Support theorem for the fundamental solution to the Schrödinger equation on certain compact symmetric spaces

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Abstract

In this paper we construct the fundamental solution to the Schrödinger equation on a compact symmetric space with even root multiplicities using shift operators of Heckman and Opdam. Next, we prove that the support of the fundamental solution becomes a lower dimensional subset at a rational time whereas its support and its singular support coincide with the whole symmetric space at an irrational time. Moreover, we also show that generalized Gauss sums appear in the expression of the fundamental solution.

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1. Introduction

It is an important problem to study when and where the fundamental solution to a given differential equation becomes zero in particular if the differential equation in question arises from physics. One famous example is Huygens’ principle for the wave equation on the odd dimensional Euclidean space. An analogous Huygens’ principle holds for the modified wave
equation on an odd dimensional Riemannian symmetric space with even root multiplicities. Roughly speaking, the above Huygens’ principle shows that waves on such a symmetric space propagate along the light cone, namely, the support of the fundamental solution to the modified wave equation coincides with the light cone. This result and some generalizations are obtained by Branson, Olafsson and Pasquale [1], Branson, Olafsson and Schlichtkrull [2], Chalykh and Veselov [3], Gonzalez [4], Helgason [9,10], Helgason and Schlichtkrull [11], Olafsson and Schlichtkrull [13], and Solomatina [17]. In this sense, wave equations on symmetric spaces have been extensively studied by a lot of people from several different points of view. However, in contrast with the wave equation, nothing is known about the support of the fundamental solution to the Schrödinger equation on a compact symmetric space. In general, we cannot expect a similar phenomenon to the Huygens’ principle for Schrödinger equations. Nonetheless surprisingly, under some conditions, the support of the fundamental solution to the Schrödinger equation becomes a lower dimensional subset.

Now let us state our main result.

Let \( M = U/K \) be a compact symmetric space with even root multiplicities and let \( A \subset M \) be a maximal torus. We assume a certain condition on the weight lattice and the coroot lattice associated with the maximal torus \( A \). (We will give this condition later in Section 2. See Definition 2.4 and Assumption 2.5.) We consider the following Cauchy problem for the Schrödinger equation on \( M \)

\[
(Sch)_M \left\{ \begin{array}{ll}
\sqrt{-1} \partial_t \psi + \Delta_M \psi = 0, & t \in \mathbb{R}, \\
\psi(0, x) = \delta_o(x), & x \in M.
\end{array} \right.
\]  

(1.1)

Here \( \delta_o(x) \) denotes the Dirac’s delta function with singularity at \( o = eK \in U/K \) and \( \Delta_M \) denotes the Laplace–Beltrami operator on \( M \) with respect to the \( U \)-invariant metric. In addition, for simplicity, we denote \( \frac{\partial}{\partial t} \) by \( \partial_t \). (We consider \( \Delta_M \) to be a non-positive operator as in the case of \( \mathbb{R}^n \).) Let \( E_M(t, x) \) be the fundamental solution to the Schrödinger equation corresponding to a free particle on \( M \), namely, the solution to the above equation \((Sch)_M\).

Then our main theorem is stated as follows.

**Theorem 1.1.** (See Theorem 4.6.) There exists a positive number \( c \) such that the following (I) and (II) hold.

(I) In the case when \( t/c \) is a rational number. Let us put \( \frac{t}{c} = \frac{p}{q} \in \mathbb{Q} \), where \( p, q \in \mathbb{Z}, q > 0 \) and \( p \) and \( q \) are coprime. Then there exists a finite subset \( G_q \) of \( A \) depending on \( q \) such that the support of \( E_M(\frac{cp}{q}, \cdot) \) (as a distribution on \( M \)) is given by

\[
\text{Supp } E_M \left( \frac{cp}{q}, \cdot \right) = \{ k \cdot a \in M | k \in K, a \in G_q \}.
\]

(We will construct the above finite set \( G_q = G_{[q; \Lambda, \Gamma_0]} \) explicitly in Section 2. See (2.24).)

(II) In the case when \( t/c \) is an irrational number. The support of \( E_M(t, \cdot) \) is given by

\[
\text{Supp } E_M(t, \cdot) = \text{Sing Supp } E_M(t, \cdot) = M.
\]

Here \( \text{Sing Supp } E_M(t, \cdot) \) denotes the singular support of \( E_M(t, \cdot) \).
Remark 1.2. There are basically four kinds of compact symmetric spaces which satisfy the assumptions of the above theorem. Such symmetric spaces are given in the following list.

(i) $M = SO(2m+2)/SO(2m+1)$ (\(\cong S^{2m+1}\)). The odd dimensional sphere.
(ii) $M = SU(2m)/Sp(m)$.
(iii) $M = E_6/F_4$.
(iv) $M = (U \times U)/\triangle U$ (\(\cong U\)) for any compact simple Lie group $U$. Here $\triangle U$ denotes the diagonal set \(\{(u, u) \in U \times U \mid u \in U\}\).

Remark 1.3. In this paper, we do not necessarily assume that $M$ is irreducible. So we need to make some condition on the weight lattice and the coroot lattice. (See Assumption 2.5.) Thereby it is implied that Theorem 1.1 requires no such condition if $M$ is irreducible. (See Corollary 4.10.)

As is well known in physics, the Schrödinger equation \((\text{Sch})_M\) describes the motion of a free particle on $M$. Then the above theorem asserts that a free particle starting from the origin $o$ at $t = 0$ exists on a lower dimensional subset of $M$ at a rational time, whereas, by contrast, such a free particle can exist anywhere on $M$ at an irrational time. In particular, we would like to stress that a free particle on $M$ distinguishes a rational number and an irrational number. In addition, as we show later, it also recognizes some number theoretical quantity such as Gauss sums.

This paper is organized as follows. In Section 2, we construct the fundamental solution to the Schrödinger equation on the maximal torus $A$ with the flat metric. Section 3 is devoted to a summary of Heckman–Opdam theory on shift operators. In Section 4, using shift operators, we construct the fundamental solution to the Schrödinger equation \((\text{Sch})_M\) on $M$ and prove the main theorem (Theorem 4.6). Then we apply our main theorem to the Schrödinger equation with smooth initial data in Section 5. In Section 6, we deal with two examples, the odd dimensional sphere and $SU(6)/Sp(3)$.

2. Schrödinger equation on the maximal torus

In this section, we study the support of the fundamental solution to the Schrödinger equation on the maximal torus with the flat metric.

We start with the notations and the settings.

Notation 2.1. Let $M = U/K$ be a compact symmetric space, where $U$ and $K$ are a compact semisimple Lie group and its closed subgroup respectively. Let $u$ and $\mathfrak{t}$ be the Lie algebras of $U$ and $K$ respectively. Let $u = \mathfrak{t} \oplus m$ be the Cartan decomposition. Let us take a maximal abelian subspace $a \subset m$. We put $d = \dim_{\mathbb{C}} a = \rank U/K$. Let $A \subset U/K$ be the maximal torus corresponding to $a$. We fix an $\Ad\cdot U$-invariant inner product \(\langle \cdot, \cdot \rangle\) on $u$, which induces inner products on $m$ and on $a$. We denote these induced inner products by the same \(\langle \cdot, \cdot \rangle\). Let $W = W(U, K)$ be the Weyl group of $U/K$. For $\alpha \in a$, let $u_\alpha = \{ X \in u^{\mathbb{C}} \mid [H, X] = \sqrt{-1}(\alpha, H)X, \text{ for } \forall H \in a\}$. $\alpha$ is called a restricted root (or a root of $(u, a)$) if $u_\alpha \neq \{0\}$. Let $R$ be the set of restricted roots. We put $m_\alpha = \dim_{\mathbb{C}} u_\alpha$ for $\alpha \in R$ and call it the multiplicity of $\alpha$. In addition, we write $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ for $\alpha \in R$. We choose the set of positive roots and denote it by $R_+$. Let $\gamma_1, \ldots, \gamma_d \in R_+$ be the simple roots.
**Assumption 2.2.** Throughout the paper, we assume the following even multiplicity condition (EMC):

\[(\text{EMC}): \quad m_\alpha \text{ is even for any restricted root } \alpha \in R.\]

**Notation 2.3.** For a lattice \( \Gamma \subset a \) and for \( H \in a \), we denote by \([H]_\Gamma\) the equivalence class of \( H \) with respect to \( \Gamma \) (namely, an element of the quotient set \( a/\Gamma \)). In other words, \([H_1]_\Gamma = [H_2]_\Gamma \Leftrightarrow H_1 - H_2 \in \Gamma\).

Now let us take

\[\Gamma = \{ H \in a \mid \exp(H) \in K \}. \quad (2.1)\]

Then the maximal torus \( A \) is identified with \( a/\Gamma \) by the mapping \( a/\Gamma \ni [H]_\Gamma \mapsto \exp(H) \in A \).

Furthermore, let

\[\Gamma_0 := \mathbb{Z}\gamma_1^\vee \oplus \cdots \oplus \mathbb{Z}\gamma_d^\vee. \quad (2.2)\]

The above lattice \( \Gamma_0 \) is called the coroot lattice. As is well known, if we assume (EMC), then the lattice \( \Gamma \) is written as \( \Gamma = \pi \Gamma_0 \). (See Takeuchi [18, Chapters 6–7].)

Let \( \Lambda \) be the dual lattice of \( \Gamma_0 \). Let us take the generators of \( \Lambda \), \( \lambda_1, \ldots, \lambda_d \in a \) such that \( \langle \lambda_i, \gamma_j^\vee \rangle = \delta_{ij} \). \( \Lambda \) is called the weight lattice and each \( \lambda_i \) \( (1 \leq i \leq d) \) is called a fundamental weight.

Next, let us consider the Schrödinger equation on \( a/\Gamma \).

\[\text{(Sch)}_{a/\Gamma} \quad \left\{ \begin{array}{ll} \sqrt{-1} \partial_t \psi + \Delta_{a/\Gamma} \psi = 0, & t \in \mathbb{R}, \\ \psi(0, [H]_\Gamma) = \delta_{a/\Gamma}([H]_\Gamma), & \text{for } [H]_\Gamma \in a/\Gamma. \end{array} \right. \quad (2.3)\]

Here \( \delta_{a/\Gamma}([H]_\Gamma) \) denotes the Dirac’s delta function on \( a/\Gamma \) with singularity at \([0]_\Gamma \in a/\Gamma \) and \( \Delta_{a/\Gamma} \) denotes the Laplacian on \( a/\Gamma \) with respect to the flat metric on \( a/\Gamma \). In other words, if we take an orthonormal basis \( H_1, \ldots, H_d \in a \), then \( \Delta_{a/\Gamma} \) is written as \( \Delta_{a/\Gamma} = \partial_{r_1}^2 + \cdots + \partial_{r_d}^2 \) for \( H = r_1 H_1 + \cdots + r_d H_d \). Note that we sometimes identify a distribution on \( a/\Gamma \) with a \( \Gamma \)-periodic distribution on \( a \).

We see easily that the fundamental solution \( E_{a/\Gamma}(t, \cdot) \) to the Schrödinger equation (Sch)\( _{a/\Gamma} \) is given by

\[E_{a/\Gamma}(t, [H]_\Gamma) = \frac{1}{\text{Vol}(a/\Gamma)} \sum_{\lambda \in \Lambda} e^{-\sqrt{-1} t \langle \lambda, H \rangle_\Gamma + 2\sqrt{-1} \langle \lambda, H \rangle}. \quad (2.4)\]

In order to study the support of \( E_{a/\Gamma}(t, \cdot) \) as a distribution on \( a/\Gamma \), we will introduce a certain condition on lattices.

**Definition 2.4.** For two lattices \( \Gamma_1 \) and \( \Gamma_2 \) in a \( d \)-dimensional real vector space, we write \( \Gamma_1 \approx \Gamma_2 \) if there exists a real number \( c \neq 0 \) such that \( c\Gamma_1 \subset \Gamma_2 \). We say that \( \Gamma_1 \) is homothetic to \( \Gamma_2 \) if \( \Gamma_1 \approx \Gamma_2 \). We call the least positive number among such \( c \)'s the homothetic ratio of \( \Gamma_1 \) to \( \Gamma_2 \) and
write $C[\Gamma_1; \Gamma_2]$. Namely,

$$C[\Gamma_1; \Gamma_2] := \min\{c \mid c > 0, \ c\Gamma_1 \subset \Gamma_2\}. \quad (2.5)$$

Suppose that $e_1, \ldots, e_d$ and $f_1, \ldots, f_d$ are the generators of $\Gamma_1$ and $\Gamma_2$ respectively. Then the above condition means that there exist $c_0 > 0$ and $T \in GL(d, \mathbb{Z})$ such that $c_0 e_j = Tf_j$ for all $j$, $1 \leq j \leq d$, from which it follows that $\Gamma_2 \approx \Gamma_1$. In fact, we have $\det T f_j = T^{(\mathrm{co})} e_j$, where $T^{(\mathrm{co})}$ denotes the cofactor matrix of $T$. Moreover, if $\Gamma_1 \approx \Gamma_2$ and $\Gamma_2 \approx \Gamma_3$, then $\Gamma_1 \approx \Gamma_3$. So “$\approx$” is an equivalence relation. We also note that the relation “$\Gamma_1 \approx \Gamma_2$” is weaker than the relation that $\Gamma_1$ is linearly equivalent to $\Gamma_2$.

**Assumption 2.5.** Now we assume that $\Lambda \approx \Gamma_0$ for the weight lattice $\Lambda$ and the coroot lattice $\Gamma_0$.

By definition, there exists a real number $c_0 = C[\Lambda; \Gamma_0] > 0$ such that

$$c_0 \lambda_1 \in \Gamma_0, \ldots, c_0 \lambda_d \in \Gamma_0. \quad (2.6)$$

Let us first consider the case when $t/\pi c_0$ is a rational number.

We put $t = \pi c_0^2 \times \frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q > 0$ and $p$ and $q$ are coprime.

**Lemma 2.6.** We define a finite set $\Gamma_0[q]$ by

$$\Gamma_0[q] := \{\ell_1\gamma_1 \vee + \cdots + \ell_d\gamma_d \vee \in \Gamma_0 \mid \ell_1, \ldots, \ell_d \in \mathbb{Z}, 0 \leq \ell_1, \ldots, \ell_d \leq q - 1\}. \quad (2.7)$$

Then, as a distribution on $\mathfrak{a}/\Gamma$, we have

$$\frac{1}{\Vol(\mathfrak{a}/\Gamma)} \sum_{\lambda \in \Lambda} e^{2\sqrt{-1}(\lambda, qH)} = q^{-d} \sum_{\mu \in \Gamma_0[q]} \delta_{\mathfrak{a}/\Gamma}(H - \pi \mu) \left[ H - \pi \mu \right]_{\Gamma'}. \quad (2.8)$$

**Proof.** Using the Poisson’s summation formula, we have

$$\frac{1}{\Vol(\mathfrak{a}/\Gamma)} \sum_{\lambda \in \Lambda} e^{2\sqrt{-1}(\lambda, qH)} = \sum_{\mu \in \Gamma_0} \delta(qH - \pi \mu) = q^{-d} \sum_{\mu \in \Gamma_0} \delta \left( H - \pi \frac{\mu}{q} \right). \quad (2.9)$$

Since $\mu \in \Gamma_0$ is written uniquely as $\mu = q\mu' + \mu'', \mu' \in \Gamma_0$, $\mu'' \in \Gamma_0[q]$, we can rewrite the right-hand side of the above equality as follows.

$$\text{R.H.S. of (2.9)} = q^{-d} \sum_{\mu' \in \Gamma_0} \sum_{\mu'' \in \Gamma_0[q]} \delta \left( H - \pi \frac{\mu'' - \mu'}{q} \right). \quad (2.10)$$

Therefore, as an equality of a distribution on $\mathfrak{a}/\Gamma$, we obtain (2.8). \qed
Similarly as in (2.7), we put
\[ \Lambda[q] := \{ \ell_1 \lambda_1 + \cdots + \ell_d \lambda_d \in \Lambda \mid \ell_1, \ldots, \ell_d \in \mathbb{Z}, 0 \leq \ell_1, \ldots, \ell_d \leq q - 1 \}. \] 
(2.11)

Then each \( \lambda \in \Lambda \) is written uniquely as \( \lambda = q \lambda' + \lambda'' \) for \( \lambda' \in \Lambda, \lambda'' \in \Lambda[q] \). Thus we have
\[ 4 \langle \lambda, \lambda \rangle t = 4 \langle q \lambda' + \lambda'', q \lambda' + \lambda'' \rangle \times \frac{\pi c_0}{2} \times \frac{p}{q} \]
\[ = 2\pi \left\{ pq \langle c_0 \lambda', \lambda' \rangle + 2p \langle c_0 \lambda', \lambda'' \rangle + \frac{p}{q} \langle c_0 \lambda'', \lambda'' \rangle \right\} \]
\[ \equiv \frac{2\pi p}{q} \langle c_0 \lambda'', \lambda'' \rangle \pmod{2\pi \mathbb{Z}}. \]
(2.12)

In the above computation, we note that \( \langle c_0 \lambda', \lambda' \rangle, \langle c_0 \lambda', \lambda'' \rangle \in \mathbb{Z} \) by the assumption that \( \Lambda \approx \Gamma_0 \).

Making use of Lemma 2.6 and (2.12), we have
\[ E_{a/\Gamma}(\frac{\pi c_0}{2} \times \frac{p}{q}, H) = \frac{1}{\text{Vol}(a/\Gamma)} \sum_{\lambda' \in \Lambda} \sum_{\lambda'' \in \Lambda[q]} e^{-\sqrt{-\frac{2\pi p}{q}} \langle c_0 \lambda'', \lambda'' \rangle + 2\sqrt{-1} \langle q \lambda' + \lambda'', H \rangle} \]
\[ = \frac{1}{\text{Vol}(a/\Gamma)} \sum_{\lambda'' \in \Lambda[q]} e^{-\sqrt{-\frac{2\pi p}{q}} \langle c_0 \lambda'', \lambda'' \rangle + 2\sqrt{-1} \langle \lambda'', H \rangle} \sum_{\lambda' \in \Lambda} e^{2\sqrt{-1} \langle \lambda', qH \rangle} \]
\[ = \varphi(H; p, q, \Lambda) \times q^{-d} \sum_{\mu \in \Gamma_0[q]} \delta_{a/\Gamma} \left( \left[ H - \frac{\pi}{q} \mu \right]_{\Gamma} \right) \]
\[ = q^{-d} \sum_{\mu \in \Gamma_0[q]} \varphi \left( \frac{\pi}{q} \mu; p, q, \Lambda \right) \delta_{a/\Gamma} \left( \left[ H - \frac{\pi}{q} \mu \right]_{\Gamma} \right). \]
(2.13)

where we put
\[ \varphi(H; p, q, \Lambda) = \sum_{\lambda'' \in \Lambda[q]} e^{-\sqrt{-\frac{2\pi p}{q}} \langle c_0 \lambda'', \lambda'' \rangle + 2\sqrt{-1} \langle \lambda'', H \rangle}. \]
(2.14)

For \( p, q \in \mathbb{Z} \) with \( p, q \) coprime, for the lattices \( \Lambda \) and \( \Gamma_0 \), and for \( \mu \in \Gamma_0[q] \), let
\[ G(p, q; \mu; \Lambda, \Gamma_0) := \varphi \left( \frac{\pi}{q} \mu; p, q, \Lambda \right). \]
(2.15)

Then we see easily that
\[ G(p, q; \mu; \Lambda, \Gamma_0) = \sum_{\lambda \in \Lambda[q]} e^{\sqrt{-1} \frac{2\pi p}{q} \langle \mu (c_0 \lambda, \lambda) - (\lambda, \mu) \rangle}. \]
(2.16)
In addition, (2.13) is written as
\[
E_{\alpha/\Gamma}\left(\frac{\pi c_0}{2} \times \frac{p}{q}, H\right) = q^{-d} \sum_{\mu \in F_0[q]} G(p, q; \mu; \Lambda, \Gamma_0) \delta_{\alpha/\Gamma}\left(H - \frac{\mu}{q}\right).
\] (2.17)

Let us call the above sum \(G(p, q, \mu; \Lambda, \Gamma_0)\) a Gauss sum associated with \(\Lambda\) and \(\Gamma_0\). In fact, it turns out that \(G(p, q, \mu; \Lambda, \Gamma_0)\) is a generalization of a quadratic Gauss sum appearing in the theory of cyclotomic fields. (See the first example in Section 6.)

Now let us consider when the Gauss sum \(G(p, q, \mu; \Lambda, \Gamma_0)\) vanishes. We start with the following lemma by Turaev.

**Lemma 2.7.** (See Turaev [19, Lemma 1].) Let \(L\) be a finite abelian group and \(Q : L \to \mathbb{Q}/\mathbb{Z}\) be a quadratic form in the sense that the associated pairing \(B_Q : L \times L \to \mathbb{Q}/\mathbb{Z}\) defined by \(B_Q(x, y) := Q(x + y) - Q(x) - Q(y)\) is bilinear. We define a Gauss sum \(G(L, Q)\) associated with \(L\) and \(Q\) by
\[
G(L, Q) := |L|^{-\frac{1}{2}} \sum_{x \in L} e^{2\pi \sqrt{-1} Q(x)}.
\] (2.18)

Let \(K\) be the kernel of the homomorphism \(L \ni x \mapsto B_Q(x, \cdot) \in \text{Hom}(L, \mathbb{Q}/\mathbb{Z})\). If \(Q(K) \neq 0\), then \(G(L, Q) = 0\). If \(Q(K) = 0\), then \(|G(L, Q)| = |K|^\frac{1}{2}\).

We identify \(\Lambda[q]\) with \(\Lambda/q\Lambda\), and define a quadratic form \(Q_\mu : \Lambda/q\Lambda \to \mathbb{Q}/\mathbb{Z}\) by
\[
Q_\mu(\lambda) := \left[\frac{1}{q} \left\{ p(c_0\lambda, \lambda) - \langle \lambda, \mu \rangle \right\} \right]_\mathbb{Z}.
\] (2.19)

Here, as in Notation 2.3, for \(r \in \mathbb{Q}\) we denote by \([r]_\mathbb{Z}\) the equivalence class of \(r\) with respect to \(\mathbb{Z}\). We apply Lemma 2.7 to the case when \(L = \Lambda/q\Lambda \cong \Lambda[q]\) and \(Q = Q_\mu\). Then in this case the corresponding bilinear form \(B_Q^\mu : \Lambda/q\Lambda \times \Lambda/q\Lambda \to \mathbb{Q}/\mathbb{Z}\) is given by \(B_Q^\mu(\lambda, \nu) = \left[\frac{2pc_0}{q}(\lambda, \nu)\right]_\mathbb{Z}\). Using these notations, \(G(p, q; \mu; \Lambda, \Gamma_0)\) is rewritten as
\[
G(p, q; \mu; \Lambda, \Gamma_0) = q^\frac{d}{2} G(\Lambda/q\Lambda, Q_\mu).
\] (2.20)

(Note that \(|L| = |\Lambda/q\Lambda| = q^d\).) Let \(K_{[p, q; \Lambda, \Gamma_0]}\) be the kernel of the homomorphism \(\Lambda/q\Lambda \cong \Lambda[q] \ni \lambda \mapsto B_Q^\mu(\lambda, \cdot)\). Then we have the following lemma.

**Lemma 2.8.** Let
\[
K_{[q; \Lambda, \Gamma_0]} := \{ \lambda \in \Lambda/q\Lambda \cong \Lambda[q] \mid 2c_0(\lambda, \nu) \equiv 0 \text{ (mod } q), \text{ for } \forall \nu \in \Lambda/q\Lambda \}. \] (2.21)

Then \(K_{[p, q; \Lambda, \Gamma_0]} = K_{[q; \Lambda, \Gamma_0]}\) for any \(p\) such that \(p\) and \(q\) are coprime. \(\Box\)
Using Lemma 2.8, let us rewrite the condition \( Q_\mu(K_{\rho,q;A,R_0}) = 0 \) as
\[
p(c_0\lambda, \lambda) - \langle \lambda, \mu \rangle \equiv 0 \pmod{q}, \quad \text{for } \forall \lambda \in K_{\rho,q;A,R_0}.
\] (2.22)

Since \( K_{\rho,q;A,R_0} \ni \lambda \mapsto p\lambda \in K_{\rho,q;A,R_0} \) is a bijection, we have
\[
(2.22) \iff p^2(c_0\lambda, \lambda) - p(\lambda, \mu) \equiv 0 \pmod{q}, \quad \text{for } \forall \lambda \in K_{\rho,q;A,R_0}
\]
\[
\iff \langle c_0 p\lambda, p\lambda \rangle - \langle p\lambda, \mu \rangle \equiv 0 \pmod{q}, \quad \text{for } \forall \lambda \in K_{\rho,q;A,R_0}
\]
\[
\iff \langle c_0\lambda, \lambda \rangle - \langle \lambda, \mu \rangle \equiv 0 \pmod{q}, \quad \text{for } \forall \lambda \in K_{\rho,q;A,R_0}.
\]

Thus the condition \( Q_\mu(K_{\rho,q;A,R_0}) = 0 \) is equivalent to
\[
\langle c_0\lambda, \lambda \rangle - \langle \lambda, \mu \rangle \equiv 0 \pmod{q}, \quad \text{for } \forall \lambda \in K_{\rho,q;A,R_0}.
\] (2.23)

Note that the above condition (2.23) does not depend on \( p \) such that \( p \) and \( q \) are coprime.

Then due to Lemma 2.7, we obtain the following.

**Proposition 2.9.** \( G(p,q,\mu;A,R_0) \neq 0 \) if and only if \( \mu \in \Gamma_0[q] \) satisfies the condition (2.23).

Now let us define a finite subset \( \mathcal{G}_{\rho,q;A,R_0} \) of \( A/\Gamma \cong A \) by
\[
\mathcal{G}_{\rho,q;A,R_0} := \left\{ \left( \frac{\pi}{q} \mu \right)_{\Gamma} \in A/\Gamma \cong A \mid \mu \in \Gamma_0[q] \text{ satisfies the condition (2.23)} \right\}. \quad (2.24)
\]

Since \( K_{\rho,q;A,R_0} \) is \( W \)-invariant, the above set \( \mathcal{G}_{\rho,q;A,R_0} \) is also \( W \)-invariant.

Taking account of (2.17) and Proposition 2.9, we see that the fundamental solution \( E_{\rho,\Gamma}(t,h) \) to the Schrödinger equation (Sch)\( \rho,\Gamma \) at \( t = \frac{\pi c_0}{2} \times \frac{p}{q} \) has a delta type singularity at \( h \in \mathcal{G}_{\rho,q;A,R_0} \).

Next, let us consider the case when \( t/\pi c_0 \) is an irrational number.

Since the Laplacian \( \Delta_{\rho,\Gamma} \) on \( \rho/\Gamma \) is self-adjoint, \( e^{\sqrt{-\Gamma} \Delta_{\rho,\Gamma}} \) is a unitary operator on \( L^2(\rho/\Gamma) \).

In addition, \( e^{\sqrt{-\Gamma} \Delta_{\rho,\Gamma}} \) is given by the convolution operator
\[
L^2(\rho/\Gamma) \ni f(h) \mapsto \int_{h'\in\rho/\Gamma} E_{\rho,\Gamma}(t,h-h') f(h') dh', \quad (2.25)
\]
on the compact group \( \rho/\Gamma \). Thus the above expression (2.25) shows that \( E_{\rho,\Gamma}(t,h) \) has a singularity at some point \( h_0 = [H_0]_{\Gamma} \in \rho/\Gamma \). In fact, if \( E_{\rho,\Gamma}(t,[H]_{\Gamma}) \in C^\infty(\rho/\Gamma) \), then the above operator (2.25) maps \( L^2(\rho/\Gamma) \) into \( C^\infty(\rho/\Gamma) \), which is a contradiction.

Here we have the following lemma.

**Lemma 2.10.**
\[
E_{\rho,\Gamma}(t,[H - 4t\lambda]_{\Gamma}) = e^{4\sqrt{-\Gamma}(\lambda,\lambda)t} - 2\sqrt{2\sqrt{-\Gamma}(\lambda,H)} E_{\rho,\Gamma}(t,h), \quad for \forall \lambda \in \Lambda, \ h = [H]_{\Gamma} \in \rho/\Gamma.
\] (2.26)
The above lemma is easily seen. In fact, it corresponds to the quasi-periodicity formula for theta functions. So we omit the proof.

Using Lemma 2.10, we see that \( E_{a/\Gamma} (t, [H]_{\Gamma}) \) has a singularity at \([H]_{\Gamma} = [H_0 - 4t\lambda]_{\Gamma}\) for any \( \lambda \in \Lambda \). By the assumption, we write \( t = \frac{\pi c_0 \omega}{4} \) for some irrational number \( \omega \). Here we note that \( \pi c_0 \lambda \in \Gamma \) for any \( \lambda \in \Lambda \) by the assumption that \( \Lambda \approx \Gamma_0 \). Since \( \omega \) is irrational, the subset

\[
\{ [H_0 - \omega \pi c_0 \lambda]_{\Gamma} \in a/\Gamma \mid \lambda \in \Lambda \} \tag{2.27}
\]

does not depend on \( p \).

Therefore, we obtain

\[
\text{Supp} \ E_{a/\Gamma} \left( \frac{\pi c_0 \omega}{4}, \cdot \right) = \text{Sing Supp} \ E_{a/\Gamma} \left( \frac{\pi c_0 \omega}{4}, \cdot \right) = a/\Gamma. \tag{2.29}
\]

Summarizing the above argument, we obtain the following.

**Theorem 2.11.** We assume that the weight lattice \( \Lambda \) is homothetic to the coroot lattice \( \Gamma_0 \). (See Definition 2.4 and Assumption 2.5.) Let \( c_0 = C_{[\Lambda:\Gamma_0]} \) the homothetic ratio of \( \Lambda \) to \( \Gamma_0 \). Then the following (I) and (II) hold for the solution \( E_{a/\Gamma} \) to the Schrödinger equation \((\text{Sch})_{a/\Gamma}\).

(I) In the case \( \frac{\pi c_0}{4} \) is a rational number. We put \( t = \frac{\pi c_0}{4} \times \frac{p}{q} \), where \( p, q \in \mathbb{Z} \), \( q > 0 \) and \( p \) and \( q \) are coprime. Then, the fundamental solution \( E_{a/\Gamma} (t, H) \) at \( t = \frac{\pi c_0}{4} \times \frac{p}{q} \) is written as

\[
E_{a/\Gamma} \left( \frac{\pi c_0}{4} \times \frac{p}{q}, H \right) = q^{-d} \sum_{\mu \in \Gamma_0[q]} G(p, q; \mu; \Lambda, \Gamma_0) \delta_{a/\Gamma} \left( \left[ H - \frac{\pi}{q} \mu \right]_{\Gamma} \right), \tag{2.30}
\]

where \( G(p, q; \mu; \Lambda, \Gamma_0) \) is a generalized Gauss sum given by (2.16). Moreover, we have

\[
\text{Supp} \ E_{a/\Gamma} \left( \frac{\pi c_0}{4} \times \frac{p}{q}, \cdot \right) = G_{[q; \Lambda, \Gamma_0]}, \tag{2.31}
\]

where \( G_{[q; \Lambda, \Gamma_0]} \) is the finite set defined by (2.24). In particular, \( \text{Supp} \ E_{a/\Gamma} \left( \frac{\pi c_0}{4} \times \frac{p}{q}, \cdot \right) \) does not depend on \( p \).

(II) In the case \( \frac{\pi c_0}{4} \) is an irrational number. Then we have

\[
\text{Supp} \ E_{a/\Gamma} (t, \cdot) = \text{Sing Supp} \ E_{a/\Gamma} (t, \cdot) = a/\Gamma. \tag{2.32}
\]

**Remark 2.12.** Assumption 2.5 is essential in the above theorem. In fact, if we remove Assumption 2.5, then such a strange phenomenon as in Theorem 2.11 does not occur. Let us consider the following example. We take a lattice \( \Gamma_0 = \mathbb{Z} \gamma_1^\vee \oplus \mathbb{Z} \gamma_2^\vee \subset \mathbb{R}^2 \) and its dual lattice...
\[ \Lambda := \varGamma_0^* = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2 \subset \mathbb{R}^2 \] such that \( \mathbb{R}\lambda_j \cap \varGamma_0 = \{0\} \) \((j = 1, 2)\). Let \( \varGamma := \pi \varGamma_0 \). Then \( \Lambda \) is no longer homothetic to \( \varGamma \).

Let \( \varGamma := \pi \varGamma_0 \). Then \( \Lambda \) is no longer homothetic to \( \varGamma \).

Let \( E_{\mathbb{R}^2/\varGamma}(t, h) \) be the fundamental solution to the Schrödinger equation

\[
\begin{align*}
\sqrt{-1} \partial_t \psi + \Delta_{\mathbb{R}^2/\varGamma} \psi &= 0, \quad t \in \mathbb{R}, \\
\psi(0, h) &= \delta_{\mathbb{R}^2/\varGamma}(h), \quad h \in \mathbb{R}^2/\varGamma.
\end{align*}
\]

Then taking account of the proof of the above theorem, we see that

\[
\text{Sing Supp } E_{\mathbb{R}^2/\varGamma}(t, \cdot) = \text{Supp } E_{\mathbb{R}^2/\varGamma}(t, \cdot) = \mathbb{R}^2/\varGamma, \quad \text{for any } t \in \mathbb{R} \setminus \{0\}.
\]

Remark 2.13. If we take into consideration the proof of Theorem 2.11(II) more carefully, we see that \( E_{a/\varGamma}(t, \cdot) \) is not continuous anywhere in \( a/\varGamma \) in the sense that for any open subset \( U \) of \( a/\varGamma \) there exists a point \( h_0 \in U \) such that \( E_{a/\varGamma}(t, \cdot) \) is not continuous at \( h = h_0 \).

3. Shift operators of Heckman and Opdam

In this section, we give a summary of Heckman–Opdam theory on shift operators. We use the framework in the book [6] by Heckman and Schlichtkrull. (See also Heckman [5] and Opdam [14–16] for the details of their theory.) As in Section 2, we assume the even multiplicity condition (EMC) on the restricted root system \( R \). Before going into the summary, we note that the multiplicity function \( k = (k_{\alpha}) \) in [6] is half of our multiplicity function and that our restricted root is half of the restricted root in [6].

Notation 3.1. For a multiplicity function \( m = (m_{\alpha})_{\alpha \in R} \), we put

\[
\Omega_a^{(m)}(H) := \prod_{\alpha \in R_+} \left( \sin \langle \alpha, H \rangle \right)^{m_{\alpha}}, \quad H \in a,
\]

\[
\rho(m) := \frac{1}{2} \sum_{\alpha \in R_+} m_{\alpha}\alpha.
\]

Since all the \( m_{\alpha} \)'s are even, \( \Omega_a^{(m)} \) is \( W \)-invariant and thus well defined as a \( W \)-invariant function on the maximal torus \( a/\varGamma \cong A \). We sometimes write \( \Omega_a^{(m)}(h), h \in a/\varGamma \).

Now let us define a differential operator \( \mathcal{L}(m) \) on \( a/\varGamma \cong A \) by

\[
\mathcal{L}(m) := \Delta_{a/\varGamma} + \sum_{\alpha \in R_+} m_{\alpha} \cot \langle \alpha, H \rangle \partial_{\alpha} - \langle \rho(m), \rho(m) \rangle,
\]

where \( \partial_{\alpha} = \langle \alpha, \text{grad} \rangle \).

Then due to Heckman and Opdam, the following theorem holds.

Theorem 3.2. There exist two \( W \)-invariant differential operators \( D_{m}^{(+)} \) and \( D_{m}^{(-)} \) on \( a/\varGamma \) satisfying the following.
\[ L_{(m)} D_{(m)}^{(+)} = D_{(m)}^{(+)} \Delta_{a/\Gamma}, \quad (3.4) \]
\[ D_{(m)}^{(-)} L_{(m)} = \Delta_{a/\Gamma} D_{(m)}^{(-)}. \quad (3.5) \]

For the above theorem, see Theorem 3.4.3 of [6]. See also Heckman [5, Theorem 3.15] and Opdam [15, Theorem 3.6].

**Remark 3.3.**

(1) The above operators \( D_{(m)}^{(\pm)} \) are called shift operators with shift \( m \) and \(-m\) respectively. Note that \( \Delta_{a/\Gamma} \) coincides with the operator \( L_{(0)} \) corresponding to the multiplicity function \( 0 \).

(2) These operators \( D_{(m)}^{(\pm)} \) are given respectively by \( D_{(m)}^{(\pm)} = c G_{\pm}(m) \) for some constant \( c \). Here the operators \( G_{+}(m) \) and \( G_{-}(m) \) are called a raising operator with shift \( m \) and a lowering operator with shift \(-m\) respectively. We will determine the above constant \( c \) later. (For the construction of \( G_{\pm}(m) \) and their detailed properties, see Heckman and Schlichtkrull [6, Part I, Chapters 1 and 3].)

(3) Taking account of Remark 3.3.8 in [6], we see that the operators \( G_{+}(m) \) and \( G_{-}(m) \) are written in the form

\[ G_{+}(m) = \text{const.} \left\{ \Omega_{\frac{1}{2}}(m) \right\}^{-1} \prod_{\alpha \in R_{+}} (\partial_{\alpha'})^{\frac{1}{2}m_{\alpha}} + \text{lower order terms}, \quad (3.6) \]
\[ G_{-}(m) = \text{const.} \Omega_{\frac{1}{2}}(m) \prod_{\alpha \in R_{+}} (\partial_{\alpha'})^{\frac{1}{2}m_{\alpha}} + \text{lower order terms}. \quad (3.7) \]

Here we remark that the above operators are elements of the Weyl algebra \( \mathcal{A} \) of polynomial differential operators on the affine space \( (a/\Gamma)^W \) and thus their coefficients do not have singularities in these \( W \)-invariant coordinates. This fact is due to [15, Lemma 3.5]. (See also the proof of Corollary 3.6.5 of [6].)

Next, following [6], we introduce Jacobi polynomials \( P_{(\lambda, m)} \). Let \( \Lambda_{+} := \{ \ell_1 \lambda_1 + \cdots + \ell_d \lambda_d \in \Lambda \mid \ell_1, \ldots, \ell_d \in \mathbb{Z}, \ \ell_1, \ldots, \ell_d \geq 0 \} \).

For \( \lambda \in \Lambda_{+} \), we put

\[ M_{\lambda}(H) := \sum_{\mu \in W_{\lambda}} e^{\sqrt{-1}(\mu, H)}, \quad H \in a. \quad (3.9) \]

Then we see easily that \( M_{\lambda} \) is \( \Gamma \)-periodic and \( W \)-invariant, and therefore well defined as a \( W \)-invariant function on \( a/\Gamma \).

**Definition 3.4.** For \( \lambda \in \Lambda_{+} \) and a multiplicity function \( m = (m_{\alpha})_{\alpha \in R} \), we define a Jacobi polynomial \( P_{(\lambda, m)} \) on \( a/\Gamma \) as follows.
\[ P(\lambda, m) = \sum_{\mu \in \Lambda_+, \mu \leq \lambda} c_{\lambda \mu}(m) M_{\mu}(\lambda), \quad c_{\lambda \lambda}(m) = 1, \quad (3.10) \]

\[ (P(\lambda, m), M_{\mu}) = 0, \quad \forall \mu \in \Lambda_+, \mu < \lambda. \quad (3.11) \]

Here in (3.11) \((\cdot, \cdot)_m\) denotes the \(L^2\) inner product on \(a/\Gamma\) defined by

\[ (F, G)_m := \int_{h \in a/\Gamma} F(h) \overline{G(h)} \Omega_{\alpha}(h) dh, \quad F, G \in C(a/\Gamma). \quad (3.12) \]

Since the Jacobi polynomial \(P(\lambda, m)\) is a function on \(a/\Gamma \cong A\), we sometimes write \(P(\lambda, m)(h), h \in a/\Gamma\) or \(P(\lambda, m)(a), a \in A\).

**Remark 3.5.** As is well known, the Jacobi polynomial \(P(\lambda, m)\) multiplied by some constant coincides with the restriction of the zonal spherical function on \(U/K\) associated with weight \(\lambda\) to the maximal torus \(A\). (See, for example, Takeuchi [18].)

The shift operators \(D_m^{(+)}\) and \(D_m^{(-)}\) in Theorem 3.2 map each Jacobi polynomial to a different Jacobi polynomial. More precisely, we have

**Theorem 3.6.** There exist polynomials \(\eta^{(+)}(m, \lambda)\) and \(\eta^{(-)}(m, \lambda)\) of \(\lambda\) such that

\[ D_m^{(+)} P(\lambda, m) = \eta^{(+)}(m, \lambda) P(\lambda + \rho(m) - 2m), \quad (3.13) \]
\[ D_m^{(-)} P(\lambda, m) = \eta^{(-)}(m, \lambda) P(\lambda + \rho(m) + m)(0) = \eta^{(-)}(m, \lambda) M_{\lambda + \rho(m)}. \quad (3.14) \]

Moreover, \(\eta^{(-)}(m, \lambda) \neq 0\) for \(\lambda \in \Lambda_+\).

**Proof.** Combining (EMC) with Corollary 3.2.4-5 and Theorem 3.3.7 in [6], we obtain (3.13) and (3.14). The last statement follows from Remark 3.3.8 of [6]. \(\square\)

In addition, the lowering operator \(G_-(m)\) satisfies the following.

**Theorem 3.7.** (See [6, Corollary 3.6.5].) For a \(W\)-invariant smooth function \(F\) on \(a/\Gamma\), we have

\[ G_-(m)(F)(0) = G_-(m)(1)(0) \cdot F(0). \quad (3.15) \]

We take the constant \(c\) in Remark 3.3(2) so that \(cG_- (m)(1)(0) = 1\), namely, we take

\[ D_m^{(-)} = \left[ G_-(m)(1)(0) \right]^{-1} G_-(m). \quad (3.16) \]

We note that \(G_-(m)(1)(0) \neq 0\). In fact, we see easily that \(\{G_-(m)P(\lambda, m)\}(0) = \eta^{(-)}(m, \lambda)|W|\). Since \(\eta^{(-)}(m, \lambda) \neq 0\) for \(\lambda \in \Lambda_+\), by taking \(F = P(\lambda, m)\) in Theorem 3.7, we have \(0 \neq \{G_-(m)P(\lambda, m)\}(0) = G_-(m)(1)(0) \cdot P(\lambda, m)(0)\).
Then by the above theorem, we obtain

**Corollary 3.8.** For a $W$-invariant smooth function $F$ on $\mathfrak{a}/\Gamma$, we have

$$\{ D^{(-)}_m F \} (0) = F(0). \quad (3.17)$$

Finally, we prove some important property of the differential operator $D^{(-)}_m$.

**Proposition 3.9.** Let $q$ be a positive integer. If $\lambda \in \Lambda_+$, $2q\lambda - \rho(m) \in \Lambda_+$, and $h \in \{ \frac{\pi \mu}{q} \Gamma \in \mathfrak{a}/\Gamma \mid \mu \in \Gamma_0[q] \}$, then we have

$$\left( D^{(-)}_m P_{(2q\lambda - \rho(m), m)} \right)(h) = \eta^{(-)}(m, 2q\lambda)|W|. \quad (3.18)$$

**Proof.** Let us write $h = \left[ \frac{\pi \mu}{q} \right] \Gamma$ for some $\mu \in \Gamma_0[q]$. By the formula (3.14) in Theorem 3.6, we have

$$\left( D^{(-)}_m P_{(2q\lambda - \rho(m), m)} \right)(h) = \eta^{(-)}(m, 2q\lambda) M_{2q\lambda} \left( \left[ \frac{\pi \mu}{q} \right] \Gamma \right)$$

$$= \eta^{(-)}(m, 2q\lambda) \sum_{w \in W} e^{\sqrt{-1} \langle 2qw \cdot \lambda, \frac{\pi \mu}{q} \rangle}$$

$$= \eta^{(-)}(m, 2q\lambda) \sum_{w \in W} e^{2\pi \sqrt{-1} \langle w \cdot \lambda, \mu \rangle}$$

$$= \eta^{(-)}(m, 2q\lambda) \sum_{w \in W} 1 = \eta^{(-)}(m, 2q\lambda)|W|. \quad (3.19)$$

In the above computation, we used the fact that $\langle w \cdot \lambda, \mu \rangle \in \mathbb{Z}$. \hfill \Box

**Remark 3.10.** Here we note that the detailed properties of Jacobi polynomials (Theorems 3.6, 3.7, Corollary 3.8, and Proposition 3.9) are also due to Opdam [15,16].

4. **Schrödinger equation on $M = U/K$**

In this section, we construct the fundamental solution to the Schrödinger equation on a compact symmetric space with the even multiplicity condition and prove the support theorem.

We start with the following theorem.

**Theorem 4.1.** (See Helgason [8, Chapter II, Proposition 3.11], Takeuchi [18, Theorem 10.4].) The radial part $\text{rad} (\Delta_{U/K})$ of the Laplacian $\Delta_{U/K}$ on $U/K$ is given by

$$\text{rad} (\Delta_{U/K}) = \Delta_{\mathfrak{a}/\Gamma} + \sum_{\alpha \in R_+} m_\alpha \cot(\alpha, H) \partial_\alpha, \quad \text{for } H \in \mathfrak{a}. \quad (4.1)$$
By (3.3) and Theorem 3.2, we see that

**Corollary 4.2.** The operator \( \text{rad}(\Delta_{U/K}) \) satisfies the following.

\[
\text{rad}(\Delta_{U/K}) = L_{(m)} + [\rho(m), \rho(m)], \tag{4.2}
\]

\[
D_{m}^{-} \text{rad}(\Delta_{U/K}) = (\Delta_{\mathfrak{a}/\Gamma} + [\rho(m), \rho(m)])D_{m}^{-}. \tag{4.3}
\]

Next, we introduce an operator \( \mathcal{M} : C^\infty(U/K) \to C^\infty(\mathfrak{a}/\Gamma) \) as follows

\[
\mathcal{M} f(H) = \int_{k \in K} f(k \exp(H)K) \text{dk}, \quad f \in C^\infty(U/K), \quad H \in \mathfrak{a}. \tag{4.4}
\]

Here in the above \( \text{dk} \) is the normalized canonical measure on \( K \). Note that the above operator \( \mathcal{M} \) maps \( C^\infty(U/K) \) to \( C^\infty(\mathfrak{a}/\Gamma)^W \) the space of \( W \)-invariant smooth functions on \( \mathfrak{a}/\Gamma \).

Then we have

**Lemma 4.3.**

\[
\mathcal{M} \Delta_{U/K} f(H) = \{\text{rad}(\Delta_{U/K}) \mathcal{M} f\}(H), \quad f \in \mathcal{M}, \quad H \in \mathfrak{a}. \tag{4.5}
\]

**Proof.** The above equality (4.5) follows easily from Weyl integration formula. \( \square \)

**Remark 4.4.** By Theorem 4.1, we see that each coefficient function of the first order term of \( \text{rad}(\Delta_{U/K}) \) has singularity on some Weyl wall. So \( \text{rad}(\Delta_{U/K})F \) is not necessarily well defined for all \( F \in C^\infty(\mathfrak{a}/\Gamma) \) on each Weyl wall. However, Lemma 4.3 shows that if \( F \in C^\infty(\mathfrak{a}/\Gamma) \) belongs to the image of the operator \( \mathcal{M} \), then \( \text{rad}(\Delta_{U/K})F \) is extended to a smooth function on \( \mathfrak{a}/\Gamma \).

As we stated in Introduction, let \( E_M(t,x) \) be the fundamental solution to the Schrödinger equation \((\text{Sch})_M\)

\[
(\text{Sch})_M \begin{cases} 
\sqrt{-1} \partial_t \psi + \Delta_M \psi = 0, & t \in \mathbb{R}, \quad x \in M, \\
\psi(0, x) = \delta_o(x),
\end{cases} \tag{4.6}
\]

where \( o = eK \in M = U/K \).

As a distribution on \( M \) with parameter \( t \), we write \( E_M(t,x) \) as

\[
E_M(t)[f] = \int_{x \in M} E_M(t,x) f(x) \text{dx}, \quad \text{for } f \in C^\infty(M). \tag{4.7}
\]

Then our first main theorem is stated as follows.

**Theorem 4.5.** Under the assumption that \( M = U/K \) satisfies the even multiplicity condition (EMC), the fundamental solution \( E_M(t) \), i.e., the solution to \((\text{Sch})_M\) is given by

\[
E_M(t)[f] = e^{\sqrt{-1}(\rho(m), \rho(m)) t} E_{\mathfrak{a}/\Gamma}(t) [D_{m}^{-} \mathcal{M} f], \quad \text{for } f \in C^\infty(M). \tag{4.8}
\]
Proof. Let us first prove that \((\sqrt{-1}\partial_t + \Delta_M)E_M(t) = 0\).

We take an arbitrary function \(f \in C^\infty(M)\) and fix it. By Corollary 4.2 and Lemma 4.3, we have

\[
\Delta_M E_M(t)[f] = E_M(t)[\Delta_M f]
\]

\[
= e^{\sqrt{-1}(\rho(m),\rho(m))t} E_{a/\Gamma}(t) [D_m^{(-)} M \Delta_M f]
\]

\[
= e^{\sqrt{-1}(\rho(m),\rho(m))t} E_{a/\Gamma}(t) [D_m^{(-)} \text{rad}(\Delta_M) M f]
\]

\[
= e^{\sqrt{-1}(\rho(m),\rho(m))t} E_{a/\Gamma}(t) [\Delta_{a/\Gamma} + (\rho(m), \rho(m))] D_m^{(-)} M f]
\]

\[
= -\sqrt{-1}\partial_t \{ e^{\sqrt{-1}(\rho(m),\rho(m))t} E_{a/\Gamma}(t) [D_m^{(-)} M f] \}
\]

\[
= -\sqrt{-1}\partial_t E_M(t)[f].
\]

In the above computation, we used the fact that \((\sqrt{-1}\partial_t + \Delta_{a/\Gamma}) E_{a/\Gamma}(t) = 0\). We also note that this computation uses the fact that \(D_m^{(-)}\) maps \(C^\infty(a/\Gamma)\) into itself and the fact that \(M\) maps \(C^\infty(U/K)\) to \(C^\infty(a/\Gamma)\).

Next, we prove the initial condition \(E_M(0) = \delta_\rho\). For \(f \in C^\infty(M)\), \(Mf\) is a \(W\)-invariant smooth function on \(a/\Gamma\). Thus by Corollary 3.8 and the initial condition of \(E_{a/\Gamma}(t)\),

\[
E_M(0)[f] = E_{a/\Gamma}(0)[D_m^{(-)} M f]
\]

\[
= (D_m^{(-)} M f)(0)
\]

\[
= Mf(0)
\]

\[
= \int_{k \in K} f(kK) dk = f(\rho),
\]

which proves the assertion. □

Next, let us go into the second main theorem.

Theorem 4.6. In addition to the even multiplicity condition (EMC), we assume that the weight lattice \(\Lambda\) is homothetic to the coroot lattice \(\Gamma_0\) (Assumption 2.5). Let \(c_0\) be the homothetic ratio of \(\Lambda\) to \(\Gamma_0\). (See Definition 2.4.) Then the support of the fundamental solution \(E_M(t)\) is given as follows.

(I) In the case when \(t/\pi c_0\) is a rational number. Let us put \(\frac{2t}{\pi c_0} = \frac{p}{q} \in \mathbb{Q}\), where \(p, q \in \mathbb{Z}, q > 0\) and \(p\) and \(q\) are coprime. Then, as a distribution on \(M\), the support of \(E_M(\frac{\pi c_0 p}{2q})\) is given by

\[
\text{Supp} E_M \left( \frac{\pi c_0 p}{2q} \right) = \{ k \cdot a \in M \mid k \in K, a \in \mathcal{G}_{[q;\Lambda,\Gamma_0]} \subseteq A \},
\]

where \(\mathcal{G}_{[q;\Lambda,\Gamma_0]}\) is a finite subset of \(A \cong a/\Gamma\) defined by (2.24).
(II) In the case when $t/\pi c_0$ is an irrational number. As a distribution on $M$, the support of $E_M(t)$ is given by

$$\text{Supp } E_M(t) = \text{Sing Supp } E_M(t) = M.$$ 

Here $\text{Sing Supp } E_M(t)$ denotes the singular support of $E_M(t)$.

**Proof.** Theorem 4.6(II) follows easily from Theorem 2.11(II). So we have only to prove (I). We define a distribution $\widetilde{E}_{a/\Gamma}(t)$ on $W \setminus a/\Gamma$ by

$$\widetilde{E}_{a/\Gamma}(t)[F] := E_{a/\Gamma}(t)\left[D_{m}^{(-)} F\right], \quad F \in C^\infty(a/\Gamma)^W.$$ (4.9)

Then by Theorem 2.11(I), $\text{Supp } \widetilde{E}_{a/\Gamma}(\pi c_0 p^2 q) \subset G[\{q; \Lambda, \Gamma_0\}]$. We will prove the opposite inclusion. Let us take any element $h_0 = [\frac{p\mu_0}{q}]_r \in G[\{q; \Lambda, \Gamma_0\}]$ and fix it. Let us also take a test function $\phi$ on $a/\Gamma$ and sufficiently small open neighborhoods $\mathcal{U}_1, \mathcal{U}_2$ of $h_0$ such that

$$\mathcal{U}_1 \subset \mathcal{U}_2, \quad \mathcal{U}_2 \cap G[\{q; \Lambda, \Gamma_0\}] = \{h_0\},$$ (4.10)

Next, let

$$\phi^W(h) := \sum_{w \in W} \phi(w \cdot h),$$

$$\mathcal{U}_j^W := \{w \cdot h \in a/\Gamma \mid w \in W, \ h \in \mathcal{U}_j\} \quad (j = 1, 2).$$ (4.11)

Then by definition, $\phi^W$ is $W$-invariant, and in addition we have

$$\mathcal{U}_1^W \subset \mathcal{U}_2^W, \quad \mathcal{U}_2^W \cap G[\{q; \Lambda, \Gamma_0\}] = \{w \cdot h_0 \in a/\Gamma \mid w \in W\},$$

$$\text{Supp } \phi^W \subset \mathcal{U}_2^W, \quad \phi^W(h) \equiv 1, \quad \text{for } h \in \mathcal{U}_1^W.$$ (4.12)

Therefore, by Proposition 3.9, we have

$$(D_{m}^{(-)} P_{(2q\lambda - \rho(m), m)} \phi^W)(h_0) = \eta^{(-)}(m, 2q\lambda) |W||W_{h_0}|,$$ (4.13)

where we put $W_{h_0} = \{w \in W \mid w \cdot h_0 = h_0\}$. Here we note that the Gauss sum $G(p, q, \mu; \Lambda, \Gamma_0)$ defined by (2.16) is $W$-invariant with respect to $\mu$, that is, $G(p, q, w \cdot \mu; \Lambda, \Gamma_0) = G(p, q, \mu; \Lambda, \Gamma_0)$ for $\forall w \in W$.

By making use of (4.12), (4.13), Theorem 2.11(I), and $W$-invariance of $D_{m}^{(-)} P_{(2q\lambda - \rho(m), m)} \phi^W$, we have

$$\widetilde{E}_{a/\Gamma}\left(\frac{\pi c_0 p}{2q}\right)[P_{(2q\lambda - \rho(m), m)} \phi^W]$$

$$= E_{a/\Gamma}\left(\frac{\pi c_0 p}{2q}\right)\left[D_{m}^{(-)} P_{(2q\lambda - \rho(m), m)} \phi^W\right]$$
\[ = q^{-d} \sum_{w \in W} G(p, q; w \cdot \mu_0 \setminus \Lambda, \Gamma_0) \left( D_m \left( P_{(2q\lambda - \rho(m),m)} \phi^W \right) (w \cdot h_0) \right) \]

\[ = q^{-d} |W| G(p, q; \mu_0 \setminus \Lambda, \Gamma_0) \left( D_m \left( P_{(2q\lambda - \rho(m),m)} \phi^W \right) (h_0) \right) \]

\[ = q^{-d} |W|^2 |W_{h_0}| G(p, q; \mu_0 \setminus \Lambda, \Gamma_0) \eta(\gamma) \left( m, 2q\lambda \right). \quad (4.14) \]

Since \( G(p, q; \mu_0 \setminus \Lambda, \Gamma_0) \neq 0 \) for \( h_0 = \left[ \frac{\pi \mu_0}{q} \right] \Gamma \in G[\frac{q}{q}; \Lambda, \Gamma_0] \),

\[ e_{\frac{\pi c_0 p}{2q}} \left( \frac{\pi c_0 p}{2q} \right) \left( P_{(2q\lambda - \rho(m),m)} \phi^W \right) \neq 0, \quad (4.15) \]

for the above \( W \)-invariant test function \( P_{(2q\lambda - \rho(m),m)} \phi^W \), which means that

\[ h_0 \in \text{Supp} \ e_{\frac{\pi c_0 p}{2q}} \left( \frac{\pi c_0 p}{2q} \right). \quad (4.16) \]

Thus we have \( \text{Supp} \ e_{\frac{\pi c_0 p}{2q}} \left( \frac{\pi c_0 p}{2q} \right) \supset G[\frac{q}{q}; \Lambda, \Gamma_0] \).

The fundamental solution \( E_M(t) \) is given by the composition of the distribution

\[ e^{\sqrt{-1} \langle \rho(m), \rho(m) \rangle t} \ e_{\frac{\pi c_0 p}{2q}} \left( \frac{\pi c_0 p}{2q} \right) \]

and the mean value operator \( \mathcal{M} \), namely, \( E_M(t)[f] = e^{\sqrt{-1} \langle \rho(m), \rho(m) \rangle t} \ e_{\frac{\pi c_0 p}{2q}} \left( \frac{\pi c_0 p}{2q} \right) \mathcal{M} f \). Therefore, we obtain the conclusion. \( \Box \)

Taking account of Theorem 2.11 and the proof of Theorem 4.6, we see that a free particle on \( M \) recognizes a generalized Gauss sum \( G(p, q, \mu; \Lambda, \Gamma_0) \) defined by (2.15) and (2.16). More precisely, we have the following.

**Corollary 4.7.** We fix a Weyl chamber \( A^+ \). Let \( h_0 = \left[ \frac{\pi \mu_0}{q} \right] \Gamma \in G[\frac{q}{q}; \Lambda, \Gamma_0] \cap A^+ \) and let \( U \) be a sufficiently small neighborhood of \( h_0 \) in \( A^+ \) such that \( U \cap G[\frac{q}{q}; \Lambda, \Gamma_0] = \{ h_0 \} \). Then we have

\[ E_M \left( \frac{\pi c_0 p}{2q} \right) [f] = q^{-d} |W|^2 G(p, q, \mu_0 \setminus \Lambda, \Gamma_0) e^{\sqrt{-1} \langle \rho(m), \rho(m) \rangle \frac{\pi c_0 p}{2q}} \left( D_m \left( \mathcal{M} f \right) \right) (h_0), \]

for \( f \in C^\infty_0(K \cdot U) \), \( \quad (4.17) \)

where \( K \cdot U \) denotes the tubular neighborhood of the \( K \)-orbit of \( \{ h_0 \} \) defined by \( \{ k \cdot a \in M \mid k \in K, a \in U \} \).

Here we have two remarks.

**Remark 4.8.** Due to (4.17), a free particle on \( M \) also recognizes the rank of \( M \) and the dimension of \( M \). In fact, if we put \( N := \text{dim} M \), then the order of \( D_m \left( \mathcal{M} f \right) \) is given by \( \frac{1}{2} \sum_{\alpha \in R_+} m_\alpha = \frac{1}{2} (N - d) \). (We put \( d = \text{rank} M \) in Section 2.)
Remark 4.9. Due to Remark 3.3(2) and (3), the differential operator $D_m^{(-)}$ is written in the form

$$D_m^{(-)} = \text{const.} \Omega^{\frac{1}{2m}}(H) \prod_{\alpha \in \mathbb{R}^+} (\partial_\alpha \vee)^{\frac{1}{2}m_\alpha} + \text{lower order terms.} \tag{4.18}$$

Moreover, as a result of Theorem 4.6, we obtain the following.

Corollary 4.10. If $M = U/K$ is one of the symmetric spaces in the list of Remark 1.2, then the same statement as in Theorem 4.6 holds.

Proof. Each symmetric space in the list satisfies the even multiplicity condition (EMC). In fact, multiplicities for symmetric spaces in the list are given as follows. (i) In the case $M = S^{2m+1}$, $m_\alpha = 2m$. (ii) In the case $M = SU(2m)/Sp(m)$, $m_\alpha = 4$ for any $\alpha \in \mathbb{R}$. (iii) In the case $M = E_6/F_4$, $m_\alpha = 8$ for any $\alpha \in \mathbb{R}$. (iv) In the case $M = (U \times U)/\Delta U \cong U$ for a compact simple Lie group $U$, $m_\alpha = 2$ for any $\alpha \in \mathbb{R}$. (See Table VI in Chapter X of Helgason [7].) It remains to prove that the weight lattice $\Lambda$ is homothetic to the coroot lattice $\Gamma_0$. Since $M$ is one of the symmetric spaces in the list of Remark 1.2, the Dynkin diagram of the corresponding restricted root system is connected. So for any two simple roots $\gamma$ and $\tilde{\gamma}$, there exists a sequence of simple roots $\{\gamma(i)\}_{i=1}^r$ such that $\gamma(1) = \gamma$, $\gamma(r) = \tilde{\gamma}$ and

$$\frac{\langle \gamma(i), \gamma(i+1) \rangle_{\gamma}}{\langle \gamma(i), \gamma(i) \rangle_{\gamma}} \in \mathbb{Q} \setminus \{0\}, \quad \text{and} \quad \frac{\langle \gamma(i), \gamma(i+1) \rangle_{\gamma}}{\langle \gamma(i+1), \gamma(i+1) \rangle_{\gamma}} \in \mathbb{Q} \setminus \{0\} \quad (1 \leq i \leq r - 1). \tag{4.19}$$

Let $B = (b_{ij})$ be the $d \times d$ matrix defined by $b_{ij} := \langle \gamma_i, \gamma_j \rangle$. Then by (4.19) we have $bB \in GL(d, \mathbb{Q})$ for $b := \langle \gamma_1, \gamma_1 \rangle \in \mathbb{R} \setminus \{0\}$. Let us write $\lambda_i = \sum_{k=1}^d c_{ik} \gamma_k$ ($1 \leq i \leq d$), and put $C = (c_{ik})$. Since $\langle \lambda_i, \gamma_j \rangle = \delta_{ij}$, we have $CB = I$. Thus $b^{-1}C \in GL(d, \mathbb{Q})$, that is, $Nb^{-1}C \in GL(d, \mathbb{Z})$ for some integer $N$. We put $c := Nb^{-1}$. Then $cc_{ij} \in \mathbb{Z}$ for any $i, j = 1, \ldots, d$, which means that

$$c\lambda_i = \sum_{k=1}^d cc_{ik} \gamma_k \in \Gamma_0, \quad \text{for } \forall i \ (1 \leq i \leq d). \tag{4.20}$$

5. Application

In this section, we apply Theorem 4.6 to the Cauchy problem for the Schrödinger equation with smooth initial data.

Let us take any point $y \in M$ and fix it. Next, let us take $u \in U$ such that $u \cdot o = y$. Since $\Delta_M$ is $U$-invariant, $\psi(t, x) = E_M(t, u^{-1} \cdot x)$ satisfies

$$\begin{cases} \sqrt{-1} \partial_t \psi + \Delta_M \psi = 0, & t \in \mathbb{R}, \\ \psi(0, x) = \delta_y(x), & x \in M. \end{cases} \tag{5.1}$$

Here $\delta_y(x)$ denotes the Dirac’s delta function with singularity at $y \in M$. 
Because of the uniqueness of the fundamental solution, the solution to (5.1) does not depend on the choice of \( u \). So let us write

\[
E_M(t, x, y) := E_M(t, u^{-1} \cdot x).
\]  

Then \( E_M(t, x, y) \) is well defined as a distribution on \( M \times M \) for each \( t \in \mathbb{R} \).

For any \( f \in C^\infty(M) \), let

\[
\psi(t, x) := \int_{y \in M} E_M(t, x, y) f(y) \, dy.
\]  

We note that the integral of the R.H.S. of (5.3) is well defined in the sense of distribution and becomes smooth as a function of \( x \).

Then we have the following.

**Theorem 5.1.** The solution to the Cauchy problem for the Schrödinger equation

\[
\begin{aligned}
\sqrt{-1} \partial_t \psi + \Delta_M \psi &= 0, \quad t \in \mathbb{R}, \\
\psi(0, x) &= f(x), \quad x \in M
\end{aligned}
\]  

is given uniquely by (5.3).

**Proof.** The above theorem easily follows from (5.1). \( \square \)

From now on, we assume that the initial data \( f \in C^\infty(M) \) of (5.4) satisfies the following condition.

**Assumption 5.2.** There exist a point \( x_0 \in M \) and a sufficiently small positive number \( \varepsilon \) such that

\[
\text{Supp } f \subset \{ x \in M \mid \text{dist}(x, x_0) < \varepsilon \}.  
\]  

As a consequence of Theorem 4.6(I), we obtain the following.

**Theorem 5.3.** We assume the same conditions as in Theorem 4.6 on \( M \). Let \( c_0 \) be the homothetic ratio of the weight lattice \( \Lambda \) to the coroot lattice \( \Gamma_0 \). In addition, we put

\[
G^{(\varepsilon)}_{[q; \Lambda, \Gamma_0]} := \{ a \in A \mid \text{dist}(a, G_{[q; \Lambda, \Gamma_0]}) < \varepsilon \}. 
\]  

Then under Assumption 5.2 the solution \( \psi \) to the Schrödinger equation (5.4) at \( t = \frac{\pi c_0 q}{2 p} \) satisfies

\[
\text{Supp } \psi \left( \frac{\pi c_0 q}{2 p}, \cdot \right) \subset \{ u_0 k \cdot a \in M \mid k \in K, \ a \in G^{(\varepsilon)}_{[q; \Lambda, \Gamma_0]} \},
\]  

where we take \( u_0 \in U \) such that \( u_0 \cdot a = x_0 \).
The meaning of Theorem 5.3 in quantum physics is that if a free particle is kept in a sufficiently small domain at \( t = 0 \) then such a free particle exists in the union of a finite number of very narrow zonal sets at a *rational* time.

6. Examples

In this section, we deal with two examples; one is the odd dimensional sphere \( S^{2m+1} \cong SO(2m+2)/SO(2m+1) \), and the other is \( SU(6)/Sp(3) \). For the above two examples, we determine the support of the fundamental solution to the Schrödinger equation at \( t = \frac{\pi c_0}{3} \times \frac{p}{q} \). Here \( p, q \in \mathbb{Z}, q > 0, \) and \( p \) and \( q \) are coprime. For the definition of the constant \( c_0 \), see Definition 2.4.

6.1. In the case \( U/K = SO(2m+2)/SO(2m+1) \cong S^{2m+1} \)

In this case, we identify \( K = SO(2m+1) \) with \( \{ \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \in SO(2m+2) \mid k \in SO(2m+1) \} \). As a maximal torus \( A \) of \( S^{2m+1} \) we choose \( A = \{ x(\theta) := (\cos \theta)e_1 + (\sin \theta)e_2 \in S^{2m+1} \mid \theta \in \mathbb{R} \} \), where \( e_i \) denotes the \( i \)-th unit vector in \( \mathbb{R}^{2m+2} \).

Obviously, rank \( S^{2m+1} = 1 \) and \( A \cong \mathbb{S}^1 \cong \mathbb{R}/2\pi \mathbb{Z} \). So \( \Gamma = 2\pi \mathbb{Z}, \Gamma_0 = 2\mathbb{Z}, \) and the weight lattice \( \Lambda = \frac{1}{2} \mathbb{Z} \). Thus, the homothetic ratio \( c_0 \) of \( \Lambda \) to \( \Gamma_0 \) is given by \( c_0 = C_{[\Lambda:\Gamma_0]} = C_{[\frac{1}{2} \mathbb{Z}:2\mathbb{Z}]} = 4 \). Then the corresponding Gauss sum is given by

\[
G(p, q; \mu; \Lambda, \Gamma_0) = G(p, q, 2k, \frac{1}{2} \mathbb{Z}, 2\mathbb{Z}) = \sum_{\ell=0}^{q-1} e^{\frac{2\pi i \sqrt{-1}}{q} (p\ell^2 - k\ell)}, \tag{6.1}
\]

for \( p, q \in \mathbb{Z} \) with \( p \) and \( q \) coprime, and for \( \mu = 2k \in \Gamma_0 = 2\mathbb{Z} \). The right-hand side of (6.1) coincides with the quadratic Gauss sum appearing in the theory of cyclotomic fields. (See, for example, Lang [12, Chapter IV].) In this case, the quadratic form \( Q = Q_k \) for \( \mu = 2k \in \Gamma_0 \) is given by

\[
Q_k(\ell) = \left[ \frac{1}{q} \left\{ p\ell^2 - k\ell \right\} \right]_{\mathbb{Z}} \quad \text{for} \quad \frac{1}{2} \ell \in \Lambda[q]. \tag{6.2}
\]

By applying Proposition 2.9 to this case, we have the following.

(i) If \( q \equiv 1, 3 \pmod{4} \), then \( \mathcal{K}_{[q; \Lambda, \Gamma_0]} = \{0\}, \) and

\[
\mathcal{G}_{[q; \Lambda, \Gamma_0]} = \left\{ \left[ \frac{2\pi k}{q} \right]_{2\pi \mathbb{Z}} \in \mathbb{R}/2\pi \mathbb{Z} \cong \mathbb{S}^1 \mid 0 \leq k \leq q - 1 \right\}. \tag{6.3}
\]

(ii) If \( q \equiv 0 \pmod{4} \), then \( \mathcal{K}_{[q; \Lambda, \Gamma_0]} = \{0, \frac{q}{4}\}, \) and

\[
\mathcal{G}_{[q; \Lambda, \Gamma_0]} = \left\{ \left[ \frac{2\pi k}{q} \right]_{2\pi \mathbb{Z}} \in \mathbb{R}/2\pi \mathbb{Z} \cong \mathbb{S}^1 \mid k = 0, 2, 4, \ldots, q - 2 \right\}. \tag{6.4}
\]
(iii) If \( q \equiv 2 \pmod{4} \), then \( K_{[q; \Lambda, J_0]} = \{0, \frac{q-2}{4}\} \), and
\[
\mathcal{G}_{[q; \Lambda, J_0]} = \left\{ \left[ \frac{2\pi k}{q} \right] \in \mathbb{R}/2\pi \mathbb{Z} \cong S^1 \mid k = 1, 3, 5, \ldots, q - 1 \right\}. \tag{6.5}
\]

Let \( E_{S^{2m+1}}(t, x) \) be the fundamental solution to the Schrödinger equation, namely, the solution to the Cauchy problem
\[
\begin{align*}
(Sch)_{S^{2m+1}} & \quad \left\{ \sqrt{-1} \partial_t \psi + \Delta_{S^{2m+1}} \psi = 0, \quad t \in \mathbb{R}, \\
\psi(0, x) &= \delta_o(x), \quad x \in S^{2m+1}.
\end{align*} \tag{6.6}
\]

Here in the above \( o = e_1 \). As a distribution on \( S^{2m+1} \), we will write the above fundamental solution as \( C^\infty(S^{2m+1}) \ni f \mapsto E_{S^{2m+1}}(t) \{ f \} \) instead of \( E_{S^{2m+1}}(t, x) \). Then \( E_{S^{2m+1}}(t) \{ f \} \) is given explicitly by
\[
E_{S^{2m+1}}(t) \{ f \} = e^{\sqrt{-1} m^2 t} E_{S^1}(t) \{ D_m M f \}, \tag{6.7}
\]
where \( E_{S^1}(t) \) is the fundamental solution of \( (Sch)_{S^1} \) given by
\[
E_{S^1}(t, \theta) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{-\sqrt{-1} n^2 t + \sqrt{-1} n \theta}. \tag{6.8}
\]

In addition, in (6.7), \( D_m \) and \( M \) are a differential operator and a mean value operator defined respectively by
\[
D_m F(\theta) = \frac{2^m m!}{(2m)!} \left( \frac{\partial_{\theta} \circ \frac{1}{\sin \theta}}{m} \right) (F(\theta) \sin^{2m} \theta), \quad F \in C^\infty(S^1), \tag{6.9}
\]
\[
M f(\theta) = \int_{k \in SO(2m+1)} f(k \cdot x(\theta)) \, dk, \quad f \in C^\infty(S^{2m+1}), \tag{6.10}
\]

where \( dk \) denotes the normalized invariant measure on \( SO(2m + 1) \). Then the support of \( E_{S^{2m+1}} \) at \( t = \frac{2\pi p}{q} \) is given by
\[
\text{Supp} \ E_{S^{2m+1}} \left( \frac{2\pi p}{q} \cdot \right) = \left\{ k \cdot x(\theta) \in S^{2m+1} \mid k \in SO(2m + 1), \ [\theta]_{2\pi \mathbb{Z}} \in \mathcal{G}_{[q; \Lambda, J_0]} \right\}, \tag{6.11}
\]
for the above finite set \( \mathcal{G}_{[q; \Lambda, J_0]} \subset S^1 \).

6.2. In the case \( M = U/K = SU(6)/Sp(3) \)

In this case, we consider \( Sp(3) \) to be the subgroup of \( SU(6) \) defined by \( \{ g \in SU(6) \mid g J_3 g = J_3 \} \). Here \( J_3 = \begin{pmatrix} 0 & -I_3 \\ I_3 & 0 \end{pmatrix} \). We choose a maximal abelian subspace \( \mathfrak{a} \) to be the space of all the
diagonal matrices of the form $H(r_1, r_2, r_3) := \sqrt{-1} \text{diag}(r_1, r_2, r_3, r_1, r_2, r_3)$ with real $r_j$'s and $r_1 + r_2 + r_3 = 0$. Then the corresponding maximal torus $A$ is given by

$$A = \left\{ \exp(H(r_1, r_2, r_3))Sp(3) \in SU(6)/Sp(3) \mid r_1 + r_2 + r_3 = 0 \right\}.$$ 

We identify $a$ with the subspace $\{r_1 + r_2 + r_3 = 0\}$ of $\mathbb{R}^3$ by the mapping $a \ni H(r_1, r_2, r_3) \mapsto (r_1, r_2, r_3) \in \mathbb{R}^3$. We take a suitable $\text{Ad-SU}(6)$-invariant metric on $\mathfrak{su}(6)$ so that the induced inner product $\langle \cdot, \cdot \rangle$ on $a$ coincides with the standard inner product on $\mathbb{R}^3$. Then the restricted roots are $\alpha_{ij} := e_i - e_j$ ($1 \leq i, j \leq 3, i \neq j$) and the multiplicities $m_{\alpha_{ij}} = 4$ for $\forall i, j$ ($i \neq j$). The simple roots are $\gamma_1 := \alpha_{12}$, $\gamma_2 := \alpha_{23}$. The corresponding fundamental weights are $\lambda_1 = \frac{1}{2}(2, -1, -1)$, $\lambda_2 = \frac{1}{2}(1, 1, -2)$. (Note that $\gamma_i^\vee = \gamma_i$ ($i = 1, 2$) in this case.) So the coroot lattice $\Gamma_0$ and the weight lattice $\Lambda$ are given respectively by $\Gamma_0 = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$ and $\Lambda = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$. Thus in this case the homothetic ratio $c_0$ of $\Lambda$ to $\Gamma_0$ is given by $c_0 = C_{[\Lambda : \Gamma_0]} = 3$. Moreover, we identify the maximal torus $A$ with $a/\Gamma$, where $\Gamma = \pi \Gamma_0$.

Therefore, for $p, q \in \mathbb{Z}$, $q > 0$ with $p$ and $q$ coprime, for $\mu = k_1\gamma_1 + k_2\gamma_2 \in \Gamma_0$, and for the above two lattices $\Gamma_0$, $\Lambda$, we have

$$G(p, q; \mu; \Lambda, \Gamma_0) = \sum_{\ell_1 = 0}^{q-1} \sum_{\ell_2 = 0}^{q-1} e^{2\pi \sqrt{-1} Q(k_1, k_2)(\ell_1, \ell_2)},$$

where

$$Q(k_1, k_2)(\ell_1, \ell_2) = \left\lfloor \frac{1}{q} \left\{ 2p(\ell_1^2 + \ell_1 \ell_2 + \ell_2^2) - (k_1 \ell_1 + k_2 \ell_2) \right\} \right\rfloor \in \mathbb{Z}.$$ 

Hence

$$\lambda = \ell_1 \lambda_1 + \ell_2 \lambda_2 \in \mathcal{K}_{[q; \Lambda, \Gamma_0]} \iff 4\ell_1 + 2\ell_2 \equiv 0, \quad 2\ell_1 + 4\ell_2 \equiv 0 \pmod{q}. \quad (6.14)$$ 

By applying Proposition 2.9 to this case, we have the following list.

(i) In the case $q \equiv 0 \pmod{6}$.

$$G_{[q; \Lambda, \Gamma_0]} = \left\{ \left[ \frac{\pi k_1}{q} \gamma_1 + \frac{\pi k_2}{q} \gamma_2 \right] \in a/\Gamma \mid 0 \leq k_1, k_2 \leq q - 1, \right\}
\begin{align*}
& \quad k_1, k_2 \in 2\mathbb{Z}, \quad k_1 + k_2 \in 6\mathbb{Z} \right\}. \quad (6.15)
\end{align*}$$

(ii) In the case $q \equiv 3 \pmod{6}$.

$$G_{[q; \Lambda, \Gamma_0]} = \left\{ \left[ \frac{\pi k_1}{q} \gamma_1 + \frac{\pi k_2}{q} \gamma_2 \right] \in a/\Gamma \mid 0 \leq k_1, k_2 \leq q - 1, \right\}
\begin{align*}
& \quad k_1, k_2 \in \mathbb{Z}, \quad k_1 + k_2 \in 3\mathbb{Z} \right\}. \quad (6.16)$$
(iii) In the case $q \equiv 2, 4 \pmod{6}$.

\[ G_{[q; \Lambda, r_0]} = \left\{ \left[ \frac{\pi k_1}{q} \gamma_1 + \frac{\pi k_2}{q} \gamma_2 \right] \in a/\Gamma \mid 0 \leq k_1, k_2 \leq q - 1, k_1, k_2 \in 2\mathbb{Z} \right\}. \quad (6.17) \]

(iv) In the case $q \equiv 1, 5 \pmod{6}$.

\[ G_{[q; \Lambda, r_0]} = \left\{ \left[ \frac{\pi k_1}{q} \gamma_1 + \frac{\pi k_2}{q} \gamma_2 \right] \in a/\Gamma \mid 0 \leq k_1, k_2 \leq q - 1, k_1, k_2 \in \mathbb{Z} \right\}. \quad (6.18) \]

Let $E_M(t, x)$ be the fundamental solution to the Schrödinger equation on $M$, namely, the solution to the Cauchy problem

\[ \begin{cases} \sqrt{-1} \partial_t \psi + \Delta_M \psi = 0, \quad t \in \mathbb{R}, \\ \psi(0, x) = \delta_0(x), \quad x \in M. \end{cases} \quad (6.19) \]

As in the first case, if we write the above fundamental solution as $E_M(t)[f]$ ($f \in C^\infty(M)$), then $E_M(t)$ is given by

\[ E_M(t)[f] = e^{32\sqrt{-1}t} E_{a/\Gamma}(t)[D_M f], \quad (6.20) \]

where $E_{a/\Gamma}(t)$ is the fundamental solution of $(\text{Sch})_{a/\Gamma}$ constructed in Section 2 and $\mathcal{M} : C^\infty(M) \to C^\infty(a/\Gamma)$ is the operator defined by (4.4). In addition, in (6.20), $D$ is a differential operator on $a/\Gamma$ of the form

\[ D = \text{const.} \left\{ \prod_{1 \leq i < j \leq 3} \sin(r_i - r_j) \right\}^2 \left\{ \prod_{1 \leq i < j \leq 3} (\partial_{r_i} - \partial_{r_j})^2 \right\} + \text{lower order terms}. \quad (6.21) \]

Here in (6.20) we note that $\rho := \frac{1}{2} \sum_{1 \leq i < j \leq 3} m_{aij} \alpha_{ij} = (4, 0, -4)$ and therefore $\langle \rho, \rho \rangle = 32$.

Using the finite set $G_{[q; \Lambda, r_0]}$ in the above list, the support of the fundamental solution $E_M(t, x)$ to the Schrödinger equation $(\text{Sch})_M$ at $t = \frac{3\pi p}{2q}$ is given by

\[ \text{Supp} E_M \left( \frac{3\pi p}{2q}, \cdot \right) = \left\{ k \cdot a \in SU(6)/Sp(3) \mid k \in Sp(3), a \in G_{[q; \Lambda, r_0]} \subset a/\Gamma \cong A \right\}. \quad (6.22) \]

7. Some remarks

7.1. In the case of product spaces

Here we consider the case when $M$ is of the form $M = M_1 \times \cdots \times M_m$, where $M_j = U_j/K_j$ ($1 \leq j \leq m$) are compact symmetric spaces which satisfy the assumptions in Theorem 4.6. Let $A_j$ be a maximal torus of $M_j$ and $c_j$ be the homothetic ratio of the weight lattice to the coroot lattice associated with $A_j$. Moreover, let $E_j(t, x)$ be the solution to the Schrödinger equation.
Here \( o_j \) is the origin of \( M_j \) and \( \Delta_{M_j} \) denotes the Laplace–Beltrami operator on \( M_j \). Then by Theorem 4.6 \( \text{Supp} \, E_j(t, \cdot) \) is a lower dimensional subset of \( M_j \) if \( t/\pi c_j \) is rational whereas \( \text{Supp} \, E_j(t, \cdot) = M_j \) if \( t/\pi c_j \) is irrational. Let us consider the Schrödinger equation on the direct product \( M = M_1 \times \cdots \times M_m \).

\[
\begin{aligned}
\left\{ \begin{array}{l}
\sqrt{-1} \partial_t \psi + \Delta_M \psi = 0, \\
\psi(0, x_j) = \delta_{o_j}(x_j), \quad x_j \in M_j.
\end{array} \right.
\end{aligned}
\tag{7.1}
\]

where \( o = (o_1, \ldots, o_m) \in M \) is the origin of \( M \) and \( \Delta_M := \sum_{j=1}^m \Delta_{M_j} \) is the Laplace–Beltrami operator on \( M \). Obviously the solution \( E_M(t, x) \) to (7.2) is given by \( E_M(t, x) = \prod_{j=1}^m E_j(t, x_j) \). Now we first consider the case when there exists a positive constant \( c \) such that \( \pi c_j / c \in \mathbb{Q} \) for \( \forall j (1 \leq j \leq m) \). Then again by Theorem 4.6 we have the following.

**Theorem 7.1.**

(I) If \( t/c \in \mathbb{Q} \), then there exist finite subsets \( G_j \subset A_j \) \((1 \leq j \leq m)\) such that

\[
\text{Supp} \, E_M(t, \cdot) = \left\{ k \cdot a K \in M \mid k \in K, \ a = (a_1, \ldots, a_m) \in G_1 \times \cdots \times G_m \right\},
\tag{7.3}
\]

where \( K = K_1 \times \cdots \times K_m \).

(II) If \( t/c \notin \mathbb{Q} \), then \( \text{Supp} \, E_M(t, \cdot) = \text{Sing Supp} \, E_M(t, \cdot) = M \).

Next, we consider the case when there is no such a positive constant \( c \) as above. Even in this case, a similar result to (I) of Theorem 7.1 occurs. In fact, if \( t/\pi c_j \in \mathbb{Q} \) for some \( j \) \((1 \leq j \leq m)\), then \( \text{Supp} \, E_M(t, \cdot) \) is included in a lower dimensional subset of \( M \). For simplicity, we assume that \( t/\pi c_1 \in \mathbb{Q} \). Then we have

\[
\text{Supp} \, E_M(t, \cdot) \subset \left\{ k \cdot a \in M \mid k \in K, \ a = (a_1, a_2, \ldots, a_m) \in G_1 \times A_2 \times \cdots \times A_m \right\}.
\tag{7.4}
\]

**7.2. Other cases**

In this paper, we assume the even multiplicity condition (EMC) on the restricted root system of \( M \) to prove the support theorem for the fundamental solution to the Schrödinger equation on \( M \). If we remove the condition (EMC), then we can no longer expect a similar support theorem. However, a similar statement is expected to hold for the singular support of the fundamental solution in the case of compact symmetric spaces of type BC.

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