This paper extends previous work on the modal logic CK as a reference system, both proof-theoretically and model-theoretically, for a correspondence theory of constructive modal logics. First, the fundamental nature of CK is discussed and compared with the intuitionistic modal logic IK which is traditionally taken to be the base line. Then, it is shown, that CK admits of a cut-free Gentzen sequent calculus G-CK which has (i) a local interpretation in constructive Kripke models and (ii) does not require explicit world labels. Finally, the paper demonstrates how non-classical modal logics such as IK, CS4, CL, or Masini’s deontic system of 2-sequents arise as theories of CK, presented both as special rules and as frame classes.

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1. Introduction

Arguably one of the most intriguing modal theories, one which has created a world of its own, is intuitionistic logic. It has first been conceived (Glivenko, Kolmogorov, Heyting) as a formalisation of constructive reasoning rejecting the principle of the Excluded Middle and was later identified (Gödel, Kripke) as a fragment of Lewis’ S4 modal system which strengthens classical implication $C \to D$ by (hereditary) necessity $C \vdash D = \Box(C \to D)$ using an S4-type $\Box$ modality. Intuitionistic logic has widespread applications in the constructive foundations of Mathematics as well as in Computer Science where the close connection between proofs and computations has set off a rich body of work on $\lambda$-calculi and type theories for programming languages. These exploit the fundamental correspondence, known as the Curry–Howard Isomorphism, which permits us to interpret proofs of constructive implications $C \vdash D$ as functional $\lambda$-programs computing values of type $D$ from values of type $C$. The simply-typed $\lambda$-calculus [1] which is isomorphic to the proof algebra of intuitionistic propositional logic (IPL) is the archetypal example from which many powerful extensions have been generated such as System F or the Calculus of Constructions.

Intuitionistic logic is an implicit modal theory encapsulating a notion of constructive proof and computation. It is natural to add modal operators on top of it to capture further intensional aspects of computation. Indeed the study of intuitionistic modal logic and specifically of modal type theories has attracted a lot of interest. The two most well-known modalised type theories are variations of computational type theory which go back to the work of Moggi [2] and variations of modal type theory initiated by Kobayashi [3], Pfenning & Wong [4] and Bierman & De Paiva [5].

Without Excluded Middle and the classical double negation duality we are looking at independent $\Box$ and $\Diamond$ modalities which may or may not have related interpretations. Defining axioms for one operator does not necessarily determine the properties of the other. While the necessity modalities are well understood, matters are not so clear cut regarding the possibility modality. Indeed, there seems to be a basic divide between intuitionistic modal logics and modal type theories
regarding the behaviour of $\Diamond$ and the question of what should be considered the constructive equivalent of system $K$. In the following we discuss the issue and then pin down the contribution of this paper.

1.1. Intuitionistic $K$ versus constructive $K$

The traditional approach in intuitionistic modal logics is to dualise the standard algebraic characterisation of $\square$ as a monotonic $\wedge$-preserving operator of intuitionistic propositional logic (IPL) and to define $\Diamond$ as a monotonic $\lor$-preserving modality. There are two equivalent axiomatisations of this idea, Plotkin and Stirling's $IK$ [6,7] and Fischer-Servi's system [8], called FS in [9]:

**Axioms (IK)**
- All theorems of IPL
- $IK1: \square(A \supset B) \supset (\square A \supset \square B)$
- $IK2: \square(A \supset B) \supset (\square A \supset \square B)$
- $IK3: \neg \square \neg$  
- $IK4: \Diamond (A \lor B) \supset (\Diamond A \lor \Diamond B)$
- $IK5: (\Diamond A \supset \square B) \supset \Diamond (A \supset B)$

**Axioms (FS)**
- All theorems of IPL
- $FS1: \square \top$
- $FS2: \square(A \land B) \equiv (\square A \land \square B)$
- $FS3: \neg \square \bot$
- $FS4: \Diamond (A \lor B) \equiv (\Diamond A \lor \Diamond B)$
- $FS5: (\Diamond A \supset \square B) \equiv (\Diamond A \supset \square B)$
- $FS6: (\Diamond A \supset \square B) \equiv (\Diamond A \supset \square B)$

**Rules**
- MP: $A$ and $B \supset B$ implies $B$
- Nec: $A$ implies $\square A$

Like classical K, the logic $IK/FS$ admits of an elementary Kripke style model theory and various extensions, such as IS4, IS4.3, IS5, may be rendered in terms of characteristic frame classes [6]. The systems $IK/FS$ arises from the standard intuitionistic semantics of the propositional connectives and the interpretation of $\square$, $\Diamond$ as universal and existential quantifiers over accessible worlds in an intuitionistic meta-theory:

$$x \models C \iff \forall y.x \preceq y \Rightarrow \forall z. yRz \Rightarrow z \models C$$
$$x \models C \iff \exists z.xRz \land z \models C$$

(1)

(2)

where $R$ is the modal accessibility relation and $\preceq$ a reflexive and transitive refinement relation capturing a notion of partial information increase. In intuitionistic logic all propositions must be closed under refinement, i.e., $x \models C$ and $x \preceq y$ implies $y \models C$. For this to hold under definition (2) of $\Diamond$, the models of $IK/FS$ need to satisfy confluence between $\preceq$ and $R$, i.e., the frame condition $\preceq^{-1}; R \subseteq R; \preceq^{-1}$ where $;$ denotes composition of relations. Many results on these intuitionistic variants of normal modal logics may be derived from the fact that they can be embedded into the classical two-dimensional modal logic $S4 \otimes K$ [9].

However, there is something odd about the “normal” interpretation. First, it is unsatisfactory that the Kripke semantics for the basic system should already require a frame condition connecting the modal and intuitionistic dimensions. The semantics given in [9] even reduces (1) back to the classical condition $x \models C \iff \forall y. yRz \Rightarrow z \models C$ in favour of introducing yet another frame condition, $\models; R \subseteq R; \models$. Such frame conditions indicate that we are not in a free and irredundant set-theoretic representation. Instead, $IK/FS$ looks more like a special theory relative to some more elementary class of models. Furthermore, definition (2) is not only in need of a frame condition, it is also directly responsible for the axioms $FS4/IK4$ as well as $FS5/IK5$, which are not unproblematic from a constructive point of view.

First, take the axiom $FS4/IK4$ of Disjunctive Distribution, viz. $\Diamond (A \lor B) \supset (\Diamond A \lor \Diamond B)$ generated by the confluence frame condition $\preceq^{-1}; R \subseteq R; \preceq^{-1}$:

- Although confluence is natural for monotonic accessibility functions $R$, it is not for accessibility relations. Suppose $R$ models a non-deterministic choice arising from partiality of information: For some computational object or state $x$ there may be two options of successors $xRy_1$ and $xRy_2$, which are undecided because of lack of information about $x$. After refining $x$ to $x'$ by additional information, i.e., $x \preceq x'$, this information deficit might be resolved so that the option $x'Ry_1$ is no longer applicable. In this case, we may not even have $x'Ry'_1$ for any refinement of $y'_1 \models y_1$, as the confluence of $R$ and $\preceq$ would require.
- Suppose $\Diamond C$ means “in some context $C$”. We may well be able to construct a proof which is guaranteed to decide $A \lor B$ “in some context” but this does not mean we can construct a proof of $\Diamond A \lor \Diamond B$ which would have to decide outright whether $\Diamond A$ or $\Diamond B$ holds, i.e., whether “in some context” $A$ or “in some context” $B$ is true. This is problematic if the decision between $A$ and $B$ in $\Diamond (A \lor B)$ depends on the contextual circumstances. If contextual reasoning involves expending computational resources then the decision $A \lor B$ can only be constructed by actually entering the context and thus cannot be anticipated.
- If $\Diamond$ reads “true in classical logic” it would certainly be constructive to assume $\Diamond (A \lor \neg A)$ but not $\Diamond A \lor \Diamond \neg A$. Or take $\Diamond$ to mean “under normal circumstances”: It would seem easy to maintain constructively that $\Diamond (I'll\ show\ up\ at\ 7am \lor I'll\ call\ you)$ but rather difficult to decide whether $\Diamond I'll\ show\ up\ at\ 7am$ or $\Diamond I'll\ call\ you$ is true.
Contextual interpretations of ◇ have been considered by Curry in the 50ies [10] and it is well-known [11] that these do not satisfy Disjunctive Distribution. Presumably the first to explicitly argue this case was Wijesekera [12,13] who observed that the natural interpretation of ◇ in constructive concurrent dynamic logic (CCDL) does not satisfy FS4/IK4. The same applies to the Beth–Kripke–Joyal cover semantics of modalities [14] which implement a notion of local truth familiar from topos theory. The lack of distribution is typical for modal type theories, specifically modal logics under the judgemental reconstruction of Pfennig [15,16].

Similar problems arise in the interaction between ◇ and □ exhibited by scheme IK5/FS5 which stems from the semantic clause (2). The following examples suggest there may not be a universal computational justification for the implication \((◇A ⊃ □B) ⊃ □(A ⊃ B)\).

- Again read ◇C generically as “in some context C” and □C as “in all contexts C”. The precondition ▷A ⊃ □B amounts to a construction that gives a proof of B in all contexts under the assumption that A holds in some context. This does not universally warrant the conclusion □(A ⊃ B) that in all contexts the assumption A would entail B. Say, if we take “context” to mean “accessible economic context”. We might be able to realise the statement “◇(financial funds are available) ⊃ □(humanitarian aid can be provided)” because in the current economic context we have a strategy to convert money from some accessible business context into food that can be used for humanitarian aid in all economic contexts, where needed. However, it is not guaranteed that in all accessible economic contexts where financial funds are available we could also provide humanitarian aid. For instance, under the conditions of the local context money cannot buy food if there is no-one who sells it.

- Let us take an interpretation from programming, related to CCDL and the realisability model of Kobayashi [3]: The possibility ◇C might be the statement that some execution of process p yields a value of type C and □C means all executions of p generate a value of type C. Additionally we stipulate that a constructive proof of ◇C or □C must guarantee that these statements are based on static information available about p and do not depend on possible further refinement of the program code p (e.g., instantiation of run-time parameters, calibrations, etc.). It may happen that p, in some run-time context, generates A but not B. Then, □(A ⊃ B) is false. At the same time there may be some other instantiation in which none of its executions generate A. Hence, ◇A is false as well. But then ▷A ⊃ □B is trivially true. This contradicts validity of IK5/FS5 which would require ▷A ⊃ □B) to be provable.

- Nanevski [17] presents a modal lambda calculus where modality □5C specifies suspended expressions which read from a dynamic store S to produce values of type C while ◇5C types expressions that first destructively update the global store and then produce a value of type C in the updated store. In this system the evaluation of an expression of type □5(A ⊃ B) may depend on dynamic store but the resulting function of type A ⊃ B must be pure. Such a pure function from input A to output B cannot be produced from an expression of type ◇5A ⊃ □5B since the latter only provides a result of type □5B which is a suspended value of type B depending on the store rather than a pure value of type B.

Finally, consider the axiom FS3/IK3. It is known that fallible worlds, in which all propositions become true, need to be added to the intuitionistic Kripke models in order to admit of a constructive meta-theory [18]. Since IPL has no modalities such fallible worlds have no influence on the class of theorems. With modalities, however, this is no longer the case as fallible worlds may be accessible from non-fallible worlds through the modal relation R and so become ‘visible’ in the form of ◇⊥ statements. Hence, with fallible worlds being present, the axiom FS3/IK3 is no longer valid. E.g., a proof/program of type ◇5⊥ may be a non-terminating process which locks up while updating the store S, which should not canonically induce (through an axiom like FS3/IK3) a program of type ⊥ which locks up straight away.

It seems to us that in computational type theories [2,19,20] or modal type theories [3,16,17,21], where constructive proofs turn into λ-programs, the schemes FS3/IK3–FS5/IK5 fail to have a uniform computational justification. On the other hand, the schemes that do appear to be computationally justified are FS1/IK1, FS2/IK2 and FS6. Restricting to these axioms yields the constructive system known as CK [22,23] with the two equivalent presentations:

<table>
<thead>
<tr>
<th>Axioms (CK-1)</th>
<th>Axioms (CK-2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>All theorems of IPL</td>
<td>All theorems of IPL</td>
</tr>
<tr>
<td>IK1: ◇(A ⊃ B) ⊃ (◇A ⊃ □B)</td>
<td>FS1: □T</td>
</tr>
<tr>
<td>IK2: □(A ⊃ B) ⊃ (◇A ⊃ □B)</td>
<td>FS2: □(A ∨ B) ⊃ (◇A ⊃ □B)</td>
</tr>
<tr>
<td>Rules</td>
<td>Rules</td>
</tr>
<tr>
<td>MP: A and A ⊃ B implies B</td>
<td>MP: A and A ⊃ B implies B</td>
</tr>
<tr>
<td>Nec: A implies □A</td>
<td>Reg: A ⊃ B implies □A ⊃ □B and ◇A ⊃ □B</td>
</tr>
</tbody>
</table>

CK is non-normal regarding ◇ because of the lack of Disjunctive Distribution and thus the general techniques for standard intuitionistic modal logics (e.g., for IntKω ◇ [9, Chapter 10]) do not directly apply. Still, constructive modal logics have adequate Kripke semantics [12,24,25,23] or topological semantics [26]. They are based on (1) for □ and the stronger interpretation of ◇ given as

\[ x \models ◇C \iff \forall y. x \ll y \Rightarrow \exists z. y R z \wedge z \models C \tag{3} \]
to replace (2). This forces \( \Box C \) to be hereditary for refinement \( < \) without a frame property. Also, this does away with schemes FS4/IK4 and FS5/IK5 which now turn into non-trivial frame properties generating proper extensions of CK. If we also add fallible worlds then FS3/IK3 is removed as well (see [23]). Alternatively, as demonstrated in [27], from a proof-theoretic perspective CK can be seen as a fragment of ICK.

CK is well known [12] as a constructive modal logic where the modality is based on an accessibility relation between partial states of information. For instance, in CCDDL [13] the accessibility relation acts on partial information about a complete machine state. CK is the normal modal structure of relational spaces, which are standard relational frames equipped with topologies [26]. Also, as indicated above, CK modalities arise from local notions of truth [14].

Note that CK (without ICK/FS3) is not the intuitionistic analogue to classical K in the sense of Fischer-Servi and Simpson [7] since adding Excluded Middle \( C \lor \neg C \equiv \top \) does not collapse the theory to give classical K. It is an open question whether there exists a true analogue of K between CK and ICK that is both constructively acceptable and returns classical K under Excluded Middle. It may even be doubted that such a constructive analogue exists, since the classical collapse criterion makes it impossible to give logical explanations for phenomena that are classically inconsistent but nevertheless constructively consistent. In any case, CK should be a reasonable common base point from which a correspondence theory for constructive modal logics can be attempted.

1.2. Towards a correspondence theory based on CK

Of course, all of the theories of ICK/FS fall into the remits of a correspondence theory based on CK. Yet, there are theories of CK, which are not at the same time extensions of ICK:

- the propositional fragment of Wijesekera’s CCDDL [12,13] or the logic of modal frames [26] which are CK plus the ICK/FS3 axiom \( \Box \bot \);
- the propositional fragment of Fitch’s \( M \) [28] which is CK plus the \( T \)-axioms \( \Box C \supset C, C \supset \Box C \);
- Masini’s I-2SC theory [29] which is CK plus axiom \( \Box \bot \) and the deontic scheme \( \Box C \supset \Box C \);
- the 54-style modal type theory CS4 [3,16,25,14], which is CK plus the \( T \)-axioms \( \Box C \supset C, C \supset \Box C \) as well as the \( 4 \)-axioms \( \Box C \supset \Box C, C \supset \Box C \) as well as the \( 3 \)-axioms \( \Box C \supset \Box C, \Box C \supset \Box C \) [3] also adds \( \Box \bot \);
- Computational Logic (CL) [30,11], also known as propositional lax logic (PLL) [24,14], which extends CS4 by the axiom \( C \supset \Box C \). This collapses \( \Box \) and strengthens \( \Box \) to a modality equipped with the tensorial strength \( (C \land D) \supset C \land D) \);
- the 55-style modal type theory Lambda 5 [31] which employs the modalities \( \Box \) and \( \Box \) to express mobile code for distributed computations. The type system is developed as a judgemental sequent calculus following Martin-Löf’s [32] notion of hypothetical judgements.

CK is a promising core system able to explain various constructive modal logics that have found applications in the literature as specialised semantic theories defined by frame classes. E.g., it is known how IK, CS4, PLL/CL arise from CK in terms of Kripke models [25]. However, a systematic CK correspondence theory has not yet been attempted and so a rich lattice of constructive modal theories still lies undiscovered.

The game is even more interesting than in the classical case: A satisfactory correspondence theory for constructive logic will not only cover the existential aspects. It should also address the computational aspects in the sense of building Curry–Howard correspondences between proof systems and \( \lambda \)-calculi. In other words, a correspondence theory for CK should not only relate theories and (Kripke) models but also logic and programming in the spirit of the propositions-as-types and proofs-as-programs paradigm. Some systems of constructive modal logic have been successfully investigated in this respect. Computational type theory [2,30], which links proofs in PLL/CL with terms in the computational \( \lambda \)-calculus, has found many applications, e.g. to accommodate non-functional (“impure”) features [33], in strictness analysis and partial evaluation [34,35] or for constraint extraction [19,20] just to name a few. PLL/CL is a specialised theory of CS4 which in turn is the logic of modal type theory [3,16,21,5,4]. Modal type theories have found use to deal with higher-order abstract syntax [36,37], meta-variables [38], staged computations [39,40], distributed computing [31,41] and much more. Recently, Nanevski [17] (see also [21, Sec. 9.1]) has demonstrated the usefulness of combining CS4-style \( \Box \)-types for capturing destructive updates with \( \Box \)-types for dynamic binding.

Outside of PLL/CL or CS4 the proof-theoretic treatment of constructive \( \Box \) and \( \Box \) and their computational interpretation of modal proofs following the Curry–Howard Isomorphism (see, e.g. the discussion in [22]) is still largely unexplored. While \( \Box \) seems well behaved, \( \Box \) poses problems.

Masini [29] presented a cut-free Gentzen system for CK \( \vdash \Box \bot \vdash \Box C \supset \Box C \) but only proofs in the \( \Box \)-fragment have been turned into an associated \( \lambda \)-calculus [42]. Because of the deontic axiom \( \Box C \supset \Box C \), the \( \Box \)-fragment of Masini’s system generates an infinity of \( \Box \)-nested theorems of the form \( \Box T, \Box \Box T, \Box \Box \Box T \) or \( \Box \Box \Box \Box T \) alternations \( \Box \Box \Box \Box T \), which form a profusion of proof constants stating the existence of implicit trees of worlds and functions on worlds. Clearly, without universal collapse axioms, like those in CS4, such proof terms are difficult to handle coherently in a logical \( \lambda \)-calculus that has to satisfy Church–Rosser, subject reduction and strong normalisation.

Yet, what about CK proper? In CK there are no such oracle proofs for constructive \( \Box \)-existence statements. Each proof of \( \Box \) must be obtained by instantiating another hypothesis of form \( \Box \). For instance, any closed proof of a proposition \( \Box C \) already amounts to a closed proof of \( C \) and any closed proof of \( \Box C \supset \Box D \) is essentially a closed proof of \( C \supset D \) together
with the identity function on contexts. Pure CK offers more coherence at the outset, so that it may be a better candidate for a constructive "Ur"-calculus which would induce the known modal λ-calculi for computational and modal type theory as specialised algebraic models.

Another important result has been obtained by Bellin, de Paiva and Ritter in [22] presenting a Curry–Howard correspondence for CK proper, in the form of a natural deduction calculus with a category-theoretical semantics. However, the proposed computational interpretation has the disadvantage that the modalities □ and ◻ are not explained independently and locally in terms of introduction (constructors) and elimination (destructor) rules. The algebraic semantics of [22] is based on the traditional sequent rule \( I, C \vdash D \rightarrow \Box I, \Box C \vdash \Diamond D \) of logic K which (i) couples an occurrence of ◻ on the right-hand side of \( \vdash \) with another occurrence of □ on the left-hand side and (ii) introduces □ \( \textit{globally} \) on all context hypotheses \( I \) at the same time. This amounts to a global treatment of context which is not fully in the spirit of Gentzen and the ultimate objective reported in [22]: "... entering a context would correspond to making an assumption in natural deduction, while exiting a context would correspond to discharging an assumption" as we understand it here.

In this paper we take a next step in this programme and present a sound and complete cut-free Gentzen calculus for CK with independent introduction and elimination rules for □ and ◻ as in Masini's work but for the pure system CK of [22]. Like the known calculi for CS4 concerning □ [16,4] our calculus is of local nature and does not need explicit worlds as labels. It is derived from the multi-sequent calculus [43] developed for multi-modal CK [44]. Though being cut-free, the system in [43] is not in proper (first-order) Gentzen format as its introduction and elimination rules involve sets rather than individual formulas. Also, the cut-elimination proof was obtained by semantic means. Thus, the system in [43] is not suited to be interpreted as a computational calculus of contexts. In this paper, we fix the problem and present a local interpretation of constructive □ and ◻ modalities. Each operator has its own intro and elim rule and both are dual similar to ∀ and ∃ in intuitionistic logic.

Formally, our sequents for CK resemble the 2-sequents of Masini [45,29]

\[
\Gamma_1 \gg \Gamma_2 \cdots \gg \Gamma_n \vdash \Phi_1 \gg \Phi_2 \cdots \gg \Phi_m
\]

which feature sequences of contexts \( \Gamma_i \) in the antecedent and sequences of contexts \( \Phi_j \) in the succedent. As Masini has shown these are a good structure for □ making it possible to avoid the explicit labelling traditionally used to obtain cut-free representations in the multi-intuitionistic modal calculi, notably IK [7] and other systems under Gabbay and Queiroz labelled deduction approach [46]. However, in contrast to Masini, our sequents offer more fine-control for handling □, ◻ by introducing the notion of a focus. The focus singles out one of the context compartments \( \Gamma_i \) in the antecedent as the "current local point of construction". This makes it possible to prevent using proofs of □C without providing a reference to some constructable context in which C can be used. In this way, □C does not entail ◻C as in the deontic system of Masini. This solution is somewhat analogous to how intuitionistic predicate logic, or type theory, prevents the (classically sound) existential 'witness': Type theory permits us to instantiate a proof \( \forall x.\phi(x) \) by a term \( t \) to obtain \( \phi(t) \) only if \( t \) is constructable in the current environment. If there are no constants in the logic then \( \exists x.\phi(x) \) is not provable from \( \forall x.\phi(x) \). The addition of a 'focus' may be seen as a structural refinement of Masini's 2-sequents. However, in contrast to the purely proof-theoretic analysis of 2-sequents [29], our sequents are equipped with a possible worlds semantics which allows us to obtain soundness and completeness theorems. While there is no proof provided in [29] for exactly what modal theory is generated by the 2-sequent system we can show that our system derives precisely the theory CK defined above. Moreover, based on the Kripke semantics and the 'focus', it is possible to show that the 2-sequents can be restricted to consist essentially of at most two context compartments on either side of the turnstile, i.e., \( n, m \leq 2 \) in (4). Thus, they are of the same outer form as the sequents used in modal type theory [16,4].

2. Syntax and semantics of CK

The multi-modal theory CK\(_n\) is set in the propositional language

\[
C, D \rightarrow A | \perp | C \land D | C \lor D | C \rightarrow D | \exists R.C | \forall R.C
\]

where \( A \in \text{Var} \) ranges over a set of propositional variables and \( R \in \text{Lab} \) over modal labels representing binary relations. The modalities may also be written with indices □\_R, ◻\_R in the tradition of classical modal logic [9]. Here, we prefer the notation ∀R, ∃R for modalities to avoid indices and remind us of their origin as quantifiers. Note, that this is the common notation for quantifiers as used in description logics [47].

Definition 1. (See [23,43].) A constructive interpretation or constructive model of CK\(_n\) is a structure \( \mathcal{I} = (\Delta^\mathcal{I}, \preceq^\mathcal{I}, \perp^\mathcal{I}, \rightarrow^\mathcal{I}) \) consisting of

- a non-empty set \( \Delta^\mathcal{I} \) of worlds, the frame universe in which each element represents a partially defined entity, process or state of knowledge;
- a refinement pre-ordering \( \preceq^\mathcal{I} \), i.e., a reflexive and transitive relation on \( \Delta^\mathcal{I} \);
• a subset \( \perp^I \subseteq A^I \) of fallible worlds closed under refinement and modal accessibility, i.e., \( x \in \perp^I \) and \( x \perp^I y \) or \( x R^I y \) implies \( y \in \perp^I \); also, \( x \in \perp^I \) implies there is \( y \) with \( x R^I y \);

• an interpretation function \( I \) mapping each modal label \( R \in \text{Lab} \) to a binary relation \( R^I \subseteq A^I \times A^I \) and each atomic proposition \( A \in \text{Var} \) to a set \( \perp^I \subseteq A^I \subseteq A^I \) which is closed under refinement, i.e., \( x \in A^I \) and \( x \perp^I y \) implies \( y \in A^I \).

Constructive models \( I \) of \( \text{CK}_n \) extend the classical two-valued models of \( \text{K}_n \) by a pre-ordering \( \succeq^I \) for capturing refinement between worlds and a notion of fallible entities \( \perp^I \) for interpreting empty information. The refinement relation \( x \succeq^I y \) is typically understood as a (potential) increment of knowledge of an ideal intuitionist mathematician as s/he performs constructions deciding the truth of sentences. Fallible elements \( b \in \perp^I \) may be thought of as over-constrained states of information, self-contradictory objects of evidence or undefined computations. A constructive interpretation induces a truth valuation on arbitrary propositions as specified in the following Definition 2:

**Definition 2.** (See [23,43]) Let \( I = (\Delta^I, \succeq^I, \perp^I, -^I) \) be a constructive model. The interpretation \( I \) is lifted from atomic \( \perp, A \) to arbitrary propositions as follows, where \( \Delta^I = \{ \text{non-fallible} \text{ elements in } I \} \):

\[
\begin{align*}
\top^I & = def \ \Delta^I \\
(\neg C)^I & = def \ \{ x | \forall y \in \Delta^I x \succeq^I y \Rightarrow y \notin C^I \} \\
(C \land D)^I & = def \ C^I \cap D^I \\
(C \lor D)^I & = def \ C^I \cup D^I \\
(C \Rightarrow D)^I & = def \ \{ x | \forall y \in \Delta^I (x \succeq^I y \land y \in C^I) \Rightarrow y \in D^I \} \\
(\exists R).C)^I & = def \ \{ x | \forall y \in \Delta^I x \succeq^I y \Rightarrow \exists z \in \Delta^I (y, z) \in R^I \land z \in C^I \} \\
(\forall R).C)^I & = def \ \{ x | \forall y \in \Delta^I x \succeq^I y \Rightarrow \forall z \in \Delta^I (y, z) \in R^I \Rightarrow z \in C^I \} \\
\end{align*}
\]

We will write \( I, x \models C \) as an abbreviation for \( x \in C^I \). This is extended to sets \( I \) of propositions, i.e., \( I, x \models I \) iff \( I, x \models C \) for all propositions \( C \in I \). When there is no confusion we will identify \( I \) with \( -^I \) and simply write \( \Delta, \succeq, \perp \) instead of \( \Delta^I, \succeq^I, \perp^I \).

We obtain the standard classical models of \( \text{K}_n \) whenever \( \succeq \) is the identity relation and \( \perp \) empty. In general, the relation \( I, x \models C \) spreads out the validity of \( C \) across many \( \succeq \)-related worlds, in the sense that if \( I, x \models C \) and \( x \succeq y \) then \( I, y \models C \). In analogous worlds \( y \in \perp \) discrimination is switched off completely by making all propositions valid. This monotonicity of truth is the characteristic feature of intuitionistic semantics. Notice that the definitions of \( (\exists R).C)^I \) and \( (\forall R).C)^I \) corresponds to the readings (3) and (1), respectively. All the other clauses of Definition 2 make up the standard Kripke semantics of \( \text{IPL} \) [48].

Apart from the philosophical importance of intuitionistic Kripke frames, partial information structures \( (\perp, \succeq) \) have found widespread applications in Computer Science. This is not the place to review them. For illustration, we will consider just a few examples (see also [43]) that suggest themselves in the context of knowledge representation. Moreover, we will show that \( \text{IK}_4/\text{FS}_4, \text{IK}_5/\text{FS}_5 \) and the deontic scheme are not universally valid in constructive models. We shall see later how they can be recovered if needed. Also we shall see how to interpret the other axioms of \( \text{CS}_4 \).

**Example 1.** The deontic axiom \( \forall R.A \supset \exists R.A \) is not sound in \( \text{CK}_n \), since it is already unsound in classical \( \text{K}_n \) and by the fact that every model in \( \text{K}_n \) is also a model in \( \text{CK}_n \). Semi-antically this can be seen by observing that the classical non-serial counter model is in particular a constructive model: Take the reflexive one-world interpretation \( \Delta^I = \{0\} \) such that \( 0 \perp 0 \) and \( R^I = \emptyset \) for modal labels \( R \in \text{Lab} \). Putting \( \perp^I = A^I = def \emptyset \) we have a constructive interpretation. Then, clearly \( 0 \models \forall R.A \) since \( 0 \) has no \( R \)-successors but at the same time it means \( 0 \models \exists R.A \).

**Example 2.** Also, the scheme \( \text{IK}_5/\text{FS}_5 \) is not part of \( \text{CK}_n \). Take \( \Delta^I = \{0, 1, 2\} \) with \( R \)-connected worlds \( R^I = \{(0, 2)\} \), refinement pre-order \( \succeq^I = \{(0, 0), (1, 1), (2, 2), (0, 1)\} \) and valuation \( C^I = \{2\} \) and \( D^I = \emptyset \). Then, \( I, 0 \models \exists R.C \supset \forall R.D \) and \( I, 2 \models C \) as well as \( I, 2 \not\models D \). The implication \( I, 0 \models \exists R.C \supset \forall R.D \) holds trivially since no world which is \( \succeq \)-reachable from \( 0 \) satisfies \( \exists R.C \). This is clear for \( I \) since it does not have an \( R \)-successor at all. Regarding world \( 0 \) it suffices to observe that its \( \succeq \)-successor 1 does not satisfy \( \exists R.C \). Thus, world \( 0 \) in this model is also a counter model showing that the proposition \( (\exists R.C \supset \forall R.D) \supset \forall R.(C \supset D) \) is not a theorem of \( \text{CK}_n \).

**Example 3.** A world \( x \in \Delta \) may be an abstraction of data records appearing in an abstract data context. Each refinement \( y \) of \( x \) has all the attributes of \( x \) and on those the same values, but possibly also additional attribute dimensions. E.g., \( x \) may be the result of suppressing information in an attempt to optimise calculations on a large data base. Every application of a projection on a data base table creates an abstraction in this sense.

More concretely, let \( a = (c, d_1) \) and \( b = (c, d_2) \) be two entries in a (relational) database that share the same first attribute but are distinguished in the second. If the attributes are referenced by relations \( \$1 \) and \( \$2 \) then the situation could be
specified by \(a \$ 1 \triangleleft c, a \$ 2 d_1, b \$ 1 c, b \$ 2 d_2\). Now let us abstract from the second attributes and consider the pairs as partially defined entities \(a^2 = (c, ?)\) and \(b^2 = (c, ?)\). Ignoring \(d_1, d_2\) means that \(a^2\) and \(b^2\) carry the same information and thus can no longer be distinguished. Since the pre-order \(\sqsubseteq\) measures the information content we get \(a^2 \sqsubseteq b^2\) and \(b^2 \sqsubseteq a^2\). This cyclic refinement relationship implies an abstract equivalence \(a^2 \equiv b^2\) but not an identity \(a^2 = b^2\) keeping in mind that both sides have incompatible realisations \(a^2 \nless a\) and \(b^2 \nless b\), respectively. The situation is depicted in Fig. 1. The dashed arrows correspond to refinement and solid arrows represent the attribute relations \(\$ 1, \$ 2\). Note that both \(a^2, b^2\) have a fallible \$2 attribute, viz. the \$2-accessible \(\bot\), which corresponds to a computational deadlock when selecting \$2 in \(a^2\) or \(b^2\).

Entities \(a^2, b^2\) are indistinguishable and share exactly the same propositions. Formally, if \(\text{Th}(x)\) denotes the set of propositions validated by entity \(x\), then \(\text{Th}(a^2) = \text{Th}(b^2)\). E.g., both \(\exists!1.C \in \text{Th}(a^2) = \text{Th}(b^2)\) and \(\exists!2.(D_1 \lor D_2) \in \text{Th}(a^2) = \text{Th}(b^2)\) since every refinement of \(a^2, b^2\) has \(c \sqsubseteq C\) as attribute for relation \(\$ 1\) and either \(d_1 \sqsubseteq D_1\) or \(d_2 \sqsubseteq D_2\) as attribute for \$2. The disjunction \(\exists!2.(D_1 \lor D_2)\) captures the choice between the two realisations of \(a^2\) as a concrete individual, viz. \((c, d_1)\) and \((c, d_2)\). On the other hand, this choice cannot be resolved at the abstract level as there is no single uniform choice of the \$2-attribute. This is reflected in the logic by the fact that \(\exists!2.D_1 \not\in \text{Th}(a^2)\) \((i = 1, 2)\) which means \(\exists!2.D_1 \lor \exists!2.D_2 \not\in \text{Th}(a^2)\).

Abstractions like this cannot be expressed in intuitionistic modal logics where \(\diamond\) distributes over \(\land\) and \(\lor\). \(\exists!2.(D_1 \lor D_2)\) is semantically equivalent to \(\exists!2.D_1 \lor \exists!2.D_2\). Also, note that the Excluded Middle \(\exists!2.D_1 \lor \neg \exists!2.D_1\) is not valid for \(a^2\).

Regarding valid propositions, in Fig. 1, \(\forall \exists!2.((D_1 \lor D_2) \lor \neg C)\) is in \(\text{Th}(a^2)\), since every reachable attribute of \$2 that is in \(D_1\) or in \(D_2\) cannot be in \(C\), which is only reachable via \$1. Similarly \(\forall \exists!1.(C \lor \neg (D_1 \lor D_2))\) is in \(\text{Th}(a^2)\), expressing that for every \$1 attribute \(c\) of every refinement of \(a^2\), it holds that if \(c \sqsubseteq C\), then \(c\) is neither contained in \(D_1\) nor \(D_2\).

### 2.1. Hilbert calculus for \(\mathcal{CK}_n\)

The Hilbert calculus is given in Table 1. Part (a) shows the usual axioms for intuitionistic propositional logic [48], specifically \(1–2\) for \(\lor\), \(3–4\) for \(\land\), \(5–6\) for \(\lor\) and \(7\) for inconsistency \(\bot\). Part (b) of Table 1 lists the two principles \(\exists!K, \forall!K\) for universal and existential context quantifications following Wijesekera’s modal rules presented in [12]. These come from generalised monotonicity depicted by the rules \(M_1, M_2\) below with the important property of having in the conclusion of each rule a universal quantifier over the context \(\Gamma\):

\[
\Gamma \vdash \exists! R.C \supset \exists! R.D \quad M_1
\]

\[
\Gamma \vdash \forall R.C 
\]

\[
\Gamma \vdash \exists! R.D \quad M_2
\]

Finally, the rules of Modus Ponens MP and Necessitation Nec are given in item (c) of Table 1.
Let the symbol ⊢ H denote Hilbert deduction, i.e., Θ ⊢ H C if there exists a derivation C0, C1, . . . , Cn such that Cn = C and each Ci (i ≤ n) is either a hypothesis Ci ∈ Θ, or a substitution instance of an axiom scheme from Table 1(a), (b) or arises from earlier propositions Cj (j < i) through MP or Nec as in Table 1(c). This is lifted to sets of propositions Φ, i.e., Θ ⊢ H Φ in the usual way.

3. The Gentzen sequent calculus G-CKn

In [23] it was shown that the constructive interpretation constitutes an adequate Kripke model theory for CK and in [43] a sound and complete tableau-style calculus was presented for its multi-modal extension CKn which, however, is not suited for computational interpretations. We are now going to reconstruct this system as a cut-free Gentzen calculus for CKn which provides a starting point for extracting an associated λ-calculus. The new calculus, G-CKn, has the same local nature and symmetrical treatment of □ and ◦, in terms of left and right introduction rules corresponding to constructors and destructors of proof terms, as the calculi for computational and modal type theory, but without prescribing any additional axioms. Moreover, all the rules have a semantic interpretation as local constructions on Kripke worlds.

G-CKn sequents have the form Σ ⊢ ψ in which Σ is the antecedent and ψ is the succedent. Like the 2-sequents of Masini [29] both are sequences of sets of propositions in which the information is separated into individual scopes. But now these scope compartments are connected by named accessibility relations and we also add a focus marker. Specifically, a general antecedent looks like

\[ \Sigma = \Gamma_1 \gg S_2 \gg \Gamma_2 \gg \cdots \gg S_f \gg \Gamma_f \gg \cdots \gg S_n \gg \Gamma_n \]  

where \( S_i \in \text{Lab} \) are modal labels. Each \( \Gamma_i \) is a scope \( \Gamma_i = [C_{i1}, C_{i2}, \ldots, C_{im}] \) containing local assumptions \( C_{ij} \). As seen in (5) one of these scopes is distinguished by a special marker \( \cdot \), called the focus. The associated set of local assumptions \( \Gamma_f \) is the scope in focus. The focus splits the antecedent into two parts \( \Sigma = \Sigma_f \cdot \Sigma_n \), the context \( \Sigma_f = \Gamma_1 \gg S_2 \gg \cdots \gg S_{f-1} \gg \Gamma_f \) before the focus and the hypotheses \( \Sigma_n = \Gamma_f \gg S_{f+1} \gg \cdots \gg S_n \gg \Gamma_n \) after the focus. It is useful to think of the focus marker \( \cdot \) as a binary operator on sequences of scopes. Note that the scope in focus \( \Gamma_f \) logically belongs to both parts of the antecedent, i.e., the context and the hypotheses.

The context part \( \Sigma_f \) of the antecedent represents the assumption that the scope in focus \( \Gamma_f \) is accessible through a connected path \( S_2, S_3, \ldots, S_f \) of modal context switches passing through scopes \( \Gamma_i \) (1 ≤ i < f). Each \( \Gamma_i \) specifies the information available in the respective scope. The focus marks the active scope relative to which the sequent’s actual judgement \( \Sigma_n \vdash \psi \) is made. The judgement informally states that if the sequence of scopes specified by the hypothesis \( \Sigma_n \) is constructable, then the sequence of assertion scopes specified by the succedent

\[ \psi = \Phi_1 \gg R_2 \gg \Phi_2 \gg R_3 \gg \Phi_3 \gg R_4 \gg \cdots \gg R_m \gg \Phi_m \]  

in some modal labels \( R_j \in \text{Lab} \), is constructable, too. Just like with other (classical or non-classical) multi-sequent calculi, the notions of constructibility left and right of \( \vdash \) are dual: While the hypothesis sets \( \Gamma_i \) in \( \Sigma_n \) and their succession are to be taken in a conjunctive sense, the assertion sets \( \Phi_j \) and their succession in \( \psi \) are interpreted disjunctively. This means that \( \psi \) as in (6) is constructable if some scope \( \Phi_j \) (1 ≤ j ≤ m) is reachable (from the focus point) through a connected path \( R_2, R_3, \ldots, R_j \) so that at least one of the propositions \( D \in \Phi_j \) is true. Constructibility on the hypothesis side \( \Sigma_n \) means that all scopes \( \Gamma_i \) (f ≤ i ≤ n) are reachable on a (single) connected path and all assumptions hold true at the respective scope position. The precise semantic interpretation is given in Definition 3 in Section 4.3.

The number of scope separators \( \gg \) in \( \Sigma_n \) to the right of the focus is called the length of the antecedent \( \Sigma \) and the number of separators \( \gg \) in \( \Sigma_n \) before the focus is called its depth. E.g., the context \( \Sigma \) in (5) has depth \( f - 1 \) and length \( n - f \). Thus, an antecedent with zero depth and zero length is a single set \( \Sigma = \cdot \Gamma \). There is no need for a focus marker in the succedent \( \psi \) since it always starts at the focus point of the antecedent. It is implicit and fixed to the first set \( \Phi_1 \). Accordingly, the succedent \( \psi \) has no depth but only a length \( m - 1 \).

Note that our sequents have only one focus alignment and thus make essentially only one \( \vdash \) judgement. In contrast, the 2-sequents of Masini [29] have more coupling between antecedent and succedent in the sense that \( \vdash \) is implicitly repeated from left to right for every scope in \( \Sigma \) or \( \psi \) up to the maximum of \( m \) and \( n \), patching up with \( \Gamma_1 = \emptyset \) and \( \Phi_f = \emptyset \) as needed. Therefore, Masini’s 2-sequents (for one modal label) are somewhat more expressive than ours. On the other hand, as we will see, this expressiveness is not needed to capture the mechanics of CK. Specifically, we will show that it suffices to consider sequents with at most two scopes on both sides, i.e., where \( n, m \leq 2 \). Moreover, the explicit introduction of a focus and the possibility to implicitly shift antecedent and succedent against each other is useful to implement a reading of \( \Box \) and \( \Diamond \) which does not presuppose the deontic scheme \( \Box C \supset \Diamond C \).

Before we can present the rules of the calculus to derive sequents \( \Sigma \vdash \psi \) we need to agree on a couple of meta-level syntactic conventions to handle sequents generically. To begin with, we will treat each scope \( \Gamma \) in the antecedent \( \Sigma \) (and similarly for \( \psi \)) as an unordered list without duplications, so that if \( C \in \Gamma \) then \( \Gamma \) is the same as \( \Gamma' \), C and \( \Gamma' \). This has the advantage that we do not need explicit structural rules for exchange and contraction implementing the meaning of the comma separator.

A similar abstraction applies to the focus separator. We may place the focus at the beginning of the respective local scope \( \Gamma' \) as done in (5) but also write \( \Gamma' \cdot \cdot \cdot \) to make it appear at the end. Furthermore, since the focus is a marking of the
Similarly, conclusion scope of the succedent. Again, this is crucial for soundness, except for rule \( \exists \). Notice, since we preserve soundness, as we shall see. Similarly, all the right rules are equivalent to our system.

We mind the difference between an empty sequence and a singleton sequence consisting of an empty set of assumptions. We point as in abstraction. This gives full freedom for splitting up antecedents at the focus scope. E.g., if \( \Sigma = \Gamma_1 \triangleright \Psi, \Lambda \triangleright \Psi \), then \( \Sigma = \Sigma_c \cup \Sigma_h \) in several ways:

\[
\Sigma_c = \Gamma_1 \triangleright \Psi, \quad \Sigma_h = [A, B, C] \triangleright \Psi
\]

Of course, at the level of scope sequences we enforce associativity of the separators \( \triangleright \) for breaking up a context at any point as in \( \Sigma = \Sigma' \triangleright \Sigma'' \) where \( \Sigma' \) and \( \Sigma'' \) are the corresponding sub-sequences. This includes the special case that one of the sub-sequences is empty. E.g., if \( \Sigma'' = \epsilon \) is empty then \( \Sigma' \triangleright \Sigma'' = \Sigma' \). This is not the same as \( \Sigma' \triangleright \emptyset \) keeping in mind the difference between an empty sequence and a singleton sequence consisting of an empty set of assumptions. We may assume that in a sequent \( \Sigma \vdash \Psi \) neither \( \Sigma \) nor \( \Psi \) will ever be empty, i.e., we have \( n, m \geq 1 \) in (5) and (6).

We will write \( \Sigma, C \) to say that the last scope contains \( C \), i.e., \( \Sigma, C = \Sigma' \triangleright \Sigma'' \triangleright \Gamma, C \), where \( \Sigma' \) is the initial sub-sequence of \( \Sigma \) without the final scope. If \( \Sigma' \) is empty then of course \( \Sigma' \triangleright \Sigma'' \triangleright \Gamma, C \) is the same as the singleton sequence \( \Gamma, C \). Similarly, \( \Sigma, C \) means that \( C \) is in the first scope, i.e., \( \Sigma = \Gamma, C \triangleright \Sigma' \triangleright \Gamma, C \). Again, \( \Sigma' \) may be the empty sequence in which case \( \Gamma, C \triangleright \Sigma' = \Gamma, C \). Finally, not surprisingly, \( \Sigma \), and \( \Sigma \) indicate that the focus is in the first and last scope of \( \Sigma \), respectively, while \( \Sigma \) says that the focus is somewhere inside \( \Sigma \) and we do not care about where.

The Gentzen calculus \( G\text{-CK}_n \) consists of the rules given in Figs. 2 and 3, separated into those dealing with the propositional connectives and those dealing with the modalities. The former, shown in Fig. 2, are essentially the well-known multi-sequent version of Gentzen’s LJ for intuitionistic logic [49] embedded into sequents with scope sequences on both sides of the turnstile. The binary operators \( \land, \lor, \supset \) have right and left introduction rules, \( \land, \lor, \supset \) require the operator to be introduced to appear in the context part of the antecedent, i.e., in the scope or to the left of it. Except for \( \land \), where it could be lifted, this constraint is essential to preserve soundness, as we shall see. Similarly, all the right rules \( \land, \lor, \supset \) introduce the main operator in the first conclusion scope of the succedent. Again, this is crucial for soundness, except for rule \( \lor, \supset \) which could be more liberal. Notice, since \( G\text{-CK}_n \) satisfies weakening and contraction, the system where \( \lor, \supset \) are combined to form the rule is equivalent to our system.

\[
\Sigma \vdash D_2, \Psi \quad \Sigma \vdash D_1 \lor D_2, \Psi \quad \lor, \supset
\]

The axiom rule \( Ax_{\exists m} \) combines both restrictions. It states that an assumption in the focus can be used to justify the same proposition in the conclusion at the focus scope.
The rules for modal operators in Fig. 3 warrant more detailed explanations and in the following we discuss them one by one. The rules are cast in the spirit of Gentzen and characterise each modal operator by way of a left and a right introduction rule. These exhibit the modalities $\forall R$ and $\exists R$ as internalisations of the scoping structure explicit in the separators $\gg R$ on the left and on the right sides of a sequent:

- Applying the right rule $\exists R$ in forward direction introduces an existential modality $\exists R. D$ on the right, wrapping up a $\gg R$ separator. Consider first the special case $\Psi = \emptyset$, i.e., the instance $\Sigma \vdash \emptyset \gg R D \Rightarrow \Sigma \vdash \exists R. D$. This says that if $D$ is constructable one $R$ step forward from the scope under the assumptions $\Sigma$ then $\exists R. D$ is constructable at the current scope under the assumptions $\Sigma$. This is the constructor rule for proofs of existential modalities. Considering the disjunctive semantics of assertions on the right of $\gg$ this inference can be extended by an arbitrary weakening $\Psi$ as seen in Fig. 3.

- The left introduction rule $\exists L$ wraps up a separator $\gg S$ in the antecedent in terms of an existential $\exists S$. Again, let $\Sigma_1 = \emptyset$ and $\Sigma = \epsilon$. Then $\exists L$ becomes $\emptyset, \gg S C \vdash \Psi \Rightarrow \exists S. C \vdash \Psi$ stating that if $\Psi$ is constructable from the assumption that $C$ is accessible through relation $S$ from the scope in focus, then $\Psi$ is constructable under the assumption $\exists S. C$. The idea is that from any hypothetical proof of $\exists S. C$ we can obtain $S$-access to a context scope where a hypothetical proof of $C$ is available. Thus, $\exists L$ is the destructor rule of existentials. The actual rule $\exists L$ in Fig. 3 generalises this by a side assumption $\Sigma_1$ in the antecedent of both the premise and conclusion, and also by a weakening $\Sigma$ in the conclusion. Note that $\exists L$ is always applied at the scope in focus.

- The left rule $\forall L$ introduces a universal modality $\forall S. C$ into a context scope from an assumption $C$ one scope to the right and reachable through modal label $S$. In the special instance where $\Sigma_1 = \emptyset$ and $\Sigma_2 = \epsilon$ this is the rule $\emptyset \gg S C \vdash \Psi \Rightarrow \forall S. C \gg S, \emptyset \vdash \Psi$ stating that if $\Psi$ is constructable under the assumption $C$ in the current focus, which happens to be accessible from some outer context using an $S$-step, then it suffices to have the assumption $\forall S. C$ in the outer scope in order to construct $\Psi$. This is justified because we may instantiate a hypothetical proof of the universal $\forall S. C$ across $S$ to give a proof of $C$. In this way the rule $\forall L$ is the destructor for proofs of $\forall$ modalities. As seen in Fig. 3 this works under arbitrary side assumptions $\Sigma_1$ and $\Sigma_2$. As we will see below it is important that the introduction rule is only applied left of the focus.

- The right introduction rule $\forall R$ plays the role of a constructor of proofs for $\forall$ modalities. Suppose we want to prove the universal proposition $\forall R. D$ under the context assumptions $\Sigma_1$, i.e., the sequent $\Sigma_1, \vdash \forall R. D$. Rule $\forall R$ permits us to reduce this to the sequent $\Sigma_1 \gg R, \emptyset, \vdash D$. I.e., we extend the original context $\Sigma_1$ by a fresh $R$-accessible scope with empty information and move the focus forward into the new scope; from there we construct a proof of $D$. If we can do that without any assumptions about the new context scope then, obviously, we have implicitly constructed a proof of the universal $\forall R. D$ relative to the original focus. The sequences $\Sigma$ and $\Psi$ in the presentation of $\forall R$ in Fig. 3 permit weakening.

- Finally, consider the rule $Ax_f$. It permits us to move the focus one position to the left across a separator $\gg S$ in the antecedent by adding a corresponding scope and separator on the right in the succedent, where the choice of the assertion set $\Phi$ in the fresh scope is arbitrary. This is a synchronous move to the left of the focus. Looking at the rule in backward direction, from the point of view of proof search, rule $Ax_f$ justifies one $\gg S$ step in the succedent by a corresponding $\gg S$ step from the antecedent. For instance, if $\Sigma_1 = \Phi = \emptyset$, then $Ax_f$ reduces the task of deriving $\emptyset, \gg S \Sigma_2, \vdash \emptyset \gg S \Psi$ to that of finding a derivation $\emptyset \gg S \Sigma_2, \vdash \Psi$, which essentially eliminates $\gg S$ from the problem. In this sense $Ax_f$ is for modal labels what $Ax_\emptyset$ is for propositions.

The rules in Fig. 3 tie up the modalities closely with context scopes and suggest a natural interpretation of proof objects as constructors and destructors of context. The existentials $\exists$ encapsulate implicit context scopes. A proof of $\exists R. C$ constructs a $R$-accessible scope and a proof of $C$ within this scope. The rules $\exists L$ and $\exists R$ tell us how to open and close scopes, respectively. A proof of $\forall R. C$ is a scope-polymorphic function which returns a proof of $C$ in any $R$-accessible scope. The rule $\forall R$ creates such a generic method and $\forall L$ instantiates a scope-polymorphic proof to a particular scope. With their left and right introduction rules the modalities $\forall R$ and $\forall R$ formally behave much like constructive existential and universal quantifiers of context scopes. This analogy has been highlighted clearly by Masini in his work on 2-sequents [45,29]. Our modal rules are refinements of those of [29] by an explicit focus marker that localises the point of construction and gives better control of the relative positions of antecedent and succedent. The rules enforce restrictions on the position of the active proposition relative to the focus marker. E.g., in $\exists L$ and $\exists R$ the existential propositions $\exists S. C$, $\exists R. D$ must be at the focus. In $\forall L$ the universal $\forall S. C$ must be on the left and in $\forall R$ the proposition $\forall R. D$ is within the scope in focus. As we shall see below, removing these restrictions would be unsound for general constructive Kripke models. In particular, the deontic axiom $\Box C \supset \Diamond C$, which is unsound, is prevented by the focus restriction on $\forall L$.

4. Soundness

In this section we show that $G$-CK$_\emptyset$ is sound for CK$_\emptyset$. We also discuss the role of the focus marker to achieve this and some natural rules for various extensions of CK$_\emptyset$, specifically IK/FS, CS4, PLL/CL and the deontic system of Masini.

Theorem 1 (Soundness). If $\emptyset, \vdash D$ is derivable in $G$-CK$_\emptyset$ then $D$ is a theorem of CK$_\emptyset$. 
The most convenient way to establish soundness is via the Kripke semantics. In [43] it was shown that CKn contains precisely the propositions valid in all constructive Kripke models as defined in Definition 2. Hence, it suffices to show that all rules of G-CKn are sound in the sense that they only generate valid sequents. It will be useful to rephrase validity in terms of refutability of sequents.

**Definition 3 (Refutability).** Let $\mathcal{I} = (\Delta^\mathcal{I}, \preceq^\mathcal{I}, \Delta^\mathcal{I}, \mathcal{I})$ be a constructive Kripke model and $\Sigma = \Gamma_1 \gg S_2, \Gamma_2 \gg S_3 \cdots \gg S_n \Gamma_n$ be a context sequence. Let $\alpha = a_1, a_2, \ldots, a_f$ be a sequence of infallible worlds $a_i \in \Delta^\alpha$ such that $f \leq n$. We say that $\alpha$ satisfies $\Sigma$, written $\alpha \vdash \Sigma$, if the following recursive conditions are fulfilled:

- if $\alpha = a$ and $\Sigma = \Gamma$ then $\mathcal{I}, a \models \Gamma$;
- if $\alpha = a$ and $\Sigma = \Gamma \gg S \Sigma'$ then $\mathcal{I}, a \models \Gamma$ and for all $\alpha' \in \Delta^\Sigma$ such that $a \preceq^\mathcal{I} \alpha'$ there exists $b \in \Delta^\mathcal{I}$ with $\alpha' \preceq^\mathcal{I} b$ and $b \vdash \Sigma'$;
- if $\alpha = a, b, \alpha'$ and $\Sigma = \Gamma \gg S \Sigma'$ then $\mathcal{I}, a \models \Gamma$, $(a, b) \in \preceq^\mathcal{I}; S^\mathcal{I}; \preceq^\mathcal{I}$ and $b, \alpha' \vdash \Sigma'$.

We say that a world $a \in \Delta^\mathcal{I}$ refutes $\Sigma$, written unsat $\Sigma$, if

- if $\Sigma = \Gamma$ then $a$ is infallible and for all $C \in \Gamma$, $\mathcal{I}, a \not\models C$; otherwise,
- if $\Sigma = \Gamma \gg R \Sigma'$ then for all $C \in \Gamma$, $\mathcal{I}, a \not\models C$ and there exists $\alpha' \in \Delta^\Sigma$ with $a \preceq^\mathcal{I} \alpha'$ such that for all $b \in \Delta^\mathcal{I}$ with $\alpha' \preceq^\mathcal{I} b$ we have unsat $\Sigma'$.

Finally, let $\Sigma, \Sigma_h \vdash \Psi$ be a sequent of depth $f - 1$. We say $\alpha$ refutes $\Sigma, \Sigma_h \vdash \Psi$, in symbols unsat $\Sigma, \Sigma_h \vdash \Psi$, if $\alpha$ satisfies $\Sigma, \Sigma_h$ and $\alpha \vdash \Psi$. The sequent is refutable (in $\mathcal{I}$) if there exists a sequence of infallible worlds $\alpha$ that refutes the sequent (in $\mathcal{I}$).

The complexity of Definition 3 is owed to the structure of general sequents. The special cases we are ultimately interested in are simple sequents of the form $\emptyset, \vdash D$. It is easy to see that such a sequent is refutable according to Definition 3 iff there exists a world $a \in \Delta^\mathcal{I}$ such that $\mathcal{I}, a \not\models D$. Thus, if we can show that $\emptyset, \not\models D$ is not refutable, then $\emptyset, \models D$ for all constructive interpretations $\mathcal{I}$. By semantic completeness of CKn [43] this means $D$ is a theorem of CKn. Accordingly, the proof of Theorem 1 (see Appendix A) shows that all rules of G-CKn only derive irrefutable sequents. The semantic proof highlights clearly the central role of the focus restrictions in the calculus and the consequences of dropping them.

An instructive alternative for proving soundness is to show that every derivation $\emptyset, \models D$ can be simulated to give $\models^H D$, where $\models^H$ denotes Haber derivability. We achieve this by a structural transformation of sequents into propositions and of sequent rules into Hilbert derivations. Specifically, we define translations $[\Sigma \vdash \Psi]^H$ for sequents, $[\Sigma]^C$ for antecedent contexts and $[\Psi]^g$ for succedent contexts as follows:

\[
\begin{align*}
[\Gamma, \vdash \Psi]^H &= df \hat{\Gamma} \supset [\Psi]^g \\
[\Gamma \gg S \vdash \Psi]^H &= df (\hat{\Gamma} \land \exists S, [\Sigma]^C) \supset [\Psi]^g \\
[\Gamma \gg S \vdash \Psi]^H &= df \hat{\Gamma} \supset \forall S, [\Sigma \vdash \Psi]^H \\
[\Gamma]^C &= df \hat{\Gamma} \\
[\Phi \gg R \vdash \Psi]^H &= df \hat{\Phi} \lor \exists R, [\Psi]^g \\
[\Phi]^g &= df \hat{\Phi}
\end{align*}
\]

where $\hat{\Gamma}$ denotes the conjunction and $\hat{\Phi}$ the disjunction of all propositions in $\Gamma$ and $\Phi$, respectively. A general sequent $\Sigma \vdash \Psi$ of the shape

\[
\Gamma_1 \gg S_2, \ldots \gg S_j, \Gamma_f \gg S_{f+1}, \Gamma_{f+1} \gg S_{f+2}, \ldots \gg S_n, \Gamma_n \vdash \Phi_1 \gg R_2, \Phi_2 \gg R_3, \ldots \gg R_m, \Phi_m
\]

turns into

\[
\begin{align*}
\hat{\Gamma}_1 &\supset \forall S_2, (\hat{\Gamma}_2 \supset \forall S_3, (\cdots \supset \Phi_1 \lor \exists R_2, (\hat{\Phi}_2 \lor \exists R_3, (\hat{\Phi}_3 \lor \cdots \lor \exists R_m, \hat{\Phi}_m) \cdots))) \\
\end{align*}
\]

or, more concretely, for sequents with two scopes in the antecedent and the succedent:

\[
\begin{align*}
[\Gamma_1 \gg S \Gamma_2 \vdash \Phi_1 \gg R \Phi_2]^H &= \hat{\Gamma}_1 \supset \forall S, (\hat{\Gamma}_2 \supset (\hat{\Phi}_1 \lor \exists R, \hat{\Phi}_2)) \\
[\Gamma_1 \gg S \Gamma_2 \vdash \Phi_1 \gg R \Phi_2]^H &= (\hat{\Gamma}_1 \land \exists S, \hat{\Gamma}_2) \supset (\hat{\Phi}_1 \lor \exists R, \hat{\Phi}_2)
\end{align*}
\]
Notice the change between $\hat{\Gamma} \supset V S$ and $\hat{\Gamma} \land \exists S$ at the focus point in the antecedent and between the conjunctive combination $\hat{\Gamma}$ in the antecedent and the disjunctive combination $\Phi$ in the succedent. In particular notice that the translation involves all operators of the language. This is a difference to the interpretation of intuitionistic 2-sequents given in [29] which does not involve the diamond modality $\exists$. The constants $\top$ and $\bot$ are used in the empty combinations $\emptyset = \bot$ and $\hat{\Gamma} = \top$. It is convenient to drop these empty cases, systematically replacing $\emptyset = \bot$ and $\emptyset = \top$ by $D$ in the syntactic compilation above.

One proves by induction on derivations that for every derivation of a sequent $\Sigma \vdash \Psi$, using the rules in Figs. 2 and 3 there exists a Hilbert proof of $[\Sigma \vdash \Psi]^{\text{fl}}$. Thus, if $\emptyset, \vdash D$ we get $\vdash_H [\emptyset, \vdash D]^{\text{fl}}$, i.e., $\vdash_H D$.

**Example 4.** Consider the rule application

$$
\frac{\emptyset \gg_R A, C, \vdash D}{\forall R.A \gg_R C, \vdash D} \text{ Ax}_f
$$

The premise sequent translates into $[\emptyset \gg_R A, C, \vdash D]^{\text{fl}} = \forall R.[A, C, \vdash D]^{\text{fl}} = \forall R.(A \land C) \supset [D]^{\text{fl}} = \forall R.(A \land C) \supset D$. The conclusion sequent yields $[\forall R.A \gg_R C, \vdash D]^{\text{fl}} = \forall R.A \supset \forall R.(C \supset [D]^{\text{fl}}) = \forall R.A \supset \forall R.(C \supset D)$. Thus, in Hilbert terms this application of $\forall L$ becomes

$$
\frac{\forall R.A \supset \forall R.(C \supset D)}{\emptyset \gg_R C, \vdash D} \text{ Ax}_f
$$

which is essentially an instance of axiom $IK1/\forall K$. The other axiom $IK2/\exists K$ is hidden in rule $Ax_f$. Specifically, the instance

$$
\frac{\emptyset \gg_R C, \vdash D}{\emptyset, \gg_R C \vdash \emptyset \gg_R D} \text{ Ax}_f
$$

has the translated premise $[\emptyset \gg_R C, \vdash D]^{\text{fl}} = \forall R.[C, \vdash D]^{\text{fl}} = \forall R.(C \supset D)$ and the conclusion becomes $[\emptyset, \gg_R C \vdash \emptyset \gg_R D]^{\text{fl}} = \exists R.C \supset [\emptyset \gg_R D]^{\text{fl}} = \exists R.C \supset \exists R.D$.

Let us call a sequent of the form $\Gamma \vdash \Phi$ with depth 0 and length 0, in which $\Gamma$ and $\Phi$ do not contain any modal operators, *intuitionistic*. It is easy to see that every derivation of an intuitionistic sequent only contains intuitionistic sequents and only involves logical rules from Fig. 2. These are precisely the standard rules of the multi-sequent Gentzen system $L$ [49]. Hence, $\text{CK}_{\forall}$ is a conservative extension of $\text{IPL}$. The intuitionistic restriction lies in the rule $\gg_R$ which requires the succedent in its premise to be a single proposition $D$. As is well known, if we relax the rule to $\gg_R^*$ to read $\Sigma_1, C \gg_R^* D, \Psi \Rightarrow \Sigma_1, \Sigma_2 \vdash C \supset D, \Psi$, then we get back classical logic and a derivation of the Excluded Middle Principle $\Sigma \vdash C \lor \neg C$.

5. Extensions of $\text{G-CK}_{\forall}$

5.1. Deontic extension [29]

In Example 1 we have seen that the deontic axiom is not generally valid. By soundness, the sequent $\emptyset, \vdash \forall R.A \supset \exists R.A$ is not derivable. The attempt to prove it yields the unique tree

$$
\frac{\forall R.C, \vdash \emptyset \gg_R C\ ?}{\forall R.C \vdash \exists R.C} \exists R
$$

which cannot be completed. The left rule $\forall L$ is not applicable at the open leaf because of the lack of a $\gg_R$ successor in the antecedent across which the move of $C$ in $\forall R.C \gg_R \emptyset$ over $R$ to yield $\emptyset \gg_R C$ could be performed. This is analogous to the situation in intuitionistic predicate logic where the sequent $\forall x.C \vdash \exists x.C$ is not derivable unless we assume all domains are non-empty. In $\text{G-CK}_{\forall}$ we can obtain $R$-seriality by way of an extra rule and then complete the proof tree as follows:

$$
\frac{\Sigma_1 \gg_R \emptyset \vdash \Psi}{\Sigma_1, \Sigma_2 \vdash \Psi} \text{ R-serial}
$$

The rule $R$-serial is sound in all constructive models that are serial, i.e., for which every world has an $R$-successor. As an axiom, $R$-serial is equivalent to assuming $\exists R.T$. 

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5.2. Disjunctive Distribution FS4/IK4

We can now explain proof-theoretically why disjunctive distribution \( \exists R.(C \lor D) \supset (\exists R.C \lor \exists R.D) \) fails. Consider the proof tree which starts like this:

\[
\frac{\exists R.(C \lor D), \vdash \exists R.C \lor \exists R.D}{\varnothing, \vdash \exists R.(C \lor D) \supset \exists R.C \lor \exists R.D} \supset R
\]

At the top we have two possible rules to proceed, \( \exists L \) and \( \lor R \). The latter is not sensible as it would break up the succedent disjunction \( \exists R.C \lor \exists R.D \) before we have split the two cases of \( C \lor D \) in the antecedent. Therefore, we apply \( \exists L \):

\[
\frac{\varnothing \gg R C \lor D, \vdash \varnothing \gg R C}{\exists R.(C \lor D), \vdash \exists R.C \lor \exists R.D} \supset R
\]

\[
\frac{\varnothing, \vdash \varnothing \gg R C \lor D}{\varnothing, \vdash \exists R.(C \lor D) \supset \exists R.C \lor \exists R.D} \supset R
\]

Now we must continue with \( \lor R \) since it is the only rule applicable. Rule \( \lor L \) for case analysis on \( C \lor D \) does not apply because it does not have the focus. There is no way to move the focus right to get \( \varnothing \gg R C \lor D, \vdash \exists R.C \lor \exists R.D \) since the rule \( Ax_f \) which would do that is not applicable at this point. To apply \( Ax_f \), the succedent would need to be of the form \( \varnothing \gg R E \). But this means we should use rule \( \exists R \) to one of the existentials \( \exists R.C \) or \( \exists R.D \). However, to do this we need to break up the disjunction \( \exists R.C \lor \exists R.D \) in the succedent using \( \lor R \) first:

\[
\frac{\varnothing \gg R C \lor D}{\varnothing, \vdash \exists R.C \lor \exists R.D} \supset R
\]

Of course, this does not help because now we must prove succedent \( C \lor D \) which is too weak. Intuitively, the construction of a proof for \( \exists R.(C \lor D) \supset (\exists R.C \lor \exists R.D) \) creates a tie between (i) accessing the disjunction \( C \lor D \) on the input side which necessarily takes us inside the \( R \)-context behind \( \exists R \) and (ii) using the case analysis in the succedent to decide the disjunction \( \exists R.C \lor \exists R.D \) which is outside the \( R \)-context. Hence, the failure of disjunctive distribution can be traced to the restriction of \( \lor L \) which does not apply to the right of the focus. If we lift this and relax \( \lor L \) to become \( \lor L^* \) also including the inference by which \( \Sigma_1 \gg R C, \Sigma_2 \vdash \psi \) and \( \Sigma_1 \gg R C, \Sigma_2 \vdash \psi \) implies \( \Sigma_1 \gg R C \lor D, \Sigma_2 \vdash \psi \), then the axiom \( \exists R.(C \lor D) \supset (\exists R.C \lor \exists R.D) \) becomes derivable. Alternatively, one can relax \( \exists R \) to \( \exists R^* \): \( \Sigma \vdash \varnothing \gg R \psi \) for case analysis on \( \psi \) to obtain this axiom scheme. Both \( \lor L^* \) and \( \exists R^* \) are sound in constructive models satisfying confluence between \( \preceq \) and \( R \), i.e., where we have \( \preceq^{-1} ; R \subseteq R ; \preceq^{-1} \). \( \Box \)

5.3. Non-fallibility FS3/IK3

The nullary form of disjunctive distribution \( \exists R.\bot \equiv \bot \), or equivalently \( \exists R.\bot \supset \bot \), is not a theorem for general constructive models due to the possible presence of fallible worlds. In the proof system it is not derivable because \( \bot \) is only applicable to inconsistencies in the context, i.e., to the left of the focus. The only possible derivation tree

\[
\frac{\varnothing, \vdash \bot \bot \bot}{\exists R.\bot \bot \bot \supset \bot} \supset R
\]

places the empty disjunction \( \bot \) as a hypothesis on the right of the focus. Under the focus restrictions this fails to justify \( \bot \) in the succedent, either by rule \( \bot L \) or by \( Ax_m \). Of course, if we use a more general rule \( \bot L^* \) deriving all instances of \( \Sigma_1 \gg R \bot, \Sigma_2 \vdash \psi \) then the above derivation for \( \exists R.\bot \supset \bot \) can be completed. Like for Disjunctive Distribution there is an alternative for enforcing \( \exists R.\bot \equiv \bot \) in terms of a right rule. We can add the rule \( R\text{-infallible} \), as seen below, to do the job:

\[
\frac{\varnothing \gg R \bot \bot}{\varnothing, \vdash \bot \bot \bot} \supset R
\]

\[
\frac{\varnothing \gg R \bot \bot}{\varnothing, \vdash \bot \bot \bot} \supset R
\]

\[
\frac{\varnothing \gg R \bot \bot}{\varnothing, \vdash \bot \bot \bot} \supset R
\]
Observe that rule $R$-infallible deriving $\neg \exists R. \bot$ is formally dual to $R$-serial which corresponds to axiom $\exists R. \top$. Both rules $\bot^*$ and $R$-infallible are sound for constructive models without fallible worlds.

5.4. Scheme FS5/IK5

The intuitionistic modal logic $\text{IK}/\text{FS}$ [8,6] is an extension of $\text{CK}$ including the axioms of Non-Fallibility, Disjunctive Distribution discussed in the previous examples, and the FS5/IK5 axiom scheme $(\exists R. C \supset \forall R. D) \supset \forall R.(C \supset D)$ which cannot be derivable in CK since it has a constructive counter model (see Example 2). Under uniform proof strategy (applying regular, invertible right rules) we obtain the derivation tree

\[
\begin{array}{c}
\exists R. C \supset \forall R. D \supset R C. \vdash D \\
\exists R. C \supset \forall R. D \supset R \emptyset. \vdash C \supset D \\
\forall R. C \supset \forall R. D \supset \forall R. (C \supset D) \supset R \\
\emptyset, \vdash (\exists R. C \supset \forall R. D) \supset \forall R. (C \supset D) \supset R
\end{array}
\]

in which the leaf sequent $\exists R. C \supset \forall R. D \supset R C. \vdash D$ expects us to prove that $D$ follows from $C$ under the assumption that in some $R$-predecessor context we have the implication $\exists R. C \supset \forall R. D$. But the focus rests in the context of the assumption $C$ and thus we cannot apply rule $\supset L$. In contrast to the examples of Disjunctive Distribution and Non-Fallibility above where the focus restriction prevents forward reasoning, here we hit the question of backward reasoning. In order to argue validity (irrefutability) of $\exists R. C \supset \forall R. D \supset R C. \vdash D$ we must be able to move our point of view into the scope in which $\exists R. C \supset \forall R. D$ resides: looking forward we would then argue that $\supset R C$ implies $\exists R. C$ which would then be used to obtain $\forall R. D$ from the implication $\exists R. C \supset \forall R. D$. This universal truth $\forall R. D$ then would permit us to propagate $D$ across $\supset R$ into the original focus to justify $D$ in the antecedent. However, the rules of $\text{CK}_a$ do not permit this speculative move of the focus. They are single threaded with a single focus which commits us to apply $\supset L$ with the information available at this point. Note that we could derive the theorem with a modified left implication rule $\supset \neg^*$

\[
\begin{array}{c}
\Sigma_1, \supset R \Sigma_2 \vdash C \\
\Sigma_1, D \supset R, \Sigma_2 \vdash \psi \supset \neg^*
\end{array}
\]

which permits us to use an implication to the left of the focus without dropping $\supset R, \Sigma_2$. The completed proof then is:

\[
\begin{array}{c}
\emptyset \supset R C, \vdash C \quad \text{Ax}_m \\
\emptyset, \supset R C \vdash \emptyset \supset R C \quad \text{Ax}_f \\
\emptyset, \supset R C \vdash \exists R C \quad \exists R \\
\forall R. D, C, \vdash D \quad \forall L \\
\exists R. C \supset \forall R. D \supset R C. \vdash D \quad \exists R \\
\forall R. C \supset \forall R. D \supset \forall R. (C \supset D) \supset R
\end{array}
\]

No surprise, $\supset \neg^*$ is not sound for arbitrary constructive models. However, $\supset \neg^*$ is valid for models satisfying the frame condition $\quad \bot^{-1} : R ; \quad \bot \subseteq R ; \quad \bot^{-1}$.

5.5. CS4

Many applications, specifically modal type theories, are based on the constructive logic CS4 [3,16] which extends CK (in a single modal label $R$, writing $\Diamond$ and $\Box$ rather than $\exists R$ and $\forall R$) by the axiom schemes

\[
\begin{array}{c}
\Box T : \Box C \supset C \\
\Diamond \Box T : C \supset \Diamond C \\
\Box 4 : \Box C \supset \Box \Box C
\end{array}
\]

CS4 is also called JS4 for Judgemental S4 [15]. As shown in [25] this logic is the CK theory of constructive models with a reflexive and transitive modal relation $R$ which satisfies the frame condition $R : \quad \bot \subseteq \bot \subseteq R$. These frame properties give rise to corresponding sequent rules seen in Fig. 4 which implement the CS4 axioms in G-CK as follows:

\[
\begin{array}{c}
C, \vdash C \quad \text{Ax}_m \\
\emptyset \supset R C, \vdash C \quad \text{Ax}_m \\
\forall R. C \supset R \Box T \\
\forall R. C, \vdash C \quad \Box T
\end{array}
\]

\[
\begin{array}{c}
C, \vdash C \quad \text{Ax}_m \\
\emptyset \supset R C, \vdash C \quad \text{Ax}_m \\
\forall R. C \supset R \Box T \\
\forall R. C, \vdash C \quad \Box T
\end{array}
\]
Soundness of $\Diamond T$ and $\Box T$ stem from reflexivity of $R$, while soundness of $\Diamond 4$ and $\Box 4$ arise from transitivity of $R$ and the frame property $R: \preceq \subseteq \leq\ R$. Observe how G-Ck reveals the symmetric nature of the axioms of CS4 and their universal role for handling context scopes on the left and on right of $\vdash$. We note that the G-Ck-rule $\Diamond 4$ (Fig. 4) is actually a rendering of the axiom scheme $\exists R.(C \lor \exists R.D) \supset \exists R.(C \lor D)$ which is the “right” way to axiomatise transitivity of $R$ in Ck. It can be derived from the combination of the Hilbert axioms $\Diamond T$ and the simpler form of $\Diamond 4$ as above.

5.6. Necessitation and lax logic PLL/CL

As we will see below, by cut admissibility, the calculus admits the rule of Modus Ponens, i.e., for every derivation $\emptyset, \vdash C \supset D$ and $\emptyset, \vdash C$ there is a derivation of the sequent $\emptyset, \vdash D$. In fact, we have the Modus Ponens in the strong form in arbitrary contexts, so that $\Sigma, \vdash C \supset D$ and $\Sigma, \vdash C$ implies $\Sigma, \vdash D$. This is nothing but the elimination rule for $\supset$ in natural deduction. We also have the (admissible) rule of Necessitation, i.e., $\emptyset, \vdash D$ implies $\emptyset, \vdash \forall R.D$. Of course, this rule depends on the context being empty. It cannot be used as a natural deduction introduction rule for $\forall$ since $\Sigma, \vdash D$ does not imply $\Sigma, \vdash \forall R.D$. E.g., we have $D, \vdash D$ but not $D, \vdash \forall R.D$. In fact, the attempt to derive this sequent

$$\frac{D \supset \emptyset, \vdash D \ \forall R}{D, \vdash \forall R.D}$$

necessarily fails because the axiom rule $\text{Ax}_m$ does not apply for sequent $D \supset \emptyset, \vdash D$ in which assumption $D$ appears in the context left of the current focus. This restriction on $\text{Ax}_m$ is crucial for soundness of the calculus since otherwise $D \supset \forall R.D$ were derivable which would trivialise the $\forall R$ modality. In classical logics this would be pointless since then $\exists R$, which is derivable from $\forall R$, trivialises as well. However, in constructive logics $\exists R$ has its own life and may well survive the $\forall$-collapse. Sometimes this is exactly what we want. For instance, this happens in lax logic (PLL) [24,30] where one replaces $\text{Ax}_m$ in CS4 by the “backward-looking” $\text{Ax}_m^*$

$$\Sigma_1, D \supset \exists R, \Sigma_2 \vdash D, \psi \supset \text{Ax}_m^*$$

ing encoding axiom $D \supset \forall R.D$ which is sound in frames satisfying $R \subseteq \preceq \Sigma$. Given $\text{Ax}_m^*$ we easily derive the tensorial strength axiom of PLL, $(C \land \exists R.D) \supset \exists R.(C \land D)$ as follows

$$\frac{C \supset \exists R.D, \vdash C \land D \ \text{Ax}_m^*}{C \supset \exists R.D, \vdash C \land D}$$

As a side remark we note that if we relaxed $\text{Ax}_m$ to become applicable to assumptions right of the focus, say $\Sigma_1 \supset \exists R C, \Sigma_2 \vdash C, \psi$ we would trivialise $\exists R$ since then $\exists R.C \supset C$ would become a theorem.
6. Completeness

In this section we show that the cut-free derivation system $G$-$CK_n$ given in Figs. 2 and 3 is complete for $CK_n$. Furthermore, as it turns out, we only ever need at most two scopes on either side of the sequent turnstile. We show that the calculus enforces some structural invariants which make it possible to work with tight sequents (Def. 4) without losing completeness. This result highlights the local nature of reasoning in $CK_n$.

To prove completeness we need some auxiliary facts on weakening and commutation of rules. Let us write $\Sigma \subseteq \Sigma'$ if $\Sigma'$ is a weakening of $\Sigma$ in the sense that it has the same hypotheses as $\Sigma$ and each scope of $\Sigma$ is a subset of the corresponding scope of $\Sigma'$. This is defined inductively by the conditions (i) $\epsilon \subseteq \epsilon'$ and (ii) if $\Gamma' \subseteq \Gamma'$ and $\Sigma \subseteq \Sigma'$ then $\Gamma' \gg R \subseteq \Gamma'' \gg R \Sigma \subseteq \Gamma'''' \gg R \Sigma'$ for all $R \in \text{Lab}$. Then, define extension $\Sigma' \cdot \Sigma_h \leq \Sigma' \cdot \Sigma_h'$ of antecedents by the condition that $\Sigma'_c = \Sigma'_c \cdot \Sigma_c''$ and $\Sigma_h' = \Sigma_h' \cdot \Sigma_h''$ so that $\Sigma_c \leq \Sigma_c'$ and $\Sigma_h \leq \Sigma_h'$. Thus, the extension $\Sigma' \cdot \Sigma_h'$ of $\Sigma_c \cdot \Sigma_h$ possibly enlarges the set of assumptions and adds further scopes to the left of the context $\Sigma_c$ and to the right of the hypotheses $\Sigma_h$. For succedents we define extension $\Psi \leq \Psi'$ as for hypotheses, i.e., $\Psi' = \Psi'_1, \Psi'_2$ with $\Psi \subseteq \Psi'_1$.

**Definition 4.** A sequent $\Sigma \vdash \Psi$ is called **tight** if the following holds:

(i) the antecedent $\Sigma$ has length $\leq 1$;
(ii) the succedent $\Psi$ has length $\leq 1$ and also contains exactly one proposition;
(iii) all scopes of the antecedent $\Sigma$ more than two places to the left of $\vdash$ are empty.

In other words, a tight sequent has one of the shapes

$$\emptyset \gg s_1 \emptyset \gg s_2 \ldots \emptyset \gg s_f \Gamma_f \cdot \gg s_{f+1} \Gamma_f,1 \vdash \Psi$$

$$\emptyset \gg s_1 \emptyset \gg s_2 \ldots \emptyset \gg s_{f-1} \Gamma_{f-1} \gg s_f \Gamma_f,1 \vdash \Psi$$

where $\Psi$ is either of the form $D$ or $\emptyset \gg R D$. A derivation is called tight if all sequents appearing in it are tight.

**Lemma 2.**

(i) For every derivation $\Sigma_1 \vdash \Psi_1$ and $\Sigma_1 \subseteq \Sigma_2$, $\Psi_1 \leq \Psi_2$ there is a derivation $\Sigma_2 \vdash \Psi_2$ of the same height or less. Moreover, if the derivation of $\Sigma_1 \vdash \Psi_1$ and the sequent $\Sigma_2 \vdash \Psi_2$ is tight then the derivation of $\Sigma_2 \vdash \Psi_2$ is also tight.
(ii) If $\Sigma \vdash \bot$ then also $\Sigma \vdash \Psi$ for arbitrary $\Psi$. If the derivation $\Sigma \vdash \bot$ and sequent $\Sigma \vdash \Psi$ are tight then the derivation of $\Sigma \vdash \Psi$ is tight.
(iii) Every (tight) derivation $D$ can be transformed into another (tight) derivation $D'$ of same height or less in which every occurrence of a rule $\exists L$ is immediately preceded by an application of $Ax_f$.

**Proof.** See Appendix A. □

Completeness depends on the admissibility cut which we state first:

**Lemma 3 (Cut admissibility).** If $\Sigma,1 \vdash D$ and $\Sigma,D,1 \vdash \Psi$ then $\Sigma,1 \vdash \Psi$. Moreover, if both $\Sigma,D,1 \vdash \Psi$ and $\Sigma,D,1 \vdash \Psi$ are tight derivations, then so is $\Sigma,1 \vdash \Psi$.

**Proof.** See Appendix A. □

**Theorem 4 (Completeness).** Every theorem $D$ of $CK_n$ has a tight derivation $\emptyset \vdash D$ in $G$-$CK_n$.

**Proof.** Using Lemma 2 we show how to simulate every Hilbert proof of $CK_n$ in $G$-$CK_n$ (see Sec. 2.1) using only tight sequents. Recall that $CK_n$ is the logic generated by the axioms of intuitionistic propositional logic $IPL$ together with the two modal axiom schemes ($R \in \text{Lab}$)

$$\forall K: \forall R.(C \supset D) \supset (\forall R.C) \supset (\forall R.D)$$

$$\exists K: \forall R.(C \supset D) \supset (\exists R.C) \supset (\exists R.D)$$

$$\forall R.(C \supset D) \supset (\forall R.C) \supset (\forall R.D)$$

$$\exists K: \forall R.(C \supset D) \supset (\exists R.C) \supset (\exists R.D)$$
and the rules MP of Modus Ponens and Necr of Necessitation. Let us see how such Hilbert proofs can be translated into G-CK. First, we generate tight derivations \( \varnothing \vdash C \) for all intuitionistic axioms \( C \) using the rules in Fig. 2. Tight derivations for \( \forall K \) and \( \exists K \) are obtained by way of the rules in Figs. 2 and 3 as follows:

Second, Modus Ponens is essentially contained in the admissibility of cut stated in Lemma 3, which preserves tightness. For suppose we have derived \( \varnothing, \vdash C \) and \( \varnothing, \vdash \Gamma \) using tight derivations. Then, by rules \( \text{Axm} \) and \( \supset \) we get \( C, \Gamma \vdash \Delta \) from which two applications of cut yield a tight derivation of \( \varnothing, \vdash D \) as desired. For Necessitation we exploit (i) of Lemma 2, i.e., the fact that our system is closed under context weakening. Specifically we use that \( \Sigma \vdash \Psi \) implies \( \varnothing \gg R \Sigma \vdash \Psi \) for all \( R \in \text{Lab} \) (preserving tightness). In particular, this implies that if \( \varnothing, \vdash D \) then \( \varnothing \gg R \varnothing, \vdash D \) and thus \( \varnothing, \vdash \forall R.D \) by rule \( \forall R \).  

7. Conclusion

In this work we present a constructive modal logic and its cut-free Gentzen calculus as a formal system to express context-dependency. The system is derived from a multi-sequent calculus for multi-modal CK, whose direct analogue in description logics is \( c^{\forall \exists}ALC \) as has been reported in [43]. The system exhibits sequents equipped with Kripke semantics which allowed us to obtain soundness and completeness theorems. It has been shown that other non-classical modal logics such as IK, CS4, PLL/CL, or Masini’s deontic system of 2-sequents arise as specialised theories of CK.

In future work we aim at introducing proof terms for our Gentzen system, especially to extract proof terms/natural deduction rules for \( \Box \) and \( \Diamond \), as an extension of the simply typed lambda calculus that expresses context-dependent computations in structured data, e.g., computational knowledge bases or databases. We hope that such a system can constitute a formal grounding for a modally typed functional programming language that finds practical adoption in the domains of knowledge representation and database processing languages. For such applications it will be important that the semantic properties of the accessibility relations (so-called “roles” in Description Logics) can be adjusted flexibly, rather than being hardwired as in CS4, PLL/CL and other specialised modal type theories like [31,21].

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Appendix A. Proofs

Theorem 1 (Soundness). If \( \varnothing, \vdash D \) is derivable in \( G\text{-CK}_\beta \) then \( D \) is a theorem of \( C\text{-CK}_\beta \).

Proof. Throughout the proof we fix an interpretation \( I \) and omit superscripts, writing \( \models \) and \( R \) for \( \models^T \) and \( R^T \), respectively, and \( \models C \) rather than \( I, \models C \). For each of the rules we assume a sequence \( \alpha = a_1, a_2, \ldots, a_{j-1}, a_j \) of non-fallible worlds which refutes the conclusion sequent and from it construct a sequence \( \beta \) refuting (one of) the premise sequent(s). This proves that all rules, when read top-down, preserve irrefutability. We will tacitly use the following simple facts about sat and unsat where \( \alpha = a_1, a_2, \ldots, a_{j-1}, a_j \) is a sequence of non-fallible worlds and a non-fallible:

- for single propositions we have \( a \text{ sat } C \iff a \models C \) and \( a \text{ unsat } C \iff a \not\models C \).
• $\alpha$ sat $\Sigma$, $C$ iff $\alpha$ sat $\Sigma$ and $a_f \models C$;
• a unsat $D$, $\Psi$ iff a unsat $\Psi$ and $a \not\models D$;
• if $\alpha$ sat $\Sigma$ and $a_f \not\leq a_f'$ where $a_f'$ non-fallible, then $(a_1, a_2, \ldots, a_{f-1}, a_f')$ sat $\Sigma$.

To prove soundness of the axioms $Ax_m$ and $\bot$ we need to argue that their conclusion instances are irrefutable. This is done by contradiction. Suppose $\alpha$ unsat $\Sigma_1$, $C$, $\Sigma_2 \vdash \Psi$. Then, by Definition 3, this means that the last world $a_f$ of $\alpha$ marking the focus point has $a_f \models C$, since $\alpha$ sat $\Sigma_1$, $C$ and at the same time $a_f \not\models C$ due to $a_f$ unsat $\Sigma, \Psi$. Obviously, this is impossible. Regarding $\bot$, assume we have $\alpha$ unsat $\Sigma_1$, $\bot$, $\Sigma_2 \vdash \Psi$ with $\alpha = a_1, a_2, \ldots, a_f$. Then, because $\bot$ appears before the focus point we must have $a_1 \models \bot$ for some $1 \leq i \leq f$. But this contradicts the assumption that all $a_i$ are irrefutable. Hence, both $Ax_m$ and $\bot$ generate irrefutable sequents. Note that the focus restriction for $\bot$ is crucial. We have seen (Section 5.3) that if occurrences of $\bot$ on the right of the focus could derive arbitrary conclusions then we would make the axiom of non-fallibility $\exists \mathcal{R}, \bot \equiv \equiv$ derivable.

The proof of soundness for rules $\land L_1$, $\land L_2$ and $\lor R_1$, $\lor R_2$ is trivial. The restriction in $\land L_1$ that the conjunction $C_1 \land C_2$ appear before the focus point can be dropped, similarly in $\lor R_1$ and $\lor R_2$ where the expansion of $D_1$ to $D_1 \lor D_2$ could take place in any scope of the succeed.

Consider rule $\lor R$ needs the conjunction to be present at the focus point, i.e., in the succeed context directly next to the turn-stile. Suppose $\alpha$ unsat $\Sigma \vdash D_1 \land D_2, \Psi$. Then, by definition, $a_f$ unsat $D_1 \land D_2, \Psi$ which implies $a_f \not\models D_1 \land D_2$. But this means $a_f \not\models D_1$ or $a_f \not\models D_2$. In the first case one shows without difficulty that $\alpha$ unsat $\Sigma \models D_1, \Psi$ and in the second case that $\alpha$ unsat $\Sigma \models D_2, \Psi$. This is what we need to argue soundness of $\lor R$. Now, observe that if the conjunction is not in the first scope but in the second, say $a_f$ unsat $\Phi_1 \models D_1 \land D_2, \Psi$, then we would only know that there is some $\ll$-successor of $a_f$ of whose $R$-successors are not in $D_1$ or not in $D_2$. But since the choice between refutation of $D_1$ or $D_2$ depends on these $R$-successors, we cannot make a static case distinction at $a_f$. In models where we have the distribution $\exists \mathcal{R}, (D_1 \land D_2) \equiv \exists \mathcal{R}, D_1 \land \exists \mathcal{R}, D_2$ this is possible but these are rather special classes of models.

Next take a look at $\lor L$. Assume $\alpha$ unsat $\Sigma_1, C_1 \supset C_2, \Sigma_2, 1, 2 \models \Psi$, i.e., $\alpha$ sat $\Sigma_1, C_1 \supset C_2, \Sigma_2, 1, 2, a_f$ sat $\Sigma_2$ and $a_f$ unsat $\Psi$. From $a_f$ sat $\Sigma_1, C_1 \supset C_2, \Sigma_2$ we conclude that $a_1 \models C_1 \supset C_2$ for $1 \leq i \leq f$. Therefore, using $a_i \not\models a_i$, we have $a_i \not\models C_1$ or $a_i \models C_2$. If the former is true we can state $a_1, a_2, \ldots, a_{f-1}$ unsat $\Sigma_1 \models C_1$ and in the latter case we get $\alpha$ unsat $\Sigma_1, C_1 \supset C_2, \Sigma_2, 1, 2 \models \Psi$.

Arguing soundness of $\lor R$ is analogous to standard intuitionistic logic where the removal of the succeed context $\Psi$ is owed to the intuitionistic refinement pre-ordering $\ll$: Suppose $\alpha$ unsat $\Sigma_1, \Sigma_2 \models C \supset D, \Psi$, or in other words, $\alpha$ sat $\Sigma_1$, $a_f$ sat $\Sigma_2, a_f$ unsat $C \supset D, \Psi$. Because of the latter $a_f \not\models C \supset D$ which means there is a $\ll$-successor of $a_f$, $a_f \ll a_f'$ such that $a_f' \models C$ and $a_f' \not\models D$. Note that the latter implies that $a_f'$ is non-fallible. Furthermore, since sat is closed under refinement, also $a_f'$ sat $\Sigma_2$. However, unsat is not closed under $\ll$ and therefore we cannot conclude that $a_f'$ unsat $\Psi$ since $a_f \not\ll a_f'$ in general. All we have is $a_f$ unsat $\Psi$. This means for the sequence $\alpha' = a_1, a_2, \ldots, a_{f-1}, a_f'$ we can state $\alpha'$ unsat $\Sigma_1, C, \Sigma_2 \models \Psi$ which is the premise of our rule $\lor R$. We do not have $\alpha'$ unsat $\Sigma_1, C, \Sigma_2 \models \Psi$ which would give us classical logic and derive $C \equiv C \equiv \equiv$. Also note that the implication $C \supset D$ must appear in the first context (pertaining to the focus) of the succeed. In a statement such as $a_f$ unsat $\Phi, C \equiv D, \Psi$ the implication would not give us any statically exploitable information about $a_f$.

Now we come to look at the modal rules. Let us start with rule $Ax_f$. Suppose $\alpha = a_1, a_2, \ldots, a_f$ refutes the conclusion sequent $\Sigma_1, \exists \mathcal{R}, D, \Psi$, i.e., $\alpha$ sat $\Sigma_1, a_f$ sat $\emptyset \supset \Sigma_2$, and $a_f$ unsat $\Phi \supset \Psi_1$. The latter $a_f$ unsat $\Phi \supset \Psi_1$ means there is $a_f' \ll a_f'$ such that for all $a_f', Sb$ we have $b$ unsat $\Psi_1$. But from $a_f$ sat $\emptyset \supset \Sigma_2$ it follows that there exists an S-successor $b$ of $a_f'$, $a_f' Sb$, such that $b$ sat $\Sigma_2$. Hence, $b$ unsat $\Psi_1$, which in particular implies $b$ is non-fallible as one can show by induction on the structure of $\Psi_1$. Now since $\ll$ is reflexive we get $(a_1, b) \in (S, \ll)$ which means for $\beta = a_1, b = (a_1, a_2, \ldots, a_{f-1}, b)$ that $b$ sat $\Sigma_1 \supset \emptyset \emptyset$. Taking all together yields that $b$ refutes the premise sequent $\Sigma_1 \supset \emptyset \emptyset$. But this is nothing but $a_f$ unsat $\emptyset \supset \emptyset D$ which in turn means that $\alpha$ refutes the premise sequent $\Sigma \emptyset \emptyset \emptyset D$.

Regarding rule $\exists L$, take $\alpha = a_1, a_2, \ldots, a_f$ refuting $\Sigma_1, \exists \mathcal{C}, \Sigma_2 \models \Psi$. From this we get $\alpha$ sat $\Sigma_1, \exists \mathcal{C}, \Sigma_2 \models \Psi$. Now $\alpha$ sat $\Sigma_1, \exists \mathcal{C}$ means $\alpha$ sat $\Sigma_1$ and $a_f \models \exists \mathcal{C}$, i.e., for all $a_f \ll a_f'$ there is $b$ with $a_f' Sb$ and $b \models C$. Since $b \models C$ is the same as $b$ sat $C$ these are precisely the conditions to state $a_f$ sat $\emptyset \supset \emptyset C$. Hence, overall, $\alpha$ unsat $\Sigma_1, \exists \mathcal{C} \models \Psi$. 
Then last rule of $D$ tightness is preserved.

Let $\Sigma$ a hypotheses of the axioms. Suppose first that $\Sigma \vdash \forall \alpha \Sigma$. which has the same structure as $\Sigma$' and the succedents are replaced by $\Sigma'$. As $\Sigma'$ has the same structure as $\Sigma$ the left rules. If we extend a tight sequent $\Sigma$ beginning and the end of the antecedent. On the succedent side, however, we can only extend at the end due to the form $\Sigma$. Clearly, in both cases all we have changed is the rule name and the succedent which is this very same $\Sigma$.

Finally, we deal with $\Box R$. A sequence $\alpha$ refuting the conclusion sequent $\Sigma, \Box \Psi$ would obtain $\alpha$ sat $\Sigma$ and $\Psi$ unsat $\Psi$. It is not difficult to see that $\alpha \vdash \forall \Sigma$ and $\alpha \vdash \Psi$. This gives us an unsat $\Sigma \vdash \Psi$ as desired.

**Lemma 2.**

(i) For every derivation $\Sigma \vdash \Psi$ and extensions $\Sigma, \Box \Psi \vdash \Psi_2$ there is a derivation $\Sigma_2 \vdash \Psi_2$ of the same height or less. Moreover, if the derivation of $\Sigma \vdash \Psi$ and the sequent $\Sigma_2 \vdash \Psi_2$ is tight then the derivation of $\Sigma_2 \vdash \Psi_2$ is also tight.

(ii) If $\Sigma, \Box \Psi \vdash \bot$ then also $\Sigma, \Box' \Psi \vdash \Psi$ for arbitrary $\Psi$. If the derivation $\Sigma, \Box \Psi \vdash \bot$ and sequent $\Sigma, \Box' \Psi \vdash \Psi$ are tight then the derivation of $\Sigma, \Box' \Psi$ is tight.

(iii) Every (tight) derivation $D$ can be transformed into another (tight) derivation $D'$ of same height or less in which every occurrence of a rule $\Box L$ is immediately preceded by an application of $\forall \alpha \Psi$.

**Proof.** (i) This is obvious by inspection of the rules. All extensions in the antecedent $\Sigma_1 \leq \Sigma_2$ and succedent $\Psi_1 \leq \Psi_2$ can be pushed through all rules of the considered derivation. Note that all rules are generic to permit extensions both at the beginning and the end of the antecedent. On the succedent side, however, we can only extend at the end due to the form of the right rules. If we extend a tight sequent $\Sigma \vdash \Psi$ to become another tight sequent $\Sigma_2 \vdash \Psi_2$ then $\Psi_1 \vdash \Psi_2$ and the extension $\Sigma_1 \leq \Sigma_2$ consists in adding empty scopes to the front of $\Sigma_1$ and possibly increasing the sets of propositions in the last two steps. We do not need to make any other changes to the sequents of a tight derivation of $\Sigma \vdash \Psi_1$, whence tightness is preserved.

(ii) Let $D$ be a derivation tree for $\Sigma, \Box \Psi$ and $\Sigma_2 \vdash \bot$. We prove $\Sigma, \Box' \Psi \vdash \Psi$ for arbitrary $\Psi$ by induction on the structure of $D$ and show that this derivation contains exactly the same sequents except that some of the antecedents are extended by $\Box'$ and the succedents are replaced by $\Psi$. Hence, tightness is preserved if $\Psi$ is of the form $D \vdash \bot \forall \Box R$. Obviously, the last rule of $D$ cannot be $\forall \alpha \Psi$ nor any of the right rules $\forall \alpha \Box R$, $\forall \Box R$, $\exists \forall \alpha \Box R$, $\forall \alpha \Psi$. So, $D$ must end in a left rule or one of the axioms. Suppose first that $\Sigma, \Box \Psi \vdash \bot$ is derived with $\Box \forall \alpha \Psi$ as the last rule. Then, $\bot \forall \Box R$ is contained in the antecedent hypotheses $\Sigma_1$ in the focus scope, so that $\Sigma_1, \Box' \Psi \vdash \Psi$ is obtained by $\Box L$. Also, if the last rule of $D$ is $\Box L$ this very same rule will do the job for arbitrary $\Psi$. Clearly, in both cases all we have changed is the rule name and the succedent which is now $\Psi$ rather than $\bot$. Now consider a left rule such as $\Box L$:

$$
\Sigma_1, \Box \Psi_1 \vdash \bot \\
\Sigma_1, \Box \Psi_1 \vdash \bot
$$

By induction hypothesis there must be a derivation $D_2^e$ of $\Sigma_1, C_2, \Sigma_1, \Box' \psi$ in which the sequents of the original $D_2$ have all succedents replaced by $\psi$. From $D_2^e$ we easily construct the desired result by application of $\Box L$:

$$
\Sigma_1, \Box \Psi_1 \vdash \bot \\
\Sigma_1, \Box \Psi_1 \vdash \bot
$$

which has the same structure as $D$ except that some of the succedents have changed to $\psi$. Similar inductive arguments apply in case of the other left rules. The point is that all left rules are ignorant of the succedent which could be $\bot$ or $\psi$ and can be applied arbitrarily left of the focus. This proves (ii).

(iii) We show that every instance of $\Box L$ can be pushed up the derivation tree (towards the leaves) until it hits an instance of $\forall \alpha \Psi$. Furthermore, it is easy to check that these transformations preserve tightness. Let $D$ be a derivation which ends in an arbitrary instance of $\Box L$:

$$
\Sigma_1, \Box \Psi_1 \vdash \bot \\
\Sigma_1, \Box \Psi_1 \vdash \bot
$$

where $\Sigma_1, \Box \Psi_1 \vdash \bot$. Then $\Sigma_1, \Box' \Psi_1 \vdash \Psi_1$ and $\Sigma_1, \Box' \Psi_1 \vdash \psi$. Similar inductive arguments apply in case of the other left rules. The point is that all left rules are ignorant of the succedent which could be $\bot$ or $\psi$ and can be applied arbitrarily left of the focus. This proves (ii).
Consider the last rule in $D$. If it is $Ax_m$ or $\bot \Gamma$ then we can derive the conclusion $\Sigma_1, \exists R.C \cdot \Sigma_2 \vdash \Psi$ straight away and thereby eliminate $\exists L$. The resulting derivation tree has height 1 which is strictly less than the original $D$ which consisted of at least two rules in sequence. All left rules $\land L, \lor L, \exists L$ at the end of $D$ can be swapped with $\exists L$, e.g., take $\exists L$.

\[
\begin{array}{c}
\vdash D' \\
\Sigma_1, C_1 \vdash C_1 \\
\Sigma_1, C_1 \supset C_2, C_2 \supset \exists R.C \cdot \Sigma_2 \vdash \Psi \\
\Sigma_1, C_1 \supset C_2, C_2 \supset \exists R.C \cdot \Sigma_2 \vdash \exists L \\
\end{array}
\]

which transforms into

\[
\begin{array}{c}
\vdash D'' \\
\Sigma_1, C_1 \vdash C_1 \\
\Sigma_1, C_1 \supset C_2, C_2 \supset \exists R.C \cdot \Sigma_2 \vdash \exists L \\
\Sigma_1, C_1 \supset C_2, C_2 \supset \exists R.C \cdot \Sigma_2 \vdash \exists L \\
\end{array}
\]

Similar commutations are possible for $\land L$ and $\lor L$. Next take a look at $\exists L$ as the last rule, which means we have two $\exists L$ in sequence. The top one overrides the one below as follows:

\[
\begin{array}{c}
\vdash D'' \\
\Sigma_1, \exists S.D, \Gamma \vdash \exists \Psi \\
\Sigma_1, \exists S.D, \Gamma, \exists R.C \cdot \Sigma_2 \vdash \exists \Psi \\
\Sigma_1, \exists S.D, \Gamma, \exists R.C \cdot \Sigma_2 \vdash \exists L \\
\end{array}
\]

\[
\begin{array}{c}
\vdash D'' \\
\Sigma_1, \exists S.D, \Gamma \vdash \exists \Psi \\
\Sigma_1, \exists S.D, \Gamma, \exists R.C \cdot \Sigma_2 \vdash \exists \Psi \\
\Sigma_1, \exists S.D, \Gamma, \exists R.C \cdot \Sigma_2 \vdash \exists L \\
\end{array}
\]

since $\Sigma_1, \exists S.D, \Gamma, \exists R.C \cdot \Sigma_2$ is the same as $\Sigma_1, \exists S.D, \Gamma, \exists R.C \cdot \Sigma_2$. An application of $\forall L$ permutes like this:

\[
\begin{array}{c}
\vdash D'' \\
\Sigma_1, S.D \supset \exists S.D, \Sigma_1, \exists S.D, \Sigma_2 \vdash \exists \Psi \\
\Sigma_1, S.D \supset \exists S.D, \Sigma_1, \exists S.D, \Sigma_2 \vdash \exists L \\
\end{array}
\]

\[
\begin{array}{c}
\vdash D'' \\
\Sigma_1, S.D \supset \exists S.D, \Sigma_1, \exists S.D, \Sigma_2 \vdash \exists \Psi \\
\Sigma_1, S.D \supset \exists S.D, \Sigma_1, \exists S.D, \Sigma_2 \vdash \exists L \\
\end{array}
\]

Note that if $\Sigma_1, S.D \supset \exists S.D, \Sigma_1, \exists S.D, \Sigma_2 \vdash \Psi$ is tight then the new interpolating sequent $\Sigma_1, S.D, \Sigma_1, \exists S.D, \Sigma_2 \vdash \Psi$ must be tight, too. It is important to observe that $\forall L$ only applies to $\supset$ separators left of the focus while $\exists L$ wraps up a separator to the right of the focus. This means these rules cannot interfere. E.g., the following is not a possible instance of $\forall L$ above $\exists L$:

\[
\begin{array}{c}
\vdash D'' \\
\Sigma_1, \forall R.D, \exists R.C \cdot \Sigma_2 \vdash \exists \Psi \\
\Sigma_1, \forall R.D, \exists R.C \cdot \Sigma_2 \vdash \exists L \\
\end{array}
\]

because $\forall R.D$ is introduced by $\forall L$ from $D$ to the right of the focus. Such an application is not needed and would not directly commute with $\exists L$. Finally we come to study the right rules at the end of $D$. It is not difficult to see that all the right rules $\land R, \lor R, \exists R$ must commute downwards as they are ignorant of the antecedent. The right rule $\forall R$ can be used to eliminate $\exists L$ completely:

\[
\begin{array}{c}
\vdash D'' \\
\Sigma_1, \exists S.D, \Psi \vdash \exists \Psi \\
\Sigma_1, \exists S.D, \Psi \vdash \exists L \\
\end{array}
\]

\[
\begin{array}{c}
\vdash D'' \\
\Sigma_1, \exists S.D, \Psi \vdash \exists \Psi \\
\Sigma_1, \exists S.D, \Psi \vdash \exists L \\
\end{array}
\]

Regarding $\supset R$ we have the direct commutation

\[
\begin{array}{c}
\vdash D' \\
\Sigma_1, D_1 \cdot \exists R.C \cdot \Sigma_2 \vdash D_2 \\
\Sigma_1, D_1 \cdot \exists R.C \cdot \Sigma_2 \vdash D_2 \\
\end{array}
\]

\[
\begin{array}{c}
\vdash D' \\
\Sigma_1, D_1 \cdot \exists R.C \cdot \Sigma_2 \vdash D_2 \\
\Sigma_1, D_1 \cdot \exists R.C \cdot \Sigma_2 \vdash D_2 \\
\end{array}
\]

Again, the new intermediate sequent $\Sigma_1, D_1, \exists R.C \cdot \Sigma_2 \vdash D_2$ in the transformed derivation is certainly tight provided $\Sigma_1, \exists R.C \cdot \Sigma_2 \vdash D_1 \supset D_2, \Psi'$ is. This completes the proof of (iii) of Lemma 2. Observe that this inductive process of pushing $\exists L$ upwards in the tree does not introduce extra rules and thus does not increase the height of the tree, as required. □
Lemma 3 (Cut admissibility). If \( \Sigma \vdash D \) and \( \Sigma, D, \Sigma' \vdash \Psi \) then \( \Sigma, \Sigma' \vdash \Psi \). Moreover, if both \( \Sigma \vdash D \) and \( \Sigma, D, \Sigma' \vdash \Psi \) are tight derivations, then so is \( \Sigma, \Sigma' \vdash \Psi \).

Proof. The proof is entirely standard. One shows by simultaneous induction on the height of the derivation trees and the size of the cut proposition that every instance of the cut is admissible in the system G-Ck. In this process instances of cut are recursively pushed up the derivation tree towards the leaves, i.e., towards instances of rules \( \text{Ax}_m \) and \( \perp \mathcal{L} \), where they disappear. It is not difficult to check, in each case, that indeed these transformations preserve tightness. Let us take an instance of cut:

\[
\begin{array}{c}
\vdash D \\
\vdash E \\
\Sigma, \Sigma' \vdash \Psi \\
\end{array}
\]

We assume by (iii) of Lemma 2 that \( \mathcal{D} \) and \( \mathcal{E} \) are normalised so every occurrence of \( \exists \mathcal{L} \) is coupled with \( \text{Ax}_f \). The transformations we identify below in order to move cut upwards into \( \mathcal{D} \) and \( \mathcal{E} \) will preserve this normal form property. We also use weakening (i) and (ii) of Lemma 2. All these are primitive recursive transformations that can be combined to eliminate cuts by local transformations on the structure of derivations. For the purposes of this proof we find it convenient to consider \( \text{Ax}_f \) along with \( \text{Ax}_m \) and \( \perp \mathcal{L} \) as an axiom rather than a rule.

(\( \mathcal{D} \) axioms) Because of the structure of the succedent the last rule of \( \mathcal{D} \) cannot be \( \text{Ax}_f \). If \( D \) is an application of \( \text{Ax}_m \) then \( D \) is contained in the focus slot of the antecedent, whence \( \Sigma, D = \Sigma \) and derivation \( \mathcal{E} \) is already a derivation of \( \Sigma, \Sigma', \vdash \Psi \). Thus the rule cut is eliminated. If \( \mathcal{D} \) ends in \( \perp \mathcal{L} \) then \( \perp \mathcal{L} \) is contained in \( \Sigma \), whence we might just as well derive the desired conclusion \( \Sigma, \Sigma', \vdash \Psi \) by \( \perp \mathcal{L} \) directly without detour, as this rule works for arbitrary succedents.

(\( \mathcal{D} \) left rules) If \( \mathcal{D} \) ends in a left rule \( \land \mathcal{L}, \lor \mathcal{L}, \exists \mathcal{L}, \forall \mathcal{L} \) then the premise sequent(s) \( \Sigma' \vdash D \) leading to \( \Sigma, \vdash D \) have the same succedent \( D \). Moreover using Lemma 2(i) one shows that \( \mathcal{D} \) and \( \mathcal{E} \) can be massaged to become a derivation \( \mathcal{D}'' \) for \( \Sigma'' \vdash D \) and \( \forall \mathcal{L} \) for \( \Sigma'', \vdash \Psi \) with the same height or smaller than \( \mathcal{D} \) and \( \mathcal{E} \), respectively. By induction hypothesis we can cut the smaller sub-derivation \( \mathcal{D}'' \) for \( \Sigma'', \vdash D \) with \( \forall \mathcal{L} \) for \( \Sigma'', D, \Sigma', \vdash \Psi \) to obtain \( \Sigma'', \Sigma', \vdash \Psi \).

Applying the left rule, which we removed from \( \mathcal{D} \), to \( \Sigma'', \Sigma', \vdash \Psi \) gives \( \Sigma, \Sigma', \vdash \Psi \) by implicit contractions. Consider the rule \( \forall \mathcal{L} \) as an example:

\[
\begin{array}{c}
\vdash D' \\
\vdash E' \\
\Sigma_1 \forall \mathcal{L}, C, \Sigma_2 \vdash D \\
\Sigma_1, \forall \mathcal{L}, C \gg D_1 \Rightarrow \Sigma_2, \Sigma', \vdash \Psi \\
\end{array}
\]

Here \( \forall \mathcal{L} \) and cut commute as follows:

\[
\begin{array}{c}
\vdash D' \text{ weakened} \\
\vdash E' \text{ weakened} \\
\Sigma_1, \forall \mathcal{L}, C \gg D_1 \Rightarrow \Sigma_2, C, \Sigma_2 \vdash D \\
\Sigma_1, \forall \mathcal{L}, C \gg D_1 \Rightarrow \Sigma_2, C, \Sigma_2, \Sigma', \vdash \Psi \\
\end{array}
\]

using contraction on \( \forall \mathcal{L}, C \) in the last rule \( \forall \mathcal{L} \). It is not difficult to convince oneself that the weakened sequents are tight whenever the original sequents were. For instance, if the addition of proposition \( \forall \mathcal{L}, C \) to the end sequent \( \Sigma_1, \forall \mathcal{L}, C \gg D_1 \) of the weakened version of derivation \( D' \) violates tightness, then also the sequent \( \Sigma_1, \forall \mathcal{L}, C \gg D_2, \Sigma', \vdash \Psi \) in the original derivation would not be tight in the first place. Similar commutations, possibly using weakening, work for all the other left rules and \( \forall \mathcal{L} \) since they are applicable to propositions in any scope to the left of the focus. Thus, these left rules commute with the application of cut. But what about \( \exists \mathcal{L} \)? By assumption the derivation \( \mathcal{D} \) is normal (Lemma 2(ii)) and thus if \( \exists \mathcal{L} \) is its last rule, the next to last rule is \( \text{Ax}_f \) and the succedent would have to be of the form \( \Phi \gg \mathcal{L} \) rather than \( \perp \mathcal{L} \). Hence this case is excluded by pre-processing.

(\( \mathcal{D} \) right rules) Thus, as regards \( \mathcal{D} \), all that remains to be considered are the right rules \( \land \mathcal{L}, \lor \mathcal{L}, \forall \mathcal{L}, \exists \mathcal{L}, \forall \mathcal{L}, \exists \mathcal{L} \). These cannot be reduced by themselves but require analysis of the last rule of \( \mathcal{E} \), which we move to next.

(\( \mathcal{E} \) axioms) \( \mathcal{E} \) cannot end in \( \text{Ax}_f \) by the form of the antecedent of its conclusion, which has length 0. If the last rule of \( \mathcal{E} \) is \( \text{Ax}_m \) involving the cut proposition \( D \), then \( \Psi = D, \psi' \). Then, sub-derivation \( \mathcal{D} \) already fits the bill since it can be weakened by Lemma 2(ii) to derive \( \Sigma, \Sigma', \vdash D, \psi' \). If \( \text{Ax}_m \) instead involves some context assumption from \( \Sigma \) or \( \Sigma' \) then we get the conclusion \( \Sigma, \Sigma', \vdash \psi \) from \( \text{Ax}_m \) right away without cut. If \( \mathcal{E} \) ends in \( \perp \mathcal{L} \) and \( \perp \mathcal{L} \) is from \( \Sigma, \Sigma' \) we can use \( \perp \mathcal{L} \) directly to obtain the conclusion of the cut. If \( D = \perp \mathcal{L} \) we argue as before, viz. that then derivation \( \mathcal{D} \) with sequent \( \Sigma, \perp \mathcal{L} \) implies \( \Sigma, \Sigma', \vdash \psi \) by Lemma 2(ii). In all these cases the derivation generated for \( \Sigma, \Sigma', \vdash \psi \) is tight if the derivation \( \mathcal{E} \) for \( \Sigma, D, \Sigma', \vdash \psi \) is.

(\( \mathcal{E} \) right rules) If \( \mathcal{E} \) ends in any of the right rules \( \land \mathcal{L}, \lor \mathcal{L}, \exists \mathcal{L}, \forall \mathcal{L}, \forall \mathcal{L}, \exists \mathcal{L} \) then its immediate sub-derivation(s) \( \mathcal{E}' \) derive(s) a sequent \( \Sigma, D, \Sigma', \vdash \psi' \). We can then apply cut to \( \mathcal{D} \) and \( \mathcal{E}' \) by induction hypothesis to produce \( \Sigma, \Sigma', \vdash \psi' \). Applying the right
rule to this sequent brings us back to $\Sigma, \Sigma', \vdash \Psi$. The two right rules for $E$ that need some extra work in the antecedent are $\forall R$ and $\supset R$. In case of $\forall R$ we have a cut like this

\[
\begin{array}{c}
\vdash D \\
\Sigma, \vdash D \\
\Sigma, \Sigma', \vdash \forall R, E
\end{array}
\]

which we can move upwards (commute with $\forall R$) as follows

\[
\begin{array}{c}
\vdash D \\
\Sigma, \Sigma', \vdash \forall R, E
\end{array}
\]

In case the last (right) rule of $E$ is $\supset R$ we have

\[
\begin{array}{c}
\vdash D \\
\Sigma, \vdash D \\
\Sigma, \Sigma', \vdash E_1 \supset E_2 \\
\vdash \exists R
\end{array}
\]

The permutation of right rules bear no risk of breaking tightness, they do not change the antecedent or only add to the focus point.

($E$ left rules) If the last rule of $E$ is an application of a left rule $\land L$, $\lor L$, $\exists L$, $\forall L$ which does not involve the cut proposition $D$ then it commutes with cut possibly using weakening Lemma 2(i) and implicit contraction. A special case in this class is $\exists L$ for which we must distinguish several cases. First, the proposition $\exists R.C$ introduced by $E$ on the left may also be part of the antecedent in $D$:

\[
\begin{array}{c}
\vdash D \\
\Sigma, \exists R.C, \vdash D \\
\Sigma, \exists R.C, \Gamma, \vdash \Psi
\end{array}
\]

where $\Sigma = \Sigma'$, $\exists R.C$ and $\Sigma' = \Gamma$. Here the application of $\exists L$ may introduce not just $\exists R.C$ on the left but also some of the propositions $D, \Gamma$ in focus. If it also introduces $D$, say

\[
\begin{array}{c}
\vdash D \\
\Sigma', \exists R.C, \vdash D \\
\Sigma', \exists R.C, \Gamma, \vdash \Psi
\end{array}
\]

in which $\Gamma' \subseteq \Gamma$, then we may eliminate the cut directly by weakening from $E'$:

\[
\begin{array}{c}
\vdash E' \\
\Sigma', \Gamma', \vdash \Psi
\end{array}
\]

which cannot destroy tightness since we expand from $\Gamma'$ to $\Gamma$ at the focus point. It remains to tackle the case where $\exists L$ does not introduce $D$ but instead $D$ is part of the assumptions in sub-derivation $E'$:
where from now on we assume that all of $\mathcal{G}$ is introduced by $\exists L$. Any subset of assumptions $\mathcal{G}' \subseteq \mathcal{G}$ already present in the conclusion of $\mathcal{E}'$ as above can be accounted for by weakening in $\mathcal{S}'$. We invoke normalisation (iii) from Lemma 2, which makes sure that the last rule of $\mathcal{E}'$ is $Ax_f$, i.e., $\Psi = \Phi \gg R \Psi_1$ and we end up with

$$\vdash \mathcal{E}'$$

$$\vdash \exists \mathcal{L}$$

This can be reordered easily using cut and a weakened version of $\mathcal{E}''$:

$$\vdash \exists \mathcal{L}$$

where in the last rule $\exists L$ we have made use of implicit contraction which identifies $\mathcal{S}'$, $\exists \mathcal{L}$, $\Gamma$ and $\mathcal{S}'$, $\exists \mathcal{L}$, $\exists \mathcal{L}$, $\Gamma$.

The other situation to be considered is where the introduced proposition $\exists \mathcal{L}$ is not part of the antecedent in $\mathcal{D}$, i.e.,

$$\vdash \exists \mathcal{L}$$

where $\mathcal{S}' = \mathcal{S}'$, $\exists \mathcal{L}$, $\Gamma$. Again by normalisation (Lemma 2(iii)) we can assume without loss of generality that $\mathcal{E}'$ ends in $Ax_f$ and $\Psi = \Phi \gg R \Psi_1$:

$$\vdash \exists \mathcal{L}$$

This permutes into another application of cut of smaller height:

$$\vdash \exists \mathcal{L}$$

One checks without difficulties that all these instances of $\exists L$ at the end of $\mathcal{E}$ are permuted with cut preserving tightness and the normal form property keeping $\exists L$ and $Ax_f$ together. Finally, we need to treat the cases of $\mathcal{D}$ ending in a right rule and $\mathcal{E}$ ending in a corresponding left rule introducing the same operator in the cut proposition $D$. This leaves 5 cases to discuss:

$$(\mathcal{D}/\wedge R, \mathcal{E}/\wedge L)$$

$$(\mathcal{D}/\vee R, \mathcal{E}/\vee L)$$

Here we apply cut between $\mathcal{D}'$ and $\mathcal{E}'$. The sub-derivation $\mathcal{D}'_j$ for $j \neq i$ is dropped.

$$(\mathcal{D}/\vee R, \mathcal{E}/\vee L)$$

Again, perform cut between $\mathcal{D}'$ and $\mathcal{E}'$, while the sub-derivation $\mathcal{E}'_j$ for $j \neq i$ is dropped.
(\(D/\supset R, E/\supset L\)) Note that the cut proposition \(D = D_1 \supset D_2\) need not be at the focus point, whence the context \(\Sigma'\) may contain any context separators:

\[
\begin{array}{c}
\vdots \\
\Sigma, D_1, \vdash D_2 \supset R \\
\Sigma, D_1 \supset D_2 \\
\Sigma, \Sigma', \vdash \psi \\
\Sigma, \Sigma', \vdash \psi
\end{array}
\]

This is solved by first applying \textit{cut} to \(E_1'\) and \(D'\) and then \textit{cut} the result with \(E_2'\):

\[
\begin{array}{c}
\vdots \\
\Sigma, D_1, \vdash D_1 \\
D' \\
\Sigma, D_1 \supset D_2 \\
\Sigma, \Sigma', \vdash \psi \\
E_2'
\end{array}
\]

Note that here the first application of \textit{cut} is at a smaller height but the second \textit{cut} need not be in general. However, this second cut is on a smaller cut proposition \(D_2\) compared to \(D_1 \supset D_2\) of the original \textit{cut}.

(\(D/\forall R, E/\forall L\))

\[
\begin{array}{c}
\vdots \\
\Sigma, \forall R, D, \vdash D \\
\Sigma, \forall R, D, \vdash D \\
\Sigma, \forall R, D, \vdash D \\
\Sigma, \forall R, D, \vdash \psi \\
\Sigma, \forall R, D, \vdash \psi
\end{array}
\]

where \(\Sigma' = \Gamma \gg R \Sigma''\). Using weakening (which happens to preserve tightness) this turns into

\[
\begin{array}{c}
\vdots \\
\Sigma, \Gamma \gg R, D, \vdash \psi \\
\Sigma, \Gamma \gg R, D, \vdash \psi \\
\Sigma, \Gamma \gg R, D, \vdash \psi
\end{array}
\]

(\(D/\exists R, E/\exists L\)) Since the cut proposition \(\exists R.D\) is always introduced by \(\exists L\) at the focus point we must have \(\Sigma' = \Gamma\) and the cut looks like this:

\[
\begin{array}{c}
\vdots \\
\Sigma, \exists R, D, \vdash \psi \\
\Sigma, \exists R, D, \vdash \psi
\end{array}
\]

Due to Lemma 2(iii) we may assume that \(E'\) ends in \(Ax_f\) and \(\Psi = \Phi \gg R \Psi_1\), i.e.,

\[
\begin{array}{c}
\vdots \\
\Sigma, \vdash \emptyset \gg R D \\
\Sigma, \vdash \emptyset \gg R D \\
\Sigma, \vdash \emptyset \gg R D \\
\Sigma, \vdash \emptyset \gg R D \\
\Sigma, \vdash \emptyset \gg R D
\end{array}
\]

at which point we need to make further distinctions according to the last rule of \(D'\). In fact, we show that every instance of the following \textit{cut}\(^*\) is admissible and preserves tightness:

\[
\begin{array}{c}
\vdots \\
\Sigma, \vdash \emptyset \gg R D \\
\Sigma, \vdash \emptyset \gg R D \\
\Sigma, \vdash \emptyset \gg R D
\end{array}
\]

from which \(\Sigma, \Gamma, \vdash \emptyset \gg R \Psi_1\) follows by weakening, Lemma 2(i).

(\(D'\) right rules) The last rule of \(D'\) cannot be any of the right rules \(\land R, \lor R, \supset R, \forall R, \exists R\) since these would not leave an empty first compartment \(\emptyset\) in the succedent \(\Sigma, \vdash \emptyset \gg R D\).

(\(D'\) axioms) Obviously, the last rule of \(D'\) can neither be \(Ax_m\) nor \(Ax_f\). If \(D'\) ends in \(\perp L\) we have \(\perp \in \Sigma\) and can get the conclusion sequent \(\Sigma, \vdash \emptyset \gg R \Psi_1\) straight away, thereby implementing this instance of \textit{cut}\(^*\).
(D' left rules) All applications of \( \land L \) at the end of D' can be pushed down across cut*: The derivation

\[
\frac{\Sigma, C_1, C_2, \Sigma_2 \vdash \psi \Rightarrow R D}{\Sigma_1, C_1 \land C_2, \Sigma_2 \vdash \psi \Rightarrow R D} \quad \frac{\Sigma_1, C_1 \land C_2, \Sigma_2 \vdash \Phi \Rightarrow R \psi_1}{\Sigma_1, C_1 \land C_2, \Sigma_2 \vdash \Phi \Rightarrow R \psi_1} \quad \text{cut*}
\]

becomes

\[
\frac{\Sigma, C_1, C_2, \Sigma_2 \vdash \psi \Rightarrow R D}{\Sigma, C_1 \land C_2, \Sigma_2 \vdash \psi \Rightarrow R D} \quad \text{cut*}
\]

Tightness is not affected since the weakening only enlarges non-empty scopes. In a similar way we proceed with \( \lor L \) and \( \lor R \). If the end rule is \( \exists L \) then we have

\[
\frac{\Sigma, \exists R C, \Gamma \vdash \psi \Rightarrow R D}{\Sigma, \exists R C, \Gamma \vdash \psi \Rightarrow R D} \quad \text{cut*}
\]

exploiting the normalisation property of derivations which links \( \exists L \) and \( Ax_f \). Now we weaken the sub-derivations and recombine like this:

\[
\frac{\Sigma, \exists R C \Rightarrow R C, \Gamma \vdash \psi \Rightarrow R D}{\Sigma, \exists R C \Rightarrow R C, \Gamma \vdash \psi \Rightarrow R D} \quad \text{cut}
\]

where cut is applied inductively to derivations D'' and \( \varepsilon' \) of strictly smaller height. Since weakening is done in the two last scopes before the turnstile, tightness is preserved. □

References