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Advances in Mathematics 226 (2011) 608–640

ADVANCES IN Mathematics

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Permutahedra and generalized associahedra

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Received 9 May 2008; accepted 9 July 2010

Available online 19 August 2010

Communicated by Sara C. Billey

Abstract

Given a finite Coxeter system (W, S) and a Coxeter element c, or equivalently an orientation of the Coxeter graph of W, we construct a simple polytope whose outer normal fan is N. Reading's Cambrian fan $\mathcal{F}_c$, settling a conjecture of Reading that this is possible. We call this polytope the c-generalized associahedron. Our approach generalizes Loday's realization of the associahedron (a type A c-generalized associahedron whose outer normal fan is not the cluster fan but a coarsening of the Coxeter fan arising from the Tamari lattice) to any finite Coxeter group. A crucial role in the construction is played by the c-singleton cones, the cones in the c-Cambrian fan which consist of a single maximal cone from the Coxeter fan.

Moreover, if W is a Weyl group and the vertices of the permutahedron are chosen in a lattice associated to W, then we show that our realizations have integer coordinates in this lattice.

MSC: 20F55; 06B99; 52B11; 05E99

Keywords: Coxeter groups; Cambrian fans; Cambrian lattices; Generalized associahedron; Cluster fans

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0001-8708/$ – see front matter
doi:10.1016/j.aim.2010.07.005

Contents

1. Introduction

Cluster fans were introduced by S. Fomin and A. Zelevinsky in their work on cluster algebras [6]. To each Weyl group there corresponds a cluster fan which encodes important algebraic information including the exchange graph of the corresponding cluster algebra. One very natural problem is to find realizations of these fans as (outer) normal fans of simple polytopes. This was first accomplished by F. Chapoton, S. Fomin, and A. Zelevinsky in [4]: for each Weyl group W, they construct a simple convex polytope whose normal fan is the cluster fan. Such a polytope is called a *generalized associahedron of type W*.

A related family of fans was introduced by N. Reading. For every finite Coxeter group W, and every choice of Coxeter element c, he defined a fan, called the c-Cambrian fan [18] and denoted $\mathcal{F}_c$. N. Reading conjectured that every Cambrian fan is the normal fan of a polytope [18, Conjecture 1.1]. In [21], N. Reading and D. Speyer showed that all Cambrian fans for a given Coxeter group are combinatorially isomorphic to the corresponding cluster fan. However, since Cambrian fans are typically not *linearly* isomorphic to cluster fans, the polytopes of [4] do not suffice to resolve N. Reading's conjecture. We call a polytope whose normal fan is the c-Cambrian fan, a *c-generalized associahedron*. Our goal in this paper is to prove N. Reading's conjecture, by constructing a c-generalized associahedron for all finite Coxeter groups W and Coxeter elements c.

Subsequently, a third construction of a fan associated to a cluster algebra was introduced by S. Fomin and A. Zelevinsky [7], the **g**-vector fan. The definition of the **g**-vectors associated to a cluster algebra is less elementary than the definition of the denominator vectors of the cluster algebra (which define the rays of the cluster fan) but in some respects, the **g**-vectors are better-behaved. In [21], N. Reading and D. Speyer conjectured that Cambrian fans are linearly isomorphic to **g**-vector fans of finite type cluster algebras with respect to an acyclic initial seed. They proved this conjecture modulo the assumption of a conjecture of [7]. The conjecture of N. Reading and D. Speyer was later proved by different methods by S.-W. Yang and A. Zelevinsky [29]. Thus, in studying Cambrian fans, we are also studying **g**-vector fans for cluster algebras of finite type with respect to an acyclic initial seed.

We now discuss further our approach to constructing polytopes having a Cambrian fan as normal fan. In order to do this, we have to go further back into the past. For symmetric groups, that is for Coxeter groups of type A, the generalized associahedron is combinatorially isomorphic to the *classical associahedron* $\mathsf{Asso}(S_n)$, whose combinatorial structure was first described by J. Stasheff in 1963 [27]. Numerous realizations of the associahedron have been given, see [4,13] and the references therein.

We will be particularly interested in a realization of $\mathsf{Asso}(S_n)$ which is closely related to the *permutahedron*, whose definition we now recall. Let (W, S) be a finite Coxeter system acting by reflections on an $\mathbb{R}$-Euclidean space. Let $\boldsymbol{a}$ be a point in the complement of the hyperplanes corresponding to the reflections in W. The convex hull of the W-orbit of $\boldsymbol{a}$ is a simple convex polytope known as a permutahedron, and denoted $\mathsf{Perm}^{\boldsymbol{a}}(W)$. The normal fan of $\mathsf{Perm}^{\boldsymbol{a}}(W)$ is the *Coxeter fan* $\mathcal{F}$.

An elegant and simple realization of $\mathsf{Asso}(S_n)$ defined by a subset of the inequalities defining the permutahedron $\mathsf{Perm}(S_n)$, is due to S. Shnider and S. Sternberg [24] (for a corrected version consider J. Stasheff and S. Shnider [28, Appendix B]). This polytopal realization of the associahedron from the permutahedron has also been studied by J.-L. Loday [13], and is often referred to as *Loday's realization*. It is this construction which we generalize here, for any finite Coxeter group.

For Coxeter groups of types A and B, the first two authors recently gave a Loday-type realization of any c-generalized associahedron [8], and they showed that Loday's realization of the associahedron is a c-generalized associahedron for a particular c. For hyperoctahedral groups, that is Coxeter groups of type B, the generalized associahedron is called a *cyclohedron*. It was first described by R. Bott and C. Taubes in 1994 [3] in connection with knot theory, and rediscovered independently by R. Simion [25]. See also [4,14,16,22,25]; none of these realizations is similar to Loday's (type A) realization.

Our construction of the c-generalized associahedron is very straightforward. Start from $\mathsf{Perm}^{\boldsymbol{a}}(W)$ and its H-representation as a non-redundant intersection of half spaces. Those half spaces correspond bijectively to rays of the Coxeter fan. The rays of the c-Cambrian fan $\mathcal{F}_c$ are a subset of those rays, and therefore determine a subset of the half spaces defining $\mathsf{Perm}^{\boldsymbol{a}}(W)$. The intersection of this subset of the half spaces is a polytope whose normal fan is the c-Cambrian fan (Theorem 4.4), and thus is by definition a c-generalized associahedron, which we denote by $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$.

We give another description of the half spaces whose intersection defines $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$, as follows. The maximal cones of the Coxeter fan are naturally identified with the elements of W. Each maximal cone of the c-Cambrian fan $\mathcal{F}_c$ is the union of one or more maximal cones of the Coxeter fan. The elements of W corresponding to cones of the Coxeter fan which are also cones of $\mathcal{F}_c$ are called c-singletons. A half space $\mathscr{H}$ of the H-representation of $\mathsf{Perm}^{\boldsymbol{a}}(W)$ is called *c-admissible* iff its boundary contains a vertex of $\mathsf{Perm}^{\boldsymbol{a}}(W)$ that corresponds to a c-singleton. It is exactly the c-admissible half spaces whose intersection defines $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$. Note that this de-

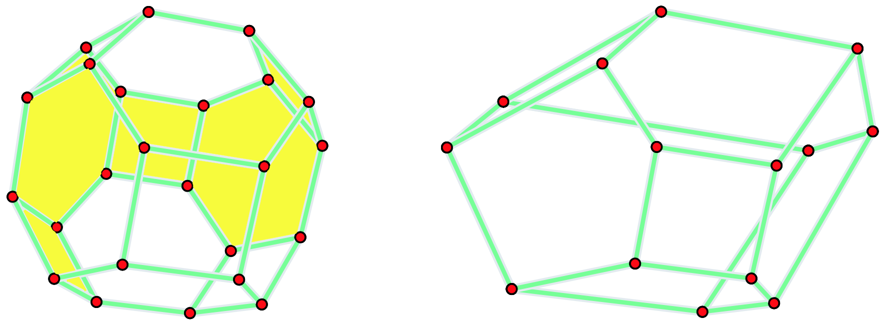

Fig. 1. Obtaining the associahedron from the permutahedron for the Coxeter group S_4 and Coxeter element $c = s_1 s_2 s_3$. The left picture shows the permutahedron with the facets contained in the boundary of c-admissible half spaces translucent and the facets contained in the boundary of non-c-admissible half spaces shaded. The picture to the right shows the associahedron obtained from the permutahedron after removal of all non-c-admissible half spaces.

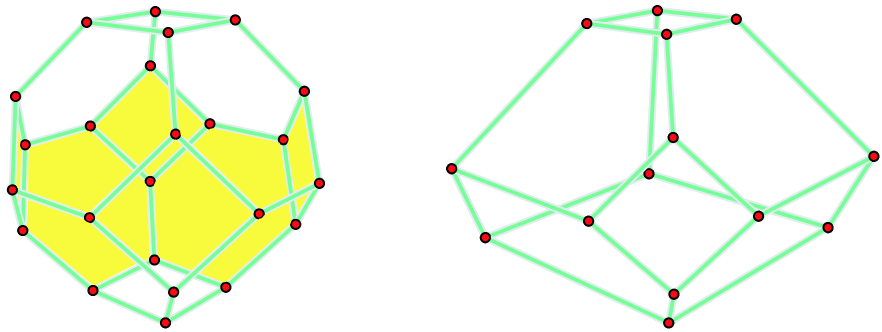

Fig. 2. Obtaining the associahedron from the permutahedron for the Coxeter group S_4 and Coxeter element $c = s_2 s_1 s_3$. The left picture shows the permutahedron with the facets contained in the boundary of c-admissible half spaces translucent and the facets contained in the boundary of non-c-admissible half spaces shaded. The picture to the right shows the associahedron obtained from the permutahedron after removal of all non-c-admissible half spaces.

scription of the half spaces used to define $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ is not needed for the proof of Theorem 4.4. However, it is crucial in the concrete combinatorial description of the c-generalized associahedra.

See Figs. 1 and 2 for illustrations of the contruction of c-generalized associahedra by intersecting the c-admissible half spaces while ignoring the non-c-admissible half spaces.

Additionally, if W is a Weyl group and the vertices of the permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W)$ are chosen in a suitable lattice associated to W, then we show that $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ has integer coordinates in this lattice (Theorem 4.16).

Another interesting aspect of this construction is that we are able to recover the c-cluster complex: relating cluster fans to quiver theory, R. Marsh, M. Reineke and A. Zelevinsky introduce in [15] what N. Reading and D. Speyer call the *c-cluster fan* in [21], and its associated simplicial complex *the c-cluster complex*. A c-cluster fan is a generalization of the cluster fan to any finite Coxeter group W and Coxeter element $c \in W$ (c bipartite is then the traditional case); its applications in quiver representations are most natural for W of types A, D and E.

By replacing the natural labeling of the maximal faces of $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ by a labeling that uses almost positive roots only, we obtain the c-cluster complex. This replacement is determined by an easy combinatorial rule determined by c-singletons as stated in Theorem 3.6. This suggests that these constructions will play an important role in the study of c-cluster complexes and related structures.

This paper is organized as follows. In Section 2, we recall some facts about finite Coxeter groups, Coxeter sortable elements, and Cambrian lattices. Additionally, the important notion of a c-singleton is defined and fundamental properties are proven. In Section 3, we recall some facts about fans, in particular Coxeter and Cambrian fans, and give a precise combinatorial description of the set of rays of Cambrian fans based on c-singletons. In Section 4, we state and prove our main result (Theorem 4.4). Finally, in Section 5 we study some specific examples of finite reflection groups. We work out the dihedral case explicitly to show that the vertex barycenters of the permutahedra and associahedra coincide and we explain how the realizations given in [8] for type A and B are particular instances of the construction described in this paper.

In a sequel [1], we describe the isometry classes of these realizations.

The construction presented in this article has been implemented as the set of functions `CAMBRIAN` to be used with the library `CHEVIE` for `GAP3` [23,5] and can be found on the first author's web page [9].

2. Coxeter-singletons and Cambrian lattices

Let (W, S) be a finite Coxeter system. We denote by e the identity of W and by $\ell : W \to \mathbb{N}$ the length function on W. Let $n = |S|$ be the rank of W. Denote by w_0 the unique element of maximal length in W.

The (*right*) *weak order* $\leqslant$ on W can be defined by $u \leqslant v$ if and only if there is a $v' \in W$ such that $v = uv'$ and $\ell(v) = \ell(u) + \ell(v')$. The *descent set* $D(w)$ of $w \in W$ is $\{s \in S \mid \ell(ws) < \ell(w)\}$. A *cover* of $w \in W$ is an element ws such that $s \notin D(w)$.

The subgroup W_I generated by $I \subseteq S$ is a (*standard*) *parabolic subgroup* of W and the set of minimal length (left) coset representatives of W/W_I is given by

$$W^I = \{x \in W \mid \ell(xs) > \ell(x),\ \forall s \in I\} = \{x \in W \mid D(x) \subseteq S \setminus I\}.$$

Each $w \in W$ has a unique decomposition $w = w^I w_I$ where $w^I \in W^I$ and $w_I \in W_I$. Moreover, $\ell(w) = \ell(w^I) + \ell(w_I)$, see [10, §5.12]. The pair (w^I, w_I) is often called the *parabolic components* of w along I. For $s \in S$ we follow N. Reading's notation and set $\langle s\rangle := S \setminus \{s\}$.

Let c be a Coxeter element of W, that is, the product of the simple reflections of W taken in some order, and fix a reduced expression for c.

2.1. *c-sortable elements*

For $I \subset S$, we denote by $c_{(I)}$ the subword of c obtained by considering only simple reflections in I. Obviously, $c_{(I)}$ is a Coxeter element of W_I. For instance, let $W = S_5$ and $S = \{s_i \mid 1 \leqslant i \leqslant 4\}$ where s_i denotes the simple transposition $(i, i+1)$. If $c = s_1s_3s_4s_2$ and $I = \{s_2, s_3\}$ then $c_{(I)} = s_3s_2$. Consider the possible ways to write $w \in W$ as a reduced subword of the infinite word $c^\infty = cccccc\cdots$. In [19, §2], N. Reading defines the *c-sorting word* of $w \in W$ as the reduced subword of c^∞ for w which is lexicographically first as a sequence of positions. The c-sorting word of w can be written as $c_{(K_1)}c_{(K_2)}\cdots c_{(K_p)}$ where p is minimal for the property:

$$w = c_{(K_1)}c_{(K_2)}\cdots c_{(K_p)} \quad \text{and} \quad \ell(w) = \sum_{i=1}^{p} |K_i|.$$

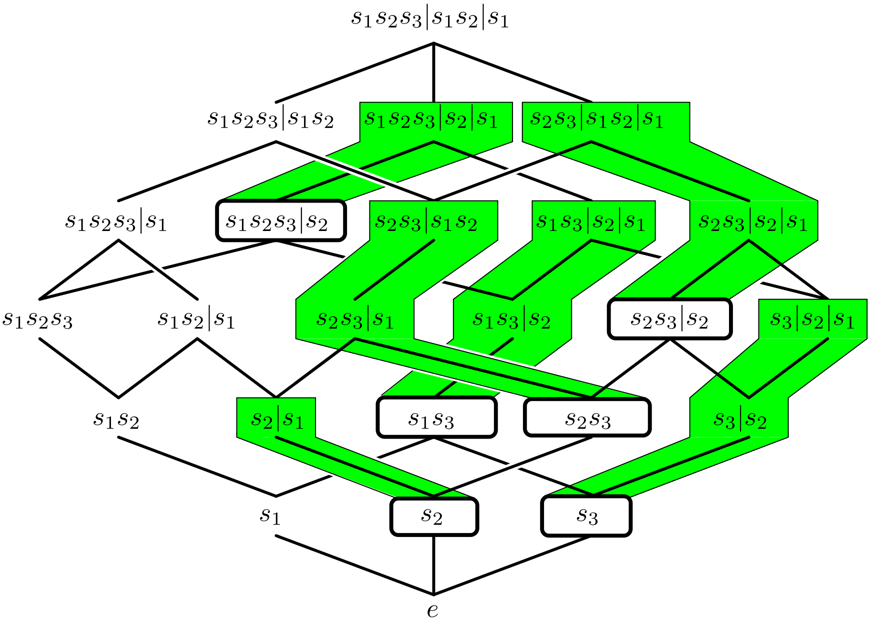

Fig. 3. $c = s_1s_2s_3$.

The sequence $c_{(K_1)}, \ldots, c_{(K_p)}$ associated to the c-sorting word for w is called *c-factorization* of w. The c-factorization of w is independent of the chosen reduced word for c but depends on the Coxeter element c. In general the c-factorization does not yield a nested sequence $K_1, \ldots, K_p$ of subsets of S. An element $w \in W$ is called *c-sortable* if and only if $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_p$. It is clear that for any chosen Coxeter element c, the identity e is c-sortable, and Reading proves in [19] that the longest element $w_0 \in W$ is c-sortable as well. The c-factorization of w_0 is of particular importance for us and is denoted by $\mathbf{w_0}$. To illustrate these notions, consider $W = S_4$ with generators $S = \{s_1, s_2, s_3\}$. The weak order of S_4 with elements represented by their c-factorization is shown in Fig. 3 for $c = s_1s_2s_3$ and Fig. 4 for $c = s_2s_1s_3$ (the delimiter '|' indicates the end of K_i and the beginning of K_{i+1}). Moreover, the background color carries additional information: The background of w is white if and only if w is c-sortable.

2.2. *c-Cambrian lattice*

N. Reading shows that the c-sortable elements constitute a sublattice of the weak order of W which is called the *c-Cambrian lattice* [18,20]. A Cambrian lattice is also a lattice quotient of the weak order on W. In particular, there is a downward projection $\pi_\downarrow^c$ from W to the c-sortable elements of W which maps w to the maximal c-sortable element below w. Hence, w is c-sortable if and only if $\pi_\downarrow^c(w) = w$ [20, Proposition 3.2]. It is easy to recover $\pi_\downarrow^c$ in Figs. 3 and 4. A c-sortable element w (white background) is projected to itself; an element w which is not c-sortable (colored background) is projected to the (maximal) boxed c-sortable element below the colored component containing w. For instance in Fig. 4, we consider $c = s_2s_1s_3$ and have $\pi_\downarrow^c(s_2s_3s_2) = s_2s_3s_2$ and $\pi_\downarrow^c(s_3s_2s_1) = \pi_\downarrow^c(s_3s_2) = \pi_\downarrow^c(s_3) = s_3$.

We say that $w \in W$ is *c-antisortable* if ww_0 is c^{-1}-sortable. We therefore have a projection $\pi_c^\uparrow$ from W to the set of c-antisortable elements of W which takes w to the minimal c-antisortable

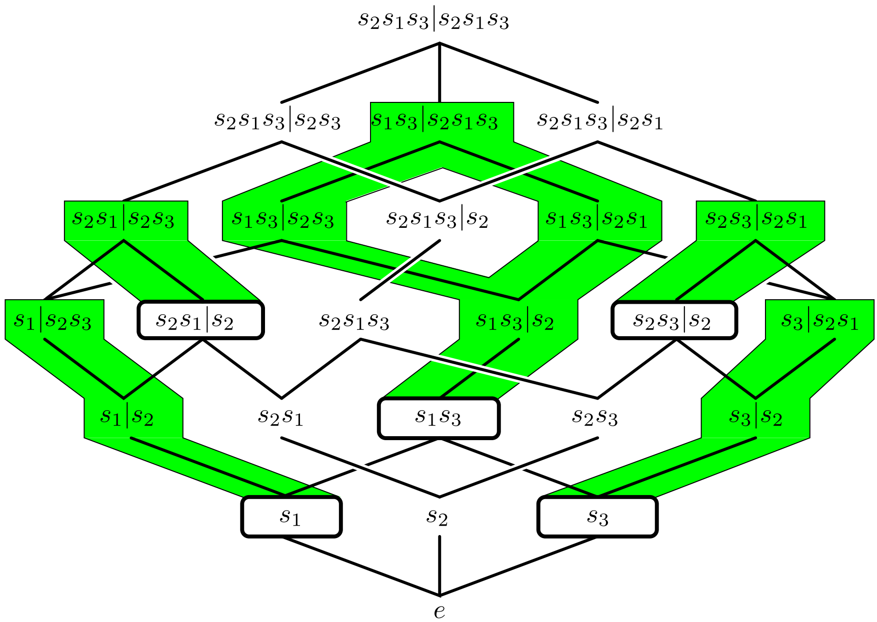

Fig. 4. $c = s_2s_1s_3$.

element above w. For example we have $\pi_c^{\uparrow}(s_1s_3) = s_1s_3s_2s_1s_3$ in Fig. 4. The maps $\pi_{\downarrow}^c$ and $\pi_c^{\uparrow}$ have the same fibers, that is,

$$\left(\pi_{\downarrow}^c\right)^{-1}\pi_{\downarrow}^c(w) = \left(\pi_c^{\uparrow}\right)^{-1}\pi_c^{\uparrow}(w).$$

These fibers are intervals in the weak order as shown by N. Reading [20, Theorem 1.1] and the fiber that contains w is $[\pi_{\downarrow}^c(w), \pi_c^{\uparrow}(w)]$.

2.3. c-singletons

We now introduce an important subclass of c-sortable elements: an element $w \in W$ is a *c-singleton* if and only if $(\pi_{\downarrow}^c)^{-1}(w)$ is a singleton. It is easy to read off c-singletons in Figs. 3 and 4: An element is a c-singleton if and only if its background color is white and it is not boxed, so, for example, $s_2s_1s_3$ in Fig. 4 is a c-singleton while neither $s_1s_3s_2$ nor $s_2s_3s_2$ are c-singletons.

We now prove some useful properties of c-singletons.

Proposition 2.1. *Let $w \in W$. The following propositions are equivalent.*

(i) *w is a c-singleton*;
(ii) *w is c-sortable and ws is c-sortable for all $s \notin D(w)$*;
(iii) *w is c-sortable and c-antisortable.*

Proof. '(i) is equivalent to (iii)' and '(i) is equivalent to (ii)' follow from the fact that the fiber containing w is $[\pi_{\downarrow}^c(w), \pi_c^{\uparrow}(w)]$ and that the map $\pi_{\downarrow}^c$ is order preserving. □

It follows that w_0 and e are c-singletons.

The word property, see [2, Theorem 3.3.1], says that any pair of reduced expressions for $w \in W$ can be linked by a sequence of braid relation transformations. In particular, the set

$$S(w) := \{s_i \in S \mid s_i \text{ appears in a reduced expression for } w\} = \bigcap_{\substack{I \subset S \\ w \in W_I}} I$$

is independent of the chosen reduced expression for w. It is clear that $w \in W_{S(w)}$ and that $S(w) = K_1$ if w is c-sortable with c-factorization $c_{(K_1)} c_{(K_2)} \cdots c_{(K_p)}$.

Two reduced expressions for $w \in W$ are equivalent *up to commutations* if they are linked by a sequence of braid relations of order 2, that is, by commutations. Let $\mathbf{u}$, $\mathbf{w}$ be reduced expressions for $u, w \in W$. Then $\mathbf{u}$ is a *prefix of* $\mathbf{w}$ *up to commutations* if $\mathbf{u}$ is the prefix of a reduced expression $\mathbf{w}'$ and $\mathbf{w}'$ is equivalent to $\mathbf{w}$ up to commutations. We now state the main result of this section. Its proof is deferred until after Proposition 2.7.

Theorem 2.2. *Let w be in W. Then w is a c-singleton if and only if w is a prefix of* $\mathbf{w_0}$ *up to commutations.*

Remark 2.3. For computational purposes, it would be interesting to find a nice combinatorial description of $\mathbf{w_0}$.

Example 2.4. Let $W = S_4$ with generators $S = \{s_i \mid 1 \leqslant i \leqslant 3\}$ and Coxeter element $c = s_2 s_1 s_3$. The c-singletons of W are

$$\begin{array}{lll} e, & s_2 s_3, & s_2 s_1 s_3 s_2 s_1, \\ s_2, & s_2 s_1 s_3, & s_2 s_1 s_3 s_2 s_3, \\ s_2 s_1, & s_2 s_1 s_3 s_2, & w_0 = s_2 s_1 s_3 s_2 s_1 s_3. \end{array}$$

We see here that $s_2 s_3$ is not a prefix of $s_2 s_1 s_3 s_2 s_1 s_3$, but it does appear as a prefix after commutation of the commuting simple reflections s_1 and s_3.

Proposition 2.5. *The c-singletons constitute a distributive sublattice of the* (*right*) *weak order on W.*

Examples of these distributive lattices for $W = S_4$ are given in Fig. 5.

Proof. Let L be the set of subsets $P \subset \{1, \ldots, \ell(w_0)\}$ with the property that the reflections at positions $i \in P$ of $\mathbf{w_0}$ can be moved by commutations to form a prefix $\mathbf{w}_P$ of $\mathbf{w_0}$. This prefix $\mathbf{w}_P$ represents $w_P \in W$. Note that $\ell(w_P) = |P|$ because $\mathbf{w}_P$ is a prefix up to commutation of a reduced word for $\mathbf{w_0}$. The set L is partially ordered by inclusion and forms a distributive lattice with $P_1 \vee P_2 = P_1 \cup P_2$ and $P_1 \wedge P_2 = P_1 \cap P_2$ according to [26, Exercise 3.48]. (In particular, $P_1 \cup P_2, P_1 \cap P_2 \in L$ if $P_1, P_2 \in L$.)

We claim that $P \mapsto w_P$ is an injective lattice homomorphism.

First we check injectivity. Suppose $w_P = w_Q$ for $P \neq Q$. Since $\mathbf{w}_P$ and $\mathbf{w}_Q$ are reduced expressions, we have $|P| = |Q| = r$. Let $P = \{p_1, \ldots, p_r\}$ and $Q = \{q_1, \ldots, q_r\}$ with $p_i < p_{i+1}$ and $q_i < q_{i+1}$. Without loss of generality, let the smallest element in $(P \cup Q) \setminus (P \cap Q)$ be p_i.

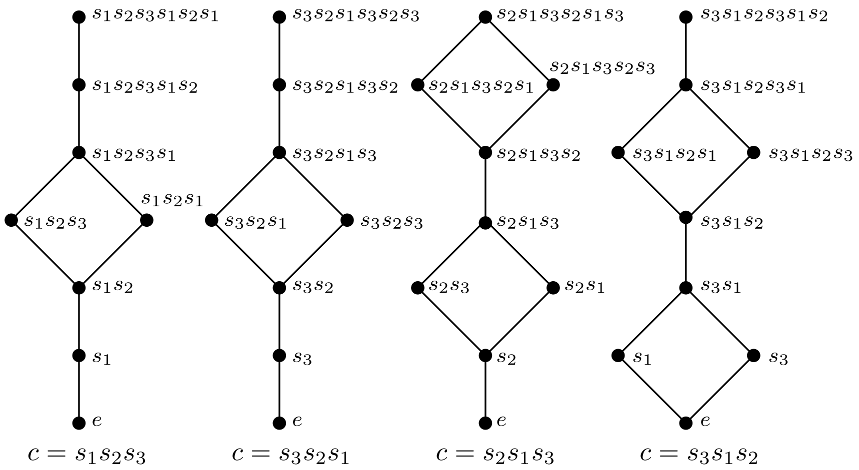

Fig. 5. There are four Coxeter elements in S_4. Each yields a distributive lattice of c-singletons.

Let $s \in S$ be the reflection at position p_i of $\mathbf{w_0}$, then s also appears at some q_j with $q_j > p_i$. When moving the reflections in Q to the front of $\mathbf{w_0}$, the s that started at q_j must pass the s at p_i, but this implies that the expression for $\mathbf{w_0}$ would not be reduced at this step, which is contrary to our assumption. Thus the map is injective.

We show that $P \mapsto w_P$ respects the lattice structures of L and W. Let $P, Q \in L$ and set $R := P \cap Q$. Since $R \in L$, $\mathbf{w}_R$ is a prefix of $\mathbf{w_0}$ up to commutations. In particular, it is also a prefix of $\mathbf{w}_P$ and $\mathbf{w}_Q$ up to commutations. Hence we have $w_R \leqslant w_P$ and $w_R \leqslant w_Q$. We obtain $\mathbf{w}_{P\setminus R}$ and $\mathbf{w}_{Q\setminus R}$ from $\mathbf{w}_P$ and $\mathbf{w}_Q$ by deletion of all reflections that correspond to an element of R and conclude $w_P = w_R w_{P\setminus R}$ and $w_Q = w_R w_{Q\setminus R}$. We have $S(\mathbf{w}_{P\setminus R}) \cap S(\mathbf{w}_{Q\setminus R}) = \varnothing$ since $\mathbf{w_0}$ is reduced; the proof is by contradiction and is similar to the proof of injectivity. Therefore, no element of W is above $\mathbf{w}_R$ and below $\mathbf{w}_P$ and $\mathbf{w}_Q$. We have shown $w_R = w_P \wedge w_Q$ with respect to the weak order on W. A similar argument proves $w_T = w_P \vee w_Q$ with respect to the weak order on W where $T = P \cup Q$: $S(\mathbf{w}_{T\setminus P}) \cap S(\mathbf{w}_{T\setminus Q}) = \varnothing$ implies that no $w \in W$ below w_R and above w_P and w_Q exists. $\square$

The following lemma characterizes the elements that cover a c-singleton.

Lemma 2.6. *Let $c_{(K_1)} \cdots c_{(K_p)}$ be the c-factorization of the c-singleton $w \in W$ and $s \notin D(w)$. Then the c-factorization of the cover ws of w is either $c_{(K_1)} \cdots c_{(K_p)} c_{(s)}$ or $c_{(K_1)} \cdots c_{(K_i \cup \{s\})} \cdots c_{(K_p)}$.*

If $ws = c_{(K_1)} \cdots c_{(K_i \cup \{s\})} \cdots c_{(K_p)}$ then i is uniquely determined and s commutes with every $r \in K_{i+1} \cup L$ where L satisfies $c_{(K_i \cup \{s\})} = c_{(K_i \setminus L)} s c_{(L)}$.

Proof. If $s \in K_p$ then $c_{(K_1)} \cdots c_{(K_p)} c_{(s)}$ is obviously the c-factorization for ws. So we assume $s \notin K_p$. As w is a c-singleton, ws is c-sortable with c-factorization $c_{(L_1)} \cdots c_{(L_q)}$ where $L_1 \supseteq \cdots \supseteq L_q$. As $s \in D(ws)$, there is a unique $1 \leqslant i \leqslant q$ and $r \in L_i$ such that

$$w = (ws)s = c_{(L_1)} \cdots c_{(L_i \setminus \{r\})} \cdots c_{(L_q)}$$

by the exchange condition.

Case 1: Suppose $i = 1$, i.e. $i = 1$ is the unique index such that

$$w = (ws)s = c_{(L_1 \setminus \{r\})} c_{(L_2)} \cdots c_{(L_q)}. \tag{1}$$

Case 1.1: $r \notin K_1$. Then $r \notin S(w)$ and $s = r$ because $r \in S(ws) = S(w) \cup \{s\}$. Since any two reduced expressions of ws are linked by braid relations according to Tits' Theorem [2, Theorem 3.3.1] and since $s \notin S(w)$, we conclude that we have to move s from the rightmost position to the left by commutation. In other words, s commutes with $K_2 \cup L$.

Case 1.2: $r \in K_1 = S(w)$. As $c_{(L_1 \setminus \{r\})} c_{(L_2)} \cdots c_{(L_q)}$ is reduced and $L_2 \supseteq \cdots \supseteq L_q$ is nested, we have $r \in L_2$. Hence

$$K_1 = S(w) = \big(L_1 \setminus \{r\}\big) \cup L_2 = L_1 \cup L_2 = L_1.$$

Thus $c_{(L_2)} \cdots c_{(L_q)}$ and $c_{(K_2)} \cdots c_{(K_p)} s$ are reduced expressions for some $\widehat{w} \in W$ and $s \in D(\widehat{w})$. The exchange condition implies the existence of a unique index $2 \leqslant j \leqslant q$ and $t \in L_j$ such that

$$\widehat{w} s = c_{(L_2)} \cdots c_{(L_j \setminus \{t\})} \cdots c_{(L_q)}.$$

In other words

$$w = c_{(L_1)} \widehat{w} = c_{(L_1)} c_{(L_2)} \cdots c_{(L_j \setminus \{t\})} \cdots c_{(L_q)}$$

is reduced. But this contradicts the uniqueness of $i = 1$ in Eq. (1). So $r \notin K_1$ and we have finished the first case.

Case 2: Suppose $i > 1$, then $K_1 = S(w) = L_1$. Set $\nu := \min(p, i-1)$ and iterate the argument for $c_{(L_1)}^{-1} w$, $c_{(L_2)}^{-1} c_{(L_1)}^{-1} w, \ldots$ to conclude $L_j = K_j$ for $1 \leqslant j \leqslant \nu$. If $\nu = p$ then $i = q = p + 1$ and $L_i \setminus \{r\} = \varnothing$. So $L_i = \{s\} \subseteq L_{i-1} = K_p$ which contradicts $s \notin K_p$. Thus $\nu = i - 1$ for some $i \leqslant p$ and $L_j = K_j$ for $1 \leqslant j \leqslant i - 1$. We may assume $i = 1$ and are done by Case 1. □

Proposition 2.7. *Let w be a c-singleton and* $\mathbf{w}$ *its c-sorting word. Any prefix of* $\mathbf{w}$ *up to commutations is a c-singleton.*

Proof. Let $c_{(K_1)} \cdots c_{(K_p)}$ denote the c-factorization of w. It is sufficient to show that the prefix w' up to commutations of length $\ell(w) - 1$ is a c-singleton. There is $1 \leqslant i \leqslant p$ and $r \in K_p$ such that $w' = c_{(K_1)} \cdots c_{(K_i \setminus \{r\})} \cdots c_{(K_p)}$ is the c-factorization of w'. It remains to show that $w's$ is c-sortable for $s \notin D(w')$.

Case 1: Suppose $s \in D(w)$. Recall the definition of the *Bruhat order* $\leqslant_{\mathcal{B}}$ *on* W: $u \leqslant_{\mathcal{B}} v$ in W if an expression for u can be obtained as a subword of a reduced expression of v, see [2, Chapter 2]. The lifting property of the Bruhat order, see [2, Proposition 2.2.7], implies $w's \leqslant_{\mathcal{B}} w$. Moreover $\ell(w's) = \ell(w') + 1 = \ell(w)$. Thus $w = w's$ and $s = r$. In particular $w's = w$ is c-sortable.

Case 2: Suppose $s \notin D(w)$, in particular $s \neq r$. So ws is c-sortable and by Lemma 2.6 there are two cases to distinguish: either $c_{(K_1)} \cdots c_{(K_p)} c_{(s)}$ or $c_{(K_1)} \cdots c_{(K_j \cup \{s\})} \cdots c_{(K_p)}$ is the c-factorization of ws.

Case 2.1: Suppose $c_{(K_1)} \cdots c_{(K_p)} c_{(s)}$ is the c-factorization of ws. Then $s \in K_p$ and the c-factorization of $w's$ is $c_{(K_1)} \cdots c_{(K_i \setminus \{r\})} \cdots c_{(K_p)} c_{(s)}$. In particular, the sequence

$$K_1 \supseteq \cdots \supseteq K_i \setminus \{r\} \supseteq \cdots \supseteq K_p \supseteq \{s\}$$

is nested and $w's$ is c-sortable.

Case 2.2: If $c_{(K_1)} \cdots c_{(K_j \cup \{s\})} \cdots c_{(K_p)}$ is the c-factorization of ws then either s and r commute or not.

Suppose first that s and r do not commute. Then $j = i$ and r appears before s in the chosen reduced expression of c, since s commutes with all simple reflections to the right of the rightmost copy of s in the c-factorization of ws by Lemma 2.6. Then $w's = c_{(K_1)} \cdots c_{(K_i \setminus \{r\} \cup \{s\})} \cdots c_{(K_p)}$ is c-sortable.

Suppose now that s and r commute. If $j \leqslant i$ then

$$w's = wrs = wsr = c_{(K_1)} \cdots c_{(K_j \cup \{s\})} \cdots c_{(K_i \setminus \{r\})} \cdots c_{(K_p)}$$

is the c-factorization of $w's$. As $K_1 \supseteq \cdots \supseteq K_j \cup \{s\} \supseteq \cdots \supseteq K_i \setminus \{r\} \supseteq \cdots \supseteq K_p$ is nested, $w's$ is c-sortable. The case $j > i$ is proved similarly.

We conclude that $w's$ is c-sortable for any $s \notin D(w')$, so w' is a c-singleton. □

Proof of Theorem 2.2. We know by Proposition 2.1 that w is a c-singleton if and only if w is c-sortable and ww_0 is c^{-1}-sortable.

Suppose w is a c-singleton. Let s be the rightmost simple reflection appearing in the c^{-1}-factorization for ww_0, so $ww_0 = us$ for some c^{-1}-sortable element u.

We have $su^{-1}w = w_0$ and therefore $u^{-1}w = sw_0$. Since $S = w_0 S w_0$, we conclude that $t := w_0 s w_0$ is a simple reflection. Now $u^{-1}wt = w_0$ implies $\ell(wt) > \ell(w)$ and wt is c-sortable by Proposition 2.1. But wt is also c-antisortable since $wtw_0 = u$ is c^{-1}-sortable. Hence, wt is a c-singleton that covers w in the weak order.

Repeating this process, we show that every c-singleton is on an unrefinable chain of c-singletons leading up to w_0. By downwards induction, every element of that chain is a prefix of w_0 up to commutations. This is clearly true for w_0. As we went up each step, though, we added a simple reflection which commuted with every reflection to its right (or was added at the rightmost end), by Lemma 2.6. Thus, when we want to remove the element we added at the last step, we can rewrite w_0 using only commutations such that this simple reflection is on the right. □

3. Coxeter fans, permutahedra, and Cambrian fans

In this section, we describe the geometry of Coxeter fans and c-Cambrian fans. We first recall some facts about the geometric representation of W. We use the notation of [10] for Coxeter groups and root systems. Let W act by reflections on an $\mathbb{R}$-Euclidean space $(V, \langle \cdot, \cdot \rangle)$.

Let Φ be a root system corresponding to W with simple roots $\Delta = \{\alpha_s \mid s \in S\}$, positive roots $\Phi^+ = \Phi \cap \mathbb{R}_{>0}[\Delta]$ and negative roots $\Phi^- = -\Phi^+$. Without loss of generality, we assume that the action of W is essential relative to V, that is, Δ is a basis of V. The set Φ^+ parametrizes the set of reflections in W: to each reflection $t \in W$ there corresponds a unique positive root $\alpha_t \in \Phi^+$ such that t maps α_t to $-\alpha_t$ and fixes the hyperplane $H_t = \{v \in V \mid \langle v, \alpha_t \rangle = 0\}$.

The *Coxeter arrangement* $\mathcal{A}$ for W is the collection of all reflecting hyperplanes for W. The complement $V \setminus (\bigcup \mathcal{A})$ of $\mathcal{A}$ consists of open cones. Their closures are called *chambers*. The chambers are in canonical bijective correspondence with the elements of W. The *fundamental chamber* $D := \bigcap_{s \in S} \{v \in V \mid \langle v, \alpha_s \rangle \geqslant 0\}$ corresponds to the identity $e \in W$ and the chamber $w(D)$ corresponds to $w \in W$.

A subset U of V is *below* a hyperplane $H \in \mathcal{A}$ if every point in U is on H or on the same side of H as D. The subset U is *strictly below* $H \in \mathcal{A}$ if U is below H and $U \cap H = \varnothing$. Similarly,

U is *above* or *strictly above* a hyperplane $H \in \mathcal{A}$. The *inversions* of $w \in W$ are the reflections that correspond to the hyperplanes H that $w(D)$ is above.

For a simple reflection $s \in S$, we have $\ell(sw) < \ell(w)$ if and only if $s \leqslant w$ in the weak order if and only if $w(D)$ is above H_s. To decide whether $w(D)$ is above or below H_s is therefore a weak order comparison. These notions will be handy in Section 4.

A *fan* $\mathcal{G}$ is a family of non-empty closed polyhedral (convex) cones in V such that

(i) every face of a cone in $\mathcal{G}$ is in $\mathcal{G}$, and
(ii) the intersection of any two cones in $\mathcal{G}$ is a face of both.

A fan $\mathcal{G}$ is *complete* if the union of all its cones is V, *essential* (or *pointed*) if the intersection of all non-empty cones of $\mathcal{G}$ is the origin, and *simplicial* if every cone is simplicial, that is, spanned by linearly independent vectors. A 1-dimensional cone is called a *ray* and a ray is *extremal* if it is a face of some cone. The set of k-dimensional cones of $\mathcal{G}$ is denoted by $\mathcal{G}^{(k)}$ and two cones in $\mathcal{G}^{(k)}$ are *adjacent* if they have a common face in $\mathcal{G}^{(k-1)}$. A fan $\mathcal{G}$ coarsens a fan $\mathcal{G}'$ if every cone of $\mathcal{G}$ is the union of cones of $\mathcal{G}'$ and $\bigcup_{C\in\mathcal{G}} C = \bigcup_{C\in\mathcal{G}'} C$. We refer to [30, Lecture 7] for more details and examples.

The chambers of a Coxeter arrangement $\mathcal{A}$ and all their faces define the *Coxeter fan* $\mathcal{F}$. The Coxeter fan $\mathcal{F}$ is known to be complete, essential, and simplicial [10, Sections 1.12–1.15]. The fundamental chamber $D \in \mathcal{F}$ is a (maximal) cone spanned by the (extremal) rays $\{\rho_s \mid s \in S\}$, where ρ_s is the intersection of D with the subspace orthogonal to the hyperplane spanned by $\{\alpha_t \mid t \in \langle s\rangle\}$.

Recall that the set of rays of $\mathcal{F}$ is partitioned into n orbits under the action of W where $n = |S|$ is the rank of W. Moreover, each orbit contains exactly one ρ_s, $s \in S$. Thus, any ray $\rho \in \mathcal{F}^{(1)}$ is $w(\rho_s)$ for some $w \in W$ where $s \in S$ is uniquely determined by ρ but w is not unique. In fact, $w(\rho_s) = g(\rho_s)$ if and only if $w \in gW_{\langle s\rangle}$.

3.1. Permutahedra

We illustrate Coxeter fans by means of permutahedra, that is, polytopes that have a Coxeter fan as outer normal fan.

Take a point $\boldsymbol{a}$ of the complement $V \setminus (\bigcup \mathcal{A})$ of the Coxeter arrangement $\mathcal{A}$, and consider its W-orbit. The convex hull of this W-orbit is a *W-permutahedron* denoted by $\mathsf{Perm}^{\boldsymbol{a}}(W)$. There is a bijection between the set of rays of $\mathcal{F}$ and the facets of $\mathsf{Perm}^{\boldsymbol{a}}(W)$: there is a half space associated to each ray $\rho \in \mathcal{F}$ such that its supporting hyperplane is perpendicular to ρ and such that the permutahedron is the intersection of these half spaces. Let us be more precise.

Let $\Delta^* := \{v_s \in V \mid s \in S\}$ be the fundamental weights of Δ, that is, Δ^* is the dual basis of Δ with respect to $\langle\cdot,\cdot\rangle$. The fundamental chamber D is spanned by the fundamental weights, that is, $D = \mathbb{R}_{\geqslant 0}[\Delta^*]$. Hence, the rays of $\mathcal{F}$ are easily expressed in terms of Δ^*: We have $\rho_s = \mathbb{R}_{\geqslant 0}[v_s]$ and therefore $w(\rho_s) = \mathbb{R}_{\geqslant 0}[w(v_s)]$ for any $w \in W$ and $s \in S$.

Without loss of generality, we choose $\boldsymbol{a} = \sum_{s\in S} a_s v_s$ in the interior of D, that is $a_s > 0$ for $s \in S$. All points $w(\boldsymbol{a})$ are distinct and the convex hull of $\{w(\boldsymbol{a}) \mid w \in W\}$ yields a realization of the W-permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W)$. It is not difficult to describe this polytope as an intersection of half spaces.

For each $\rho = w(\rho_s) \in \mathcal{F}^{(1)}$, we define the closed half space

$$\mathscr{H}_\rho^{\boldsymbol{a}} := \left\{ v \in V \;\middle|\; \langle v, w(v_s)\rangle \leqslant \langle \boldsymbol{a}, v_s\rangle \right\}.$$

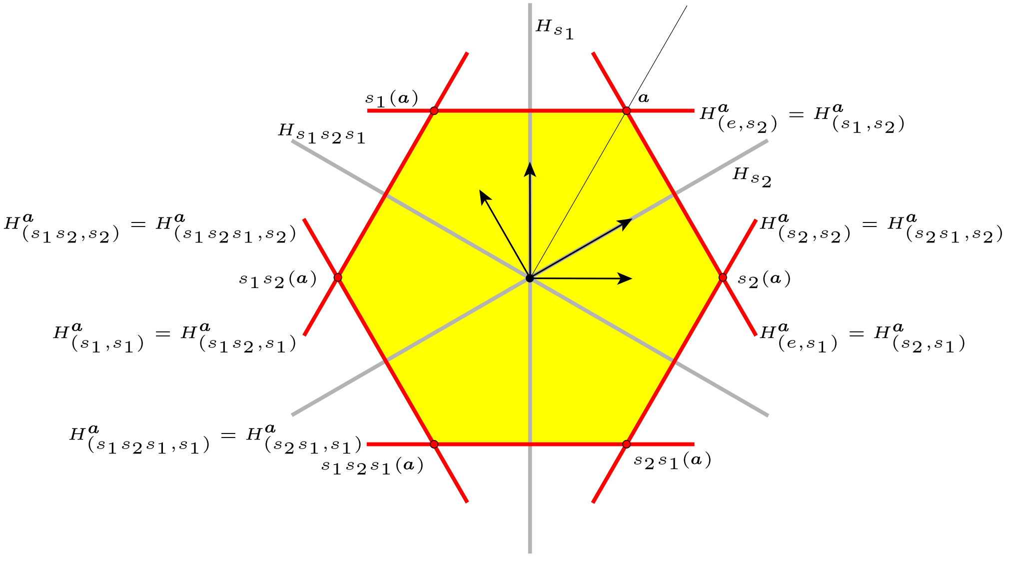

Fig. 6. The permutahedron $\mathsf{Perm}(S_3)$ obtained as the convex hull of the S_3-orbit of $\boldsymbol{a}$ or as the intersection of the half spaces $\mathscr{H}^{\boldsymbol{a}}_{(x,s)}$.

This definition does not depend on the choice of $w \in W$ such that $\rho = w(\rho_s)$, but only on the coset $W/W_{\langle s\rangle}$. The open half space $\mathscr{H}^{\boldsymbol{a},+}_{\rho}$ and the hyperplane $H^{\boldsymbol{a}}_{\rho}$ are defined in a similar manner, using strict inequality and equality, respectively. Now, the permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W)$ is given by

$$\mathsf{Perm}^{\boldsymbol{a}}(W) = \bigcap_{\rho \in \mathcal{F}^{(1)}} \mathscr{H}^{\boldsymbol{a}}_{\rho}.$$

We also write $\mathscr{H}^{\boldsymbol{a}}_{(w,s)}$, $\mathscr{H}^{\boldsymbol{a},+}_{(w,s)}$, or $H^{\boldsymbol{a}}_{(w,s)}$ instead of $\mathscr{H}^{\boldsymbol{a}}_{\rho}$, $\mathscr{H}^{\boldsymbol{a},+}_{\rho}$, or $H^{\boldsymbol{a}}_{\rho}$ where we implicitly assume $\rho = w(\rho_s)$. As for the rays of the Coxeter fan, w and s are not necessarily uniquely determined, but we have $H^{\boldsymbol{a}}_{\rho} = H^{\boldsymbol{a}}_{(w,s)} = H^{\boldsymbol{a}}_{(w',s)}$ if and only if $\rho = w(\rho_s) = w'(\rho_s)$ and $w \in w'W_{\langle s\rangle}$. Moreover, $w(\boldsymbol{a}) \in H^{\boldsymbol{a}}_{(w',s)}$ if and only if $H^{\boldsymbol{a}}_{(w',s)} = H^{\boldsymbol{a}}_{(w,s)}$. A simple description of the vertex $w(\boldsymbol{a})$ of the permutahedron follows:

$$w(\boldsymbol{a}) = \bigcap_{s \in S} H^{\boldsymbol{a}}_{(w,s)}.$$

Example 3.1 *(Realization of* $\mathsf{Perm}(S_3)$*).* We consider the Coxeter group $W = S_3$ of type A_2 acting on $\mathbb{R}^2$. The reflections s_1 and s_2 generate W and the simple roots that correspond to s_1 and s_2 are α_1 and α_2. They are normal to the reflecting hyperplanes H_{s_1} and H_{s_2}. The fundamental weight vectors that correspond to the simple roots are the vectors v_1 and v_2. Let $\boldsymbol{a} = a_1v_1 + a_2v_2$ be a point of the interior of D. Then $a_1, a_2 > 0$. We obtain the permutahedron $\mathsf{Perm}(S_3)$ as the convex hull of the W-orbit of $\boldsymbol{a}$. Alternatively, the permutahedron is described as the intersection of the half spaces $\mathscr{H}^{\boldsymbol{a}}_{(x,s)}$ with bounding hyperplanes $H^{\boldsymbol{a}}_{(x,s)}$ for $x \in W$ and $s \in S$. All objects are indicated in Fig. 6.

3.2. Cambrian fans

A *lattice congruence* Θ on a lattice L is an equivalence relation on the elements of L which respects the join and meet operations in L, that is, $a_1 \Theta a_2$ and $b_1 \Theta b_2$ implies $(a_1 \wedge b_1)\Theta(a_2 \wedge b_2)$ and $(a_1 \vee b_1)\Theta(a_2 \vee b_2)$. For any lattice congruence Θ of the weak order on W, N. Reading constructs a complete fan $\mathcal{F}_\Theta$ that coarsens the Coxeter fan $\mathcal{F}$ [17]. A maximal cone $C_\vartheta \in \mathcal{F}_\Theta$ corresponds to a congruence class ϑ of Θ and C_ϑ is the union of the chambers of $\mathcal{A}$ that correspond to the elements of ϑ. In [17, Section 5] N. Reading proves that these unions are indeed convex cones and that the collection $\mathcal{F}_\Theta$ of these cones and their faces is a complete fan.

The *c-Cambrian fan* $\mathcal{F}_c$ of W is obtained by this construction if we consider the lattice congruence with congruence classes $[\pi_\downarrow^c(w), \pi_c^\uparrow(w)]$ for $w \in W$ and chosen Coxeter element c. The n-dimensional cone that corresponds to the c-sortable element w is denoted by $C(w)$. It is the union of the maximal cones of $\mathcal{F}$ that correspond to the elements of $(\pi_\downarrow^c)^{-1}\pi_\downarrow^c(w) = [\pi_\downarrow^c(w), \pi_c^\uparrow(w)]$. In particular, $C(w)$ is a maximal cone of $\mathcal{F}_c$ and of $\mathcal{F}$ if and only if w is a c-singleton.

In [21], N. Reading and D. Speyer define a bijection between the set of rays of $\mathcal{F}_c$ and the set of *almost positive roots* $\Phi_{\geqslant -1} := \Phi^+ \cup (-\Delta)$. To describe this labeling of the rays, we first define a set of almost positive roots for any c-sortable w. For $s \in S(w)$, let $1 \leqslant j_s \leqslant \ell(w)$ be the unique integer such that s_{j_s} is the rightmost occurrence of s in the c-sorting word $s_1 \cdots s_{\ell(w)}$ of w and define

$$\mathsf{Lr}_s(w) := \begin{cases} s_1 \cdots s_{j_s - 1}(\alpha_s) & \text{if } s \in S(w) \\ -\alpha_s & \text{if } s \notin S(w), \end{cases} \quad \text{and} \quad \mathrm{cl}_c(w) := \bigcup_{s \in S} \mathsf{Lr}_s(w).$$

Example 3.2. To illustrate these maps, we consider the Coxeter group $W = S_3$ with generators $S = \{s_1, s_2\}$ as shown in Fig. 6. Choose $c = s_1 s_2$ as Coxeter element. It is easy to check that $w \in W \setminus \{s_2 s_1\}$ is c-sortable and that $w \in W \setminus \{s_2, s_2 s_1\}$ is a c-singleton. From the above definition it follows that

$$\begin{aligned} &\mathsf{Lr}_{s_1}(e) = \mathsf{Lr}_{s_1}(s_2) = -\alpha_1, && \mathsf{Lr}_{s_2}(e) = \mathsf{Lr}_{s_2}(s_1) = -\alpha_2, \\ &\mathsf{Lr}_{s_1}(s_1) = \mathsf{Lr}_{s_1}(s_1 s_2) = \alpha_1, && \mathsf{Lr}_{s_2}(s_1 s_2 s_1) = \mathsf{Lr}_{s_2}(s_1 s_2) = \alpha_1 + \alpha_2, \\ &\mathsf{Lr}_{s_1}(s_1 s_2 s_1) = \alpha_2, && \mathsf{Lr}_{s_2}(s_2) = \alpha_2, \end{aligned}$$

and therefore

$$\begin{aligned} &\mathrm{cl}_c(e) = \{-\alpha_1, -\alpha_2\}, && \mathrm{cl}_c(s_1 s_2) = \{\alpha_1, \alpha_1 + \alpha_2\}, \\ &\mathrm{cl}_c(s_1) = \{\alpha_1, -\alpha_2\}, && \mathrm{cl}_c(s_1 s_2 s_1) = \{\alpha_2, \alpha_1 + \alpha_2\}, \\ &\mathrm{cl}_c(s_2) = \{-\alpha_1, \alpha_2\}. \end{aligned}$$

N. Reading and D. Speyer use the cluster map cl_c to prove that c-Cambrian fans and cluster fans have the same combinatorics: the maximal cone $C(w)$ of the c-Cambrian fan represented by the c-sortable element w is mapped to the set $\mathrm{cl}_c(w)$ of almost positive roots. The cardinality of $\mathrm{cl}_c(w)$ matches the number of extremal rays of $C(w)$ and cl_c induces a bijection f_c between the set of rays of $\mathcal{F}_c$ and the almost positive roots by extending cl_c to intersections of cones: $\mathrm{cl}_c(C_1 \cap C_2) := \mathrm{cl}_c(C_1) \cap \mathrm{cl}_c(C_2)$. To put it slightly differently, N. Reading and D. Speyer showed the following theorem.

Theorem 3.3. *(See Reading and Speyer [21, Theorem 7.1].) There is a bijective labeling $f_c : \mathcal{F}_c^{(1)} \to \Phi_{\geqslant -1}$ of the rays of the c-Cambrian fan $\mathcal{F}_c$ by almost positive roots such that the extremal rays of $C(w)$ are labeled by* $\mathrm{cl}_c(w)$.

We now aim for an explicit description of f_c that relates nicely to c-singletons, but first we need the following two lemmas.

Lemma 3.4. *For $\beta \in \Phi_{\geqslant -1}$, there exists a c-singleton w and a simple reflection s such that $\mathsf{Lr}_s(w) = \beta$.*

Proof. The identity e is a c-singleton and $\mathrm{cl}_c(e) = \Phi_{\geqslant -1} \setminus \Phi^+$, so we are done if β is a negative simple root. Suppose that $\beta \in \Phi^+$ and consider the longest element w_0 with c-sorting word $\mathbf{w_0} = s_{j_1} s_{j_2} \cdots s_{j_N}$. Since $w_0(D)$ is above all reflecting hyperplanes,

$$\Phi^+ = \left\{ s_{j_1} s_{j_2} \cdots s_{j_{p-1}} (\alpha_{s_{j_p}}) \;\middle|\; 1 \leqslant p \leqslant N \right\}$$

and $\beta = s_{j_1} \cdots s_{j_{i-1}} (\alpha_{s_{j_i}})$ for some $1 \leqslant i \leqslant N$. Since $w = s_{j_1} \cdots s_{j_i}$ is a prefix of $\mathbf{w_0}$, it is a c-singleton and $\mathsf{Lr}_{s_{j_i}}(w) = \beta$. □

Lemma 3.5. *Let $\rho \in \mathcal{F}_c^{(1)}$. There is a c-singleton w such that ρ is an extremal ray of $C(w)$.*

Proof. Pick $\rho \in \mathcal{F}_c^{(1)}$. According to Theorem 3.3, $f_c(\rho) = \beta$ for some almost positive root β. By Lemma 3.4, there is a c-singleton w and a simple reflection $s \in S$ such that $\mathsf{Lr}_s(w) = \beta$. This implies that ρ is an extremal ray of $C(w)$. □

If w is a c-singleton, then $C(w) \in \mathcal{F}_c^{(n)}$ is the maximal cone $w(D)$ which is spanned by the set of rays $\{w(\rho_s) \mid s \in S\}$. The main result of this section is

Theorem 3.6. *Let $\rho \in \mathcal{F}_c^{(1)}$. There is a unique simple reflection $s \in S$ and there is a c-singleton w such that $\rho = w(\rho_s)$ and $f_c(w(\rho_s)) = \mathsf{Lr}_s(w)$.*

Proof. The uniqueness of $s \in S$ follows from the fact that any ray of the Coxeter fan is of the form $w(\rho_s)$ where $s \in S$ is uniquely determined (but w is not!).

The first claim follows directly from Lemma 3.5. We proceed by induction on the length of w. If $\ell(w) = 0$ then $w = e$. In particular, e is a c-singleton and $s = es$ is c-sortable for any $s \in S$. Fix some $s \in S$. Since $\mathrm{cl}_c(e) = -\Delta = \{-\alpha_t \mid t \in S\}$ and $\mathrm{cl}_c(s) = \{-\alpha_t \mid t \in \langle s \rangle\} \cup \{\alpha_s\}$, we conclude $f_c(e(\rho_s)) = -\alpha_s$ as $s(D) \subset C(s)$ and the set of rays of $\mathcal{F}$ in $s(D) \cap D$ are $\{\rho_t \mid t \in \langle s \rangle\}$.

Suppose that $\ell(w) > 0$ and let $t \in S$ be the last simple reflection of the c-sorting word of w. By Proposition 2.7, wt is a c-singleton with $\ell(wt) < \ell(w)$. By induction, $f_c(wt(\rho_s)) = \mathsf{Lr}_s(wt)$ for some $s \in S$. If $s \neq t$ then $t \in W_{\langle s \rangle}$ and we conclude that $wt(\rho_s) = w(\rho_s)$ and $\mathsf{Lr}_s(wt) = \mathsf{Lr}_s(w)$. Now suppose $s = t$. We have $C(w) \cap C(ws) = w(D) \cap ws(D)$. The extremal rays of this cone are $\{w(\rho_t) \mid t \in \langle s \rangle\}$, and their image under f_c is

$$\mathrm{cl}_c(w) \cap \mathrm{cl}_c(ws) = \left\{ \mathsf{Lr}_t(w) \;\middle|\; t \in \langle s \rangle \right\} = \mathrm{cl}_c(w) \setminus \left\{ \mathsf{Lr}_s(w) \right\}.$$

So $f_c(w(\rho_s)) = \mathsf{Lr}_s(w)$. □

4. Realizing generalized associahedra

4.1. A general result

A fan has to satisfy some obvious conditions in order to be the normal fan of a full-dimensional polytope. In particular, the fan has to be pointed and complete. These conditions are far from sufficient and in general it is quite hard to decide whether a given fan is the normal fan of a polytope or not. To illustrate this, we give an example of a family of pointed and complete fans none of which is the normal fan of any polytope.

First, we recall the notion of the face fan of a polytope. Let P be a full-dimensional polytope containing the origin in its relative interior. The *face fan* of P is the set of cones spanned by all proper faces of P. As is true of the normal fan, the face fan is always pointed and complete. There is a family of simplicial fans none of which is the face fan of any polytope, see [30, Example 7.5]. Equivalently, no fan of this family is the normal fan of any polytope, since the face fan of P equals the normal fan of the polar polytope of P (and vice versa), see [30, Exercise 7.1].

We now aim for a sufficient criterion to decide whether a given fan is the normal fan of a polytope. Our notation is inspired by Section 3.

Consider a pointed, complete, and simplicial fan $\mathcal{G} \subseteq \mathbb{R}^n$ with d-dimensional cones $\mathcal{G}^{(d)}$. To $\rho \in \mathcal{G}^{(1)}$ we associate a vector v_ρ such that $\rho = \mathbb{R}_{\geqslant 0}[v_\rho]$. Suppose that we are given a collection of positive real numbers λ_ρ, one for each $\rho \in \mathcal{G}^{(1)}$. We then define a hyperplane

$$H_\rho = \{x \in \mathbb{R}^n \mid \langle x, v_\rho \rangle = \lambda_\rho\}$$

and a half space

$$\mathscr{H}_\rho := \{x \in \mathbb{R}^n \mid \langle x, v_\rho \rangle \leqslant \lambda_\rho\}.$$

We write $\mathscr{H}_\rho^+$ if the inequality is strict. Since $\mathcal{G}$ is simplicial, we have for every maximal cone $C \in \mathcal{G}^{(n)}$ a point $x(C)$ defined by $\{x(C)\} := \bigcap_{\rho \in C^{(1)}} H_\rho$. Then

$$P := \operatorname{ConvexHull}\{x(C) \mid C \in \mathcal{G}^{(n)}\} \quad \text{and} \quad \widetilde{P} := \bigcap_{\rho \in \mathcal{G}^{(1)}} \mathscr{H}_\rho$$

are well-defined polytopes of dimension at most n.

For example, the W-permutahedron constructed from the Coxeter fan $\mathcal{F}$ as explained in Section 3.1 fits nicely in this context: $x(w(D))$ is by definition $w(\boldsymbol{a})$ and the half spaces $\mathscr{H}_\rho$ are precisely the half spaces $\mathscr{H}_\rho^{\boldsymbol{a}}$, for $\rho \in \mathcal{F}^{(1)}$. In this case the two polytopes P and $\widetilde{P}$ coincide.

Let $C \in \mathcal{G}^{(n)}$ and let $f \in C^{(n-1)}$ be an $(n-1)$-dimensional face of C. An *outer normal of C relative to f* is a vector v normal to f, that is, normal to the hyperplane spanned by f, and such that $C \subseteq \{x \in \mathbb{R}^n \mid \langle x, v \rangle \leqslant 0\}$.

Let $C_i, C_j \in \mathcal{G}^{(n)}$ be two adjacent maximal cones in $\mathcal{G}$, that is, $C_i \cap C_j \in \mathcal{G}^{(n-1)}$. A vector u is said to be *pointing to C_i from C_j* if there is an outer normal v of C_j relative to $C_i \cap C_j$ such that $\langle u, v \rangle > 0$. In particular, observe that:

(i) Any outer normal of C_j relative to $C_i \cap C_j$ is pointing to C_i from C_j;
(ii) If $x_i \in C_i$ and $x_j \in C_j$ are points in the interior of these cones, then the vector $x_i - x_j$ is pointing to C_i from C_j;

(iii) Any vector not contained in the span of $C_i \cap C_j$ is either pointing to C_i from C_j or pointing to C_j from C_i.

Notice that the vector $x(C_i) - x(C_j)$ is a normal vector to $C_i \cap C_j$, but not necessarily pointing to C_i from C_j, since $x(C_i)$ is not necessarily a point in C_i.

Theorem 4.1. *Use the notation as above and suppose that $x(C_i) - x(C_j)$ points to C_i from C_j whenever $C_i, C_j \in \mathcal{G}^{(n)}$ with $C_i \cap C_j \in \mathcal{G}^{(n-1)}$. Then $P = \widetilde{P}$ has (outer) normal fan $\mathcal{N}(P) = \mathcal{G}$ and is of dimension n.*

Remark 4.2. The hypothesis of Theorem 4.1 is satisfied in (at least) three cases.

First, the case of W-permutahedra constructed from the Coxeter fan $\mathcal{F}$. Indeed, the point $w(\boldsymbol{a})$ is strictly inside the cone $w(D)$. So $w(\boldsymbol{a}) - w'(\boldsymbol{a})$ points to $w(D)$ from $w'(D)$ whenever $w(D)$ and $w'(D)$ are adjacent cones.

Second, the case of the parallelepiped constructed from the fan $\mathcal{G}$ which is the skew coordinate hyperplane arrangement obtained from the hyperplanes which bound the fundamental chamber in the Coxeter arrangement. (Note that this fan corresponds to the usual construction of the Boolean lattice as a quotient of weak order, via the descent map, see [12].)

Third, as we shall show, the case of Cambrian fans described on the two next sections.

Remark 4.3. A similar theorem, which includes the assumption that $\mathcal{G}$ should be a coarsening of a Coxeter fan, appears as [11, Theorem A.3]. That theorem would suffice for our purposes, but we prefer to give the following independent proof of the more general theorem.

Proof of Theorem 4.1. Let us prove first that $P \subseteq \widetilde{P}$. It suffices to prove $\langle x(C), v_\rho \rangle < \lambda_\rho$ for $C \in \mathcal{G}^{(n)}$ and $\rho \in \mathcal{G}^{(1)} \setminus C^{(1)}$.

Let $C \in \mathcal{G}^{(n)}$ and $\rho \in \mathcal{G}^{(1)} \setminus C^{(1)}$. We will show that there is a finite sequence

$$C_0 := C, \ldots, C_k = C'$$

of maximal cones of $\mathcal{G}$ such that $\rho \subseteq C'$, $C_i \cap C_{i+1} \in \mathcal{G}^{(n-1)}$ and v_ρ is pointing to C_{i+1} from C_i, for $0 \leqslant i < k$.

For x in C, we write $x + \rho$ for the half line $\{x + \lambda v_\rho \mid \lambda \geqslant 0\}$ parallel to ρ and starting at x. Write C_ρ for the union of all maximal cones of $\mathcal{G}$ that contain ρ. Since $\mathcal{G}$ is a pointed complete fan, C_ρ contains n-dimensional balls of arbitrary diameter centered at points of ρ. In particular, C_ρ contains such a ball of diameter d, where d is the distance between the lines containing ρ and $x + \rho$. So $(x + \rho) \cap C_\rho \neq \varnothing$ for any point $x \in C$. Hence there is a maximal cone C' of $\mathcal{G}$ such that ρ is an extremal ray of C' and $(x + \rho) \cap C' \neq \varnothing$ for any point $x \in C$. For any $x \in C$, the line segment between C and C' on $x + \rho$ determines a sequence of cones $C_0 = C, C_1, \ldots, C_p = C'$ of $\mathcal{G}$ of arbitrary dimension, namely, the cones that $x + \rho$ meets between C and C' in the natural order on the C_i induced by the order of points of $x + \rho$ given by the parametrization of this half line. We would like this sequence to be of the form $C = C_0, C_{0,1}, C_1, C_{1,2}, \ldots, C_{k-1,k}, C_k = C'$ such that C_i is a maximal cone and $C_{i,i+1} = C_i \cap C_{i+1}$ is a cone of codimension 1. Since the number of cones in $\mathcal{G}$ is finite, the number of cones met by all possible half lines $x + \rho$ for $x \in C$ is finite. Since C is a full-dimensional cone, we may move x in C and then may assume that $x + \rho$ does not intersect any cone of $\mathcal{G}$ of codimension larger than 1. In other words, there is a finite sequence $C_0 = C, C_1, \ldots, C_k = C'$ of maximal cones of $\mathcal{G}$ such that $C_{i,i+1} = C_i \cap C_{i+1} \in \mathcal{G}^{(n-1)}$ and $(x + \rho) \cap C_i \neq \varnothing$. Pick y_i in the interior of C_i and in $x + \rho$. So $y_{i+1} - y_i$ points to C_{i+1}

from C_i. Since the cones $C_0, \dots, C_k$ have the same order as the points on $x + \rho$, the distance from x to y_i is strictly smaller than the distance from x to y_{i+1}. This means the vector $y_{i+1} - y_i = \kappa v_\rho$ with $\kappa > 0$. Hence v_ρ points to C_{i+1} from C_i.

Now, consider the piecewise linear path from $x(C_0)$ to $x(C_k)$ that traverses from $x(C_i)$ to $x(C_{i+1})$. Since $x(C_{i+1}) - x(C_i)$ points to C_{i+1} from C_i, the vector $x(C_{i+1}) - x(C_i)$ is an outer normal to C_i relative to $C_i \cap C_{i+1}$, and since v_ρ points to C_{i+1} from C_i for $0 \leqslant i < k$, we conclude that $\langle x(C_{i+1}) - x(C_i), v_\rho\rangle > 0$. Hence

$$\big\langle x(C_0), v_\rho\big\rangle < \cdots < \big\langle x(C_k), v_\rho\big\rangle = \lambda_\rho.$$

This proves $P \subseteq \widetilde{P}$.

Fix a cone $C \in \mathcal{G}^{(n)}$. Define $Q = \bigcap_{\rho\in C^{(1)}} \mathscr{H}_\rho$. This is a convex cone pointed at $x(C)$. Since the hyperplanes H_ρ for $\rho \in C^{(1)}$ are facet-supporting for $\widetilde{P}$, we know that $Q \supseteq \widetilde{P}$. It follows that $x(C)$ is an extremal point of both $\widetilde{P}$ and P, and thus it is a vertex of each.

We next show that, near $x(C)$, the three regions P, $\widetilde{P}$, and Q all agree. Let $D \in \mathcal{G}^{(n)}$ be adjacent to C. Since $D^{(1)}$ and $C^{(1)}$ have all but one element in common, $x(D)$ lies along one of the extremal rays of Q from $x(C)$. Bear in mind that $x(D) \in P$. Each of the n maximal cones of $\mathcal{G}$ adjacent to C yields a point in P along one of the rays of Q; thus, near $x(C)$, we have that P and Q coincide, and therefore so does P'.

Thus, we have established that the facets of P and of P' that intersect $x(C)$ are exactly those supported by H_ρ with $\rho \in C^{(1)}$. By definition, the outer normals to these facets are $C^{(1)}$, and the outer normal cone to for both P and P' at $x(C)$ is therefore exactly C. Thus, the outer normal fan to P is exactly $\mathcal{G}$.

We still want to check that $P = \widetilde{P}$. It suffices to check that they have the same set of vertices. We know that all the vertices of P are also vertices of $\widetilde{P}$, but we have not yet ruled out the possibility that $\widetilde{P}$ could have some extra vertices, that is to say, vertices not of the form $x(C)$. However, since the outer normal cones of $\widetilde{P}$ at the vertices of the form $x(C)$ are known to be exactly the maximal cones of $\mathcal{G}$, and thus to cover all of $\mathbb{R}^n$, it is impossible for $\widetilde{P}$ to have any additional vertices, so $P = \widetilde{P}$.

The claim that $\dim(P) = n$ follows from the fact that $\lambda_\rho > 0$ for all $\rho \in \mathcal{G}^{(1)}$: a neighborhood of 0 is contained in P. □

4.2. Realizations of generalized associahedra

We apply Theorem 4.1 to show how c-Cambrian fans $\mathcal{F}_c$ and associahedra $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ relate. The associahedron is described as the intersection of certain facet-supporting half spaces $\mathscr{H}_\rho^{\boldsymbol{a}}$ of the permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W)$ determined by the set of rays of $\mathcal{F}_c$, and the common vertices of $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ and $\mathsf{Perm}^{\boldsymbol{a}}(W)$ are characterized in terms of c-singletons. The proof of Theorem 4.4 is deferred to Section 4.3.

Theorem 4.4. *Let c be a Coxeter element of W and choose a point $\boldsymbol{a}$ in the interior of the fundamental chamber D, to fix a realization of the permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W)$.*

(i) *The polyhedron*

$$\mathsf{Asso}_c^{\boldsymbol{a}}(W) = \bigcap_{\rho\in\mathcal{F}_c^{(1)}} \mathscr{H}_\rho^{\boldsymbol{a}}$$

is a simple polytope of dimension n with c-Cambrian fan $\mathcal{F}_c$ as normal fan.

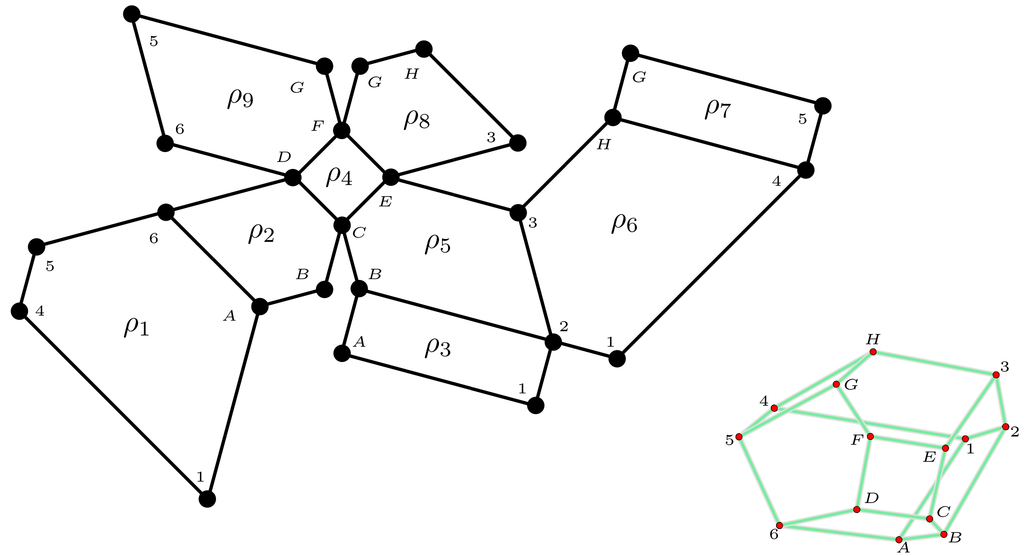

Fig. 7. An unfolding of the associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(S_4)$ with $c = s_1s_2s_3$. The 2-faces are labelled by $\rho_i \in \mathcal{F}_c^1$ for the facet-defining hyperplane $H_{\rho_i}^{\boldsymbol{a}}$.

(ii) *The vertex sets* $V(\mathsf{Asso}_c^{\boldsymbol{a}}(W))$ *and* $V(\mathsf{Perm}^{\boldsymbol{a}}(W))$ *satisfy*

$$V\big(\mathsf{Asso}_c^{\boldsymbol{a}}(W)\big) \cap V\big(\mathsf{Perm}^{\boldsymbol{a}}(W)\big) = \big\{w(\boldsymbol{a}) \bigm| w \text{ is a } c\text{-singleton}\big\}.$$

The first statement implies that every facet-supporting half space of the associahedron is also a facet-supporting half space of the permutahedron. We mentioned this in the introduction in the context of c-admissible half spaces. A facet-supporting half space $\mathscr{H}_\rho^{\boldsymbol{a}}$ of the permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W)$ is *c-admissible* if $w(\boldsymbol{a}) \in H_\rho^{\boldsymbol{a}}$ for some c-singleton w. We rephrase the first statement as follows:

Corollary 4.5. *The associahedron* $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ *is the intersection of all c-admissible half spaces of* $\mathsf{Perm}^{\boldsymbol{a}}(W)$.

We illustrate these results with a basic example.

Example 4.6. The first statement of Theorem 4.4 claims that the intersection of a subset of the half spaces $\mathscr{H}_\rho^{\boldsymbol{a}}$ of $\mathsf{Perm}^{\boldsymbol{a}}(W)$ yields a generalized associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ if we restrict to half spaces such that ρ is a ray of the c-Cambrian fan $\mathcal{F}_c$. Figs. 7 and 8 illustrate this for $W = S_4$ generated by $S = \{s_1, s_2, s_3\}$. We use the following conventions: The point $\boldsymbol{a}$ used to fix a realization of $\mathsf{Perm}^{\boldsymbol{a}}(W)$ is labeled A. A facet of the associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ is labeled by the ray $\rho_j \in \mathcal{F}_c$ that is perpendicular to that facet. Recall that each ray ρ can be written as $w(\rho_{s_i})$ for some (non-unique) c-singleton w and some (unique) simple reflection s_i by Lemma 3.5. In Fig. 7 we chose the Coxeter element $c = s_1s_2s_3$ and can express the ray $\rho_i \in \mathcal{F}_c$ that corresponds to the (c-admissible) half space $\mathscr{H}_{\rho_i}^{\boldsymbol{a}}$ as follows:

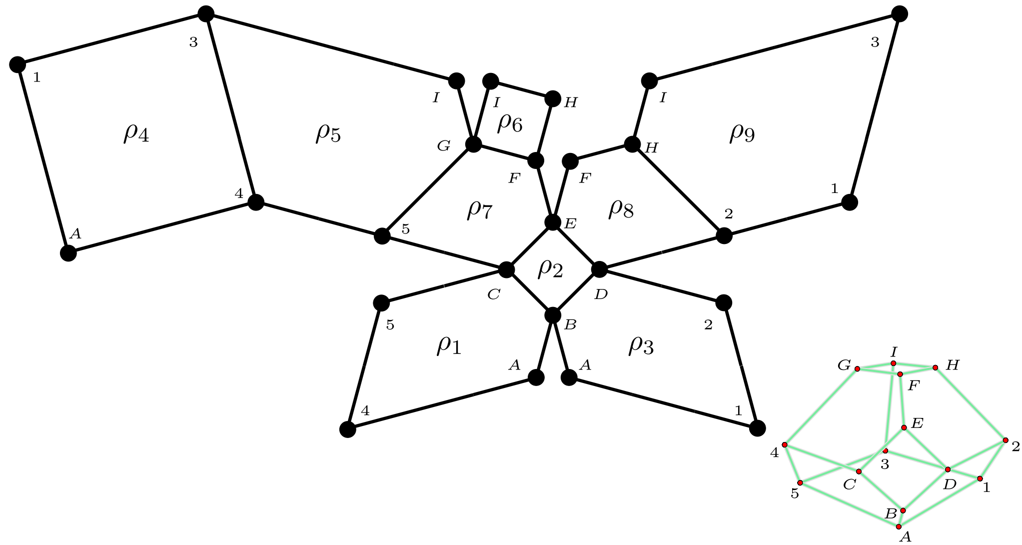

Fig. 8. An unfolding of the associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(S_4)$ with $c = s_2s_1s_3$. The 2-faces are labeled by $\rho = (w, s) \in \mathcal{F}_c^1$ for the facet-defining hyperplane $H_\rho^{\boldsymbol{a}}$.

$$\rho_1 = e(\rho_{s_1}),$$

$$\rho_2 = e(\rho_{s_3}) = s_1(\rho_{s_3}) = s_1s_2(\rho_{s_3}) = s_1s_2s_1(\rho_{s_3}),$$

$$\rho_3 = e(\rho_{s_2}) = s_1(\rho_{s_2}),$$

$$\rho_4 = s_1s_2(\rho_{s_2}) = s_1s_2s_3(\rho_{s_2}) = s_1s_2s_1(\rho_{s_2}) = s_1s_2s_3s_1(\rho_{s_2}),$$

$$\rho_5 = s_1(\rho_{s_1}) = s_1s_2(\rho_{s_1}) = s_1s_2s_3(\rho_{s_1}),$$

$$\rho_6 = s_1s_2s_3s_1s_2s_1(\rho_{s_1}),$$

$$\rho_7 = s_1s_2s_3s_1s_2(\rho_{s_2}) = s_1s_2s_3s_1s_2s_1(\rho_{s_2}),$$

$$\rho_8 = s_1s_2s_3(\rho_{s_3}) = s_1s_2s_3s_1(\rho_{s_3}) = s_1s_2s_3s_1s_2(\rho_{s_3}) = s_1s_2s_3s_1s_2s_1(\rho_{s_3}), \quad \text{and}$$

$$\rho_9 = s_1s_2s_1(\rho_{s_1}) = s_1s_2s_3s_1(\rho_{s_1}) = s_1s_2s_3s_1s_2(\rho_{s_1}).$$

The claim of the second statement of Theorem 4.4 is that the common vertices of $\mathsf{Perm}^{\boldsymbol{a}}(W)$ and $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ are the points $w(\boldsymbol{a})$ for w a c-singleton. It is straightforward to verify this claim directly if $c = s_1s_2s_3$ in Fig. 7: The common vertices of $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ and $\mathsf{Perm}^{\boldsymbol{a}}(W)$ are labeled A through H and we have

$$\begin{aligned} &A = \boldsymbol{a}, && B = s_1(\boldsymbol{a}), && C = s_1s_2(\boldsymbol{a}), \\ &D = s_1s_2s_1(\boldsymbol{a}), && E = s_1s_2s_3(\boldsymbol{a}), && F = s_1s_2s_3s_1(\boldsymbol{a}), \\ &G = s_1s_2s_3s_1s_2(\boldsymbol{a}), && H = s_1s_2s_3s_1s_2s_1(\boldsymbol{a}). \end{aligned}$$

If the Coxeter element is $c = s_2s_1s_3$ (Fig. 8) then we have the following list of expressions for $\rho_i \in \mathcal{F}_c$ (we do not list all possible expressions for ρ_i):

$$\rho_1 = e(\rho_{s_3}), \qquad \rho_2 = s_2(\rho_{s_2}), \qquad \rho_3 = e(\rho_{s_1}),$$
$$\rho_4 = e(\rho_{s_2}), \qquad \rho_5 = s_2s_1s_3s_2s_3(\rho_{s_3}), \qquad \rho_6 = s_2s_1s_3s_2s_1s_3(\rho_{s_2}),$$
$$\rho_7 = s_2s_1(\rho_{s_1}), \qquad \rho_8 = s_2s_3(\rho_{s_3}), \qquad \rho_9 = s_2s_1s_3s_2s_1(\rho_{s_1}).$$

The common vertices of the permutahedron and associahedron are labeled A through I and we have

$$A = \boldsymbol{a}, \qquad B = s_2(\boldsymbol{a}), \qquad C = s_2s_1(\boldsymbol{a}),$$
$$D = s_2s_3(\boldsymbol{a}), \qquad E = s_2s_1s_3(\boldsymbol{a}), \qquad F = s_2s_1s_3s_2(\boldsymbol{a}),$$
$$G = s_2s_1s_3s_2s_3(\boldsymbol{a}), \qquad H = s_2s_1s_3s_2s_1(\boldsymbol{a}), \qquad I = s_2s_1s_3s_2s_1s_3(\boldsymbol{a}).$$

4.3. Proof of Theorem 4.4

The proof of Theorem 4.4 is based on Theorem 4.1. Let $c \in W$ be a Coxeter element and let $\boldsymbol{a}$ be in the interior of D. We use the following notation. Set

$$\mathcal{R} := \{ w(v_s) \mid w \in W,\ s \in S \}$$

so that the set of rays of the Coxeter fan is

$$R_{\geqslant 0}\mathcal{R} := \{ \lambda v \mid \lambda \geqslant 0,\ v \in \mathcal{R} \}.$$

For $v = w(v_s) \in \mathcal{R}$, we write $\lambda(v) := \langle \boldsymbol{a}, v_s \rangle = \langle w(\boldsymbol{a}), v \rangle > 0$ which depends only on s. Let ρ be a ray of the Cambrian fan $\mathcal{F}_c$. By Theorem 3.6, there is a c-singleton w and a unique $s \in S$ such that $\rho = w(\rho_s) = \mathbb{R}_{\geqslant 0}[w(v_s)]$. With these notations, the equations of the hyperplane $H_\rho^{\boldsymbol{a}}$ and the half space $\mathscr{H}_\rho^{\boldsymbol{a}}$ attached to ρ can be rewritten as

$$H_\rho^{\boldsymbol{a}} = \{ x \in \mathbb{R}^n \mid \langle x, w(v_s) \rangle = \lambda(v_s) \} \quad \text{and} \quad \mathscr{H}_\rho^{\boldsymbol{a}} = \{ x \in \mathbb{R}^n \mid \langle x, w(v_s) \rangle \leqslant \lambda(v_s) \}.$$

We use the same notations as in Section 4.1 applied to the c-Cambrian fan $\mathcal{F}_c$ and denote by $x(C)$ the intersection point of the hyperplanes $H_\rho^{\boldsymbol{a}}$ for ρ the extremal rays of a maximal cone C of $\mathcal{F}_c$. (It is convenient here, if $\rho = \mathbb{R}_{\geqslant 0}[v]$, to use the notation $\lambda(v)$ instead of using the notation λ_ρ as in Section 4.1.)

Let w and w' be distinct c-sortable elements such that the associated maximal cones $C := C(w)$ and $C' := C(w')$ of the Cambrian fan $\mathcal{F}_c$ intersect in a cone of codimension 1. So either w is a cover of w' or w' is a cover of w in the lattice of c-sortable elements. Without loss of generality, we may assume that w is a cover of w'. To meet the requirements of Theorem 4.1, we have to prove that the vector $x(C) - x(C')$ points to C from C'.

Remark 4.7. We saw in Section 3 that $C(w)$ is a maximal cone of the Cambrian fan $\mathcal{F}_c$ and of the Coxeter fan $\mathcal{F}$ if and only if w is a c-singleton. Therefore, if we meet the requirement of Theorem 4.1, then $x(C(w))$ is a vertex of $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ and of $\mathsf{Perm}^{\boldsymbol{a}}(W)$ if and only if w is a c-singleton, which will prove the second part of Theorem 4.4.

The intersection $C \cap C'$ is contained in a hyperplane H_t for some reflection $t \in W$ since $\mathcal{F}_c$ is a coarsening of the Coxeter fan $\mathcal{F}$. We now show which of the two roots associated to H_t is an outer normal to C' relative to $C' \cap C$.

Lemma 4.8. *Let $w, w' \in W$ be c-sortable elements such that w is a cover of w' in the lattice of c-sortable elements. Suppose the $(n-1)$-dimensional cone $C(w) \cap C(w')$ of $\mathcal{F}_c$ lies on H_t for some reflection $t \in W$. Let β be a root for W that is perpendicular to H_t. If β is an outer normal to C' relative to $C' \cap C$, then β is a negative root.*

Proof. Let $\widetilde{w} \in (\pi_{\downarrow}^{c})^{-1}(w')$ such that w is a cover of $\widetilde{w}$ in the right weak order. Then

$$w(D) \cap \widetilde{w}(D) \subset H_t$$

is an $(n-1)$-dimensional cone of the Coxeter fan $\mathcal{F}$. Since β is an outer normal for $C(w')$ relative to $C(w) \cap C(w')$, it is an outer normal for $\widetilde{w}(D)$ with respect to $\widetilde{w}(D) \cap w(D)$.

Case 1: Suppose that $w' = \widetilde{w} = e$. Then $w = s$ for some $s \in S$. The hyperplane dividing $w(D)$ from D is H_s, perpendicular to α_s. D lies on the side of H_s having positive inner product with α_s. Thus the outer normal $\beta = -\alpha_s$ is a negative root.

Case 2: Suppose $\widetilde{w} \neq e$. Then $\widetilde{w}^{-1} w = s \in S$ since w covers $\widetilde{w}$ in right weak order. By the previous case, $\widetilde{w}(-\alpha_s)$ is an outer normal. Since $\ell(w) = \ell(\widetilde{w}s) > \ell(\widetilde{w})$, we have that $\beta = \widetilde{w}(-\alpha_s)$ is a negative root, as desired. $\square$

We have that $C \cap \mathbb{R}_{\geqslant 0}\mathcal{R} = \{\rho_{u_1}, \ldots, \rho_{u_p}\}$ and since $\mathcal{F}_c$ is simplicial, we may assume that the extremal rays of C are the first $n := |S|$ rays. Similarly, we may assume $\{\rho_{u'_1}, \rho_{u_2}, \ldots, \rho_{u_n}\}$ are the extremal rays of C'. Hence, H_t is spanned by $\{u_2, \ldots, u_n\}$. As we have $x(C) := \bigcap_{i=1}^{n} H^{\boldsymbol{a}}_{\rho_{u_i}}$ and $x(C) - x(C') \in \bigcap_{i=2}^{n} H^{\boldsymbol{a}}_{\rho_{u_i}}$, we are able to conclude $x(C) - x(C') = \mu\beta$ for some $\mu \in \mathbb{R}$ and β a negative root. Thus, $x(C) - x(C')$ is pointing to C from C' if and only if $x(C) - x(C') = \mu\beta$ with $\mu > 0$.

Lemma 4.9. *Let $w, w' \in W$ be c-sortable elements such that w covers w' in the lattice of c-sortable elements and $C(w) \cap C(w') \subset H_t$ is an $(n-1)$-dimensional cone of $\mathcal{F}_c$ for a reflection t. Let the extremal rays of $C := C(w)$ and $C' := C(w')$ be generated by $\{u_1, \ldots, u_n\}$ and $\{u'_1, u_2, \ldots, u_n\}$ and suppose $u_1 + u'_1 = \sum_{i=2}^{n} b_i u_i \in H_t$ with $b_i \in \mathbb{R}$. Then the following statements are equivalent*:

(i) $x(C) - x(C')$ *is pointing to C from C'*;
(ii) $x(C) - x(C') = \mu\beta$ *with $\beta \in \Phi^-$ and $\mu > 0$*;
(iii) $\langle x(C) - x(C'), u_1 \rangle > 0$;
(iv) $\lambda(u_1) + \lambda(u'_1) > \sum_{i=2}^{n} b_i \lambda(u_i)$.

Proof. The first equivalence follows from Lemma 4.8 and the preceding discussion.

As $u_1 \in C$ and C is spanned by the vectors in $C \cap C'$ together with u_1, we have $\langle \beta, u_1 \rangle > 0$ if and only if β is an outer normal of C' relative to $C \cap C'$. This shows the second equivalence.

The last equivalence follows from

$$\bigl\langle x(C) - x\bigl(C'\bigr), u_1 \bigr\rangle = \lambda(u_1) - \Bigl\langle x\bigl(C'\bigr), -u'_1 + \sum_{i=2}^{n} b_i u_i \Bigr\rangle = \lambda(u_1) + \lambda\bigl(u'_1\bigr) - \sum_{i=2}^{n} b_i \lambda(u_i). \qquad \square$$

We apply Theorem 4.1 to conclude:

If one of the equivalent conditions in Lemma 4.9 is achieved for all pairs of adjacent cones in $\mathcal{F}_c$, then $\mathcal{F}_c$ is the outer normal fan of $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$, which proves Theorem 4.4.

The final step of the proof of Theorem 4.4 consists of showing that each pair of adjacent cones in $\mathcal{F}_c$ satisfies the equivalent conditions of Lemma 4.9, which we do in Lemma 4.14 below.

We first consider the special case that e is covered by $s \in S$ and there is a reduced expression for c that starts with s, that is, s is *initial* in c.

Lemma 4.10. *Let $s \in S$ be initial in c. Then $C(e) \cap C(s) \subseteq H_s$, $u_1' = v_s$, $u_1 = s(v_s)$, and*

$$u_1' + u_1 = \sum_{r \neq s} b_r u_r \in H_s$$

with $b_r = -2\frac{\langle \alpha_s, \alpha_r \rangle}{\langle \alpha_s, \alpha_s \rangle} \geqslant 0$. Moreover, $\lambda(u_1') + \lambda(u_1) > \sum_{r \neq s} b_r \lambda(u_r)$.

Proof. Without loss of generality, assume $S = \{s_1, \dots, s_n\}$ and $s = s_1$. Since $s \in S$ is initial, s is a c-singleton. The maximal cones $C(e)$ and $C(s)$ of $\mathcal{F}_c$ are therefore maximal cones of the Coxeter fan with extremal rays generated by $\{v_{s_1}, \dots, v_{s_n}\}$ and $\{s(v_{s_1}), v_{s_2}, \dots, v_{s_n}\}$. Since $\alpha_r = \sum_{i=1}^n \langle \alpha_r, \alpha_i \rangle v_i$, we have

$$s_1(v_{s_1}) = v_{s_1} - 2\frac{\langle \alpha_{s_1}, v_{s_1} \rangle}{\langle \alpha_{s_1}, \alpha_{s_1} \rangle}\alpha_{s_1} = -v_{s_1} + \sum_{i=2}^{n} \left(-2\frac{\langle \alpha_{s_1}, \alpha_{s_i} \rangle}{\langle \alpha_{s_1}, \alpha_{s_1} \rangle}\right) v_{s_i}.$$

In particular, $s(v_{s_1}) + v_{s_1} \in H_{s_1}$ and $\langle \alpha_{s_1}, \alpha_{s_i} \rangle \leqslant 0$ for $s_1 \neq s_i$. As $\boldsymbol{a}$ is a vertex of $\mathsf{Perm}^{\boldsymbol{a}}(W)$, we conclude

$$\lambda\big(s_1(u_1)\big) = \langle \boldsymbol{a}, v_{s_1} \rangle > \big\langle \boldsymbol{a}, s_1(v_{s_1}) \big\rangle = -\lambda\big(s_1(v_{s_1})\big) + \sum_{i=2}^{n} b_i \lambda(s_i). \qquad \square$$

Some terminology and results due to N. Reading and D. Speyer are needed to prove Lemma 4.14 (and therefore to finish the proof of Theorem 4.4). To distinguish objects related to a Cambrian fan with respect to different Coxeter elements, we use the Coxeter element as an index. For example, if we use the Coxeter element scs instead of c, then $C_{scs}(w)$ denotes the maximal cone that corresponds to the scs-sortable element w. If $s \in S$ is initial in c then $\mathcal{F}_{sc}$ is the sc-Cambrian fan for the Coxeter element sc of $W_{\langle s \rangle}$.

Lemma 4.11. *(See [21, Lemmas 4.1, 4.2].) Let c be a Coxeter element and s initial in c.*

(i) *Let $w \in W$ such that $\ell(sw) < \ell(w)$. Then w is c-sortable if and only if sw is scs-sortable.*
(ii) *Let $w \in W$ such that $\ell(sw) > \ell(w)$. Then w is c-sortable if and only if $w \in W_{\langle s \rangle}$ and w is sc-sortable.*

Note that $\ell(sw) < \ell(w)$ if and only if the chamber $w(D)$ of the Coxeter arrangement corresponding to w lies above H_s. In this case, the maximal cone $C(w)$ of $\mathcal{F}_c$ is above the hyperplane H_s because w is minimal in its fiber $(\pi_\downarrow^c)^{-1}\pi_\downarrow^c(w) = [\pi_\downarrow^c(w), \pi_c^\uparrow(w)]$ for the c-Cambrian congruence. On the other hand, if $\ell(sw) > \ell(w)$, then $w(D)$ is below H_s in the Coxeter arrangement. In this case, we know that the maximum element of the fiber $[\pi_\downarrow^c(w), \pi_c^\uparrow(w)]$ for w, and thus all of $C(w)$, is below H_s, by [21, Lemma 4.11]. It follows that the hyperplane H_s separates the cones of $\mathcal{F}_c$ into two families and it never intersects a maximal cone of $\mathcal{F}_c$ in its interior. For $\rho \in \mathcal{F}_c^{(1)}$ we define

$$\zeta_s(\rho) := \begin{cases} s(\rho) & \text{if } \rho \neq \rho_s, \\ -\rho_s & \text{otherwise.} \end{cases}$$

We abuse notation and consider ζ_s also as a map on the set of vectors generating the set of rays $\mathcal{F}_c^{(1)}$. The following lemma is a consequence of [21, Lemma 6.5] and [21, Theorem 1.1]. Compare also the comments after [21, Corollary 7.3], from which the last statement is taken.

Lemma 4.12. *(See [21].) Let $s \in S$ be initial in the Coxeter element c. If $\rho_1, \dots, \rho_n$ are the extremal rays of the maximal cone $C(w) \in \mathcal{F}_c$ then $\zeta_s(\rho_1), \dots, \zeta_s(\rho_n)$ are the extremal rays of a maximal cone of $\mathcal{F}_{scs}$. If $\ell(sw) < \ell(w)$, then these extremal rays are the extremal rays of the maximal cone $C(sw)$ that corresponds to the scs-sortable element sw.*

Before we finish the proof of Theorem 4.4 with Lemma 4.14 we make an observation that will be useful also in Section 4.4.

Lemma 4.13. *Let c be a Coxeter element, $s \in S$ initial in c, and $w \in W_{\langle s\rangle}$ sc-sortable. Then the maximal cone $C(w) \in \mathcal{F}_c$ is spanned by $C_{sc}(w) \in \mathcal{F}_{sc}$ and the ray $\rho_s \in \mathcal{F}_c$.*

Proof. The ray ρ_s is the unique ray of $\mathcal{F}_c$ that is strictly below H_s by [21, Lemma 6.3]. From Lemma 4.11 it follows that $C(w)$ is below H_s and has ρ_s as an extremal ray. Hence $C(w)$ is spanned by ρ_s and a maximal cone $E(w) := C(w) \cap H_s \in \mathcal{A}_{\langle s\rangle}$.

Now consider all inversions t of w, that is, all hyperplanes $H_t \in \mathcal{A}$ such that $C(w)$ is above H_t. Since $w \in W_{\langle s\rangle}$ we conclude $t \in W_{\langle s\rangle}$. Hence, the inversions of $E(w)$ and $C_{sc}(w)$ coincide and $E(w) = C_{sc}(w)$. □

Lemma 4.14. *Let $w, w' \in W$ be c-sortable elements such that w covers w' in the lattice of c-sortable elements and $C(w) \cap C(w') \subset H_t$ is an $(n-1)$-dimensional cone of $\mathcal{F}_c$ for a reflection t. Let the extremal rays of $C := C(w)$ and $C' := C(w')$ be generated by $\{u_1, \dots, u_n\}$ and $\{u'_1, u_2, \dots, u_n\}$.*

Then $u'_1 + u_1 = \sum_{i=2}^n b_i u_i$ with $b_i \in \mathbb{R}$ and $\lambda(u'_1) + \lambda(u_1) > \sum_{i=2}^n b_i \lambda(u_i)$.

Proof. The proof is an induction on the rank $n = |S|$ and the length $\ell(w)$.

If $|S| = 1$ then the result is clear, so assume that $S = \{s_1, \dots, s_n\}$ with $n > 1$ and $\ell(w) = 1$. Assume without loss of generality that $w = s_1$, and since w covers w', $w' = e$. If w is initial for c then we are done by Lemma 4.10. So assume that w is not initial for c. For $2 \leqslant i \leqslant n$ we have $u_i = v_{s_i}$. Moreover, we have $u'_1 = v_{s_1}$ and $u_1 = u$ for some $u \in \mathcal{R}$. Thus the maximal cones $C(e)$ and $C(w)$ are generated by $\{v_{s_1}, v_{s_2}, \dots, v_{s_n}\}$ and $\{u, v_{s_2}, \dots, v_{s_n}\}$. For the sake of definiteness, suppose $s = s_2$ is initial in c. Then $C(e)$ and $C(w)$ are both below H_s. By Lemma 4.13, we have maximal cones $C_{sc}(e) = C(e) \cap H_s$ and $C_{sc}(w) = C(w) \cap H_s$ in the sc-Cambrian fan $\mathcal{F}_{sc}$

of $W_{\langle s\rangle}$ and these cones are generated by $\{v_{s_1}, v_{s_3}, \ldots, v_{s_n}\}$ and $\{u, v_{s_3}, \ldots, v_{s_n}\}$. So by induction on the rank of $|S|$, we obtain the claim with $b_2 = 0$.

For the induction, we assume that the claim is true whenever $\widetilde{w}$ is $\widetilde{c}$-sortable for a Coxeter group generated by $\widetilde{S}$ with $|\widetilde{S}| < |S|$ or $\widetilde{w}$ is a c-sortable element satisfying $\ell(\widetilde{w}) < \ell(w)$.

Assume $w, w' \in W$ are c-sortable with $\ell(w) > 1$ and w covers w' in the lattice of c-sortable elements. Let $s \in S$ be initial in c. We split into cases based on the positions of $C(w)$ and $C(w')$ relative to H_s. Note that it is impossible for $C(w)$ to lie below H_s and $C(w')$ to lie above H_s simultaneously, since w covers w' in the c-Cambrian lattice.

Case 1: Suppose $C(w)$ and $C(w')$ are above H_s. The ray ρ_s is strictly below H_s by [21, Lemma 6.3], so $v_s \notin \{u'_1, u_1, \ldots, u_n\}$. Moreover, we conclude from Lemma 4.12 that the maximal cones $C_{scs}(sw)$ and $C_{scs}(sw')$ in $\mathcal{F}_{scs}$ are generated by $\{s(u_1), \ldots, s(u_n)\}$ and $\{s(u'_1), s(u_2), \ldots, s(u_n)\}$ since $\ell(sw) < \ell(w)$, $\ell(sw') < \ell(w')$ and $w, w' > s$ in the right weak order. We have $C_{scs}(sw) \cap C_{scs}(sw') \subset H_{sts}$ because $C(w) \cap C(w') \subset H_t$. By induction on the length, we have

$$s(u_1) + s\big(u'_1\big) = \sum_{i=2}^{n} b_i s(u_i) \quad \text{and} \quad \lambda\big(s(u_1)\big) + \lambda\big(s\big(u'_1\big)\big) > \sum_{i=2}^{n} b_i \lambda\big(s(u_i)\big)$$

with $b_i \in \mathbb{R}$.

Applying s to these (in)equalities yields

$$u_1 + u'_1 = \sum_{i=2}^{n} b_i u_i \in H_t \quad \text{and} \quad \lambda(u_1) + \lambda\big(u'_1\big) > \sum_{i=2}^{n} b_i \lambda(u_i) \quad \text{with } b_i \in \mathbb{R},$$

since $\lambda(u)$ depends only on the orbit of u under the action of W.

Case 2: $C(w)$ and $C(w')$ are below H_s. Since w is c-sortable and $\ell(sw) > \ell(w)$, we have that $w \in W_{\langle s\rangle}$, and similarly for w'. The ray ρ_s is the only ray of $\mathcal{F}_c$ strictly below H_s by [21, Lemma 6.3], hence we may assume that $u_2 = v_s$. Now $\{u_1, u_3, \ldots, u_n\}$ and $\{u'_1, u_3, \ldots, u_n\}$ generate the extremal rays of maximal cones $C_{sc}(\widetilde{w}), C_{sc}(\widetilde{w}') \subset H_s$ of the sc-Cambrian fan $\mathcal{F}_{sc}$ with $\widetilde{w}, \widetilde{w}' \in W_{\langle s\rangle}$. The claim follows by induction on the rank $|S|$.

Case 3: $C(w)$ is above H_s and $C(w')$ is below H_s. Hence $C(w)$ and $C(w')$ are separated by H_s, so we have $s = t$. Hence $u'_1 = v_s$ (ρ_s is the only ray of $\mathcal{F}_c$ below H_s) and there is a maximal cone $C_{scs}(g)$ for some scs-sortable element $g \in W$ which is generated by the extremal rays $\zeta_s(u'_1), \zeta_s(u_2), \ldots, \zeta_s(u_n)$. Now, observe that

$$\begin{aligned}
&\zeta_s(u_1) = s(u_1),\\
&\zeta_s\big(u'_1\big) = -u'_1 = -v_s, \quad \text{and}\\
&\zeta_s(u_i) = u_i \quad \text{for } 2 \leqslant i \leqslant n.
\end{aligned}$$

Thus the extremal rays of the maximal cones $C_{scs}(g)$ and $C_{scs}(sw)$ are generated by

$$-v_s, u_2, \ldots, u_n \quad \text{and} \quad s(u_1), u_2, \ldots, u_n.$$

Moreover, $C_{scs}(g) \cap C_{scs}(sw) \subseteq H_s$.

We first show that $g = w$. The definition of Cambrian fans implies $sw(D) \subset C_{scs}(sw)$. From $C_{scs}(sw) \cap C_{scs}(g) \subseteq H_s$ we deduce that $w(D) \subset C_{scs}(g)$. An equivalent statement is $w \in (\pi_\downarrow^{scs})^{-1}(g)$. Now $g > sw$ implies $h > sw$ for all $h \in (\pi_\downarrow^{scs})^{-1}(g)$. But $C(w)$ is above H_s, so $h < w$ implies $h \notin (\pi_\downarrow^{scs})^{-1}(g)$. Hence w is the minimal element of $[\pi_\downarrow^{scs}(g), \pi_{scs}^\uparrow(g)]$ and we have $w = g$.

Though $C_{scs}(w) \cap C_{scs}(sw) \subseteq H_s$, it is not possible to apply the induction hypothesis immediately, since the length $\ell(w)$ has not been reduced, but we claim that for any $z \in S$ initial in scs either $C_{scs}(w)$ and $C_{scs}(sw)$ are both above H_z or both below H_z. Indeed, from $z \in S \setminus \{s\}$ we conclude that $v_z \in H_s$. We know that $u_2, \ldots, u_n \in H_s$ and $u_1', u_1, s(u_1) \notin H_s$. So $v_z \in C_{scs}(w)$ if and only if $v_z \in C_{scs}(sw)$. Since v_z is the only ray of $\mathcal{F}_{scs}$ below the hyperplane H_z, we have shown that $C_{scs}(w)$ and $C_{scs}(sw)$ are on the same side of H_z.

This implies that we are now in Case 1 or Case 2 where w covers sw, both are scs-sortable and z is initial in scs. Since $\{\zeta_s(u_1), \zeta_s(u_2) = u_2, \ldots, \zeta_s(u_n) = u_n\}$ generates $C_{scs}(w)$ and $\{\zeta_s(u_1'), \zeta_s(u_2) = u_2, \ldots, \zeta_s(u_n) = u_n\}$ generates $C_{scs}(w)$, the argument of the relevant case yields

$$\zeta_s(u_1) + \zeta_s\big(u_1'\big) = \sum_{i=2}^{n} b_i' u_i \in H_s \quad \text{with } b_i' \in \mathbb{R}.$$

We can re-express this quantity

$$\zeta_s(u_1) + \zeta_s\big(u_1'\big) = s(u_1) - v_s = u_1 - s(v_s) = u_1 + u_1' - \big(v_s + s(v_s)\big)$$

where the second equality follows because H_s is fixed under the action of s, and the third equality follows from the fact that $u_1' = v_s$. Since $v_s + s(v_s) \in H_s$, we conclude that $u_1 + u_1' \in H_s$, and can therefore be written as $\sum_{i=2}^{n} b_i u_i$ with $b_i \in \mathbb{R}$.

It remains to prove $\lambda(u_1) + \lambda(u_1') > \sum_{i=2}^{n} b_i \lambda(u_i)$. By Lemma 4.9 it is sufficient to show that $\langle x(C(w')) - x(C(w)), u_1' \rangle > 0$.

Recall that $t = s \in S$. Pick a maximal chain in the c-Cambrian lattice

$$y_0 \lessdot y_1 \lessdot \cdots \lessdot y_p$$

with $y_0 = s$ and $y_p = w$. Then $s \leqslant y_i$ for $0 \leqslant i \leqslant p$, so $C(y_i)$ is above H_s for $0 \leqslant i \leqslant p$. So for the pair $\widetilde{w}' = y_{i-1}$ and $\widetilde{w} = y_i$ we have $z_i := x(C(y_i)) - x(C(y_{i-1})) = \mu_i \beta_i$ with $\mu_i > 0$ and $\beta_i \in \Phi^-$ by Lemma 4.9 and the proof in Case 1 above. Now $\langle \beta_i, v_s \rangle$ is the coefficient of the simple root α_s in the simple root expansion of β_i. Since β_i is a negative root, $\langle \beta_i, v_s \rangle \leqslant 0$. In particular we have

$$\big\langle x\big(C(y_{i-1})\big), v_s\big\rangle \geqslant \big\langle x\big(C(y_{i-1})\big), v_s\big\rangle + \langle z_i, v_s\rangle = \big\langle x\big(C(y_i)\big), v_s\big\rangle$$

for $1 \leqslant i \leqslant p$. Hence

$$\big\langle x\big(C(e)\big), v_s\big\rangle > \big\langle x\big(C(s)\big), v_s\big\rangle \geqslant \big\langle x\big(C(y_2)\big), v_s\big\rangle \geqslant \cdots \geqslant \big\langle x\big(C(w)\big), v_s\big\rangle,$$

where the first inequality is Lemma 4.10. As $u_1' = v_s$ we have

$$\big\langle x\big(C(e)\big), v_s\big\rangle = \lambda(v_s) = \lambda\big(u_1'\big) = \big\langle x\big(C\big(w'\big)\big), v_s\big\rangle.$$

Thus $\langle x(C(w')) - x(C(w)), u_1' \rangle > 0$. $\square$

4.4. On integer coordinates

Suppose that W is a Weyl group and that the root system Φ for W is crystallographic, that is, for any two roots $\alpha, \beta \in \Phi$ we have $s_\alpha(\beta) = \beta + \lambda\alpha$ for some $\lambda \in \mathbb{Z}$. The simple roots Δ span the lattice L and the fundamental weights v_s, $s \in S$, span a lattice L^* which is dual to L. For $\beta \in L$ and $v \in L^*$ we have $\langle \beta, v\rangle \in \mathbb{Z}$. In fact, $\beta \in L$ if and only if $\langle \beta, v\rangle \in \mathbb{Z}$ for all $v \in L^*$. For each ray $\rho \in \mathcal{F}_c$, we denote by $v_\rho \in L^*$ the lattice point on ρ closest to the origin.

Lemma 4.15. *Let Φ be a crystallographic root system for the Weyl group W and c be a Coxeter element of W. The set $\{v_\rho \mid \rho$ an extremal ray of $C\}$ forms a basis of L^* for each maximal cone $C \in \mathcal{F}_c$.*

Proof. Let $C = C(w)$ denote the maximal cone of $\mathcal{F}_c$ for some c-sortable $w \in W$. The proof is by induction on $\ell(w)$ and the rank of W. Let s be initial in c. To apply Lemma 4.11, we distinguish the two cases $\ell(sw) < \ell(w)$ and $\ell(sw) > \ell(w)$.

Suppose that $\ell(sw) < \ell(w)$. Then sw is scs-sortable and $C(w) = s(C_{scs}(sw))$. Since the simple reflection s preserves the lattice, the result follows by induction.

Suppose on the other hand that $\ell(sw) > \ell(w)$. Then the cone $C(w)$ lies below the hyperplane H_s and $w \in W_{\langle s\rangle}$ is sc-sortable. Let $C_{\langle s\rangle}(w)$ denote the maximal cone that corresponds to w in the Cambrian fan $\mathcal{F}_{\langle s\rangle} \subset H_s$ for $W_{\langle s\rangle}$. Then $C_{\langle s\rangle}(w) = C(w) \cap H_s$ by Lemma 4.13. The induction hypothesis implies that the extremal rays of $C_{\langle s\rangle}(w)$ form a basis for the lattice $L^*_{\langle s\rangle} \subset H_s$ and ρ_s is the unique extremal ray of $C(w)$ not contained in H_s by Lemma 4.13. Since the fundamental weights v_t, $t \in S$, span L^* it follows that L^* is spanned by v_s and $L^*_{\langle s\rangle} = L^* \cap H_s$. Hence, the extremal rays of $C(w)$ span L^*. □

Theorem 4.16. *Let Φ be a crystallographic root system for the Weyl group W and c be a Coxeter element of W. Suppose that $\boldsymbol{a} \in L$. Then the vertex sets $V(\mathsf{Perm}^{\boldsymbol{a}}(W))$ and $V(\mathsf{Asso}_c^{\boldsymbol{a}}(W))$ are contained in L.*

Proof. The result for the permutahedron is obvious, since by definition the vertices of the permutahedron are the W-orbit of $\boldsymbol{a}$, which is in L by assumption, and the action of W preserves L.

Let $w \in W$ be c-sortable, $x(w)$ be the vertex of $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ contained in the maximal cone $C(w) \in \mathcal{F}_c$, and ρ_i, $1 \leqslant i \leqslant n$ be the extremal rays of $C(w)$. Denote the lattice point on ρ_i closest to the origin by y_i. The point $x(w)$ satisfies $\langle x(w), y_i\rangle = c_i$ for some integer c_i since $\boldsymbol{a} \in L$. Because $\{y_i\}$, $1 \leqslant i \leqslant n$, is a basis of L^*, this set of equations for $x(w)$ has an integral solution. In other words, $x(w) \in L$. □

5. Observations and remarks

5.1. Recovering the c-cluster complex from the c-singletons

It is possible to obtain polytopal realizations of the c-cluster complex from the construction of generalized associahedra which we have presented, as follows. Suppose that we are given a W-permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W)$, a Coxeter element c, and the c-sorting word $\mathbf{w_0}$ of w_0. We can easily compute all c-singletons using the characterization given in Theorem 2.2. The associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ is now obtained from $\mathsf{Perm}^{\boldsymbol{a}}(W)$ by keeping all the admissible inequalities, that is all inequalities $\langle v, w(v_s)\rangle \leqslant \langle \boldsymbol{a}, v_s\rangle$ for c-singleton w. We label the facet $\langle v, w(v_s)\rangle \leqslant \langle \boldsymbol{a}, v_s\rangle$

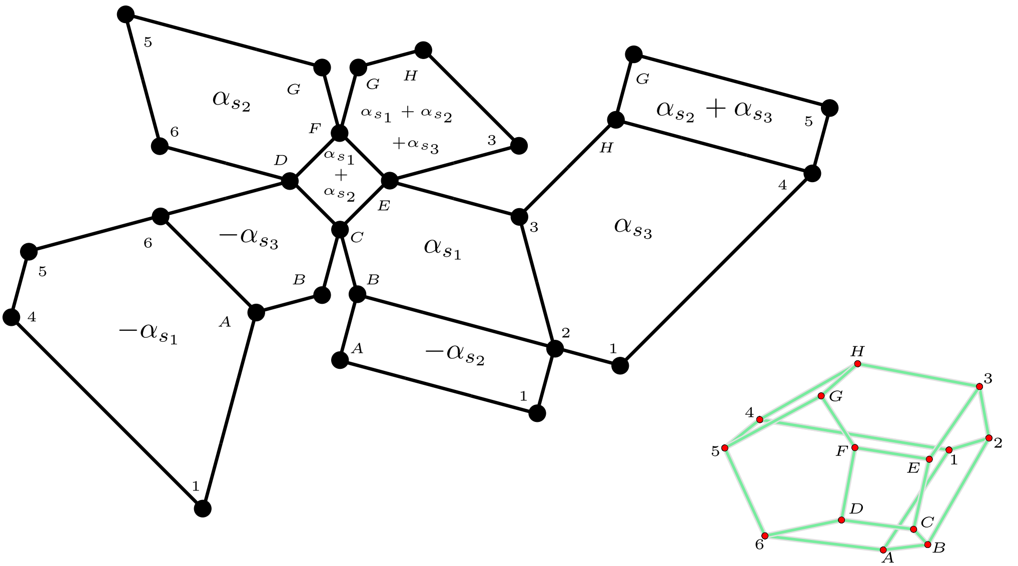

Fig. 9. An unfolding of the associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(S_4)$ with $c = s_1 s_2 s_3$, the polar of the c-cluster complex. The 2-faces are labeled by replacing the labels $w(\rho_s)$ in Fig. 7 by the almost positive root $\mathsf{Lr}_s(w)$.

of $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ by the almost positive root $\mathsf{Lr}_s(w)$ and extend this labeling to the Hasse diagram of $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ as follows: if a face f is the intersection of facets $F_1, \ldots, F_k$ then assign f the union of the almost positive roots assigned to $F_1, \ldots, F_k$. By Theorem 3.6, this labeling matches the labeling of the c-Cambrian fan by almost positive roots given by Reading and Speyer. Therefore, the opposite poset of this labeled Hasse diagram is the face poset of the c-cluster complex because it is the face poset of the c-Cambrian fan $\mathcal{F}_c$. The polar of $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ is therefore a polytopal realization of the c-cluster complex. In particular, a set of almost positive roots is c-compatible (see [19]) if and only if it can be obtained as the intersection of some facets of $\mathsf{Asso}_c^{\boldsymbol{a}}$ by the process described above.

We illustrate the recovery of the c-cluster complex for $W = S_4$. Fig. 9 corresponds to the Coxeter element $c = s_1 s_2 s_3$ and Fig. 10 corresponds to the Coxeter element $c = s_2 s_1 s_3$. We use the polar of the c-cluster complex for the illustration.

First, consider the Coxeter element $c = s_1 s_2 s_3$. The facets are labeled by almost positive roots as indicated. The vertices correspond to clusters as follows:

$$
\begin{aligned}
&A = \{-\alpha_{s_1}, -\alpha_{s_2}, -\alpha_{s_3}\}, && B = \{\alpha_{s_1}, -\alpha_{s_2}, -\alpha_{s_3}\},\\
&C = \{\alpha_{s_1}, \alpha_{s_1} + \alpha_{s_2}, -\alpha_{s_3}\}, && D = \{\alpha_{s_2}, \alpha_{s_1} + \alpha_{s_2}, -\alpha_{s_3}\},\\
&E = \{\alpha_{s_1}, \alpha_{s_1} + \alpha_{s_2}, \alpha_{s_1} + \alpha_{s_2} + \alpha_{s_3}\}, && F = \{\alpha_{s_2}, \alpha_{s_1} + \alpha_{s_2}, \alpha_{s_1} + \alpha_{s_2} + \alpha_{s_3}\},\\
&G = \{\alpha_{s_2}, \alpha_{s_2} + \alpha_{s_3}, \alpha_{s_1} + \alpha_{s_2} + \alpha_{s_3}\}, && H = \{\alpha_{s_3}, \alpha_{s_2} + \alpha_{s_3}, \alpha_{s_1} + \alpha_{s_2} + \alpha_{s_3}\},\\
&1 = \{-\alpha_{s_1}, -\alpha_{s_2}, \alpha_{s_3}\}, && 2 = \{\alpha_{s_1}, -\alpha_{s_2}, \alpha_{s_3}\},\\
&3 = \{\alpha_{s_1}, \alpha_{s_1} + \alpha_{s_2} + \alpha_{s_3}, \alpha_{s_3}\}, && 4 = \{-\alpha_{s_1}, \alpha_{s_2} + \alpha_{s_3}, \alpha_{s_3}\},\\
&5 = \{-\alpha_{s_1}, \alpha_{s_2}, \alpha_{s_2} + \alpha_{s_3}\}, && 6 = \{-\alpha_{s_1}, \alpha_{s_2}, -\alpha_{s_3}\}.
\end{aligned}
$$

Next, consider the Coxeter element $c = s_2 s_1 s_3$. The facets are labeled by almost positive roots as indicated and the vertices correspond to clusters as follows:

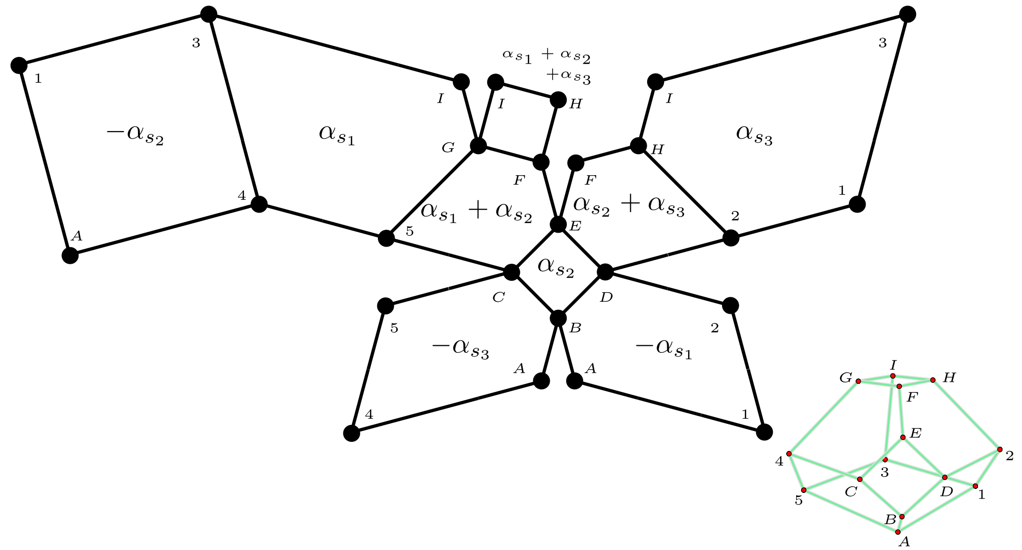

Fig. 10. An unfolding of the associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(S_4)$ with $c = s_2 s_1 s_3$, the polar of the c-cluster complex. The 2-faces are labeled by replacing the labels $w(\rho_s)$ in Fig. 8 by the almost positive root $\mathsf{Lr}_s(w)$.

$$A = \{-\alpha_{s_1}, -\alpha_{s_2}, -\alpha_{s_3}\}, \qquad B = \{-\alpha_{s_1}, \alpha_{s_2}, -\alpha_{s_3}\},$$
$$C = \{\alpha_{s_1}+\alpha_{s_2}, \alpha_{s_2}, -\alpha_{s_3}\}, \qquad D = \{-\alpha_{s_1}, \alpha_{s_2}, \alpha_{s_2}+\alpha_{s_3}\},$$
$$E = \{\alpha_{s_1}+\alpha_{s_2}, \alpha_{s_2}, \alpha_{s_2}+\alpha_{s_3}\}, \qquad F = \{\alpha_{s_1}+\alpha_{s_2}, \alpha_{s_1}+\alpha_{s_2}+\alpha_{s_3}, \alpha_{s_2}+\alpha_{s_3}\},$$
$$G = \{\alpha_{s_1}, \alpha_{s_1}+\alpha_{s_2}, \alpha_{s_1}+\alpha_{s_2}+\alpha_{s_3}\}, \qquad H = \{\alpha_{s_3}, \alpha_{s_2}+\alpha_{s_3}, \alpha_{s_1}+\alpha_{s_2}+\alpha_{s_3}\},$$
$$I = \{\alpha_{s_1}, \alpha_{s_1}+\alpha_{s_2}+\alpha_{s_3}, \alpha_{s_3}\}, \qquad 1 = \{-\alpha_{s_1}, -\alpha_{s_2}, \alpha_{s_3}\},$$
$$2 = \{-\alpha_{s_1}, \alpha_{s_2}+\alpha_{s_3}, \alpha_{s_3}\}, \qquad 3 = \{\alpha_{s_1}, -\alpha_{s_2}, \alpha_{s_3}\},$$
$$4 = \{\alpha_{s_1}, -\alpha_{s_2}, -\alpha_{s_3}\}, \qquad 5 = \{\alpha_{s_1}, \alpha_{s_1}+\alpha_{s_2}, -\alpha_{s_3}\}.$$

5.2. A conjecture about vertex barycenters

J.-L. Loday mentions in [13] that F. Chapoton observed the following: the vertex barycenters of the permutahedron and associahedron coincide in the case of Loday's original realization of the (classical) type A associahedron. The first two authors observed the same phenomenon for the realizations of type A and B associahedra described in [8]. None of these observations have been proven so far. Checking numerous examples in GAP [23], we observed that the vertex barycenter of $\mathsf{Perm}^{\boldsymbol{a}}(W)$ and $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ coincide for $\boldsymbol{a} = \sum_{s\in S} a v_s$, $a > 0$. The cases checked include types A_n for $n \leqslant 7$, B_n and D_n for $n \leqslant 5$, F_4, H_3, H_4, and dihedral groups $I_2(m)$. The experiments can be summarized in the following conjecture.

Conjecture 5.1. *Let W be a Coxeter group and $c \in W$ a Coxeter element. Choose a real number $a > 0$ and set $\boldsymbol{a} = \sum_{s\in S} a v_s$ to fix a realization of the permutahedron* $\mathsf{Perm}^{\boldsymbol{a}}(W)$. *Then the vertex barycenters of* $\mathsf{Perm}^{\boldsymbol{a}}(W)$ *and* $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ *coincide.*

It is straightforward to prove this conjecture for a special family, the dihedral groups G_m of order $2m$. We outline the proof.

Let $m \geqslant 2$ be an integer. The dihedral group G_m of order $2m$ is the finite Coxeter group of type $I_2(m)$ generated by the two reflections s and t with st having order m. For any m, the action of G_m on $V = \mathbb{R}^2$ is essential and we identify $\mathbb{R}^2$ with the complex numbers. If we define

$$v_s := \frac{1 + e^{i\frac{\pi}{m}}}{2} \quad \text{and} \quad v_t := \frac{1 + e^{-i\frac{\pi}{m}}}{2}$$

then G_m is generated by the reflections with respect to the hyperplanes spanned by v_s and v_t. We choose

$$\boldsymbol{a} := \frac{v_s + v_t}{1 + \cos(\frac{\pi}{m})} = 1$$

and follow our earlier notation where $w(\boldsymbol{a})$ denotes the point obtained by the action of $w \in G_m$ on $\boldsymbol{a}$. Then

$$w(\boldsymbol{a}) = \begin{cases} e^{i\ell(w)\frac{\pi}{m}} & \text{if } \ell(sw) < \ell(w), \\ e^{-i\ell(w)\frac{\pi}{m}} & \text{if } \ell(sw) > \ell(w). \end{cases}$$

The convex hull of the points $w(\boldsymbol{a})$, $w \in G_m$, is the permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(G_m)$ which is a regular $2m$-gon. It is easy to verify that the origin is the vertex barycenter of $\mathsf{Perm}^{\boldsymbol{a}}(G_m)$.

We consider the Coxeter element $c = st$; if $c = ts$, the reasoning is similar. The c-singletons are e and all $w \in G_m$ with $\ell(sw) < \ell(w)$. The generator t is the only c-sortable element which is not a c-singleton. Denote the intersection of the line through $\boldsymbol{a}$ and $t(\boldsymbol{a})$ and the line through $w_0(\boldsymbol{a})$ and $sw_0(\boldsymbol{a})$ by P. The associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(G_m)$ is the convex hull of P and the points $w(\boldsymbol{a})$ where $w \in G_m$ is a c-singleton. A straightforward computation yields

$$P = \frac{i \sin(\frac{\pi}{m})}{\cos(\frac{\pi}{m}) - 1}.$$

It is not hard to verify that

$$\sum_{\substack{w \in G_m \\ \text{not } c\text{-singleton}}} w(\boldsymbol{a}) = \sum_{k=1}^{m-1} \left(e^{-i\frac{\pi}{m}}\right)^k = P,$$

so the vertex barycenters of $\mathsf{Perm}^{\boldsymbol{a}}(G_m)$ and $\mathsf{Asso}_c^{\boldsymbol{a}}(G_m)$ coincide.

5.3. Recovering the realizations of [8] for types A and B

5.3.1. Type A

Let $\mathcal{B} = \{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. The symmetric group S_n acts naturally on $\mathbb{R}^n$ by permutation of the coordinates. We set

$$\Delta := \{e_{i+1} - e_i \mid 1 \leqslant i \leqslant n-1\} \quad \text{and} \quad \Phi^+ := \{e_j - e_i \mid 1 \leqslant i < j \leqslant n\}.$$

Then $\Phi = \Phi^+ \cup (-\Phi^+)$ is a root system of type A_{n-1} with simple roots Δ. Moreover, we recall that the reflection group S_n acts essentially on

$$V := \mathbb{R}[\Delta] = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \;\Big|\; \sum_{i=1}^{n} x_i = 0 \right\} \subset \mathbb{R}^n.$$

Let s_i be the simple reflection that maps the simple root $e_{i+1} - e_i$ to $e_i - e_{i+1}$. The dual basis Δ^* of Δ is described by

$$v_{s_i} := \frac{i-n}{n} \sum_{k=1}^{i} e_k + \frac{i}{n} \sum_{k=i+1}^{n} e_k \in V.$$

We choose $\boldsymbol{a} := \sum_{i=1}^{n-1} v_{s_i}$, so $\boldsymbol{a} = \sum_{k=1}^{n} (k - \frac{n+1}{2}) e_k$.

There is a bijection between Coxeter elements $c \in S_n$ and orientations of the Coxeter graph of S_n: if s_i appears before s_{i+1} in a reduced expression of c then the edge between s_i and s_{i+1} is oriented from s_i to s_{i+1}. The orientation is from s_{i+1} to s_i if s_i appears after s_{i+1} in a reduced expression of c. Given an oriented Coxeter graph, we can apply the construction described earlier and obtain a permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(S_n)$ and an associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(S_n)$.

Consider the affine subspace $\mathcal{V} \subset \mathbb{R}^n$ that is a translate of V by $v_G = \frac{n+1}{2} \sum_{i=1}^{n} e_i$:

$$\mathcal{V} = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \;\Big|\; \sum_{i=1}^{n} x_i = \frac{n(n+1)}{2} \right\}.$$

Translate $\mathsf{Perm}^{\boldsymbol{a}}(S_n) \subset V$ by v_G to obtain $\mathsf{Perm}^{\boldsymbol{a}}(S_n) + v_G \subset \mathcal{V}$. The vertices of the translate $\mathsf{Perm}^{\boldsymbol{a}}(S_n) + v_G$ are the orbit of $\boldsymbol{a} + v_G = \sum_{i=1}^{n} i e_i$ under the action of S_n; in other words, we have

$$w(\boldsymbol{a}) + v_G = \sum_{i=1}^{n} w^{-1}(i) e_i$$

for $w \in S_n$. The permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(S_n) + v_G$ was described in [8] but the vertices were labeled differently.

Proposition 5.2. *Consider a Coxeter element $c \in S_n$ or equivalently an orientation of the Coxeter graph and let v_G and $\boldsymbol{a}$ be as above. The associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(S_n) + v_G$ is the associahedron Asso_c constructed in* [8].

Proof. In [8, Proposition 1.3], it was proved that the c-singletons are the common vertices of the permutahedron and the associahedron and that the normal fan of the latter is $\mathcal{F}_c$. In other words, the realization of the associahedron in [8] precisely matches the description of $\mathsf{Asso}_c^{\boldsymbol{a}}(S_n)$ given in Corollary 4.5. □

5.3.2. Type B

Consider the simple root system of type B given by

$$\Delta' := \{e_{n+1} - e_n\} \cup \{e_{i+1} - e_i + e_{2n+1-i} - e_{2n-i} \mid 1 \leqslant i \leqslant n-1\} \subset \mathbb{R}^{2n}.$$

If we set $V' := \mathbb{R}[\Delta']$ then V' is an n-dimensional subspace of $\mathbb{R}^{2n}$ which is contained in V, the span of the type A_{2n-1} root system as in 5.3.1. Denote the simple reflection that corresponds to $e_{n+1} - e_n$ by s_0. For $1 \leqslant i \leqslant n-1$, we denote the simple reflection that corresponds to $(e_{i+1} - e_i) + (e_{2n+1-i} - e_{2n-i})$ by s_{n-i}. The *hyperoctahedral group* W_n (or Coxeter group of type B_n) is generated by these reflections. It is easy to see that $V' = V \cap \bigcap_{i=1}^{n-1} V_i^B$ where

$$V_i^B := \{x \in \mathbb{R}^{2n} \mid x_i + x_{2n+1-i} = 0\}.$$

In particular we have $\boldsymbol{a} \in V'$.

The claim that $\boldsymbol{a}$ is in the open cone spanned by the fundamental weights of Δ' follows from the fact that the scalar product of $\boldsymbol{a}$ with any element of Δ' is strictly positive.

A Coxeter element $c \in W_n$ is related to an orientation of the Coxeter graph of W_n as in type A: If s_i appears before (resp. after) s_{i+1} in a reduced expression of c then the edge between s_i and s_{i+1} is oriented from s_i to s_{i+1} (resp. from s_{i+1} to s_i). A Coxeter element or an orientation of the Coxeter graph therefore yields a permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W_n)$ as described in Section 3.1 and an associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(W_n)$. The orientation of the Coxeter graph of W_n determines a *symmetric* orientation of the Coxeter graph of S_{2n}, that is, an orientation of the Coxeter graph of type A_{2n-1} where the edges $\{s_i, s_{i+1}\}$ and $\{s_{2n-i-1}, e_{2n-i}\}$ have opposite orientations. This orientation determines a Coxeter element $\widetilde{c}$ of S_{2n} and we have

$$\mathsf{Perm}^{\boldsymbol{a}}(W_n) = \mathsf{Perm}^{\boldsymbol{a}}(S_{2n}) \cap V' \quad \text{and} \quad \mathsf{Asso}_c^{\boldsymbol{a}}(W_n) = \mathsf{Asso}_{\widetilde{c}}^{\boldsymbol{a}}(S_{2n}) \cap V'.$$

The following proposition is a direct consequence of the construction in [8].

Proposition 5.3. *Consider a Coxeter element $c \in W_n$ or equivalently an orientation of the Coxeter graph of type B_n. Let v_G and $\boldsymbol{a}$ be as above for type A. The translated associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(W_n) + v_G$ is the cyclohedron constructed in* [8] *that corresponds to the orientation of the Coxeter graph determined by c.*

Acknowledgments

The authors thank Nathan Reading and David Speyer for having made their results in [21] available to us during their work and for their helpful remarks. In particular, we thank David Speyer for having conjectured Theorem 4.1 while reading an earlier version of this article.

The authors wish to thank the Fields Institute for its hospitality. It turned out to be the focal point of our work. C.H. was partially supported by an NSERC Discovery Grant and a grant from FQRNT. C.L. was partially supported by a DFG grant (Forschergruppe 565 *Polyhedral Surfaces*). H.T. was partially supported by an NSERC Discovery Grant.

Finally, the authors thank the two anonymous referees who made invaluable comments to improve the presentation of this article.

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