A unified definition of stress intensity factors for cracks/corners/interface cracks/interface corners in anisotropic/piezoelectric/viscoelastic materials

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Abstract

To have a definition covering all the possible situations of singular orders (real or complex, distinct or repeated), the matrix form near tip solutions of interface corners written by the Stroh’s complex variable formalism of anisotropic elasticity were employed in this paper. Starting from the matrix form near tip solutions for anisotropic elastic materials, identical mathematical expressions were applied to the cases of piezoelectric materials and viscoelastic materials by the ways such as the expansion of the elastic properties to include the piezoelectric effects, and the use of the Laplace transform to make the viscoelastic stress-strain relations look like linear elastic materials. The stress intensity factors defined through the matrix form expression are therefore unified for several different situations. With this unified definition, some special cases are reduced and compared with the definitions proposed in the literature.

Keywords: stress intensity factor; interface corner; interface crack; anisotropic; piezoelectric; viscoelastic

1. Introduction

Consider a crack or a corner (or called notch) in a homogeneous material, a crack lying along the interface between two dissimilar materials, or an interface corner (or called junction) of a multi-material wedge. Due to the

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discontinuity of geometry and material properties, the stress singularity usually occurs at the tips of cracks, corners, interface cracks, or interface corners if this problem is treated by a linear elastic mathematical model. Although the cracks, corners, and interface cracks are special cases of interface corners, most of the definitions of stress intensity factors proposed in the literature are not consistent with each other. Most common definition considers only the orders of stress singularity with real numbers. When the singular orders are complex or repeated, oscillatory or logarithmic singularity may occur and hence different definitions seem to become indispensable. Owing to the possible difference of the singular orders for different interface corners, one may encounter different definitions and units for different corners or cracks. Thus, a direct comparison of the magnitude of stress intensity factors for different interface corners is meaningless, and hence the fracture toughness or fracture criterion established for a certain interface corners (e.g. a crack specimen) cannot be applied directly to the other interface corners.

To provide a unified definition of the stress intensity factors for several different possible conditions, recently we rewrote the near tip solutions of interface corners in a matrix power function form (Hwu, 2012). Through this matrix form solution, a unified definition was proposed in (Hwu and Huang, 2012) for the general cases of anisotropic materials. The advantages of the unified definition were then illustrated through the numerical examples of interface corners (including cracks, interface cracks and corners) whose stress singularities change among exponential (real), oscillatory (complex) and logarithmic (repeated) types when the corner angles or material properties vary. With the success of the unified definition proposed for the anisotropic materials, further extension is now pursued for the piezoelectric materials and viscoelastic materials. Hope this extension is helpful for bridging the problems of cracks, corners, interface cracks and interface corners with anisotropic, piezoelectric and viscoelastic materials.

2. A unified definition of stress intensity factors

Consider an interface corner between two dissimilar anisotropic elastic materials (Fig. 1). The interface is assumed to be perfectly bonded and the two edges of the corner are traction-free. By employing the Stroh’s complex variable formalism for anisotropic elasticity, the solutions of the stresses near the corner tip have been found (Hwu, 2010) and can be further expressed in matrix power function form as (Hwu, 2012)

$$\sigma(r, \theta) = \frac{1}{\sqrt{2\pi \ell}} (r / \ell)^{\Delta(\theta)} k(\theta)$$

(1)

in which the polar coordinate \((r, \theta)\) is taken with the origin located at the corner tip. \(\sigma(r, \theta)\) is a vector of stresses at point \((r, \theta)\), \(\Delta(\theta)\) is called the matrix of singular orders at angle \(\theta\), \(k(\theta)\) is a vector of stress intensity factors along the radial direction \(\theta = \text{const}\), and \(\ell\) is a reference length. If the direction along the interface is taken to be \(\theta = 0\), the near tip stresses along the interface can be expressed as

$$\sigma(r, 0) = \frac{1}{\sqrt{2\pi \ell}} (r / \ell)^{\Delta} k,$$

(2)

where \(\Delta = \Delta(0)\) and \(k = k(0)\). With the near tip solutions (2), the stress intensity factor \(k\) can be defined as

$$k = \lim_{r \to 0} \sqrt{2\pi \ell} (r / \ell)^{\Delta} \sigma(r, 0).$$

(3)

The generalized stress intensity factor \(k(\theta)\) is defined as

$$k(\theta) = \lim_{r \to 0} \sqrt{2\pi \ell} (r / \ell)^{\Delta(\theta)} \sigma(r, \theta).$$

(4)
Although the near tip solutions are derived for the interface corner with traction-free corner edges, it is not difficult to prove that the matrix power function form, eqn.(1), is still valid for the general multi-material wedges with other boundary conditions (Hwu, 2012). Furthermore, through proper matrix expansion (Hwu and Ikeda, 2008; Hwu and Kuo, 2010) and Laplace transformation (Kuo and Hwu, 2013; Hwu and Kuo, 2013), it can be proved that the same function form is also applicable to the cases with piezoelectric materials and viscoelastic materials, and the combinations of any two different kinds of materials.

Even the near tip solutions are derived based upon the Stroh formalism whose material eigenvalues are required to be distinct, through suitable modification on the formalism it has been proved that the matrix power function form, eqn.(1), is also valid for the so-called degenerate materials such as isotropic materials whose material eigenvalues are repeated. Thus, the applicability to anisotropic, piezoelectric, and viscoelastic materials can be enhanced by including all the possible degenerate materials.

In general, more than one orders of stress singularity may occur in the structures of interface corners, and hence it is possible to have repeated roots of singular orders. Also, it is possible to have real or complex roots of singular orders. While the real value of singular order shows the stresses approaches to infinity exponentially, its complex value reveals the oscillatory singularity. When the singular orders are repeated, the logarithmic singularity may occur. In the literature, the definition of the stress intensity factors was given according to its multiplicity, real or complex, distinct or repeated. Thus, it is difficult to communicate each other if the singular orders encounter a change in their characteristics. Here, use of the matrix of singular orders instead of the singular orders themselves eliminates the difference among them and provides a unified definition of stress intensity factors for all possible cases of singular orders.

Since the values of the near tip stresses calculated from eqn.(1) are not allowed to be affected by the selection of the reference length $\ell$, the values of stress intensity factors may be changed if different reference length is chosen. If no reference length is introduced in the definition of stress intensity factors, an awkward physical unit may occur when the singular order is complex, and different units will be used for different singular orders. Due to the inconsistency of the definitions and units, a direct comparison of the magnitude of the stress intensity factors for different interface corners is meaningless. It is therefore important to keep the reference length in the definition and to study further whether the reference length is simply an auxiliary parameter or a material property for the general interface corners. The stress intensity factors defined in this way has a unified unit ($\text{Pa} \sqrt{\text{m}}$) and will vary smoothly even when the stress singularity changes among real, complex and logarithmic types.

To know the connection of this unified definition with the traditional definitions commonly used in the literature, some special cases are discussed below.

### 3. Anisotropic/piezoelectric/viscoelastic materials

As stated in the previous section, the near tip solutions and the definitions of the stress intensity factors are all applicable for the general anisotropic, piezoelectric and viscoelastic materials. The difference is just the size and content of the vectors and matrices used in the equations. To show them clearly, the definition (3) is rewritten in component form as follows.
In equation (5), $K_I$, $K_{II}$, $K_{III}$, and $K_{IV}$ are, respectively, the stress intensity factors of opening mode, shearing mode, tearing, and electric mode; $\sigma_{r\theta}, \sigma_{\theta\theta}, \sigma_{\theta3}$ and $D_{\theta}$ are the polar components of the stresses and electric displacement; $t$ is the time variable denoting the time dependence of viscoelastic materials.

From equation (5) we see that even the definition has a unified mathematical form, it depends on the matrix of singular orders $\Delta$. Thus, it is important for us to have advanced study on this matrix. Moreover, because the matrix form solution shown in (1) preserves the characteristics of scalar form solution for cracks and can cover all the possible situations of interface corners, it deserves to have a further analytical study on this matrix form near tip solution. Therefore, in our recent study (Hwu, 2012) most of the efforts were put on the analytical derivation of the matrix power function form near tip solutions. Since this matrix power function form is valid for all possible cases of interface corners, all kinds of singularities including the negative power exponential, the oscillatory and the logarithmic singularity were discussed. Most of the discussions are related to the determination of the matrix of singular orders $\Delta(\theta)$. To cover all the possible conditions, the solutions of $\Delta(\theta)$ are categorized into the following three parts.

Case 1: No matter the singular orders $\alpha=1, 2, 3$, are real or complex, repeated or distinct, their associated eigenfunctions $\lambda_{\alpha}(\theta), \alpha=1, 2, 3$ are independent each other, i.e., $\lambda_1(\theta) \neq \lambda_2(\theta) \neq \lambda_3(\theta)$.

Case 2: One of the singular orders $\alpha=1, 2, 3$, is a double root and no enough independent eigenfunctions exist, i.e., if $\alpha = 1, 2$, $\lambda_1(\theta) = \lambda_2(\theta)$.

Case 3: One of the singular orders $\alpha=1, 2, 3$, is a triple root and no enough independent eigenfunctions exist, i.e., if $\alpha = 1, 2, 3$, $\lambda_1(\theta) = \lambda_2(\theta) = \lambda_3(\theta)$.

According to the above three categories, the matrix of singular orders $\Delta(\theta)$ can be determined as follows (Hwu, 2012).

$$\Delta(\theta) = \Lambda^*(\theta) < \delta^* \Lambda^{-1}(\theta), \quad \Lambda^*(\theta) = \Omega(\theta)\Lambda(\theta), \text{ for case 1},$$

$$\Delta(\theta) = \tilde{\Lambda}^*(\theta) < \delta^* \tilde{\Lambda}^{-1}(\theta), \quad \tilde{\Lambda}^*(\theta) = \Omega(\theta)\tilde{\Lambda}(\theta), \text{ for case 2},$$

$$\Delta(\theta) = \tilde{\Lambda}^*(\theta) < \delta^* \tilde{\Lambda}^{-1}(\theta), \quad \tilde{\Lambda}^*(\theta) = \Omega(\theta)\tilde{\Lambda}(\theta), \text{ for case 3},$$

where
\[ \mathbf{\Lambda}(\theta) = [\lambda_1(\theta) \; \lambda_2(\theta) \; \lambda_3(\theta)], \quad \Omega(\theta) = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]  
\[ \hat{\mathbf{\Lambda}}(\theta) = [\hat{\lambda}_1(\theta) \; \hat{\lambda}_2(\theta) \; \hat{\lambda}_3(\theta)], \quad \tilde{\mathbf{\Lambda}}(\theta) = [\tilde{\lambda}_1(\theta) \; \tilde{\lambda}_2(\theta) \; \tilde{\lambda}_3(\theta)], \]  
(6b)

and

\[ <\delta_a> = [\begin{array}{ccc} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{array}], \quad <\delta_a> = [\begin{array}{ccc} \delta_1 & 1 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & \delta_3 \end{array}], \quad <\delta_a> = [\begin{array}{ccc} \delta_1 & 1 & 0 \\ 0 & \delta_1 & 2 \\ 0 & 0 & \delta_3 \end{array}]. \]  
(6c)

The over dot means differentiation with respect to the singular order, i.e.,

\[ \dot{\lambda}_1(\theta) = \frac{\partial}{\partial \delta_1} \{\lambda_1(\theta)\}, \quad \ddot{\lambda}_1(\theta) = \frac{\partial^2}{\partial \delta_1^2} \{\lambda_1(\theta)\}. \]  
(7)

In matrix operation, it is known that if \( f(\mathbf{\Lambda}) = \sum_{n=0}^{\infty} c_n \mathbf{\Lambda}^n \) converges, and if \( \mathbf{\Lambda} \) is similar to a diagonal matrix, such as \( \mathbf{\Lambda}^{-1} \mathbf{\Lambda}^\alpha = <\delta_\alpha> \) shown in (6c), then \( f(\mathbf{\Lambda}) = \mathbf{\Lambda}^\alpha <\delta_\alpha> \mathbf{\Lambda}^{-1}. \) With this understanding, the matrix power function \((r/\ell)^{\mathbf{\Lambda}(\theta)}\) appeared in (1) can be calculated by

\[ \begin{align*}
(\ell/r)^{\mathbf{\Lambda}(\theta)} <\delta_\alpha> & \mathbf{\Lambda}^{-1}(\theta), & \text{for case 1,} \\
(\ell/r)^{\mathbf{\Lambda}(\theta)} <\delta_\alpha> & \mathbf{\Lambda}^{-1}(\theta), & \text{for case 2,} \\
(\ell/r)^{\mathbf{\Lambda}(\theta)} <\delta_\alpha> & \mathbf{\Lambda}^{-1}(\theta), & \text{for case 3.}
\end{align*} \]  
(8)

From (6), we see that the matrix of singular orders is constructed based upon the singular orders \( \delta_\alpha \) and their associated eigenfunctions \( \lambda_\alpha(\theta), \alpha = 1, 2, 3 \), whose explicit solutions can be found in (Hwu, 2010) for the general multi-material wedges. It has also been proved that \( k(\theta) \) is related to \( k \) by (Hwu, 2012)

\[ k(\theta) = \Omega(\theta)\mathbf{\Lambda}(\theta)\mathbf{\Lambda}^{-1}k, \]  
(9a)

where

\[ \mathbf{\Lambda} = \mathbf{\Lambda}(0). \]  
(9b)

4. Cracks/corners/interface cracks/interface Corners

Since cracks in homogeneous materials are special cases of interface cracks with two identical materials, and interface cracks are special cases of interface corners with two flat corner angles, it is believed that a unified definition with a unified unit for the stress intensity factors is the best way to build a connection among cracks, interface cracks, corners and interface corners even their singular orders may have different types. To see more clearly, we now reduce the matrix form solution to some special and familiar cases.
(i) Cracks in homogeneous anisotropic elastic materials

When a crack is located in a homogeneous anisotropic elastic material, its singular orders have been obtained analytically as

\[ \delta_1 = \delta_2 = \delta_3 = 0.5. \]  \hspace{1cm} (10)

Thus, no matter how the associated eigenfunction matrix \( \Lambda(\theta) \) looks like, according to \( (6a) \) the matrix of singular orders \( \Lambda(\theta) \) will always be \( \Lambda(\theta) = 0.5I \) where \( I \) is an identity matrix, and hence

\[ (r/\ell)^{-\Lambda(\theta)} = (r/\ell)^{-\Lambda} = r^{-0.5} = 1/\sqrt{r}. \]  \hspace{1cm} (11)

The definition given in \( (3) \) can then be reduced to the conventional definition

\[ k = \lim_{r \to 0} \sqrt{2\pi r} \sigma(r, 0). \]  \hspace{1cm} (12)

Using this result together with the eigenfunction matrix \( \Lambda(\theta) \) for the isotropic materials (Hwu, 2012), the near tip solution \( (1) \) and the generalized stress intensity factors \( (4) \) can be reduced to

\[
\begin{align*}
\sigma_{\theta\theta}(r, \theta) &= \frac{K_1(\theta)}{\sqrt{2\pi r}}, \quad \sigma_{\varphi\varphi}(r, \theta) = \frac{K_{II}(\theta)}{\sqrt{2\pi r}}, \quad \sigma_{\varphi\varphi}(r, \theta) = \frac{K_{III}(\theta)}{\sqrt{2\pi r}},
\end{align*}
\]

(13a)

where

\[
\begin{align*}
K_1(\theta) &= \frac{1}{2} \cos \frac{\theta}{2} \{K_1(1 + \cos \theta) - 3K_{II} \sin \theta\},
K_{II}(\theta) &= \frac{1}{2} \cos \frac{\theta}{2} \{K_1 \sin \theta + K_{II}(3\cos \theta - 1)\}, \quad K_{III}(\theta) = K_{III} \cos \frac{\theta}{2}.
\end{align*}
\]

(13b)

Equation \( (15) \) is exactly the same as the well-known near tip solutions (Broek, 1974).

(ii) Interface cracks between two dissimilar orthotropic materials

When a crack is located on the interface between two dissimilar orthotropic materials, the singular orders and their associated eigenfunction matrix for orthotropic bi-materials can be found in (Hwu, 2011). Using the known results of \( \delta_{\sigma} \) and \( \Lambda \), the matrix of singular order \( \Lambda \) can be obtained by \( (6) \) and the definition of the stress intensity factors \( (3) \) can be reduced to

\[
\begin{align*}
K_1 &= \lim_{r \to 0} \sqrt{2\pi r} \left\{ -\sqrt{D_{11}/D_{22}} \sin[\epsilon \ln(r/\ell)]\sigma_{\theta\theta}(r, 0) + \cos[\epsilon \ln(r/\ell)]\sigma_{\varphi\varphi}(r, 0) \right\},
K_{II} &= \lim_{r \to 0} \sqrt{2\pi r} \left\{ \cos[\epsilon \ln(r/\ell)]\sigma_{\theta\theta}(r, 0) + \sqrt{D_{22}/D_{11}} \sin[\epsilon \ln(r/\ell)]\sigma_{\varphi\varphi}(r, 0) \right\},
K_{III} &= \lim_{r \to 0} \sqrt{2\pi r} \sigma_{\varphi\varphi}(r, 0),
\end{align*}
\]

(14)

where \( \epsilon \), \( D_{11} \) and \( D_{22} \) are, respectively, the oscillatory index and the components of the bi-material matrix \( D \) whose solutions for the general anisotropic bi-materials can be found in (Ting, 1986; Hwu, 2010). The definition \( (14) \) is exactly the same as that proposed by Rice (1988) for isotropic bi-material interface cracks whose \( D_{11}/D_{22} = 1 \).

(iii) Interface corners between two dissimilar isotropic materials
For the interface corners between two dissimilar isotropic materials, the explicit expressions of the key matrix \( \hat{N} \) can be found in (Hwu, 2012). Through this key matrix, the singular orders and eigenfunctions can be obtained analytically. Since it is too complicated to write down the explicit solutions of \( \delta_\theta \) and \( \lambda_{\alpha}(\theta) \) for the general cases of interface corners, only the results of interface cracks are shown below.

\[
\Delta(\theta)=\begin{bmatrix}
0.5+i\epsilon f_1(\theta) & -2i\epsilon f_2(\theta) & 0
\\
2i\epsilon f_1(\theta) & 0.5-i\epsilon f_1(\theta) & 0
\\
0 & 0 & 0.5
\end{bmatrix}, \quad k(\theta)=\begin{bmatrix}
[iK_y(\lambda_{11}^+ - \lambda_{11}^-) + K_{II}(\lambda_{12}^+ + \lambda_{12}^-)] / 2
\\
[iK_y(\lambda_{22}^+ - \lambda_{22}^-) + K_{II}(\lambda_{22}^+ + \lambda_{22}^-)] / 2
\\
K_{II}\lambda_{33}^+
\end{bmatrix},
\]

(15a)

where

\[
f_1(\theta) = (\lambda_{11}^+\lambda_{12}^- + \lambda_{12}^+\lambda_{11}^-) / \lambda_0, \quad f_2(\theta) = \lambda_{11}^+\lambda_{12}^- / \lambda_0, \quad f_3(\theta) = \lambda_{22}^+\lambda_{21}^- / \lambda_0, \quad \lambda_0 = \lambda_{11}^+\lambda_{12}^- - \lambda_{12}^+\lambda_{11}^-.
\]

(15b)

\( \lambda_y^+, i, j = 1, 2 \) are the components of the rotated eigenfunction matrix \( \Lambda^+ \). With the results of (15), the stresses near the tip of interface cracks between two dissimilar isotropic materials can be obtained by the matrix form solution (1), which can be proved to be identical to the explicit solutions presented in (Hwu, 2012).

5. Conclusions

From the analytical expressions shown in this paper we see that the matrix power function not only provides a compact matrix form near tip solution, but also gives a unified definition of the stress intensity factors valid for all possible multi-material interface corners. The materials can be anisotropic, piezoelectric, viscoelastic and/or all the degenerate cases of these three kinds of materials. The interface corners can be any combination of multi-material wedges, and include corners, cracks, and interface cracks as special cases.

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References


