = 0$ is necessary for commutation with A and it is easy to see that (up to scaling) $\mathscr{C}(A)$ is the convex hull of A and A^{-1}, hence consists of matrices of the form $I + eZ$ with $|e| \leqslant |b|$.

Next we prove item (iib). Direct calculation gives for $\mathscr{E}ll(A)$ the ellipse $\alpha^2 + \beta^2 = 1/(1 + b^2)$, a concentric disk, see Fig. 1. To show that $A \leqslant B = B(c, d, e)$ it is sufficient to demonstrate that the symmetric matrix $Q_b = SB + B^tS$ has non-negative trace and determinant on $\mathscr{E}ll(A)$. Minimizing $\frac{1}{4}$ trace$(Q_b) = 1 + \alpha c + \beta d$ on an ellipse in (α, β) is a simple linear programming problem; direct calculation shows that $1 + \alpha c + \beta d$ is positive on $\mathscr{E}ll(A)$ if and only if $1 + b^2 > c^2 + d^2$. Similarly, minimizing $\frac{1}{4}$ det$(Q_b) = 1 + (\alpha c + \beta d)^2 - (\alpha^2 + \beta^2 + (c - \beta e)^2 + (d + \alpha e)^2)$ on $\mathscr{E}ll(A)$ is a simple quadratic programming problem (using Lagrange multipliers). This minimum is calculated to be $b^2 - (|e| + \sqrt{(1 + b^2)(c^2 + d^2)})^2$, and its positivity yields (13) (the trace condition $1 + b^2 > c^2 + d^2$ calculated earlier turns out to be redundant), and item (iib) is established. □

Denoting by $|\arg(\lambda_j(M))|$ the absolute value of the argument of the jth eigenvalue of $M \in \mathbb{R}^{n\times n}$ one has the following.

Corollary 13. *If $A, B \in \mathbb{R}^{2\times 2}$ have regular inertia and $A \leqslant B$, then for $j = 1, \ldots, n$ $|\arg(\lambda_j(B))| \leqslant |\arg(\lambda_j(A))|$.*

For real eigenvalues there is nothing to prove, and for non-real eigenvalues this follows from direct calculation of the eigenvalues of $B(c, d, e)$ in (iib) above satisfying (13). □

We wish to examine further the geometric aspects of case (ii) in Lemma 12. Here, $A = \begin{pmatrix} 1 & b \\ -b & 1 \end{pmatrix}$ with $0 \neq b \in \mathbb{R}$, and the inclusion $\mathscr{C}(A) \subset \mathscr{C}_{\mathfrak{L}}(A)$ is strict. Set $B(c, d, e) - I = (B(c, d, 0) - I) + eB(0, 0, 1)$. Thus, the inequality in (13) may be interpreted as

$$\|eB(0, 0, 1)\| + t\|B(c, d, 0) - I\| \leqslant \|bB(0, 0, 1)\|$$

in the trace inner product norm (see Section 3) and $t = \sqrt{1 + b^2}$. Since symmetric and skew-symmetric matrices are orthogonal in the trace inner product, this inequality is satisfied if and only if $B(c, d, e)$ is in the convex hull of the two sets $\mathscr{C}(A) = \{e'B(0, 0, 1) : |e'| \leqslant |b|\}$ and $\widehat{\mathscr{B}} := \{B(c', d', 0) : t(c'^2 + d'^2) \leqslant |b|\}$. But $\widehat{\mathscr{B}}$ is the convex hull of

$$\mathscr{B} = \{B(c', d', 0) : t(c'^{\,2} + d'^{\,2}) = |b|\}.$$

So one can conclude that $\mathscr{C}_{\mathfrak{L}}(A)$ is the convex hull of $\mathscr{C}(A)$ and $\mathscr{B}$.

The matrix $B_o := B(c, 0, 0)$ with $c = \frac{b}{\sqrt{1+b^2}}$ belongs to the set $\mathscr{B}$. Direct calculation shows that $\mathscr{E}ll(B_o)$ is the ellipse $1 > \alpha^2 + \beta^2(1 + b^2)$ which touches $\mathscr{E}ll(A)$ tangentially at the points $\alpha = 0, \beta = \frac{\mp 1}{\sqrt{1+b^2}}$, see Fig. 1. In this example, none of the matrices in $\mathscr{C}_{\mathfrak{L}}(A) \setminus \mathscr{C}(A)$ commutes with A (this is consistent with Theorem 6). Some of these matrices have real spectrum (for example, the symmetric matrices in $\mathscr{B}$) or non-real spectrum (for example, $A + \epsilon B$ with $B \in \mathscr{B}$ and ϵ small). Some are non-diagonalizable, for example $B_1 := B(c, 0, e)$ with $c = e = \frac{|b|}{1+\sqrt{1+b^2}}$. The ellipse $\mathscr{E}ll(B_1)$ touches $\mathscr{E}ll(A)$ at a single point, $\alpha = 0$, $\beta = \frac{1}{\sqrt{1+b^2}}$, see Fig. 1.

Proof of Lemma 11(ii). Let A be Lyapunov regular with $\mathscr{C}(A) = \mathscr{C}_{\mathfrak{L}}(A)$, and assume by contradiction that A has at least two non-real (say, antistable) conjugate eigenvalues. Up to real similarity we may put A in the form $\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ with A_1 a 2×2 matrix of type d in Lemma 12. If $\mathscr{C}(A) = \mathscr{C}_{\mathfrak{L}}(A)$ then by restriction to the first block one would have $\mathscr{C}(A_1) = \mathscr{C}_{\mathfrak{L}}(A_1)$, contradicting Lemma 12. $\square$

7. Invariant subspaces

Since $\mathscr{C}_{\mathfrak{L}}(A)$ is not necessarily commutative, we cannot expect the relation $A \leqslant B$ to imply that B has the same invariant subspaces as A. However, it does have the same *real*-invariant subspaces.

Theorem 14. *Assuming that A has regular inertia and $\mathbb{S}(A) \subset \mathbb{S}(B)$, the following holds*:

(i) $B \in \{A\}_r$, *namely, real A-invariant subspaces are B-invariant*;
(ii) *Real A-reducing subspace pairs are B-reducing*;
(iii) *Assume that A is upper block triangular. Then B is conformably upper block triangular and the diagonal blocks satisfy $\emptyset \neq \mathbb{S}(A_{ii}) \subset \mathbb{S}(B_{ii})$.*

If A, B are Lyapunov regular we actually have $A \leqslant B$; in view of Lemma 5. By restriction, the same holds w.r.t. A_{ii}, B_{ii} in item (iii).

Proof. We shall use the fact that $\mathbb{S}(A) \neq \emptyset$ if and only if A has regular inertia [20, Theorem 2.4.10]:

(i) By joint similarity we may put A, B in the block form $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ with $A_{21} = 0$. We need to show that $B_{21} = 0$. Up to an additional joint similarity $T = \operatorname{diag}\{I, \epsilon I\}$ $(\epsilon > 0)$ we may replace (A, B) by

$$A_\epsilon := T^{-1}AT = \begin{pmatrix} A_{11} & \epsilon A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad B_\epsilon := T^{-1}BT = \begin{pmatrix} B_{11} & \epsilon B_{12} \\ \frac{1}{\epsilon} B_{21} & B_{22} \end{pmatrix}. \tag{14}$$

We still have $\emptyset \neq \mathbb{S}(A_\epsilon) \subset \mathbb{S}(B_\epsilon)$, implying by restriction that $\emptyset \neq \mathbb{S}(A_{ii}) \subset \mathbb{S}(B_{ii})$ $(i = 1, 2)$. Now choose any block-diagonal matrix $S = \operatorname{diag}\{S_1, S_2\}$ with $S_i \in \mathbb{S}(A_{ii})$. For ϵ very small, the dominant part of $SA_\epsilon + A_\epsilon^t S$ is positive definite and block-diagonal, hence $S \in \mathbb{S}(A_\epsilon) \subset \mathbb{S}(B_\epsilon)$. In other words, $Q := SB_\epsilon + B_\epsilon^t S$ is positive definite. The 2, 1-block of Q is $\epsilon^{-1} S_2 B_{12}$, and for ϵ very small Q cannot be positive definite unless this term vanishes; so $S_2 B_{21} = 0$. Since S_2 is non-singular, we conclude that $B_{21} = 0$.
(ii) Follows directly from (i).
(iii) Assume that A is upper block triangular and with diagonal blocks (of necessarily regular inertia) A_{ii}. Item (i) implies that B is upper triangular under the same block structure.
By restriction we get $\emptyset \neq \mathbb{S}(A_{ii})$. We claim that $\mathbb{S}(A_{ii}) \subset \mathbb{S}(B_{ii})$. Indeed, choose conformably $S(t) = \operatorname{diag}\{t^i S_{ii}\}$ with $S_{ii} \in \mathbb{S}(A_{ii})$ arbitrary. A simple recursive Schur complement argument on $S(t)A + A^t S(t)$ shows that $S(t) \in \mathbb{S}(A) \subset \mathbb{S}(B)$ for t sufficiently large. So $Q(t) := S(t)B + B^t S(t)$, as well as its iith block $t^i(S_{ii}B_{ii} + B_{ii}^t S_{ii})$, is positive definite, implying that $S_{ii} \in \mathbb{S}(B_{ii})$. Since there was no restriction of S_{ii} other than $S_{ii} \in \mathbb{S}(A_{ii})$, we are done. $\square$

8. Spectral properties

With the help of Theorem 14 we can use real similarity to put A, B in compatible block form and study their spectral properties. These include eigenvalue clustering and commutation of the "real spectral parts" of A and B.

We shall use the polar representation for eigenvalues of A and B and study the decrease in their argument. To this end, we denote by $m_\pm(A,a)$ $(a \geqslant 0)$ the number of eigenvalues of A, counting multiplicities, in a sector symmetric with respect to the real axis, i.e.

$$m_\pm(A,a) := \{x + iy \in \mathbb{C} : \ \pm x > 0, \ ax \leqslant |y|\}.$$

We shall denote by $\mathscr{V}_r, \mathscr{V}_{nr}, \mathscr{V}_s, \mathscr{V}_{as}$ the invariant subspaces related to the real, non-real, Hurwitz stable and antistable parts of the spectrum. Clearly, by Theorem 15 below the pairs $\mathscr{V}_r, \mathscr{V}_{nr}$ and $\mathscr{V}_s, \mathscr{V}_{as}$ are A-reducing, hence also B-reducing. We denote by A_r, A_{nr}, A_s, A_{as} the restriction of A to these spaces. Similarly, we use B_r, etc. for the restrictions of B to the subspaces $\mathscr{V}_r$ etc. defined by A.

Theorem 15. *Assuming that $A, B \in \mathbb{R}^{n\times n}$ have regular inertia and that $\mathbb{S}(A) \subset \mathbb{S}(B)$:*

(i) *For all $a \geqslant 0$ we have $m_\pm(A,a) \geqslant m_\pm(B,a)$;*
(ii) *$B_r \in \{A_r\}''$; hence, by Theorem 6(ii), $B_r \in \mathscr{C}(A_r)$.*
(iii) *$\mathbb{S}(A_r) \subset \mathbb{S}(B_r)$ and $\mathbb{S}(A_{nr}) \subset \mathbb{S}(B_{nr})$;*
(iv) *If A, B are Lyapunov regular, $A_r \leqslant B_r$, and $A_{nr} \leqslant B_{nr}$;*
(v) *$A_s \leqslant B_s$ and $A_{as} \leqslant B_{as}$;*
(vi) $\mathrm{Inertia}(A) = \mathrm{Inertia}(B)$ *and* $\mathrm{Sign}(A) = \mathrm{Sign}(B)$.[6]

Proof. (i) Up to (orthogonal) similarity we may assume that A is in real Schur canonical form: namely, upper block triangular and with A_{ii} either a real number or a real 2×2 matrix of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ $(a, b \neq 0)$. By item (ii), B has conformal upper block triangular structure, and B_{ii} is a 1×1 or a 2×2 matrix, conformable in size and stability type with A_{ii}, but not necessarily commuting with A_{ii}.
It is enough to prove item (i) for each pair A_{ii}, B_{ii}. In the 1×1 case the eigenvalues are real and there is nothing to prove, and in the 2×2 case this follows directly from Corollary 13.

(ii) Up to real similarity we may assume that $A_r = \mathrm{diag}\{A_i\}$ is in real Jordan upper canonical form. By Theorem 14, $B_r = \mathrm{diag}\{B_i\}$ conformably with B_i upper triangular and $\mathbb{S}(A_i) \subset \mathbb{S}(B_i)$. To show that $B_r \in \{A_r\}''$, we need to show that (1) B_i is upper Toeplitz; (2) whenever A_i is a minor of A_j also B_i is a minor of B_j.

First we show that B_i is upper Toeplitz. Setting $T_i = I - t(A_i - \lambda_i I)$ we have $T_i \in \{A_i\}'$. Since $A_i \leqslant B_i$, by similarity $T_i^{-1}A_iT_i = A_i \leqslant B_i + tC_i = T^{-1}B_iT$. Here, C_i is strictly upper triangular and $C_i = 0$ exactly when B_i is Toeplitz. Consider the matrix $S_i = \mathrm{diag}\{t, t^2, \ldots\}$ for $t \gg 1$ fixed. $S_iA_i + A_i^tS_i$ is diagonal dominant, hence $S_i \in \mathbb{S}(A_i)$. On the other hand, if $C_i \neq 0$ then $Q := S_iC_i + C_i^tS_i \in \mathbb{S}_{n_i}$ is non-zero and has zero diagonal, hence has a negative eigenvalue. But then for $t \gg 0$ the matrix $S_i(B_i + tC_i) + (B_i + tC_i)^tS_i = Q_o + tQ$ (with $Q_o := S_iB_i + B_i^tS_i$) has a negative eigenvalue, hence $S_i \in \mathbb{S}(A) \setminus \mathbb{S}(B_i + tC_i)$. This would contradict $A_i \leqslant B_i + tC_i$, unless $C_i = 0$. So B_i is upper Toeplitz.

Assume that the $k_i \times k_i$ Jordan cell A_i is a minor of the $k_j \times k_j$ Jordan cell A_j. We want to show that B_i is a minor of B_j. Let T be the identity matrix with the $k_i \times k_j$ matrix $E_{ij} = [I_{k_i}, 0]$ added in block ij. We have $T \in \{A_r\}'$ and it is enough to show that also $[B_r, T] = 0$. Let C' be the zero matrix with $B_iE_{ij} - E_{ij}B_j$ added in block ij. By similarity, $A_r = T^{-1}A_rT \leqslant T^{-1}B_rT = B_r + C'$.

Use the diagonal matrices $S_i \in \mathbb{S}(A_i)$ and $S_j \in \mathbb{S}(A_j)$ constructed earlier in the proof. Let $S' \in \mathbb{S}(A_r)$ be the zero matrix with S_i, S_j added in block ii, jj resp. If $C' \neq 0$ then $Q' := S'C' + C'^tS' \in \mathbb{S}$ is non-zero and has zero diagonal, hence Q' has a negative eigenvalue. But then for $1 \gg \epsilon > 0 S'(C' + \epsilon B_r) + (C' + \epsilon B_r)^tS' = Q' + \epsilon Q_o$ (with $Q_o' := S'B_r + B_r^tS'$) still has a negative eigenvalue, hence $S' \in \mathbb{S}(A_r) \setminus \mathbb{S}(C' + \epsilon B_r)$. This contradicts $A_r \leqslant B_r + C'$, so $C' = 0$.

Item (iii) follows from Theorem 14. Items (iv) and (v) follow from the equivalence of items (i) and (ii) in (9) (Proposition 5). Note that stable and antistable matrices are always Lyapunov regular. (vi) follows directly from the common stable–antistable structure of A and B. $\square$

[6] It is known, e.g. [7, Proposition 2.6] that $\mathrm{Sign}(A)$ is the only involution in $\mathscr{C}(A)$. In view of (6) we may conclude that $\mathrm{Sign}(A)$ is also the only involution in $\mathscr{C}_{\mathfrak{Q}}(A)$.

9. The Lyapunov order respects controllability

With every pair $(X, u) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n$ the controllability matrix is defined as

$$\mathrm{Cont}(X, u) = [u, Xu, \ldots, X^{n-1}u].$$

The columns of $\mathrm{Cont}(X, v)$ span the associated controllable subspace and its rank will be denoted by $r(X, u)$. By a classical definition, (X, u) is controllable if and only if $r(X, u) = n$. More generally, we shall call the pair (X, u) *maximally controllable* if $r(X, u) \geqslant r(X, v)$ for all $v \in \mathbb{R}^n$. Note that every matrix X has maximally controllable vectors which form an open dense set in $\mathbb{R}^n$.[7] Moreover, a maximally controllable pair satisfies $r(X, u) = n$ if and only if X is non-derogatory. Note that for a pair $(X, V) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, $m = 1, \ldots, n$ the $n \times (nm)$ controllability matrix $\mathrm{Cont}(X, V)$ is similarly defined and its range spans the controllable subspace associated with the pair (X, V).

Proposition 5 has the following consequence.

Proposition 16. *Assume that $A, B \in \mathbb{R}^{n\times n}$ are Lyapunov regular and $A \leqslant B$:*

(i) *If $V \in \mathbb{R}^{n\times m}$ and the pair (B, V) is controllable then so is the pair (A, V),*
(ii) *If $Q \in \overline{\mathbb{P}}$ and the pair (B, Q) is controllable then so is the pair (A, Q).*

Version (ii) is connected with the inertia theorem for the Lyapunov equation [6,13,34].

Proof. As versions (i) and (ii) are easily shown to be equivalent via the map $V \to Q = VV^t$, we shall only prove (ii). Using Proposition 5(II) we shall replace A, B by A^t, B^t throughout. First assume that A, hence also B, is antistable. Choose $Q \in \overline{\mathbb{P}}$ so that (B^t, Q) is controllable. According to the Lyapunov controllability theorem, e.g. [6], [20, Theorem 2.4.7], [34] $Q \in \mathbb{Q}(B)$. Assuming $A \leqslant B$, by Proposition 5(iv) we get $Q \in \mathbb{Q}(A)$. Thus, $Q = SA + A^tS$ for some $S \in \mathbb{P}$. A being antistable, the Lyapunov controllability theorem implies that the pair (A^t, Q) is controllable.

Now, consider the more general case where A and B are Lyapunov regular. Assume that $Q \in \overline{\mathbb{P}}(A)$ and (B^t, Q) is controllable. This occurs if and only if no eigenvector of B lies in the null space of Q [30, Theorem 13.3]. Now let $E = \mathrm{Sign}(A) = \mathrm{Sign}(B)$. Lyapunov regularity guarantees that B and EB have the same set of eigenvectors, hence by the same result (E^tB^t, Q) is controllable. Since EB, EA are antistable and $EA \leqslant EB$, by the first part of the proof we conclude that (E^tA^t, Q) is controllable, and (using the same eigenvalue argument) so is (A^t, Q). $\square$

10. The weak Pick test via extreme rays

Given $A, B \in \mathbb{R}^{n\times n}$ and $Q \in \overline{\mathbb{P}}$, the matrix $W = W(Q) = \mathfrak{L}_B({\mathfrak{L}_A}^{-1}(Q))$ defined by

$$SA + A^tS = Q, \quad SB + B^tS = W$$

is well defined if A is Lyapunov regular. To verify the relation $A \leqslant B$ directly, one needs to check that $W \in \overline{\mathbb{P}}$ for all Q, which is clearly not practical. As it turns out, this task simplifies for general pairs (A, B), and especially when $B \in \{A\}''$.

For general pairs (A, B) (i.e. B is not necessarily in $\{A\}''$) we show that it is enough to consider matrices Q of the form $Q = uu^t$ where $u \in \mathbb{R}^n$ is such that the pair (A^t, u) is maximally controllable (see Section 9). We refer to this test as the weak Pick test. Clearly the weak Pick test is a necessary condition for $A \leqslant B$. In this section, we show that it is also sufficient.

If, moreover, $B \in \{A\}''$, a sharper result can be obtained: it is enough to apply the Pick test on a *single* matrix of the above type. This criterion is, in fact, a basis-free formulation of the classical Pick test. This sharper result and its relationship with complex interpolation is discussed in the sequel paper [12].

[7] The reason is that the vanishing of all minors of order $r + 1$ of the controllability matrix defines an algebraic variety.

10.1. Controllability gramians as extreme directions in $\overline{\mathbb{S}}(A)$

Compare the three sets:

$$\overline{\mathbb{S}}(A) = \{S \in \mathbb{S} : SA + A^tS \in \overline{\mathbb{P}}\},$$
$$\mathbb{S}_{\rm ext}(A) = \{S = S(A,u) \in \mathbb{S} : SA + A^tS = uu^t \text{ for some } u \in \mathbb{R}^n\};$$
$$\mathbb{S}_{\rm cont}(A) = \{S(A,u) \in \mathbb{S}_{\rm ext}(A) : (A^t,u) \text{ is maximally controllable}\}.$$

$\mathbb{S}_{\rm cont}(A)$ consists of a small set on the topological boundary of $\overline{\mathbb{S}}(A)$. If A is antistable and (A^t,u) is controllable then $S(A^t,u) \in \mathbb{P}$ and is known in the control literature as the *controllability gramian* of (A^t,u) [30, Theorem 9.2 and Exercise 9.4]. With a slight abuse of terminology, we refer to all members of $\mathbb{S}_{\rm cont}(A)$ as controllability gramians.

The relation $A \leqslant B$ means that $\overline{\mathbb{S}}(A) \subset \overline{\mathbb{S}}(B)$. The weak Pick test for the pair (A,B) amounts to checking that all the controllability gramians for A are in $\overline{\mathbb{S}}(B)$. Obviously, the weak Pick test is a necessary test for $A \leqslant B$; we show that it is also sufficient. We shall denote by $\mathscr{E}(K)$ the set of extreme directions of a cone K.

Theorem 17. *Assume that $A \in \mathbb{R}^{n\times n}$ is Lyapunov regular. Then:*

(i) *The set $\overline{\mathbb{S}}(A)$ is a closed convex cone and $\mathscr{E}(\overline{\mathbb{S}}(A)) = \mathbb{S}_{\rm ext}(A)$.*
(ii) *The set $\mathbb{S}_{\rm cont}(A)$ is a generic (relatively open and dense) subset of $\mathbb{S}_{\rm ext}(A)$.*

Proof. (i) $\overline{\mathbb{P}} \subset \mathbb{S}$ is a closed convex cone whose extreme directions consist of the matrices $uu^t (u \in \mathbb{R}^n)$. If A is Lyapunov regular the linear operator $\mathfrak{L}_A : \overline{\mathbb{S}}(A) \to \overline{\mathbb{P}}$ is a cone isomorphism, hence by convexity theory $\mathscr{E}(\overline{\mathbb{S}}(A)) = \mathscr{E}(\mathfrak{L}_A^{-1}(\overline{\mathbb{P}})) = \mathfrak{L}_A^{-1}(\mathscr{E}(\overline{\mathbb{P}})) = \mathbb{S}_{\rm ext}(A)$.

(ii) Given A, let r be the maximal controllability rank for a pair (A^t,u) with $u \in \mathbb{R}^n$. u is maximally A^t-controllable if and only if at least one of the $r \times r$ submatrices of the controllability matrix does not vanish. It is easy to see that the set of maximally controllable vectors for A^t is open and dense. Now, $\mathbb{S}_{\rm cont}(A)$ is the inverse image under $\mathfrak{L}_A$ of a dense relatively open subset of rank-one positive semidefinite matrices, hence is a dense relatively open subset of $\mathbb{S}_{\rm ext}(A)$. $\square$

If A is not Lyapunov regular, the Lyapunov operator has a non-void kernel, $\mathbb{S}_o(A) := Ker(\mathfrak{L}_A)$. Since $\overline{\mathbb{S}}(A)$ is invariant under translations by elements of the non-void affine set $\mathbb{S}_o(A)$, by definition it cannot have extreme rays. However, $\mathbb{S}_{\rm ext}(A)$ in Theorem 17(i) may or may not be void. For example, it is void if $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ but not if $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If it is not void, we can still prove that $\mathbb{S}_{\rm cont}(A)$ is relatively dense (though typically not open) in $\mathbb{S}_{\rm ext}(A)$. Indeed, let $\mathbb{S}'(A)$ be the orthogonal projection (in the Frobenius norm) of $\mathbb{S}(A)$ along (i.e. perpendicular to) $\mathbb{S}_o(A)$. Then $\mathbb{S}(A)$ is the algebraic sum of the cones $\mathbb{S}_o(A)$ and $\mathbb{S}'(A)$. First one shows that $\mathbb{S}'(A) \cap \mathbb{S}_{\rm cont}(A)$ is a relatively open and dense set in $\mathbb{S}'(A) \cap \mathbb{S}_{\rm ext}(A)$; then, summing $\mathbb{S}_o(A)$ to both sides, denseness (but not openness) is extended to the entire cone.

For further information on the boundary structure of cones related to the Lyapunov and Riccati equations see [28,2].

10.2. Pick matrices and the weak Pick test

Here we consider an equivalent formulation which is closer in spirit to the original Pick test, i.e. is given in terms of Pick-like matrices.

Definition 18. Let the three cones $\overline{\Pi}(A,B)$, $\Pi_{\rm ext}(A,B)$, $\Pi_{\rm cont}(A,B)$ be defined as the respective images of the sets $\overline{\mathbb{S}}(A)$, $\mathbb{S}_{\rm ext}(A)$, $\mathbb{S}_{\rm cont}(A)$ under the Lyapunov operator $\mathfrak{L}_B$. We shall refer to matrices in these cones as nominal, extremal or maximally controllable Pick matrices for (A,B), respectively.

Extremal Pick matrices will be denoted by $P = P(A, B, u)$. They are defined by $SA + A^tA = uu^t$, $SB + B^tS = P$. They are maximally controllable if (A^t, u) is maximally controllable. The geometry of the closed convex cone $\overline{\Pi}(A, B)$ of Pick matrices is similar to that of $\overline{\mathbb{S}}(A)$ and is described next.

Corollary 19. *Let $A, B \in \mathbb{R}^{n\times n}$ be Lyapunov regular. Then $\mathscr{E}(\overline{\Pi}(A, B)) = \Pi_{\text{ext}}(A, B)$; and $\Pi_{\text{cont}}(A, B)$ is a generic (relatively open and dense) subset of $\Pi_{\text{ext}}(A, B)$.*

Indeed, $\mathfrak{L}_B$ maps $\overline{\mathbb{S}}(A)$ isometrically onto $\overline{\Pi}(A, B)$, and Theorem 17 can be applied. □

Corollary 19 partly extends to matrices without the Lyapunov regularity condition on both A and B. If just A is not Lyapunov regular, we use the extension of Theorem 17 to the non-Lyapunov regular case. Lyapunov regularity of B is not needed to imply that $\mathscr{E}(\overline{\mathbb{S}}(A)) \subset \mathbb{S}_{\text{ext}}(A)$ and that $\mathbb{S}_{\text{cont}}(A)$ is dense in $\mathbb{S}_{\text{ext}}(A)$; linearity of $\mathfrak{L}_B$ is sufficient.

Based on the above analysis, we consider the following two criteria, given A, B:

(T1) (*The Lyapunov order*): Every Pick matrix for (A, B) is positive semidefinite;
(T2) (*The weak Pick test*): Every maximally controllable Pick matrix $P(A, B, u)$ is positive semidefinite.

Also define the **cic**s

$$\overline{\mathscr{C}}_{\mathfrak{L}}(A) = \{B \in \mathbb{R}^{n\times n} : \overline{\Pi}(A, B) \subset \overline{\mathbb{P}}\},$$
$$\overline{\mathscr{C}}_{\mathfrak{L},\text{cont}}(A) = \{B \in \mathbb{R}^{n\times n} : \Pi_{\text{cont}}(A, B) \subset \overline{\mathbb{P}}\}. \tag{15}$$

Clearly, $\overline{\mathscr{C}}_{\mathfrak{L}}(A)$ is the same as defined in (3); and the statements (T1) and (T2) are equivalent to $B \in \overline{\mathscr{C}}_{\mathfrak{L}}(A)$ and $B \in \overline{\mathscr{C}}_{\mathfrak{L},\text{cont}}(A)$, respectively.

Theorem 20. *For all $A, B \in \mathbb{R}^{n\times n}$ we have $\overline{\mathscr{C}}_{\mathfrak{L}}(A) = \overline{\mathscr{C}}_{\mathfrak{L},\text{cont}}(A)$; as a result, conditions (T1) and (T2) are equivalent. In other words, the weak Pick test is a necessary and sufficient test for $A \leqslant B$.*

Proof. Since $\mathbb{S}_{\text{cont}}(A) \subset \overline{\mathbb{S}}(A)$, we get the inclusion $\overline{\mathscr{C}}_{\mathfrak{L}}(A) \subset \overline{\mathscr{C}}_{\mathfrak{L},\text{cont}}(A)$. To show the opposite inclusion, assume that $B \in \overline{\mathscr{C}}_{L,\text{cont}}(A)$. The continuous linear operator $\mathfrak{L}_B$ maps $\mathbb{S}_{\text{cont}}(A)$ into $\overline{\mathbb{P}}$. By the density claim in Theorem 17 it follows that $\mathfrak{L}_B$ maps $\mathbb{S}_{\text{ext}}(A) = \mathscr{E}(\overline{\mathbb{S}}(A))$ into $\overline{\mathbb{P}}$. By convexity, $\mathfrak{L}_B$ maps $\overline{\mathbb{S}}(A)$ into $\overline{\mathbb{P}}$, i.e. $B \in \overline{\mathscr{C}}_{\mathfrak{L}}(A)$ as required. □

11. Comments and open problems

1. The time domain implications of Properties I–X in Section 1, and especially of Theorem 15, are still not clear, but seem to be related to transient behavior of the type discussed in [18] and in problem 6.3 in [3].
2. If A is Lyapunov regular and $A \leqslant B$ then the linear map $\mathfrak{L}_B\mathfrak{L}_A^{-1} : \mathbb{S} \to \mathbb{S}$ is a non-negativity preserver, i.e. takes $\overline{\mathbb{P}}$ into itself. We do not know whether every linear non-negativity preserver in $\mathbb{S}$ is of this form.
3. Theorem 6 plus (11) imply that for all $S \in \overline{\mathbb{S}}(A)$

$$\mathscr{C}(A) \subset \overline{\mathbb{A}}(S) \cap \{A\}''. \tag{16}$$

It would be interesting to characterize matrices $S \in \mathbb{S}(A)$ for which (16) holds with equality. It is natural to conjecture that S should be on an extreme ray in the cone $\overline{\mathbb{S}}(A)$, namely $\text{rank}(SA + A^tS) = 1$.

Consider, for example, $A = \text{diag}\{1, 2\}$ and $S = \begin{pmatrix} 6 & 4 \\ 4 & 3 \end{pmatrix} \in \overline{\mathbb{S}}(A)$. The set $\{A\}''$ consists of all real diagonal matrices and $\mathscr{C}(A) = \{r \cdot \text{diag}\{1, a\} : \ r > 0, \ 1/2 \leqslant a \leqslant 2\}$. We have $SA + A^tS = 12uu^t$ where $u = (1, 1)^t$, hence $S \in \overline{\mathbb{S}}(A)$. $\overline{\mathbb{A}}(S)$ contains non-diagonal matrices, for example, $B = \begin{pmatrix} -4 & -3 \\ 6 & 4 \end{pmatrix}$ which satisfies $SB + B^tS = 0$. However, direct calculation shows that a diagonal matrix $B = \text{diag}\{b_1, b_2\} \in \{A\}''$ satisfies $SB + BS \geqslant 0$ exactly when $B \in \mathscr{C}(A)$. Thus (16) holds with equality.

4. Given $A, B \in \mathbb{R}^{n\times n}$, what are the necessary and sufficient conditions for the existence of $T \in GL(n, \mathbb{R})$ such that $A \sim T^{-1}BT$? or such that $A \leqslant T^{-1}BT$?
5. The following question is related to Theorem 15: Is it true that $A \leqslant B$ together with $B \in \{A\}_c$ (see (6)) implies $B \in \{A\}''$, or at least $B \in \{A\}'$?
6. Theorem 15(i) raises the question of characterizing matrix flows which have similar sector behavior with respect to the eigenvalues, such as the MSF (matrix sign function) algorithm mentioned in (8).
7. Can one define a proper distance function d so that $A \leqslant B$ with A Lyapunov regular, implies that $d(A, E) \geqslant d(B, E)$, where $E = \text{Sign}(A) = \text{Sign}(B)$?
8. The inequality $A \leqslant B$ does not imply that $A + B \leqslant B$. Similarly, the relation $B \in \mathscr{C}(A)$ does not imply that $B \in \mathscr{C}(A + B)$. As a counterexample for both statements take $A = I + N$ and $B = I - N = A^{-1}$ where $N \neq 0$ and $N^2 = 0$. We cannot have $A + B \leqslant B$ since $A + B$ is a scaled involution and B is not. Similarly, we cannot have $B \in \mathscr{C}(A + B)$ since $\mathscr{C}(A + B) = \{aI : a > 0\}$.
9. The matrix Lyapunov relation $SA + A^tS \in \overline{\mathbb{P}}$ may be interpreted as the LMI analogue of cone duality. Namely, the formal relation $(A, S) := SA + A^tS$ is interpreted as a "matrix-valued inner product" in $\mathbb{R}^{n\times n} \times \mathbb{S}$. In this language, the set $\mathbb{S}(A) \subset \mathbb{S}$ is the "dual cone" of $\mathbb{R}_+\{A\} \in \mathbb{R}^{n\times n}$; and, in the other direction, the set $\mathbb{A}(S) \subset \mathbb{R}^{n\times n}$ in (11) is the "dual cone" of $\mathbb{R}_+\{S\} \in \mathbb{S}$.
10. The relation (I) $\mathbb{S}(A) \subset \mathbb{S}(B)$ studied here stands in contrast with the relation (II) $\mathbb{S}(A) \cap \mathbb{S}(B) = \emptyset$ studied in e.g. [10,23] and to (III) $\text{Sign}(A) = \text{Sign}(B)$. Clearly (I) $\Longrightarrow$ (II) $\Longrightarrow$ (III) and in general none of the converse implications holds. For 2×2 matrices the relation with (III) is of interest. If A, B are antistable, one can take $A = \begin{pmatrix} 0 & -1 \\ 0.1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 2 & 0.5 \end{pmatrix}$ [10, Example 5.3] showing that (III)$\not\Rightarrow$(II). However, in the case of mixed inertia (III) $\Longrightarrow$ (I): Indeed up to similarity we may assume that A, B as well as $E = \text{Sign}(A)$ are diagonal, and $E \in \mathbb{S}(A) \cap \mathbb{S}(B)$. In fact, direct calculation implies one of the inclusions $\mathbb{S}(A) \subset \mathbb{S}(B)$ or $\mathbb{S}(B) \subset \mathbb{S}(A)$.
11. In this paper, we restricted the scope to (quadratic) Lyapunov functions associated with finite-dimensional linear time-invariant systems of the form $\dot{x} = Ax$. It appears that extensions in many directions are possible.

Acknowledgements

The authors wish to thank the anonymous referee for his/her thorough review, improving the quality of this work.

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