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Rational finite elements of class C^k over a quadrangulation

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ABSTRACT

In this paper we give a constructive method for rational C^k finite element method in $\mathbb{R}^2$ for quadrilateral. For $k \geqslant 2$, the usual polynomials splines require some supersmoothness at the vertices of the quadrilateral, even when dealing with macro-elements. This is no more a requirement for our rational approach. The process of the construction is illustrated with the help of two examples namely the C^1 case and the C^2 case.

1. Introduction

Let Ω be a connected polygonal bounded domain in $\mathbb{R}^2$ and $\diamond$ be a quadrangulation of Ω. Assuming all quadrilaterals in $\diamond$ are strictly convex, let $\diamondplus$ be the triangulated quadrangulation of Ω by adding two diagonals to each quadrilateral of $\diamond$. See [9,10,12] and the references cited therein for some existing methods on how to construct quadrangulation of Ω.

Let $\mathcal{V} = \{v_i,\ i = 1, \ldots, N_v\}$ be the set of the vertices and $\mathcal{E} = \{e_j,\ j = 1, \ldots, N_e\}$ the set of all the edges of $\diamond$.

The spline space

$$\mathcal{S}_d^k(\diamondplus) = \{s \in C^k(\Omega),\ s|_T \in \mathbb{P}_d,\ \forall T \in \diamondplus\}$$

where $\mathbb{P}_d$ denotes the set of bivariate polynomials of total degree at most d, has been investigated.

It is our purpose to study the space

$$\mathcal{RS}_{r,q}^{k,t}(\diamondplus) = \{s \in C^k(\Omega),\ s|_T \in \mathcal{RQP}_{k,q,r}^t,\ \forall T \in \diamondplus\}$$

where $\mathcal{RQP}_{k,q,r}^t$ is a set of bivariate rational functions defined in Section 3.

Our motivation comes mainly from the fact that, for $k \geqslant 2$, $\mathcal{S}_d^k(\diamondplus)$ is in fact some kind of super-spline space, the order of the derivatives needed at each vertex $v_i \in \mathcal{V}$ being higher than k, contrary to our space $\mathcal{RS}_{r,q}^{k,t}(\diamondplus)$ (see Section 3).

The paper is laid out as follows. In Section 2 we introduce some notations and recall some basic results. In Section 3, we construct the quadrilateral rational finite elements.

2. Preliminaries

For a function $S \in C^r(\Omega)$, we denote respectively by $C^p(v_i)$ and $C^q(e_j)$ the fact that S is C^p around the vertex $v_i \in \mathcal{V}$, and C^q across the edge $e_j \in \mathcal{E}$.

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Let us recall from Ciarlet [3] that a finite element is a triple $(K, E, \mathcal{L})$ where K is a polygon, E a space of real-valued functions defined over K and $\mathcal{L}$ is a finite set of linearly independent linear forms defined over E (the set of degrees of freedom), that enables us to build up functions in E.

We also recall that $\mathcal{L} = \{l_i,\ i = 1, \dots, \dim E\}$ is E-unisolvent if and only if $\#\mathcal{L} = \dim E$ and $\mathcal{L}$ satisfies the following property: if $v \in E$ such that $l_i(v) = 0$, $i = 1, \dots, \dim E$, then $v \equiv 0$.

Moreover, [3] a finite element $(K, E, \mathcal{L})$ is of class C^r if, whenever it is used to define the restriction to K of a (global) function S, then S has (global) smoothness r.

The finite elements are special spline functions as bivariate spline functions can be of variable smoothness across the domain, as we can use various smoothness conditions across each interior edge of triangulation Δ, while finite elements have only a fixed order of smoothness. However, when using finite element, an explicit construction can be made for each element individually.

For this reason, polynomial finite elements have been heavily studied. Unfortunately, one of the drawbacks of the polynomial splines (defined on triangles as well as quadrilaterals) lays in the following theorem which is an obvious modification of [11, Theorem 10.1]

Theorem 1. *Suppose that K is a polygon and v_1 one of its vertices, connected to v_2 and v_N. Let Δ_K be a refinement of K such that there are $n \geqslant 0$ interior edges connected to the vertex v_1. Let S be a polynomial spline of degree d and smoothness r defined on Δ_K. Then the cross derivatives of S up to order r on the edges $e_1 = [v_1, v_2]$ and $e_N = [v_1, v_N]$ can be specified independently only if we require $S \in C^\rho(v_1)$ with*

$$\rho \geqslant \left\lceil \frac{(n+2)r - n}{n+1} \right\rceil.$$

Therefore, whenever K is a quadrilateral triangulated via its diagonals, we have to enforce supersmoothness of order ρ with

$$\rho \geqslant \left\lceil \frac{3r-1}{2} \right\rceil = \begin{cases} 3m & \text{if } r = 2m, \\ 3m+1 & \text{if } r = 2m+1 \end{cases}$$

at each vertex v_i of K.

This implies that, for C^1 interpolation, one has to prescribe derivatives up to order 1 at each vertices, but as soon as $r \geqslant 2$, the data should invoke extra smoothness at each vertex.

We conclude this section with some notations that will be useful later.

Let σ be a non-degenerated triangle with vertices B_1, B_2, and B_3. And let $\lambda_\sigma(M) = (\lambda_{1,\sigma}(M), \lambda_{2,\sigma}(M), \lambda_{3,\sigma}(M))$ denote the barycentric coordinates of a point M relative to σ. If no confusion is possible, we note λ, λ_1, λ_2, λ_3 in lieu of λ_σ, $\lambda_{1,\sigma}$, $\lambda_{2,\sigma}$, $\lambda_{3,\sigma}$. Moreover, for any $i = 1, 2, 3$, σ_i will denote the edge opposite to the vertex B_i.

Given $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, let $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, and $\lambda^\alpha = \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}$ where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$.

For q, p and s positive integers with $s \leqslant q + p - 1$, let

$$\Gamma^{(2)}(q, p) = \{\alpha \in \mathbb{N}^3,\ |\alpha| = 2q + p + 1\}$$

and

$$\Gamma_s^{(2)}(q, p) = \{\alpha \in \Gamma^{(2)}(q, p),\ s + 1 \leqslant \alpha_i \leqslant q + p,\ i = 1, 2, 3\}.$$

As usual, for d a nonnegative integer, $\mathbb{P}_d$ denotes the space of all bivariate polynomials of total degree $\leqslant d$.

In the *PPR* (Polynomial Plus Rational) finite element method, we expose here, the space of interpolation consist of two parts: a polynomial part and a pure rational part.

Then we define the polynomial part of the space by

$$\mathbb{P}_{2q+p+1}(q, p, s, 0) = \operatorname{span}\{\lambda^\alpha,\ \alpha \in \Gamma^{(2)}(q, p) \setminus \Gamma_s^{(2)}(q, p)\}.$$

It is easy to see that

$$\dim \mathbb{P}_{2q+p+1}(q, p, s, 0) = \frac{3}{2}\big[(q+2)(q+1) + (s+2p)(s+1)\big].$$

Remark 2. Note that $\mathbb{P}_{2q+p+1}(q, p, s, 0) \subset \mathbb{P}_{2q+p+1}$.

Now given l and m positive integers such as $1 \leqslant l \leqslant m \leqslant q$, let

$$\mathbb{P}_{2q+p+1}(q, p, s, 0, m) = \mathbb{P}_{2q+p+1}(q, p, s, 0) \oplus \mathbb{R}_q(s+1, m, p)$$

where

$$\mathbb{R}_q(l, m, p) = \mathbb{R}_{q,l,p} \oplus \mathbb{R}_{q,l+1,p} \oplus \cdots \oplus \mathbb{R}_{q,m,p}$$

and

$$\mathbb{R}_{q,l,p} = \operatorname{span}\{r_{i,n}^{l,q,p},\ n = 1, \ldots, l+p\}.$$

We recall that these rational basis functions are defined as follows:

$$r_{i,m}^{l,k,p} = \tau_i^{m,l,p} \rho_i^{l,k}$$

where $\tau_i^{m,l,p}$, $m = 1, \ldots, l$, are polynomials of degree $l-1$ and $\rho_i^{l,k}$ are bubble functions.
For more details on rational PPR triangular finite element we refer to [1,5].

3. Rational quadrilateral finite elements

3.1. Introduction and notations

Quadrilateral finite elements using polynomial functions have been studied in [2–4,7–10,13–15]. The special case of quadrilateral finite using rational functions has received less attention. Let us, however, mention that we are aware of some of the investigation on the topic as [6,16]. Our aim is to give a method for this type of construction for rational functions. The construction proposed here is based on the PPR finite element technique.

Let K be a non-degenerated quadrilateral with vertices A_1, A_2, A_3 and A_4. Let also, for $i = 1, 2, 3$, Σ_i denote the triangle deduced from K with vertices A_i, A_{i+1} and A_{i+2}. The point of intersection of the diagonals of K will be denoted A_0. Let K_i, $i = 1, 2, 3, 4$, be the triangle deduced from K with vertices A_0, A_i and A_{i+1}.

3.2. Rational quadrilateral finite element of C^k

Let k be a nonnegative integer. Then, for any $n = 1, \ldots, k$ and for $m = 1, \ldots, n$, let $Q_{ij}^{n,m}$ denote n distinct interior points of the side A_iA_j, for fixed i, j in $\{1, 2, 3, 4\}$, $i < j$.

Thus, totally, one needs $n \times k$ points. For each n, the points $Q_{ij}^{n,m}$ are distinct. However, among $n = 1, \ldots, k$, $Q_{ij}^{n,m}$ may not be distinct.

Let

$$\mathcal{L}_k = \left\{ \partial^\alpha f(A_i),\ |\alpha| \leqslant k,\ i = 1, 2, 3, 4;\ \frac{\partial^n}{\partial \nu_{j\,j+1}^n} f\left(Q_{j\,j+1}^{n,m}\right),\ n = 1, 2, \ldots, k,\ m = 1, \ldots, n,\ j \in \{1, 2, 3, 4\} \right\} \tag{1}$$

be a set of degrees of freedom defined on K, where $\frac{\partial}{\partial \nu_{jl}}$ denotes the normal derivative relative to the side A_jA_l.

Proposition 3. $\#\mathcal{L}_k = 4(k+1)^2$.

Proof. It suffices to do a simple count. $\square$

We recall that the purpose is to construct a function S in $C^k(K)$ which interpolate degree of freedom at data given on K.
First of all we construct three functions s_1, s_2 and s_3 which belong to $\mathbb{P}_{2k+1}(k, 0, 0, 0, k)$ such as:
s_1 satisfies

$$\left.\begin{array}{l} \partial^\alpha s_1(A_1) = \partial^\alpha f(A_1) \\ \partial^\alpha s_1(A_2) = \partial^\alpha f(A_2) \\ \partial^\alpha s_1(A_3) = 0 \end{array}\right\}, \quad |\alpha| \leqslant k,$$

$$\left.\begin{array}{l} \dfrac{\partial^j}{\partial \nu_{12}^j} s_1\left(Q_{12}^{j,m}\right) = \dfrac{\partial^j}{\partial \nu_{12}^j} f\left(Q_{12}^{j,m}\right) \\ \dfrac{\partial^j}{\partial \nu_{23}^j} s_1\left(Q_{23}^{j,m}\right) = \dfrac{\partial^j}{\partial \nu_{23}^j} f\left(Q_{23}^{j,m}\right) \\ \dfrac{\partial^j}{\partial \nu_{13}^j} s_1\left(Q_{13}^{j,m}\right) = 0 \end{array}\right\}, \quad j = 1, 2, \ldots, k,\ m = 1, \ldots, j$$

and s_2 satisfies

$$\left.\begin{array}{l} \partial^\alpha s_2(A_3) = \partial^\alpha f(A_3) \\ \partial^\alpha s_2(A_2) = \partial^\alpha s_2(A_4) = 0 \end{array}\right\}, \quad |\alpha| \leqslant k,$$

$$\left.\begin{array}{l} \dfrac{\partial^j}{\partial \nu_{34}^j} s_2\big(Q_{34}^{j,m}\big) = \dfrac{\partial^j}{\partial \nu_{34}^j} f\big(Q_{34}^{j,m}\big) - \dfrac{\partial^j}{\partial \nu_{34}^j} s_1\big(Q_{34}^{j,m}\big) \\ \dfrac{\partial^j}{\partial \nu_{23}^j} s_2\big(Q_{23}^{j,m}\big) = \dfrac{\partial^j}{\partial \nu_{24}^j} s_2\big(Q_{24}^{j,m}\big) = 0 \end{array}\right\}, \quad j = 1, 2, \ldots, k, \ m = 1, \ldots, j$$

and s_3 to verify

$$\left.\begin{array}{l} \partial^\alpha s_3(A_4) = \partial^\alpha f(A_4) - \partial^\alpha s_1(A_4) \\ \partial^\alpha s_3(A_1) = \partial^\alpha s_3(A_3) = 0 \end{array}\right\}, \quad |\alpha| \leqslant k,$$

$$\left.\begin{array}{l} \dfrac{\partial^j}{\partial \nu_{14}^j} s_3\big(Q_{14}^{j,m}\big) = \dfrac{\partial^j}{\partial \nu_{14}^j} f\big(Q_{14}^{j,m}\big) - \dfrac{\partial^j}{\partial \nu_{14}^j} s_1\big(Q_{14}^{j,m}\big) \\ \dfrac{\partial^j}{\partial \nu_{13}^j} s_3\big(Q_{13}^{j,m}\big) = \dfrac{\partial^j}{\partial \nu_{34}^j} s_3\big(Q_{34}^{j,m}\big) = 0 \end{array}\right\}, \quad j = 1, 2, \ldots, k, \ m = 1, \ldots, j.$$

Let us define the following space of functions:

$$\begin{aligned} \mathcal{RQP}_{0,k,k}^0 = \big\{ & S \in C^k(K),\ \exists s_1, s_2, s_3 \in \mathbb{PR}_{2k+1}(k, 0, 0, 0, k);\ S|_{K_1} = s_1, \\ & S|_{K_2} = s_1 + s_2,\ s|_{K_3} = s_1 + s_2 + s_3,\ S|_{K_4} = s_1 + s_3 \big\}. \end{aligned} \tag{2}$$

We have

Proposition 4. $\dim \mathcal{RQP}_{0,k,k}^0 = 4(k+1)^2$.

Proof. It suffices to note that the number of coefficients nonzero to determine explicitly, respectively s_1, s_2 and s_3, are $2(k+1)^2$, $(k+1)^2$ and $(k+1)^2$.

Proposition 5. *The set of degrees of freedom $\mathcal{L}_k$ is $\mathcal{RQP}_{0,k,k}^0$-unisolvent and $(K, \mathcal{RQP}_{0,k,k}^0, \mathcal{L}_k)$ is a finite element of class C^k.*

Proof. First we note that $\dim \mathcal{P}_k = 4(k+1)^2 = \#\mathcal{L}_k$. Therefore to prove that $\mathcal{L}_k$ is $\mathcal{RQP}_{0,k,k}^0$-unisolvent, it suffices to prove that if $S \in \mathcal{RQP}_{0,k,k}^0$ then:

$$l(S) = 0, \quad \forall l \in \mathcal{L}_k \quad \Longrightarrow \quad S \equiv 0.$$

So we have to prove that for $S \in \mathcal{RQP}_{0,k,k}^0$ which satisfies

$$\partial^\alpha S(A_i) = 0, \quad |\alpha| \leqslant k,\ i = 1, 2, 3, 4, \tag{3}$$

$$\frac{\partial^n}{\partial \nu_{j\,j+1}^n} S\big(Q_{j\,j+1}^{n,m}\big) = 0, \quad j \in \{1, 2, 3, 4\},\ n = 1, 2, \ldots, k,\ m = 1, \ldots, n, \tag{4}$$

it comes $S = 0$.

The result derives immediately from the construction. Actually, relations (3), (4) lead to $s_1 \equiv 0$, $s_2 \equiv 0$ and $s_3 \equiv 0$. Thus $S \equiv 0$. So, we conclude that $\mathcal{L}_k$ is $\mathcal{RQP}_{0,k,k}^0$-unisolvent.

It remains to prove that S is in $C^k(K)$. The result is obtained if we prove that, for all $\alpha \in \mathbb{N}^2$, with $|\alpha| \leqslant k$, we have

$$\begin{aligned} & \partial^\alpha (s_1 + s_3)|_{K_3 \cap K_4} = \partial^\alpha (s_1 + s_2 + s_3)|_{K_3 \cap K_4}, \\ & \partial^\alpha (s_1)|_{K_1 \cap K_4} = \partial^\alpha (s_1 + s_3)|_{K_1 \cap K_4}, \\ & \partial^\alpha (s_1)|_{K_1 \cap K_2} = \partial^\alpha (s_1 + s_2)|_{K_1 \cap K_2}, \\ & \partial^\alpha (s_1 + s_2)|_{K_2 \cap K_3} = \partial^\alpha (s_1 + s_2 + s_3)|_{K_2 \cap K_3}. \end{aligned} \tag{5}$$

These relations yield

$$\begin{aligned} & \partial^\alpha (s_1 + s_2 + s_3)|_{K_3 \cap K_4} - \partial^\alpha (s_1 + s_3)|_{K_3 \cap K_4} = \partial^\alpha (s_2)|_{K_3 \cap K_4}, \\ & \partial^\alpha (s_1 + s_3)|_{K_1 \cap K_4} - \partial^\alpha (s_1)|_{K_1 \cap K_4} = \partial^\alpha (s_3)|_{K_1 \cap K_4}, \\ & \partial^\alpha (s_1 + s_2)|_{K_1 \cap K_2} - \partial^\alpha (s_1)|_{K_1 \cap K_2} = \partial^\alpha (s_2)|_{K_1 \cap K_2}, \\ & \partial^\alpha (s_1 + s_2 + s_3)|_{K_2 \cap K_3} - \partial^\alpha (s_1 + s_2)|_{K_2 \cap K_3} = \partial^\alpha (s_3)|_{K_2 \cap K_3}. \end{aligned} \tag{6}$$

So, it suffices to prove that

$$\partial^\alpha(s_2)|_{K_3\cap K_4} = \partial^\alpha(s_3)|_{K_1\cap K_4} = \partial^\alpha(s_2)|_{K_1\cap K_2} = \partial^\alpha(s_3)|_{K_2\cap K_3} = 0$$

to get the result.

In fact, $K_3 \cap K_4$ and $K_1 \cap K_2$ are both included in $[A_2, A_4]$, and s_2 restricted to $[A_2, A_4]$, is a polynomial of degree $2k+1$ in a variable t, with the following degree of freedom:

$$\begin{cases} \dfrac{\partial^j}{\partial t^j} s_2(A_i) = 0, \quad i = 2, 4,\ j = 0, 1, \ldots, k, \\ \dfrac{\partial^m}{\partial \nu_{24}^m} s_2\big(Q_{24}^{m,n}\big) = 0, \quad m = 1, 2, \ldots, k,\ n = 1, \ldots, m. \end{cases}$$

Therefore, s_2 and $\frac{\partial^m s_2}{\partial \nu_{24}^m}$, $m = 1, 2, \ldots, k$, vanish along the side $[A_2, A_4]$.

So, it comes immediately

$$\partial^\alpha(s_2)|_{K_1\cap K_2} = \partial^\alpha(s_2)|_{K_3\cap K_4} = 0.$$

In a similar manner, we note that $K_2 \cap K_3$ and $K_1 \cap K_4$ are both included in $[A_1, A_3]$, and s_3 restricted to $[A_1, A_3]$, is also a polynomial of degree $2k+1$ in a variable u, with the following degree of freedom:

$$\begin{cases} \dfrac{\partial^j}{\partial u^j} s_3(A_i) = 0, \quad i = 1, 3,\ j = 0, 1, \ldots, k, \\ \dfrac{\partial^m}{\partial \nu_{13}^m} s_3\big(Q_{13}^{m,n}\big) = 0, \quad m = 1, 2, \ldots, k,\ n = 1, \ldots, m; \end{cases}$$

we conclude that s_3 and $\frac{\partial^m s_3}{\partial \nu_{24}^m}$, $m = 1, 2, \ldots, k$, vanish along the side $[A_1, A_3]$.

The following relations come immediately:

$$\begin{aligned} &\partial^\alpha(s_1+s_2+s_3)|_{K_3\cap K_4} = \partial^\alpha(s_1+s_3)|_{K_3\cap K_4}, \\ &\partial^\alpha(s_1+s_3)|_{K_1\cap K_4} = \partial^\alpha(s_1)|_{K_1\cap K_4}, \\ &\partial^\alpha(s_1+s_2)|_{K_1\cap K_2} = \partial^\alpha(s_1)|_{K_1\cap K_2}, \\ &\partial^\alpha(s_1+s_2+s_3)|_{K_2\cap K_3} = \partial^\alpha(s_1+s_2)|_{K_2\cap K_3}. \end{aligned} \tag{7}$$

Consequently, it comes that S is in $C^k(K)$. $\square$

As shown in this section the functions S constructed lead to the definition of the space

$$\mathcal{RS}_{k,k}^{0,0}(\diamondplus) = \big\{S \in C^k(\Omega),\ S|_T \in \mathcal{RQP}_{0,k,k}^0,\ \forall T \in \diamondplus\big\}.$$

Thus, for all quadrangulation $\diamond$ one can define a set of bivariate rational splines S as follows

$$\mathcal{RS}_{k,k}^{0,0}(\diamond) = \big\{S \in C^k(\Omega),\ S|_Q \in \mathcal{RS}_{k,k}^{0,0}(\diamondplus),\ \forall Q \in \diamond\big\}.$$

Proposition 6. *Given a quadrangulation $\diamond$ of Ω, every function S defined on each quadrilateral $\diamondplus$ of $\diamond$ such as $S|_{\diamondplus}$ is an element of $\mathcal{RS}_{k,k}^{0,0}(\diamondplus)$, which itself is an element of $\mathcal{RS}_{k,k}^{0,0}(\diamond)$.*

Proposition 7. *The space $\mathcal{RS}_{k,k}^{0,0}(\diamond)$ is of dimension $\frac{(k+2)(k+1)}{2} \times \#\mathcal{V} + \frac{(k+1)k}{2} \times \#\mathcal{E}$.*

3.3. Rational quadrilateral finite element of class C^1

Let

$$\mathcal{L}_1 = \left\{\partial^\alpha f(A_i),\ |\alpha| \leqslant 1,\ i = 1, 2, 3, 4;\ \frac{\partial}{\partial \nu_{j\,j+1}} f\big(Q_{j\,j+1}^{1,1}\big),\ j \in \{1, 2, 3, 4\}\right\}$$

be a set of degrees of freedom defined on K, where $\frac{\partial}{\partial \nu_{jl}}$ denotes the normal derivative relative to the side $A_j A_l$.

Our aim is to construct a function S in $C^1(K)$ which interpolates degree of freedom at data given on K.

First of all we construct three functions s_1, s_2 and s_3 which belong to $\mathbb{P}_3(1, 0, 0, 0, 1)$ such as:

s_1 satisfies

$$\left.\begin{array}{l}\partial^\alpha s_1(A_1)=\partial^\alpha f(A_1)\\ \partial^\alpha s_1(A_2)=\partial^\alpha f(A_2)\\ \partial^\alpha s_1(A_3)=0\end{array}\right\},\quad |\alpha|\leqslant 1,$$

$$\left.\begin{array}{l}\dfrac{\partial}{\partial\nu_{12}}s_1\big(Q_{12}^{1,1}\big)=\dfrac{\partial}{\partial\nu_{12}}f\big(Q_{12}^{1,1}\big)\\ \dfrac{\partial}{\partial\nu_{23}}s_1\big(Q_{23}^{1,1}\big)=\dfrac{\partial}{\partial\nu_{23}}f\big(Q_{23}^{1,1}\big)\\ \dfrac{\partial}{\partial\nu_{13}}s_1\big(Q_{13}^{1,1}\big)=0\end{array}\right\}.$$

The following degree of freedom is associated with s_2:

$$\left.\begin{array}{l}\partial^\alpha s_2(A_3)=\partial^\alpha f(A_3)\\ \partial^\alpha s_2(A_2)=\partial^\alpha s_2(A_4)=0\end{array}\right\},\quad |\alpha|\leqslant 1,$$

$$\frac{\partial}{\partial\nu_{34}}s_2\big(Q_{34}^{1,1}\big)=\frac{\partial}{\partial\nu_{34}}f\big(Q_{34}^{1,1}\big)-\frac{\partial}{\partial\nu_{34}}s_1\big(Q_{34}^{1,1}\big),$$

$$\frac{\partial}{\partial\nu_{23}}s_2\big(Q_{23}^{1,1}\big)=\frac{\partial}{\partial\nu_{24}}s_2\big(Q_{24}^{1,1}\big)=0$$

and s_3 satisfies

$$\left.\begin{array}{l}\partial^\alpha s_3(A_4)=\partial^\alpha f(A_4)-\partial^\alpha s_1(A_4)\\ \partial^\alpha s_3(A_1)=\partial^\alpha s_3(A_3)=0\end{array}\right\},\quad |\alpha|\leqslant 1,$$

$$\frac{\partial}{\partial\nu_{14}}s_3\big(Q_{14}^{1,1}\big)=\frac{\partial}{\partial\nu_{14}}f\big(Q_{14}^{1,1}\big)-\frac{\partial}{\partial\nu_{14}}s_1\big(Q_{14}^{1,1}\big),$$

$$\frac{\partial}{\partial\nu_{34}}s_3\big(Q_{34}^{1,1}\big)=0=\frac{\partial}{\partial\nu_{13}}s_3\big(Q_{13}^{1,1}\big).$$

Functions s_1, s_2 and s_3 have the following expressions:

$$\begin{aligned}s_1&=C_1(3,0,0)\lambda_{1,1}^3+C_1(0,3,0)\lambda_{2,1}^3+3C_1(2,0,1)\lambda_{1,1}^2\lambda_{3,1}+3C_1(2,1,0)\lambda_{1,1}^2\lambda_{2,1}\\&\quad+3C_1(1,0,2)\lambda_{1,1}\lambda_{3,1}^2+3C_1(0,1,2)\lambda_{1,1}\lambda_{3,1}^2+D_1(2)r_{2,1}^{1,1,1}+D_1(3)r_{3,1}^{1,1,1},\\ s_2&=C_2(0,3,0)\lambda_{2,2}^3+3C_2(1,2,0)\lambda_{1,2}\lambda_{2,2}^2+3C_2(0,2,1)\lambda_{2,2}^2\lambda_{3,2}+D_2(1)r_{1,2}^{1,1,1},\\ ss_3&=C_3(0,3,0)\lambda_{2,3}^3+3C_3(1,2,0)\lambda_{1,3}\lambda_{2,3}^2+3C_3(0,2,1)\lambda_{2,3}^2\lambda_{3,3}+D_3(1)r_{1,3}^{1,1,1}.\end{aligned}\tag{8}$$

Where $C_l(\alpha)$ and $D_l(i)$ denote the coefficients relatively to Σ_l. We also let $r_{i,l}$ be the functions r_i relative to Σ_l.
Let us define the following space

$$\begin{aligned}\mathcal{RQP}_{0,1,1}^0=\big\{&S\in C^1(K),\ \exists s_1,s_2,s_3\in\mathbb{PR}_3(1,0,0,0,1);\ S|_{K_1}=s_1,\\ &S|_{K_2}=s_1+s_2,\ S|_{K_3}=s_1+s_2+s_3,\ S|_{K_4}=s_1+s_3\big\}.\end{aligned}\tag{9}$$

We have

Proposition 8. *The set of degrees of freedom $\mathcal{L}_1$ is $\mathcal{RQP}_{0,1,1}^0$-unisolvent and $(K,\mathcal{RQP}_{0,1,1}^0,\mathcal{L}_1)$ is a finite element of class C^1.*

3.4. Rational quadrilateral finite element of class C^2

Let

$$\mathcal{L}_2=\left\{\partial^\alpha f(A_i),\ |\alpha|\leqslant 2,\ i=1,2,3,4;\ \frac{\partial^n}{\partial\nu_{j\,j+1}^n}f\big(Q_{j\,j+1}^{n,m}\big),\ n=1,2,\ m=1,\dots,n,\ j\in\{1,2,3,4\}\right\}\tag{10}$$

be a set of degrees of freedom defined on K, where $\frac{\partial}{\partial\nu_{jl}}$ denotes the normal derivative relative to the side A_jA_l.

Our aim is to construct a function S in $C^2(K)$ which interpolates degree of freedom at data given on K.

First of all we construct three functions s_1, s_2 and s_3 which belong to $\mathbb{P}_5(2,0,0,0,2)$ such as:

s_1 satisfies

$$\left.\begin{array}{l}\partial^\alpha s_1(A_1)=\partial^\alpha f(A_1)\\ \partial^\alpha s_1(A_2)=\partial^\alpha f(A_2)\\ \partial^\alpha s_1(A_3)=0\end{array}\right\}, \quad |\alpha|\leqslant 2,$$

$$\left.\begin{array}{l}\frac{\partial^j}{\partial\nu_{12}^j}s_1\big(Q_{12}^{j,m}\big)=\frac{\partial^j}{\partial\nu_{12}^j}f\big(Q_{12}^{j,m}\big)\\ \frac{\partial^j}{\partial\nu_{23}^j}s_1\big(Q_{23}^{j,m}\big)=\frac{\partial^j}{\partial\nu_{23}^j}f\big(Q_{23}^{j,m}\big)\\ \frac{\partial^j}{\partial\nu_{13}^j}s_1\big(Q_{13}^{j,m}\big)=0\end{array}\right\}, \quad j=1,2,\ m=1,\dots,j,$$

s_2 satisfies

$$\left.\begin{array}{l}\partial^\alpha s_2(A_3)=\partial^\alpha f(A_3)\\ \partial^\alpha s_2(A_2)=\partial^\alpha s_2(A_4)=0\end{array}\right\}, \quad |\alpha|\leqslant 2,$$

$$\left.\begin{array}{l}\frac{\partial^j}{\partial\nu_{34}^j}s_2\big(Q_{34}^{j,m}\big)=\frac{\partial^j}{\partial\nu_{34}^j}f\big(Q_{34}^{j,m}\big)-\frac{\partial^j}{\partial\nu_{34}^j}s_1\big(Q_{34}^{j,m}\big)\\ \frac{\partial^j}{\partial\nu_{23}^j}s_2\big(Q_{23}^{j,m}\big)=\frac{\partial^j}{\partial\nu_{24}^j}s_2\big(Q_{24}^{j,m}\big)=0\end{array}\right\}, \quad j=1,2,\ m=1,\dots,j$$

and s_3 satisfies

$$\left.\begin{array}{l}\partial^\alpha s_3(A_4)=\partial^\alpha f(A_4)-\partial^\alpha s_1(A_4)\\ \partial^\alpha s_3(A_1)=\partial^\alpha s_3(A_3)=0\end{array}\right\}, \quad |\alpha|\leqslant 2,$$

$$\left.\begin{array}{l}\frac{\partial^j}{\partial\nu_{14}^j}s_3\big(Q_{14}^{j,m}\big)=\frac{\partial^j}{\partial\nu_{14}^j}f\big(Q_{14}^{j,m}\big)-\frac{\partial^j}{\partial\nu_{14}^j}s_1\big(Q_{14}^{j,m}\big)\\ \frac{\partial^j}{\partial\nu_{13}^j}s_3\big(Q_{13}^{j,m}\big)=\frac{\partial^j}{\partial\nu_{34}^j}s_3\big(Q_{34}^{j,m}\big)=0\end{array}\right\}, \quad j=1,2,\ m=1,\dots,j.$$

Functions s_1, s_2 and s_3 have the following expressions:

$$\begin{aligned}
s_1 &= C_1(5,0,0)\lambda_{1,1}^5+5C_1(4,0,1)\lambda_{1,1}^4\lambda_{3,1}+5C_1(4,1,0)\lambda_{1,1}^4\lambda_{2,1}\\
&\quad+10C_1(3,0,2)\lambda_{1,1}^3\lambda_{3,1}^2+20C_1(3,1,1)\lambda_{1,1}^3\lambda_{2,1}\lambda_{3,1}+10C_1(3,2,0)\lambda_{1,1}^3\lambda_{2,1}^2\\
&\quad+C_1(0,5,0)\lambda_{2,1}^5+5C_1(1,4,0)\lambda_{1,1}\lambda_{2,1}^4+5C_1(0,4,1)\lambda_{2,1}^4\lambda_{3,1}\\
&\quad+10C_1(2,3,0)\lambda_{1,1}^2\lambda_{2,1}^3+20C_1(1,3,1)\lambda_{1,1}\lambda_{2,1}^3\lambda_{3,1}+10C_1(0,3,2)\lambda_{2,1}^3\lambda_{3,1}^2\\
&\quad+D_1(3,1,1)r_{3,1}^{(1),1,2,0}+D_1(3,2,1)r_{3,1}^{(1),2,2,0}+D_1(3,2,2)r_{3,2}^{(1),2,2,0}\\
&\quad+D_1(1,1,1)r_{1,1}^{(1),1,2,0}+D_1(1,2,1)r_{1,2}^{(1),2,2,0}+D_1(1,2,2)r_{1,3}^{(1),2,2,0},\\
s_2 &= C_2(0,5,0)\lambda_{2,2}^5+5C_2(1,4,0)\lambda_{1,2}\lambda_{2,2}^4+5C_2(0,4,1)\lambda_{2,2}^4\lambda_{3,2}\\
&\quad+10C_2(2,3,0)\lambda_{1,2}^2\lambda_{2,2}^3+20C_2(1,3,1)\lambda_{1,2}\lambda_{2,2}^3\lambda_{3,2}+10C_2(0,3,2)\lambda_{2,2}^3\lambda_{3,2}^2\\
&\quad+D_2(1,1,1)r_{1,1}^{(2),1,2,0}+D_2(1,2,1)r_{1,2}^{(2),2,2,0}+D_2(1,2,2)r_{1,3}^{(2),2,2,0},\\
s_3 &= C_3(0,5,0)\lambda_{2,3}^5+5C_3(1,4,0)\lambda_{1,3}\lambda_{2,3}^4+5C_3(0,4,1)\lambda_{2,3}^4\lambda_{3,3}\\
&\quad+10C_3(2,3,0)\lambda_{1,3}^2\lambda_{2,3}^3+20C_3(1,3,1)\lambda_{1,3}\lambda_{2,3}^3\lambda_{3,3}+10C_3(0,3,2)\lambda_{2,3}^3\lambda_{3,3}^2\\
&\quad+D_3(1,1,1)r_{1,1}^{(3),1,2,0}+D_3(1,2,1)r_{1,2}^{(3),2,2,0}+D_3(1,2,2)r_{1,3}^{(3),2,2,0},
\end{aligned}\tag{11}$$

where $C_l(\alpha)$ and $D_l(i,s,n)$ denote the coefficients relatively to Σ_l. We also let $r_{i,s}^{(l),n,2,0}$ be the functions $r_{i,s}^{n,2,0}$ relative to Σ_l.

Let us define the space

$$\mathcal{RQP}^0_{0,2,2} = \big\{S \in C^2(K),\ \exists s_1, s_2, s_3 \in \mathbb{PR}_5(2,0,0,0,2);\ S|_{K_1} = s_1,\\ S|_{K_2} = s_1 + s_2,\ S|_{K_3} = s_1 + s_2 + s_3,\ S|_{K_4} = s_1 {+} _3\big\}. \tag{12}$$

Proposition 9. *The set of degrees of freedom $\mathcal{L}_2$ is $\mathcal{RQP}^0_{0,2,2}$-unisolvent and $(K, \mathcal{RQP}^0_{0,2,2}, \mathcal{L}_2)$ is a finite element of class C^2.*

3.5. Generalization

3.5.1. Introduction

In the previous section, the polynomials used for constructing finite element of class C^k, are all of odd degree $2k+1$. Furthermore, having at our disposal derivatives up to k in each vertex, we systematically have constructed finite elements of class C^k. In this section, we show how to construct finite elements of class C^q in the same way, if we have derivatives up to order k on the edges, with $1 \leqslant q \leqslant k$, and at the same time we generalize the construction by the use of polynomials of even degree.

3.5.2. Construction of finite elements

Suppose given three positive integers k, q and p such as $q \leqslant k$.

Then we consider the set $\mathcal{L}_{p,q,k}$ of the following degrees of freedom defined on the quadrilateral K with vertices A_1, A_2, A_3 and A_4. Let us define

$$\mathcal{L}_{p,q,k} = \left\{\partial^\alpha f(A_i),\ |\alpha| \leqslant k,\ i = 1,2,3,4;\ \frac{\partial^n}{\partial \nu^n_{j\,j+1}} f\big(Q^{n,m}_{j\,j+1}\big),\ n = 0,1,2,\dots,q,\ m = 1,\dots,n+p,\ j \in \{1,2,3,4\}\right\} \tag{13}$$

where for any $n = 0,1,2,\dots,k$ and for $m = 1,\dots,n+p$, $Q^{n,m}_{ij}$ denote $n+p$ interior distinct points of the side A_iA_j, for i, j fixed in $\{1,2,3,4\}$, with $i < j$. In fact, one needs only that if $n_1 \neq n_2$, it implies $Q^{n_1,m_1}_{ij} \neq Q^{n_2,m_1}_{ij}$.

For convenience we define for any nonnegative integer t, the subset $\mathcal{L}^t_{p,q,k}$ of $\mathcal{L}_{p,q,k}$, dedicated to the rational part in the construction, by:

$$\mathcal{L}^t_{p,q,k} = \left\{\frac{\partial^n}{\partial \nu^n_{j\,j+1}} f\big(Q^{n,m}_{j\,j+1}\big),\ n = t+1, t+2, \dots, q,\ m = 1,\dots,n+p,\ j \in \{1,2,3,4\}\right\}. \tag{14}$$

Now we are going to construct a function S of $C^q(K)$ such as restricted to the sub-triangles K_i, $i = 1, 2, 3, 4$, it belongs to the space $\mathbb{PR}_{2k+p+1}(k, p, t, 0, q)$.

The functions s_1, s_2 and s_3 used in the construction of S are defined as follows:

s_1 satisfies the following relations

$$\left.\begin{aligned} &\partial^\alpha s_1(A_1) = \partial^\alpha f(A_1)\\ &\partial^\alpha s_1(A_2) = \partial^\alpha f(A_2)\\ &\partial^\alpha s_1(A_3) = 0 \end{aligned}\right\},\quad |\alpha| \leqslant k,$$

$$\left.\begin{aligned} &\frac{\partial^j}{\partial \nu^j_{12}} s_1\big(Q^{j,m}_{12}\big) = \frac{\partial^j}{\partial \nu^j_{12}} f\big(Q^{j,m}_{12}\big)\\ &\frac{\partial^j}{\partial \nu^j_{23}} s_1\big(Q^{j,m}_{23}\big) = \frac{\partial^j}{\partial \nu^j_{23}} f\big(Q^{j,m}_{23}\big)\\ &\frac{\partial^j}{\partial \nu^j_{13}} s_1\big(Q^{j,m}_{13}\big) = 0 \end{aligned}\right\},\quad j = 0,1,2,\dots,q,\ m = 1,\dots,j+p,$$

and s_2 is such as

$$\left.\begin{aligned} &\partial^\alpha s_2(A_3) = \partial^\alpha f(A_3)\\ &\partial^\alpha s_2(A_2) = \partial^\alpha s_2(A_4) = 0 \end{aligned}\right\},\quad |\alpha| \leqslant k,$$

$$\left.\begin{aligned} &\frac{\partial^j}{\partial \nu^j_{34}} s_2\big(Q^{j,m}_{34}\big) = \frac{\partial^j}{\partial \nu^j_{34}} f\big(Q^{j,m}_{34}\big) - \frac{\partial^j}{\partial \nu^j_{34}} s_1\big(Q^{j,m}_{34}\big)\\ &\frac{\partial^j}{\partial \nu^j_{23}} s_2\big(Q^{j,m}_{23}\big) = \frac{\partial^j}{\partial \nu^j_{24}} s_2\big(Q^{j,m}_{24}\big) = 0 \end{aligned}\right\},\quad j = 0,1,2,\dots,q,\ m = 1,\dots,j+p,$$

and s_3 satisfies

$$\left.\begin{array}{l}\partial^\alpha s_3(A_4)=\partial^\alpha f(A_4)-\partial^\alpha s_1(A_4)\\ \partial^\alpha s_3(A_1)=\partial^\alpha s_3(A_3)=0\end{array}\right\}, \quad |\alpha|\leqslant k,$$

$$\left.\begin{array}{l}\dfrac{\partial^j}{\partial\nu_{14}^j}s_3\big(Q_{14}^{j,m}\big)=\dfrac{\partial^j}{\partial\nu_{14}^j}f\big(Q_{14}^{j,m}\big)-\dfrac{\partial^j}{\partial\nu_{14}^j}s_1\big(Q_{14}^{j,m}\big)\\ \dfrac{\partial^j}{\partial\nu_{13}^j}s_3\big(Q_{13}^{j,m}\big)=\dfrac{\partial^j}{\partial\nu_{34}^j}s_3\big(Q_{34}^{j,m}\big)=0\end{array}\right\}, \quad j=0,1,2,\ldots,q,\ m=1,\ldots,j+p.$$

Definition 10. Let us define the function S to be an element of the following space

$$\begin{aligned}\mathcal{RQP}^t_{p,q,k}=\big\{&S\in C^q(K),\ \exists s_1,s_2,s_3\in\mathbb{PR}_{2k+p+1}(k,p,t,0,q);\ S|_{K_1}=s_1,\\ &S|_{K_2}=s_1+s_2,\ S|_{K_3}=s_1+s_2+s_3,\ S|_{K_4}=s_1+s_3\big\}\end{aligned}\tag{15}$$

with t a nonnegative integer such as $t<q$.

This function S corresponds to our expectation. It is easy to show that, following the same technique of the previous section, S verifies the degree of freedom defined on K.

Furthermore, we have the following propositions.

Proposition 11. *We have* $\dim\mathcal{RQP}^t_{p,q,k}=2[(q+1)(q+2p)+(k+2)(k+1)]$.

Proposition 12. *The set of degrees of freedom $\mathcal{L}_{p,q,k}$ is $\mathcal{RQP}^t_{p,q,k}$-unisolvent and the triple $(K,\mathcal{RQP}^t_{p,q,k},\mathcal{L}_{p,q,k})$ is a finite element of class C^q.*

Remark 13. The construction of the previous section has been generalized here and we obtain the result of this section with $p=0$, $q=k$ and $t=0$.

So, we have constructed the following space:

$$\mathcal{RS}^{p,t}_{k,q}(\diamondplus)=\big\{S\in C^q(\Omega),\ S|_T\in\mathcal{RQP}^t_{2k+p+1},\ \forall T\in\diamondplus\big\}.$$

We are now in a position to define for all quadrangulation $\diamond$ the set of rational bivariate splines S:

$$\mathcal{RS}^{p,t}_{k,q}(\diamond)=\big\{S\in C^q(\Omega),\ S|_Q\in\mathcal{RS}^{p,t}_{k,q}(\diamondplus),\ \forall Q\in\diamond\big\}.$$

Proposition 14. *Given a quadrangulation $\diamond$ of Ω, all function S defined on each quadrilateral $\diamondplus$ of $\diamond$ such as $S|_{\diamondplus}$ is an element of $\mathcal{RS}^{p,t}_{k,q}(\diamondplus)$, which itself is an element of $\mathcal{RS}^{p,t}_{k,q}(\diamond)$.*

Proposition 15. *The dimension of the space $\mathcal{RS}^{p,t}_{k,q}(\diamond)$ is $\frac{(k+2)(k+1)}{2}\times\#\mathcal{V}+\frac{(q+1)(q+2p)}{2}\times\#\mathcal{E}$.*

3.5.3. Examples

We illustrate the construction with the three following examples.

Example 3.1. For $p=q=k=1$ and $t=0$, we construct an element of class C^1. In this case the dimension of the space $\mathcal{RS}^{1,0}_{1,1}(\diamondplus)$ is equal to 24 and we use as set of degrees of freedom $\mathcal{L}_{1,1,1}$ and for the subset dedicated to the rational part the set $\mathcal{L}^0_{1,1,1}$ where

$$\mathcal{L}_{1,1,1}=\left\{\partial^\alpha f(A_i),\ |\alpha|\leqslant 1,\ i=1,2,3,4;\ \frac{\partial^n}{\partial\nu^n_{j\,j+1}}f\big(Q^{n,m}_{j\,j+1}\big),\ n=0,1,\ m=1,\ldots,n+1,\ j\in\{1,2,3,4\}\right\}$$

and

$$\mathcal{L}^0_{1,1,1}=\left\{\frac{\partial}{\partial\nu_{j\,j+1}}f\big(Q^{1,m}_{j\,j+1}\big),\ m=1,2,\ j\in\{1,2,3,4\}\right\}.$$

The functions s_1, s_2 and s_3 which define the function S are of the form:

$$\begin{aligned}
s_1 &= C_1(4,0,0)\lambda_{1,1}^4 + 4C_1(3,1,0)\lambda_{1,1}^3\lambda_{2,1} + 6C_1(2,2,0)\lambda_{1,1}^2\lambda_{2,1}^2 + 4C_1(1,3,0)\lambda_{1,1}\lambda_{2,1}^3 + C_1(0,4,0)\lambda_{2,1}^4 \\
&\quad + 4C_1(0,3,1)\lambda_{2,1}^3\lambda_{3,1} + 6C_1(0,2,2)\lambda_{2,1}^2\lambda_{3,1}^2 + 4C_1(3,0,1)\lambda_{1,1}^3\lambda_{3,1} + D_1(1,1,1)r_{1,1,1}^{1,1,1} \\
&\quad + D_1(1,1,2)r_{1,2,1}^{1,1,1} + D_1(3,1,1)r_{3,1,1}^{1,1,1} + D_1(3,1,2)r_{3,2,1}^{1,1,1}, \\
s_2 &= 4C_2(1,3,0)\lambda_{1,2}\lambda_{2,2}^3 + C_2(0,4,0)\lambda_{2,2}^4 + 4C_2(0,3,1)\lambda_{2,2}^3\lambda_{3,2} \\
&\quad + 6C_2(0,2,2)\lambda_{1,2}^2\lambda_{2,2}^2 + D_2(1,1,1)r_{1,1,2}^{1,1,1} + D_2(1,1,2)r_{1,2,2}^{1,1,1}, \\
s_3 &= 4C_3(1,3,0)\lambda_{1,3}\lambda_{2,3}^3 + C_3(0,4,0)\lambda_{2,3}^4 + 4C_3(0,3,1)\lambda_{2,3}^3\lambda_{3,3} \\
&\quad + 6C_3(0,2,2)\lambda_{1,3}^2\lambda_{2,3}^2 + D_3(1,1,1)r_{1,1,3}^{1,1,1} + D_3(1,1,2)r_{1,2,3}^{1,1,1}.
\end{aligned} \tag{16}$$

Example 3.2. Now let $p=0$, $q=1$, $k=2$ and $t=0$, we also obtain an element of class C^1. The space $\mathcal{RS}_{2,1}^{0,0}(\diamondplus)$ used is of dimension 28 and the sets of degrees of freedom are the following $\mathcal{L}_{0,1,2}$ and $\mathcal{L}_{0,1,2}^0$ for the subset dedicated to the rational part, where

$$\mathcal{L}_{0,1,2} = \left\{ \partial^\alpha f(A_i),\ |\alpha| \leqslant 2,\ i=1,2,3,4;\ \frac{\partial^n}{\partial \nu_{j\,j+1}^n} f\big(Q_{j\,j+1}^{n,1}\big),\ n=0,1,\ m=1,\dots,n+1,\ j\in\{1,2,3,4\} \right\}$$

and

$$\mathcal{L}_{0,1,2}^0 = \left\{ \frac{\partial}{\partial \nu_{j\,j+1}} f\big(Q_{j\,j+1}^{1,m}\big),\ m=1,2,\ j\in\{1,2,3,4\} \right\}.$$

The functions s_1, s_2 and s_3 which determine the function S have the following expressions:

$$\begin{aligned}
s_1 &= C_1(5,0,0)\lambda_{1,1}^5 + 5C_1(4,1,0)\lambda_{1,1}^4\lambda_{2,1} + 10C_1(3,2,0)\lambda_{1,1}^3\lambda_{2,1}^2 + 5C_1(1,4,0)\lambda_{1,1}\lambda_{2,1}^4 + C_1(0,5,0)\lambda_{2,1}^5 \\
&\quad + 5C_1(0,4,1)\lambda_{2,1}^4\lambda_{3,1} + 10C_1(0,3,2)\lambda_{2,1}^3\lambda_{3,1}^2 + 20C_1(1,3,1)\lambda_{1,1}\lambda_{2,1}^3\lambda_{3,1} \\
&\quad + 5C_1(4,0,1)\lambda_{1,1}^4\lambda_{3,1} + 10C_1(3,0,2)\lambda_{1,1}^3\lambda_{3,1}^2 + 20C_1(3,1,1)\lambda_{1,1}^3\lambda_{2,1}\lambda_{3,1} \\
&\quad + D_1(1,1,1)r_{1,1,1}^{1,2,1} + D_1(3,1,1)r_{3,1,1}^{1,2,1}, \\
s_2 &= 10C_2(2,3,0)\lambda_{1,2}^2\lambda_{2,2}^3 + C_2(1,4,0)\lambda_{1,2}\lambda_{2,2}^4 + C_2(0,5,0)\lambda_{2,2}^5 + 5C_2(0,4,1)\lambda_{2,2}^4\lambda_{3,2} \\
&\quad + 10C_2(0,3,2)\lambda_{2,2}^3\lambda_{3,2}^2 + 20C_2(1,3,1)\lambda_{1,2}\lambda_{2,2}^3\lambda_{3,2} + D_2(1,1,1)r_{1,1,2}^{1,2,0}, \\
s_3 &= 10C_3(2,3,0)\lambda_{1,3}^2\lambda_{2,3}^3 + C_3(1,4,0)\lambda_{1,3}\lambda_{2,3}^4 + C_3(0,5,0)\lambda_{2,3}^5 + 5C_3(0,4,1)\lambda_{2,3}^4\lambda_{3,3} \\
&\quad + 10C_3(0,3,2)\lambda_{2,3}^3\lambda_{3,3}^2 + 20C_3(1,3,1)\lambda_{1,3}\lambda_{2,3}^3\lambda_{3,3} + D_3(1,1,1)r_{1,1,3}^{1,2,0}.
\end{aligned} \tag{17}$$

Example 3.3. When we choose $p=1$, $q=2$, $k=2$ and $t=0$, the finite element obtained is of class C^2. It is constructed while considering a polynomial finite element of class C^0. In this case the dimension of the space $\mathcal{RS}_{2,2}^{1,0}(\diamondplus)$ is 48 and we use as set of degrees of freedom $\mathcal{L}_{1,2,2}$ and for the subset dedicated to the rational part the set $\mathcal{L}_{1,2,2}^0$ with

$$\mathcal{L}_{1,2,2} = \left\{ \partial^\alpha f(A_i),\ |\alpha| \leqslant 2,\ i=1,2,3,4;\ \frac{\partial^n}{\partial \nu_{j\,j+1}^n} f\big(Q_{j\,j+1}^{n,m}\big),\ n=0,1,2,\ m=1,\dots,n,\ j\in\{1,2,3,4\} \right\}$$

and

$$\mathcal{L}_{1,2,2}^0 = \left\{ \frac{\partial^n}{\partial \nu_{j\,j+1}^n} f\big(Q_{j\,j+1}^{n,m}\big),\ n=1,2,\ m=1,\dots,n+1,\ j\in\{1,2,3,4\} \right\}.$$

The function S is defined with the help of the functions s_1, s_2 and s_3 whose expressions are:

$$\begin{aligned}
s_1 &= C_1(6,0,0)\lambda_{1,1}^6 + 6C_1(5,1,0)\lambda_{1,1}^5\lambda_{2,1} + 15C_1(4,2,0)\lambda_{1,1}^4\lambda_{2,1}^2 + 20C_1(3,3,0)\lambda_{1,1}^3\lambda_{2,1}^3 \\
&\quad + 15C_1(2,4,0)\lambda_{1,1}^2\lambda_{2,1}^4 + 6C_1(1,5,0)\lambda_{1,1}\lambda_{2,1}^5 + C_1(0,6,0)\lambda_{2,1}^6 + 6C_1(0,5,1)\lambda_{2,1}^5\lambda_{3,1} \\
&\quad + 15C_1(0,4,2)\lambda_{2,1}^4\lambda_{3,1}^2 + 30C_1(1,4,1)\lambda_{1,1}\lambda_{2,1}^4\lambda_{3,1} + 20C_1(0,3,3)\lambda_{2,1}^3\lambda_{3,1}^3 + 6C_1(5,0,1)\lambda_{1,1}^5\lambda_{3,1} \\
&\quad + 15C_1(4,0,2)\lambda_{1,1}^4\lambda_{3,1}^2 + 30C_1(4,1,1)\lambda_{1,1}^4\lambda_{2,1}\lambda_{3,1} + D_1(1,1,1)r_{1,1,1}^{1,2,1} \\
&\quad + D_1(1,1,2)r_{1,2,1}^{1,2,1} + D_1(3,1,1)r_{3,1,1}^{1,2,1} + D_1(3,1,2)r_{3,2,1}^{1,2,1} + D_1(1,2,1)r_{1,1,1}^{2,2,1} \\
&\quad + D_1(1,2,2)r_{1,2,1}^{2,2,1} + D_1(3,2,1)r_{3,1,1}^{2,2,1} + D_1(3,2,2)r_{3,2,1}^{2,2,1} + D_1(3,2,2)r_{3,3,1}^{2,2,1},
\end{aligned}$$

$$\begin{aligned}
s_2 &= 15C_2(2,4,0)\lambda_{1,2}^2\lambda_{2,2}^4 + C_2(0,6,0)\lambda_{2,2}^6 + 6C_2(0,5,1)\lambda_{2,2}^5\lambda_{3,2} + 15C_2(0,4,2)\lambda_{1,2}^4\lambda_{2,2}^2 \\
&\quad + 30C_2(1,4,1)\lambda_{1,2}\lambda_{2,2}^4\lambda_{3,2} + 20C_2(0,3,3)\lambda_{2,2}^3\lambda_{3,2}^3 + 6C_2(1,5,0)\lambda_{1,2}\lambda_{2,2}^5 + D_2(1,1,1)r_{11,2}^{1,2,1} \\
&\quad + D_2(1,1,2)r_{12,2}^{1,2,1} + D_2(1,2,1)r_{1,1,2}^{2,2,1} + D_2(1,2,2)r_{1,2,2}^{2,2,1} + D_2(1,2,3)r_{1,3,2}^{2,2,1}, \\
s_3 &= 15C_3(2,4,0)\lambda_{1,3}^2\lambda_{2,3}^4 + C_3(0,6,0)\lambda_{2,3}^6 + 6C_3(0,5,1)\lambda_{2,3}^5\lambda_{3,3} + 15C_3(0,4,2)\lambda_{1,3}^4\lambda_{2,3}^2 \\
&\quad + 30C_3(1,4,1)\lambda_{1,3}\lambda_{2,3}^4\lambda_{3,3} + 20C_3(0,3,3)\lambda_{2,3}^3\lambda_{3,3}^3 + 6C_3(1,5,0)\lambda_{1,3}\lambda_{2,3}^5 + D_3(1,1,1)r_{11,3}^{1,2,1} \\
&\quad + D_3(1,1,2)r_{12,3}^{1,2,1} + D_3(1,2,1)r_{1,1,3}^{2,2,1} + D_3(1,2,2)r_{1,2,3}^{2,2,1} + D_3(1,2,3)r_{1,3,3}^{2,2,1}.
\end{aligned} \tag{18}$$

4. Concluding remarks

a) Rational finite elements over triangulated quadrilaterals are better than rational finite elements over triangulations. Suppose $\triangle$ is a triangulation based on the same vertices $\mathcal{V}$ and let us denote by $\mathcal{E}_T$ the set of edges of this triangulation. As mentioned in [5] the dimension of the space $\mathcal{RS}_{2k+1}^k(\triangle)$ is $\frac{(k+2)(k+1)}{2} \times \#\mathcal{V} + \frac{(k+1)k}{2} \times \#\mathcal{E}_T$. While, for $p=0$ and $q=k$, the dimension of the space $\mathcal{RS}_{k,k}^{0,t}(\diamond)$ is $\frac{(k+2)(k+1)}{2} \times \#\mathcal{V} + \frac{(k+1)k}{2} \times \#\mathcal{E}$. So, as the cardinal of $\mathcal{E}$ is smaller than the cardinal of $\mathcal{E}_T$, we have $\dim \mathcal{RS}_{k,k}^{0,t}(\diamond) \leqslant \dim \mathcal{RS}_{2k+1}^k(\triangle)$.

b) Notice that these constructions of rational quadrilateral finite elements and the rational triangular finite elements are compatible among themselves. Moreover, if it is possible to make up a triangulation $\diamondplus$, it is better to use quadrilateral finite elements instead of triangles taken separately.

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