Toponogov's triangle comparison theorem in model spaces of nonconstant curvature

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Abstract: So far there exist two versions of Toponogov's triangle comparison theorem with surfaces of revolution as model spaces. U. Abresch has considered surfaces of nonpositive curvature and D. Elerath has investigated embedded surfaces of positive curvature. As the classical theorem, these two extensions are proved applying Rauch's comparison theorems. In this paper we present a new version of Toponogov's theorem generalizing all comparison theorems mentioned above. The new proof is based on estimates for the second fundamental tensor of distance spheres and handles the rigidity as well.

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1. Introduction

An important tool for studying Riemannian manifolds with lower curvature bound is Toponogov's triangle comparison theorem:

Theorem 1.1. Let $M^n$ be a complete Riemannian manifold with $K \geq k$. Suppose that $\triangle = (\gamma_0, \gamma_1, \gamma_2)$ is a triangle in $M^n$, where $\gamma_1$ and $\gamma_2$ are minimizing geodesics from $p_1$ resp. $p_2$ to $p_0$. $\gamma_0$ is a geodesic from $p_1$ to $p_2$, and $|\gamma_0| \leq |\gamma_1| + |\gamma_2|$. If $k > 0$, let in addition $|\gamma_0| \leq \pi/\sqrt{k}$.
Then there is a comparison triangle $\tilde{\Delta}$ in $M^n_k$ (the space of constant curvature $k$) with edges of the same length, i.e., $|\gamma_i| = |\tilde{\gamma}_i|$ for $i = 0, 1, 2$ and, moreover, the following holds:

(i) $\langle \tilde{\gamma}_i \rangle \leq \langle \gamma_i \rangle$ for $i \in \{1, 2\}$ (angle comparison);
(ii) dist$(\tilde{\gamma}_0, \tilde{\gamma}_0(t)) \leq$ dist$(\gamma_0, \gamma_0(t))$ for $t \in (0, |\gamma_0|)$ (distance comparison).

If equality holds in (i) or in (ii), $\tilde{\Delta}$ can be totally geodesically embedded into $M^n$ and is isometric to $\Delta$.

The following hinge version is equivalent to the angle comparison of Theorem 1.1.

Theorem 1.2. Let $(\gamma_0, \gamma_1, \alpha_1)$ be a hinge in $M^n$ (i.e., $\langle \gamma_0(0), \gamma_1(0) \rangle = \alpha_1$), where $\gamma_1$ is minimizing. If $k > 0$, let in addition $|\gamma_0| \leq \pi/\sqrt{k}$. Then any comparison hinge $(\tilde{\gamma}_0, \tilde{\gamma}_1, \alpha_1)$ in $M^n_k$ satisfies

$$\text{dist} (\tilde{\gamma}_1(|\gamma_1|), \tilde{\gamma}_0(|\gamma_0|)) \geq \text{dist} (\gamma_1(|\gamma_1|), \gamma_0(|\gamma_0|)).$$

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A proof for both theorems can be found in [2]. These theorems have been generalised by Abresch [1] and Elerath [4]. Both authors consider surfaces of revolution as model spaces, which correspond to a lower curvature bound for $M^n$ that depends only on the distance to a fixed point and is controlled by a continuous function $k : [0, \infty) \to \mathbb{R}$. U. Abresch investigates surfaces of nonpositive curvature, while D. Elerath considers surfaces of positive curvature, which are embedded in $\mathbb{R}^3$. In this paper we provide a unified approach to the preceding theorems. The crucial point is to find an appropriate set of surfaces of revolution that are admissible as model space. Actually, our theorem even holds for model spaces with a singular point.

The paper is organized as follows: In Section 2 we introduce the relevant model spaces. They have to obey condition (E), which means that the cut locus of any point is contained in the opposite meridian. Thereafter we are able to present all results of the paper. The purpose of Section 3 is to study the properties of geodesics and triangles in the model space. We prove the uniqueness of the comparison triangle and we observe that condition (E) is necessary. Therefore our class of model spaces is as large as possible. In Section 4 we prove the theorem with estimates for the second fundamental tensor of distance spheres, while the proofs of all theorems mentioned above use the Rauch Comparison Theorems. The new method was suggested by Gromov first and was worked out by Karcher [7] and Meyer [9] to prove Theorem 1.1. In Section 5 we handle the equality case. So far, this has only been done for the constant curvature case. We obtain a ruled surface consisting of minimizing geodesics, which join one vertex and the opposite edge of the triangle in the manifold. The interior of this surface is totally geodesically embedded and isometric to the interior of the triangle in the model space. The construction of the surface is done by an iterative process, for which the function $k$ has to be locally Lipschitz continuous. Looking at two examples in the lens space $L(4, 1)$ we discuss the properties of the ruled surface.

1.1. Notation

$M^n$ will denote a complete Riemannian manifold with metric $\langle \cdot, \cdot \rangle$ of class $C^2$ and Levi-Civita connection $\nabla$. $B_\delta(p)$ will denote the ball of radius $\delta$ around $p$ and $C(p)$ the cut locus of $p$. $T^1_p M$ denotes the unit sphere and $C_p$ the cut locus in $T_p M$. $R_N$ is the tensor $R(\cdot, N)N$, where $R$ is the curvature tensor of $\nabla$ and $N$ is a vector field. We write $|c|$ for the length of the curve $c$ and $c^-$ for the curve with the opposite direction. If $X$ is a vector field along the curve $t \mapsto c(t)$, let $X' := \nabla_{\partial/\partial t} X$. We call $X$ normal, if $\langle X, \dot{c} \rangle$ vanishes. Sometimes we shall let the same symbol denote both a curve and its image. All geodesics will be assumed to be normal. A triangle is a triple of geodesics with the same start and endpoints. If the joining geodesics are unique, we call also a triple of points a triangle.

2. Model spaces and Toponogov's theorem

2.1. Definition of the model space

Each continuous function $k : [0, \infty) \to \mathbb{R}$ determines a simply connected surface of revolution $M(k)$ with curvature

$$K = k \circ \text{dist}_{p_0},$$

(1)
Toponogov's Theorem in models of nonconstant curvature

where \( \widetilde{p}_0 \) denotes the pole of \( M(k) \). In polar coordinates \((r, \varphi)\) the metric is given by

\[
g = dr^2 + (y \circ r)^2 \, d\varphi^2,
\]

where \( y \) is the solution of the one-dimensional Jacobi field equation

\[
y'' + ky = 0, \quad y(0) = 0, \quad y'(0) = 1.
\]

We can extend \( g \) to a \( C^2 \)-tensor field for \( \mathbb{R}^2 \). In order to get a metric, we restrict \( g \) to the ball \( \widetilde{M}(k) := B_L(\widetilde{p}_0) \), where \( L \) is the first positive zero of \( y \). If \( L = \infty \), this is already our model space \( M(k) \), a complete, simply connected Riemannian manifold diffeomorphic to \( \mathbb{R}^2 \). We call the rays starting in \( \widetilde{p}_0 \) meridians. Using polar coordinates we write \( \Psi : (0, L) \times [0, 2\pi) \to \widetilde{M}(k) \) and \( \mu_\alpha = \Psi(\cdot, \alpha) \). If \( p = \Psi(t, \alpha) \neq \widetilde{p}_0 \), we set \( \mu_p := \mu_\alpha \) and \( \widetilde{\mu}_p := \mu_\alpha + \pi \).

2.2. The complete length space \( M(k) \)

If \( L < \infty \), then \( M(k) \) is the metric completion of \( \widetilde{M}(k) \). In this case \( M(k) \) is homeomorphic to the sphere \( S^2 \) and possibly has one singular point, the cut locus of the pole. The metric of \( \widetilde{M}(k) \) extends continuously to an inner metric on \( M(k) \). Summarizing we have

**Proposition 2.1** (Basic Properties of the Model Space). \( (\widetilde{M}(k), g) \) is a simply connected Riemannian manifold obeying (1). The group of isometries fixing \( \widetilde{p}_0 \) equals \( O(2) \). The subgroup of rotations induces a Killing vector field, which we call \( V \). \( M(k) \) is a complete, simply connected length space with inner metric. The diameter of \( M(k) \) equals \( L \).

There exist minimizing curves between any pair of points, even if \( M(k) \) is not a Riemannian manifold [6, Théorème 1.10].

2.3. Condition (E)

For the model spaces used in [4] the function \( y \) is strictly increasing, while the function \( k \) is positive and decreasing. These properties imply that the cut locus of any point \( p \) is contained in the opposite meridian \( \widetilde{\mu}_p \). This condition, which we call condition (E), also holds in the spaces of constant curvature and in the model spaces used in [1].

In order to extend this condition to model spaces with \( L < \infty \), let \( H \) be given by \( H(p, q) := 2L - r(p) - r(q) - \text{dist}(p, q) \) for \( p, q \in M(k) \). \( H \) is nonnegative and the strong inequality \( H(p, q) > 0 \) holds, if and only if all minimizing geodesics between \( p \) and \( q \) are smooth and contained in \( \widetilde{M}(k) \).

**Definition.** A model space \( M(k) \) obeys condition (E), if between any points \( p \) and \( q \) in \( \widetilde{M}(k) \setminus \{\widetilde{p}_0\} \) with \( H(p, q) > 0 \) and \( q \not\in \widetilde{\mu}_p \) there is only one minimizing geodesic and this geodesic has no conjugate point.

If \( L = \infty \), we have \( H \equiv \infty \) and condition (E), as before, means that the cut locus of any point is contained in the opposite meridian. In fact condition (E) is too strong. It is sufficient to require that the conjugate locus is contained in the opposite meridian. This implies condition (E) (cf. [8]).
2.4. Toponogov's theorem for the model space $M(k)$

Let $D$ be the map $D(l_1, l_2) = \text{dist}(\mu_0(l_1), \mu_\pi(l_2))$, $l_1, l_2 \in [0, L)$. If $\Delta = (\gamma_0, \gamma_1, \gamma_2)$ is a triangle, we write $l_i$ for the lengths of the edges and use the notations illustrated by Figure 1.

![Figure 1.](image)

Theorem 2.2 (Toponogov's Triangle Comparison Theorem). Let $M^n$ be a complete Riemannian manifold with

$$K(\sigma_p) \geq k \circ \text{dist}_{p_0}(p) \quad (2)$$

for all planes $\sigma_p \subset T_pM^n$, where $k$ is a continuous function. Suppose the model space $M(k)$ obeys condition (E) and $\widetilde{p}_0$ denotes its pole. Let $\Delta$ be a triangle, where $\gamma_1$ and $\gamma_2$ are minimizing and

$$l_0 < D(l_1, l_2) \quad (U)$$

holds. Then there is a triangle $\tilde{\Delta} = (\tilde{p}_0, \tilde{p}_1, \tilde{p}_2)$ in $M(k)$ with $l_i = \tilde{l}_i$. This comparison triangle is unique up to isometry and the following holds:

(i) $\tilde{\alpha}_i \leq \alpha_i$ for $i \in \{1, 2\}$ (angle comparison),

(ii) $\text{dist}(\tilde{p}_0, \tilde{\gamma}_0(t)) \leq \text{dist}(p_0, \gamma_0(t))$ for $t \in (0, l_0)$ (distance comparison).

Remarks 2.3. (a) Any triangle $\tilde{\Delta} = (\tilde{p}_0, \tilde{p}_1, \tilde{p}_2)$ in $M(k)$ with edges of the same length as $\Delta$ is called a comparison triangle.

(b) Inequality (U) implies the strong triangle inequality $l_0 < l_1 + l_2$ and thus $U < 2L = 2 \cdot \text{diam}(M(k))$, where $U$ is the circumference of the triangle $\Delta$. For the classical theorem only the weak triangle inequality is required and $U \leq 2L$ is proved in [9]. But if $U = 2L$, the comparison triangle is not necessarily unique.

(c) If the comparison triangle is not unique, the angle comparison fails: We can apply Theorem 2.2 to noncongruent comparison triangles and exchange their roles.

(d) In Proposition 3.9 we shall construct noncongruent triangles with edges of the same length in model spaces that do not obey condition (E).

(d) If $\gamma_0$ is minimal too, the angle comparison holds for $\alpha_0$ as well. A proof for this can be found in [8].
Corollary 2.4 (Hinge Version). Suppose the complete Riemannian manifold $M^n$ obeys inequality (2) and condition (E) holds for the model space $M(k)$. Let $(\gamma_0, \gamma_1, \alpha_1)$ be a hinge in $M^n$, where $\gamma_1$ is minimizing and $\gamma_1(l_1) = p_0$. We set $l_2 := \text{dist}_{p_0}(\gamma_0(l_0))$. Suppose

(i) $l_0 < L - l_1$,
(ii) $l_0 < D(l_1, l_2)$ (inequality (U)),
(iii) $l_0 < \text{conj}(\mu_0(l_1))$. Then any comparison hinge $(\gamma_0', \gamma_1', \alpha_1')$ in $M(k)$ satisfies

$\text{dist}(\gamma_0(l_0), \gamma_1(l_0)) \geq \text{dist}(p_0, \gamma_0'(l_0))$.

Remarks 2.5. (a) We call $(\gamma_1', \gamma_0', \alpha_1')$ a comparison hinge, if $\alpha_1' = \alpha_1$ for $i = 0, 1$ and if $\gamma_1'(l_1)$ is the pole of $M(k)$. A comparison hinge is unique up to isometry.

(b) If $k$ is nonpositive, then (i), (iii) and the weak inequality in (ii) hold for any hinge.

(c) At the end of this section, in Remark 2.9b, we will observe that it is possible to omit condition (E) and to relax inequality (U) to $l_0 \leq l_1 + l_2$.

Theorem 2.6 (Rigidity). Suppose equality holds under the assumptions of Theorem 2.2, i.e.,

$\alpha_i = \alpha_i$ for one $i \in \{1, 2\}$ or

$\text{dist}(\gamma_0, \gamma_i(t)) = \text{dist}(p_0, \gamma_i(t))$ for one $t \in (0, l_0)$.

Then

$\text{dist}_{p_0} \circ \gamma_0' = \text{dist}_{p_0} \circ \gamma_0$.

If, in addition, $k$ is locally Lipschitz continuous or decreasing on $(0, L)$, there is a ruled surface consisting of minimizing geodesics joining $p_0$ and $\gamma_0$ with the following property: The interior is totally geodesically embedded and isometric to the interior of the triangle in the model space. If $\alpha_1 = \tilde{\alpha}$, the surface can be constructed containing $\gamma_1$. For $\hat{K}$, the curvature of the surface, the following holds:

$\hat{K} = k \circ \text{dist}_{p_0}$.

Remarks 2.7. (a) If equality holds in the hinge version (Corollary 2.4), the claims of Theorem 2.6 are true for any triangle $(\gamma_0, \gamma_1, \gamma_2)$, where $\gamma_2$ is minimizing.

(b) In Section 5.3 we shall construct two triangles in the lens space $L(4, 1)$ to observe that the results of Theorem 2.6 are optimal. In the first example the ruled surface is unique, but we have the strong inequality in the angle comparison for both angles. This shows that equation (5) does not imply equation (3). In addition, none of the geodesics $\gamma_1$ and $\gamma_2$ is contained in the ruled surface. In the second example the geodesic $\gamma_0$ is not injective. Therefore only the interior of the triangle is embedded. Moreover, equality holds for both angles and there is a ruled surface that contains neither $\gamma_1$ nor $\gamma_2$.

2.5. Additional versions of Toponogov’s theorem

First we deduce the well-known versions from Theorem 2.2:

(1) We can relax inequality (U) in Theorem 2.2 to the weak inequality

$\text{dist}_{p_0} \circ \gamma_0' = \text{dist}_{p_0} \circ \gamma_0$.

(U') $l_0 \leq D(l_1, l_2)$.
if \( D(l_1, l_2) = l_1 + l_2 \) and if the comparison triangle is still unique in the case \( l_0 = D(l_1, l_2) \).

Then the comparison triangle is simply a geodesic and the distance comparison follows from the triangle inequality.

(2) If \( k \) is nonpositive, the conditions under (1) are satisfied and hence Theorem 2.2 and Corollary 2.4 hold with inequality (U) replaced by inequality (U'). Thus we have proved [1, Theorem 3.2]. In particular the classical theorem for nonpositive curvature follows.

(3) The classical theorem for positive curvature follows from Theorem 2.2 as well: If \( U < 2\pi / \sqrt{k} \), the conditions under 1) are satisfied and the general case can be shown using a limiting argument.

(4) The comparison theorems in [4] follow, if we require the uniqueness of the comparison triangle.

Now we state a comparison theorem dealing with thin triangles, for which the model space does not have to obey condition (E).

**Theorem 2.8.** Suppose, in the context of Theorem 2.2, the following holds:

(i) \( l_0 < L - l_1 \),
(ii) \( l_0 \leq l_1 + l_2 \) instead of inequality (U),
(iii) \( l_0 < \text{conj}(\mu_0(l_1)) \).

Then the claims of Theorem 2.2 and Theorem 2.6 are true, even if \( M(k) \) does not obey condition (E).

**Remarks 2.9.** (a) This result was already obtained in [3] by Eschenburg. He uses a slightly different proof, which is not so useful for the equality discussion.

(b) Under the assumptions of this theorem the length of \( \gamma_2 \) depends monotonically on \( \alpha_1 \). Therefore the corresponding hinge version holds as well. Hence Corollary 2.4 remains valid, if we omit condition (E) and relax inequality (U) to \( l_0 \leq l_1 + l_2 \).

3. Properties of the model space

3.1. Geodesics in \( \tilde{M}(k) \)

Let \( T := \text{grad dist}_{p_0} \) and \( V = \Psi_\ast(\partial / \partial \varphi) \) be the Killing vector field induced by the rotations around the pole. For the Levi-Civita connection of \( M(k) \) we obtain

\[
\nabla_T T = 0, \quad \nabla_T V = \nabla_V T = \left( \frac{y'}{y} \circ r \right) V \quad \text{and} \quad \nabla_V V = (-y'y \circ r) T. \tag{7}
\]

If \( t \mapsto \gamma(t) \) is a geodesic in \( \tilde{M}(k) \setminus \{p_0\} \), we write \( r = \text{dist}_{p_0} \circ \gamma \) and \( \varphi = \varphi \circ \gamma \). The tangent vector of \( \gamma \) is given by \( \dot{\gamma} = \dot{r}(T \circ \gamma) + \dot{\varphi}(V \circ \gamma) \). \( \gamma \) is a normal geodesic, if and only if the functions \( r \) and \( \varphi \) satisfy \( 1 = \dot{r}^2 + (y \circ r)^2 \dot{\varphi}^2 \) and

\[
\ddot{r} = \dot{\varphi}^2(y \circ r), \quad \ddot{\varphi} + 2\dot{\varphi}(\frac{y'}{y} \circ r) = 0. \tag{8}
\]

The last equation implies the following theorem, where we write \( \langle \cdot, \cdot \rangle \) instead of \( g(\cdot, \cdot) \).

**Theorem 3.1** (Clairaut's Theorem). The angular momentum \( \langle \dot{\gamma}, V \circ \gamma \rangle \) of any geodesic \( \gamma \) is constant: \( \langle \dot{\gamma}, V \circ \gamma \rangle = (y \circ r)^2 \dot{\varphi} =: c_1 \).
Furthermore, we obtain the Law of Sines in the model space \( M(k) \):
\[
(y \circ r)^2(1 - r^2) = (y \circ r)^2 \approx c_1^2.
\] (9)

**Proposition 3.2** (Characterization of Minimizing Geodesics in \( \tilde{M}(k) \)). The geodesic \( \gamma : [0, l] \to \tilde{M}(k) \) is minimizing, if and only if the following holds

(i) \( H_\gamma(t) := H(\gamma(0), \gamma(t)) > 0 \) for \( 0 \leq t < l \),
(ii) \( y|_{[0, l]} \) is the only minimizing geodesic from \( \gamma(0) \) to \( \gamma(t) \) for \( 0 \leq t < l \),
(iii) \( \gamma \) has no conjugate point in \([0, l)\).

**Proof.** \( H_\gamma(t) > 0 \) implies that every minimizing geodesic from \( \gamma(0) \) to \( \gamma(t) \) is contained in \( \tilde{M}(k) \) and smooth. Now the proposition can be proved in the same way as [2, Lemma 5.2].

**Corollary 3.3.** Let \( M(k) \) obey condition (E) and \( \gamma \) be a geodesic in \( \tilde{M}(k) \) with \( H_\gamma > 0 \). If \( \gamma \) is not a meridian, then \( \gamma \) is minimizing exactly as long as it does not meet the opposite meridian \( \mu_\gamma(0) \).

### 3.2. Comparison triangles in \( M(k) \)

**Proposition 3.4.** Let the comparison triangle \( \tilde{\Delta} \) satisfy inequality (U) and the model space condition (E). Then \( \tilde{\gamma}_0 \) is minimizing and contained in \( \tilde{M}(k) \).

**Proof.** Inequality (U) implies \( \tilde{\Delta} \subset \tilde{M}(k) \) and \( H_{\tilde{\gamma}_0} > 0 \). Suppose \( \tilde{\gamma}_0 \) is not a meridian and \( q = \tilde{\gamma}_0(t) \) is the first point of \( \tilde{\gamma}_0 \) on \( \mu_{\tilde{\gamma}_0}(0) \). Let \( l := \text{dist}(\tilde{p}_0, q) \). The triangle inequality implies \( D(l_1, l_2) \leq t + |l_2 - l| \) and \( |l_2 - l| \leq (l_0 - t) \). This contradicts inequality (U) and hence \( \tilde{\gamma}_0 \) does not meet \( \mu_{\tilde{\gamma}_0}(0) \). Now the claim follows from Corollary 3.3.

**Proposition 3.5.** Let \( \tilde{p}_1 = \mu_0(l_1) \) and \( p_2(\alpha) = \mu_\alpha(l_2) \), \( l_1, l_2 \in (0, L) \). Then the map \( F : [0, \pi] \to \mathbb{R} \), \( \alpha \mapsto \text{dist}(p_2(\alpha), \tilde{p}_1) \) is increasing with range \([|l_1 - l_2|, D(l_1, l_2)]\).

**Proof.** \( d := 2L - l_1 - l_2 \) is an upper bound for \( F \). We define the open set \( A := \{ \alpha \in [0, \pi] | F(\alpha) < d \} \). \( F \) is strictly increasing on any interval contained in \( A \). Hence the proof is finished, if \( A = [0, \pi] \). Otherwise let \( \alpha_m := \inf([0, \pi] \setminus A) \). \( F \) is strictly increasing on \([0, \alpha_m)\) and we conclude \( F|_{[\alpha_m, \pi]} \equiv d \), again finishing the proof.

In order to show that \( F \) is strictly increasing on the interval \( J \subset A \), we define, for every \( \alpha_0 \) in the interior of \( J \), a differentiable support function \( \tilde{F} \) for \( F \) with \( \tilde{F} \geq F, \tilde{F}(\alpha_0) = F(\alpha_0) \) and \( \tilde{F}'(\alpha_0) > 0 \): Let \( \gamma \) be a minimizing geodesic from \( \tilde{p}_1 \) to \( p_2(\alpha_0) \). \( F(\alpha_0) < d \) implies \( H(\tilde{p}_1, p_2(\alpha_0)) > 0 \). Thus \( \gamma \) is contained in \( \tilde{M}(k) \). We define \( \tilde{F}'(\alpha) := \eta + \text{dist}_g(p_2(\alpha)) \), where \( q = \gamma(\eta) \) is an inner point of \( \gamma \). The first variation formula yields \( \tilde{F}'(\alpha_0) = (V \circ \gamma, \gamma)'|_\gamma \). This is the angular momentum of \( \gamma \) and is positive, since \( \gamma \) is not a meridian.

**Remarks 3.6.** (a) \( H(\mu_0(l_1), \mu_\alpha(l_2)) > 0 \) holds for \( \alpha \in [0, \alpha_m) \).

(b) \( F|_{[\alpha_m, \pi]} \) is identically \( D(l_1, l_2) \).

(c) If \( (\alpha_m, \pi) \) is not the empty set, then \( F \equiv 2L - l_1 - l_2 \) on this interval. In particular \( D(l_1, l_2) = 2L - l_1 - l_2 \) and \( H(\mu_0(l_1), \mu_\alpha(l_2)) = 0 \) for \( \alpha \in [\alpha_m, \pi] \).

Proposition 3.5 and Proposition 3.4 yield the first claim of Theorem 2.2:
Proposition 3.7. Under the assumptions of Theorem 2.2 there exists a comparison triangle in \( M(k) \). This comparison triangle is unique up to isometry.

3.3. Monotonicity properties for the angle at the vertex \( \tilde{p}_1 \)

Let \( \gamma^\alpha \) be the geodesic that starts in \( \tilde{p}_1 = \mu_0(l_1), \ l_1 > 0 \) with \( \alpha(-T(\tilde{p}_1), \gamma^\alpha(0)) = \alpha \) and positive angular momentum. We investigate the function \( G(\alpha) := \text{dist}(\tilde{p}_0, \gamma^\alpha(l_0)) \). The vector field \( (\partial/\partial\alpha)\gamma^\alpha(t) \) along the geodesic \( \gamma^\alpha \), \( 0 < \alpha_1 < \pi \) is a Jacobi field \( J = zX \), where \( X \) is a parallel, normal vector field.

\[
G'(\alpha_1) = z(l_0)(X(l_0), T(\gamma^\alpha(l_0))). \tag{10}
\]

This proves the following lemma:

Lemma 3.8. Let \( l_0 < \text{conj}(\mu_0(l_1)) \) and \( 0 < l_0 < L - l_1 \). Then the function \( G(\alpha) = \text{dist}(\tilde{p}_0, \gamma^\alpha(l_0)) \) is strictly increasing on \([0, \pi]\).

Proposition 3.9. Suppose \( M(k) \) does not obey condition (E). Then there are two noncongruent triangles with edges of the same length satisfying inequality (U) and hence Theorem 2.2 fails.

Proof. There are points \( p_1 = \mu_0(l_1) \) and \( q = \mu_0(l_2) \) with \( l_1, l_2 > 0, \alpha_0 \in (0, \pi) \) and \( H(p_1, q) > 0 \), such that

1) there are two different minimizing geodesics from \( p_1 \) to \( q \), or
2) \( q \) is a conjugate point of a minimizing geodesic \( \gamma \) from \( p_1 \) to \( q \).

The proof is finished in the first case. In the second case let \( l := ||\gamma|| \). Proposition 3.5 implies \( |l_1 - l_2| < l < D(l_1, l_2) \). There is a nontrivial Jacobi field \( J = zX \) with zeros in 0 and \( l \), where \( X \) is again a parallel, normal vector field along \( \gamma \). The function \( (X, T \circ \gamma) \) has no zero, since \( \gamma \) is not a meridian. We can assume that \( J \) is the vector field \( (\partial/\partial\alpha)\gamma^\alpha(t) \) along the geodesic \( \gamma = \gamma^\alpha \). Let \( \epsilon > 0 \) be sufficiently small. Then \( z \) is monotonous on \([l - \epsilon, l + \epsilon]\) and

\[
l \pm \epsilon < D(l_1, l_2^\pm), \quad \text{where} \quad l_2^\pm := \text{dist}_{\tilde{p}_0}(\gamma(l \pm \epsilon)). \tag{11}
\]

We can also assume \( \text{dist}_{\tilde{p}_0}(\gamma^0(l \pm \epsilon)) < l_2^\pm \), since \( \text{dist}_{\tilde{p}_0}(\gamma^0(l)) = |l_1 - l| < l_2 \). Now we look at the functions \( G^\pm(\alpha) := \text{dist}_{\tilde{p}_0} \circ \gamma^\alpha(l \pm \epsilon) \). We know \( G^+(0) < G^+(\alpha_1) \) and formula (10) yields \( (G^\pm)'(\alpha_1) = z(l \pm \epsilon)(X, T \circ \gamma)|_{l \pm \epsilon} \). Therefore one of the functions \( G^\ast, \ast \in \{+, -\} \) is strictly decreasing in \( \alpha_1 \). Thus there is \( \alpha_1 \in (0, \alpha_1) \) with \( G^\ast(\alpha_1) = G^\ast(\alpha_0(l_1)) = l_2^\ast \). Hence we have found two noncongruent triangles, which satisfy inequality (U) by inequality (11).

4. Proof of Toponogov's triangle comparison theorem

4.1. Deformation of the comparison triangle

We construct a family \( \Delta^t, t \in [0, 1] \), where \( \Delta = \Delta^1 \). This deformation has the following properties:

I. \( l_i^t < l_i \) for \( t < 1, \ i = 1, 2 : \gamma_1^t \subset \mu_0 \).

II. \( \gamma_0^t \) is the union of two geodesics \( \gamma_0'^t \) and \( \gamma_0''t \) and for the lengths of these geodesics we have
III. The deformation is continuous, i.e., the function \((t, s) \mapsto f_t(s) := \text{dist}_{\gamma_t(0)}(\gamma_t(s))\) is continuous.

IV. \(f^0 < f := \text{dist}_{\gamma_0}(\gamma_0)\) (i.e., the distance comparison holds for \(\Delta^0\)).

In the first part of the deformation we shorten \(\gamma_1\) and \(\gamma_2\) by defining \(p_1^t := \mu_0(t l_1)\). We choose \(t_1 > 0\) such that \(D(t l_1, t l_2) > l_0\) for \(t \in [t_1, 1]\). According to Proposition 3.5 for any \(t \in [t_1, 1]\) there is a unique \(\alpha_0^t \in [0, \pi]\) with \(\text{dist}(p_1^t, \mu_0^t(\gamma_1^t)) = l_0\). By condition (E) the minimizing geodesic \(\gamma_0^t\) from \(p_1^t\) to \(p_2^t := \mu_0^t(t l_2)\) is unique and contained in \(\tilde{M}(k)\). III (continuity) holds, since \(\alpha_0^t\) and \(\gamma_0^t\) are unique.

In the second part of the deformation, which is shown in Figure 2, the geodesic \(\gamma_1^t\) is fixed and we set \(\alpha_0^t := t \alpha_0^t / t_1\), \(p_2^t := \mu_0^t(t l_2)\). The function \(F_t(s) := (\text{dist}_{\gamma_0^t}(\gamma_1^t(s)) + \text{dist}_{\gamma_2^t}(\gamma_2^t(s)))\) is strictly increasing. Hence there is a unique \(s^t\) with \(F_t(s^t) = l_0\). Let \(\gamma_0^t\) be the meridian from \(p_1^t\) to \(\gamma_0^t(\gamma_1^t(s^t))\) and \(\gamma_2^t\) be the unique minimizing geodesic from \(\gamma_1^t(s^t)\) to \(p_2^t\). Then III holds again by uniqueness and IV follows from I by an application of the triangle inequality.

Let \(t_0 := \sup\{t \in [0, 1] \mid f^t \leq f\}\). This number is positive by IV and by III \(f^{t_0} \leq f\). If \(t_0 = 1\), this is the distance comparison and the proof is finished. Let us hence assume \(t_0 < 1\) in the following. In this case there is \(s_0 \in (0, l_0)\) with \(f^{t_0}(s_0) = f(s_0)\) and thus the function \(h := f - f^{t_0}\) vanishes in its inner minimum \(s_0\). \(\gamma_0^t(s_0)\) is an inner point of the geodesic \(\gamma_0^t\) and therefore we will assume that \(\gamma_0^t\) is smooth.

4.2. The strong curvature inequality

In this section we assume \(K(\gamma_0) \geq k \circ \text{dist}_{\gamma_0}(p) + \varepsilon\), where \(\varepsilon\) is a positive constant. Let \(\gamma\) be a minimizing geodesic from \(p_0\) to \(p := \gamma_0(s_0)\) and \(\tilde{\gamma}\) be the corresponding geodesic in the model space. Inequality (U) implies \(l := f^{s_0}(s_0) = f(s_0) \in (0, L)\). Since \(f\) is possibly nonsmooth, we define differentiable, upper support functions \(f^{(n)} := F^{(n)} \circ \gamma_0\), where \(F^{(n)} := \text{dist}_{\gamma_0} + n\). Therefore we need the solutions \(y^n\) of the modified Jacobi field equations \((y^n)^{''} + ky^n = 0\), \(y^n(\eta) = 0\), \((y^n)'(\eta) = 1\).

Lemma 4.1. Let \(S^n\) be the second fundamental tensor of the distance spheres around \(\gamma(\eta)\) with respect to the normal field \(N = \text{grad dist}_{\gamma(\eta)}\), \(\eta \in (0, 1/4]\). Then there is a positive \(\delta\) independent
of $\eta$, such that
\[ \tau_i^n \leq \frac{(y^n)'(l)}{y^n}(l) - \delta \]
holds for the eigenvalues $\tau_1^n, \ldots, \tau_{n-1}^n$ of $S^n_\eta$.

**Proof.** Let $I$, $\tilde{I}$ be the index forms of the geodesics $\gamma|_{[\eta,t]}$ resp. $\tilde{\gamma}|_{[\eta,t]}$ and $J$ be a Jacobi field along $\gamma|_{[\eta,t]}$ with $J(\eta) = 0$ such that $J(l)$ is a normal eigenvector to the eigenvalue $\tau_i^n$. Then $S^n_\eta J = J'$.

The corresponding Jacobi field in the model space is given by $\tilde{J}(t) = y^n(t) \frac{\gamma'(t)}{y^n(l)y(t)}$. Let $X, \tilde{X}$ be parallel vector fields along $\gamma$ resp. $\tilde{\gamma}$ with $X(l) = J(l), \tilde{X}(l) = J(l)$ and $\iota: T_{\gamma(\eta)}M(k) \to T_{\gamma(\eta)}M^n$ be an isometric map with $\iota\tilde{X}(\eta) = X(\eta), \iota\gamma'(\eta) = \tilde{\gamma}(\eta)$. As in [5, Chapter 6.1] we define the map $\Phi(a\tilde{\gamma} + b\tilde{X}) := ay + bX$. Now we compute
\[
\tilde{I}(\tilde{J}, \tilde{J}) - I(\Phi \tilde{J}, \Phi \tilde{J}) = \int_\eta^l (R(\Phi \tilde{J}, \gamma')\gamma, \tilde{J})_t - (R(\gamma', \gamma')\gamma, \tilde{J})_t) dt
\]
\[
\geq \varepsilon \int_{l/2}^l \|\tilde{J}(t)\|^2 dt = \varepsilon \int_{l/2}^l \left(\frac{y^n(t)}{y^n(l)}\right)^2 dt.
\]
The last term can be estimated from below by a constant $\delta > 0$ independent of $\eta$, since $(\eta, t) \mapsto y^n(t)$ is continuous and positive. We compute
\[
\tau_i^n = \langle J', J \rangle_t = I(J, J)
\]
\[
\leq I(\Phi \tilde{J}, \Phi \tilde{J}) \leq \tilde{I}(\tilde{J}, \tilde{J}) - \delta = \langle \tilde{J}', \tilde{J} \rangle_t - \delta = \frac{(y^n)'(l)}{y^n}(l) - \delta,
\]
where we used the fact that Jacobi fields minimize the index form.

$\tau_1^n, \ldots, \tau_{n-1}^n$ are the eigenvalues of the restriction of $\text{Hess}(\text{dist}_{\gamma(\eta)} \rho)$ to $N(p)^1$, while $N(p)$ is an eigenvector for the eigenvalue zero. In order to assimilate the eigenvalues, we introduce the strictly increasing function $Y(t) := \int_0^t y(r) dr$, $t \in [0, L)$. Then
\[
\text{Hess}(Y \circ F^{(n)})_\rho(v) = (y \circ F^{(n)}) \text{Hess}(F^{(n)})(v) + (y' \circ F^{(n)}) (N, v) N.
\]
From this formula we conclude that $y'(l)$ is the eigenvalue of $\text{Hess}(Y \circ F^{(n)})_\rho$ to the eigenvector $N(p)$. Using Lemma 4.1 the other eigenvalues $\lambda_i^n$ can be estimated:
\[
\lambda_i^n = y(l)\tau_i^n < y'(l),
\]
provided $\eta$ is small enough. Since $f^{(n)}$ is an upper support function for $f$, the function $h^{(n)} := Y \circ f^{(n)} - Y \circ f^0$ has a minimum in $s_0$ too. But $(Y \circ f^{(n)})(s_0) = (\text{Hess}(Y \circ F^{(n)})_\rho y_0(s_0), y_0(s_0)) < y'(l)$, since $y_0(s_0)$ and $N(p)$ are linearly independent, and $(Y \circ f^{(n)})(s_0) = y'(l)$ by formula (7). Therefore $h^{(n)}$ cannot have a minimum in $s_0$ and our assumption $t_0 < 1$ was wrong.

### 4.3. The general curvature inequality

Let $k_\varepsilon(t) := k(t) - \varepsilon$ and $y_\varepsilon$ be the solution of the corresponding Jacobi equation
\[
y''_\varepsilon + k_\varepsilon y_\varepsilon = 0, \quad y_\varepsilon(0) = 0, \quad y'_\varepsilon(0) = 1.
\]
Now we apply the results of the previous section in the model spaces $M(k_{t})$. According to Lemma 4.1 we obtain $\eta = \eta (\varepsilon ) \in (0, 1/4]$ such that $\tau^{n}_{t} \leqslant y'_{t}(l)/y_{t}$, $y'_{t}(l)$ is the eigenvalue of $\text{Hess}(Y \circ F^{(n)})$ to the eigenvector $N(p)$ and the other eigenvalues are given by

$$\lambda^{n}_{t} = y(l)\tau_{t}^{n} < y(l)\frac{y'_{t}(l)}{y_{t}}.$$ 

The right hand side converges to $y'(l)$, if $\varepsilon \to 0$. Hence for any $\zeta > 0$ we can choose $\varepsilon > 0$ and then $\eta = \eta (\varepsilon ) \in (0, 1/4]$ such that $\text{Hess}(Y \circ F^{(n)}) \leqslant y'(l) + \zeta$. More generally, let $s \in (0, l_{0})$ and $\gamma$ be a minimizing geodesic from $p_{0}$ to $y_{0}(s)$. Then for any $\zeta > 0$ there is $\eta > 0$ such that $\text{Hess}(Y \circ F^{(n)}) \leqslant y'(f(s)) + \zeta$. The function $f^{(n)} := F^{(n)} \circ y_{0}$ is an upper support function for $f$ in $s$ with

$$(Y \circ f^{(n)})''(s) \leqslant y'(f(s)) + \zeta.$$  

(12)

Let $R \in (0, L)$ with $\Delta \subseteq B_{R}(p_{0})$. Next we apply a maximum principle similar to [10, page 6].

**Lemma 4.2.** Suppose $g : [0, R] \to \mathbb{R}$ has a continuous derivative. Then also the following function is (uniformly) continuous.

$$G(t, u) := \begin{cases} \frac{g(t) - g(u)}{t - u} - g'(u) & \text{for } t \neq u, \\ 0 & \text{for } t = u, \end{cases} \quad t, u \in [0, R].$$

Let $C := 1 + \max_{t \in [0, R]} |k(t)|$, $m := \min_{t \in [0, R]} y \circ f^{(0)}(s)$. Applying Lemma 4.2 to the functions $y'$ and $Y$, we obtain $\varepsilon > 0$ such that

$$\left| \frac{y'(t) - y'(u)}{t - u} - y''(u) \right| < \frac{m}{2}, \quad \left| \frac{Y(t) - Y(u)}{t - u} - y(u) \right| < \frac{m}{2C}$$

hold for $t, u \in [0, R]$ with $0 < |t - u| < \varepsilon$. Therefore

$$y'(t) - y'(u) \leqslant \left( y''(u) + \frac{m}{2} \right) (t - u) \quad \text{and}$$

$$Y(u) - Y(t) \leqslant \left( \frac{m}{2C} - y(u) \right) (t - u)$$

(13)

(14)

hold for $t, u \in [0, R]$ with $0 \leqslant t - u < \varepsilon$. $h = f - f^{(0)}$ is nonnegative and satisfies $h(l_{0}) > 0$, $h(s_{0}) = 0$. Without loss of generality we can assume $h(s) > 0$ for $s \in (s_{0}, l_{0})$. Now we choose a positive $\delta$ such that

$$|f(s) - f^{(0)}(s)| < \varepsilon \quad \text{for } s \in I := [s_{1}, s_{2}] := [s_{0} - \delta, s_{0} + \delta] \subseteq (0, l_{0}).$$

$\hat{h} := Y \circ f - Y \circ f^{(0)}$ is positive in $s_{2}$. Let $z(s) := 1 - e^{C(t-s_{0})}$ and $\rho > 0$ be small. Then $g := \hat{h} + \rho z$ is positive in $s_{2}$ as well. Since $g(s_{0}) = 0$ and $g(s_{1}) > 0$, the minimum of $g|_{I}$ is nonpositive and occurs at a point $\hat{s} \in (s_{1}, s_{2})$. Let $f^{(n)}$ be an upper support function for $f$ in $\hat{s}$ such that formula (12) holds for $\zeta := \rho/2$. Then the function $g^{(n)} := Y \circ f^{(n)} - Y \circ f^{(0)} + \rho z$ has a local minimum in $\hat{s}$ with $g^{(n)}(\hat{s}) \leqslant 0$ too. With $t = f(\hat{s})$ and $u = f^{(0)}(\hat{s})$ we have $0 \leqslant t - u < \varepsilon$. 

Using inequality (13) and inequality (14) we compute for the function $h(n) = Y \circ f(n) - Y \circ f^0$:

\[
h(n)(\tilde{s}) - Ch^N(\tilde{s}) \leq y'(t) + \frac{1}{2} \rho - y'(u) + C(Y(u) - Y(t))
\]
\[
\leq \left[ y''(u) + \frac{m}{2} \right] (t - u) + C \left[ \frac{m}{2C} - y(u) \right] (t - u) + \frac{1}{2} \rho 
\]
\[
\leq [-k(u)y(u) + m - (1 - k(u))y(u)](t - u) + \frac{1}{2} \rho \leq \frac{1}{2} \rho.
\]

Taking into account that $z'' - Cz \leq -1$ we obtain

\[
g(n)(\tilde{s}) - Cg(n)(\tilde{s}) \leq \frac{1}{2} \rho + \rho \cdot (-1) < 0.
\]

Thus $g(n)(\tilde{s})$ is negative, which is a contradiction. Hence we have finished the proof for the distance comparison of Theorem 2.2. Now the proof of the angle comparison is simply an application of the first variation formula, if we use again an upper support function for $f$. The hinge version (Corollary 2.4) follows, since, under its assumptions, $|\tilde{\gamma}_2|$ depends monotonically on $\tilde{\alpha}_1$ by Lemma 3.8.

4.4. Proof of Theorem 2.8

We lower the angle $\tilde{\alpha}_1$ and define the family $\Delta^t$ by $\gamma_0^t := \gamma(\Delta^t)$. Then the function $(t, s) \mapsto f^t(s) = \text{dist}_{\gamma} \circ \gamma_0^t(s)$ is $C^1$-differentiable and satisfies

I. $(f^t)'(0) < (f^n)'(0)$ for $t < t_1$,
II. $f^t(0) = l_1$ for $t \in [0, 1]$,
III. $f^t(l_0) < f^1(l_0)$ for $t < 1$,
IV. $f^0 \leq f$.

Once more we choose the triangle $\Delta^b$ by setting $t_0 := \sup\{t \in [0, 1] \mid f^t \leq f\}$ and to prove the distance comparison, we have to show that the assumption $t_0 < 1$ is wrong, which is done in the following. We conclude $h = f - f^b > 0$ on $(0, l_0]$, since, according to the previous section, this function cannot have an inner minimum of nonpositive value. Now we choose $\varepsilon$ in the same way as in the previous section and $s_2 > 0$ such that $h|_{[0, s_2]} < \varepsilon$. We set $z(s) = 1 - e^{Cs}$. If $\rho > 0$ is small enough such that $g := \frac{1}{2} \rho + \rho z := Y \circ f - Y \circ f^b + \rho z$ is positive in $s_2$, we conclude, as in the previous section, $g|_{[0, s_2]} \geq 0$. Let $t_1 \in (t_0, 1]$ such that $(Y \circ f^t)'(0) < (Y \circ f^b - \rho z)'(0) - \frac{1}{2} \rho C$ holds for $t \in [t_0, t_1]$. Since $f^t(s)$ is $C^1$-differentiable, we find an interval $J = [0, s_1]$, $0 < s_1 < s_2$ with $Y \circ f^t|_J \leq (Y \circ f^b - \rho z)|_J$ for $t \in [t_0, t_1]$. Therefore, using the fact that $g$ is nonnegative on $J$, we conclude $f^t|_J \leq f|_J$ for $t \in [t_0, t_1]$. Eventually we find $t_2 \in (t_0, t_1]$ with $f^t \leq f$, which contradicts the choice of $t_0$.

5. Proof of Theorem 2.6 (Rigidity)

In this section we do not use condition (E) and hence we prove also the claim of Theorems 2.8 concerning the rigidity.

First of all we show that equation (3) or equation (4) implies equation (5), which can also be written as $f \equiv \tilde{f} := \text{dist}_{\gamma_0} \circ \gamma_0$. Let us first assume equation (3) holds, e.g., $\alpha_1 = \tilde{\alpha}_1$. Suppose $s_1 := \sup\{s \mid (f - \tilde{f})|_{[0, s]} = 0\} < l_0$. Let $\gamma_3$ be a minimizing geodesic from $\gamma_0(s_1)$ to $\rho_0$ and $b_1 := \angle(\gamma_3(0), -\gamma_0(s_1)), b_2 := \angle(\gamma_3(0), \gamma_0(s_1))$. If $s_1 > 0$, we have two triangles satisfying
the distance comparison. Therefore the angle comparison $\tilde{\beta}_1 \leq \beta_1$ holds as well. We conclude $\tilde{\beta}_2 = \beta_2$, which is, by assumption, also true, if $s_1 = 0$. Now we choose $\delta \in (0, l_0 - s_1)$ such that

1) $f(s_1 + \delta) < f(s_1 + \delta)$.

2) $\delta \leq f(s_1) + f(s_1 + \delta)$.

3) Lemma 3.8 is valid, if we replace $l_0$ by $\delta$ and $l_1$ by $f(s_1)$.

Then the function $G(\beta) = \text{dist}_{\gamma_0}(\gamma^\beta(\delta))$ is strictly increasing, where $\gamma^\beta$ is the geodesic starting in $\gamma_0(s_1)$ and making the angle $\beta$ with the meridian. According to Theorem 2.8 the angle comparison holds for the triangle $\triangle_1 = (\gamma_0([s_1, s_1 + \delta], \gamma_3, \gamma)$, if $\gamma$ is any minimizing geodesic from $\gamma_0(s_1 + \delta)$ to $p_0$. Therefore $\tilde{\beta}_2 \leq \beta_2 = \beta_2$, where $\beta_2$ denotes the angle in the comparison triangle for $\triangle_1$ corresponding to $\beta_2$. On the other hand 1) and the fact that $G$ is monotonous imply $\beta_2 > \beta_2$.

Hence $s_1 = l_0$, which means $f \equiv \tilde{f}$.

Now suppose equation (4) holds. Then, once more, we have two triangles satisfying the angle comparison and, as above, we conclude $f \equiv \tilde{f}$, since equality holds in the angle comparison for these two triangles.

Remarks 5.1. (a) An example in Section 5.3 shows that equation (5) does not imply equation (3).

(b) Any minimizing geodesic from $\gamma_0((0, l_0))$ to $p_0$ makes the same angles with $\gamma_0$ as the corresponding geodesic in the model space with $\tilde{\gamma}_0$.

(c) In the following we will assume that the angles $\beta_i$ discussed above are contained in $(0, \pi)$, since otherwise, it does not make sense to construct a ruled surface.

Proposition 5.2. Suppose $f \equiv \tilde{f}$ and let $\gamma : [0, 1] \to M$ be a minimizing geodesic from $p_0$ to $p = \gamma_0(s), s \in (0, l_0)$. Let $\sigma$ be the plane spanned by $\gamma(l)$ and $\gamma_0(l)$. Then $K(P_t, \sigma) = k(t)$ for $t \in [0, l]$, where $P_t$ is the parallel transport along $\gamma$ from $T_pM$ to $T_{\gamma(t)}M$.

Proof. Once more we use the upper support functions $f^{(n)} = F^{(n)} \circ \gamma_0, \eta > 0$. The function $Y \circ f^{(n)} - Y \circ \tilde{f}$ has a minimum in $s$ and hence

$$0 \leq (Y \circ f^{(n)} - Y \circ \tilde{f})'(s) = \langle \text{Hess}(Y \circ F^{(n)})_{\gamma_0(s)}, \gamma_0(s) \rangle - y'(l).$$

Let $v$ be a normal vector in $\sigma$ orthogonal to $\gamma(l)$. Then we can write $\gamma_0(s) = av + b\gamma(l)$, where $a \neq 0$. We extend $v$ to the parallel vector field $X$ and have to show $(R(X, \gamma_0) \gamma_0, X)|_{t} = k(t)$ for $t \in (0, l)$. Suppose this equation is not true. Then we can assume $K(X, \gamma_0) \geq k + \epsilon, \epsilon > 0$ on an interval $[t_1, t_2]$ with $0 < t_1 < t_2 < l$. If $\eta$ is small enough (e.g. $\eta < t_1/2$), we obtain, as in Lemma 4.1, a constant $\delta$ independent of $\eta$ such that $I(J, \gamma_0) - I(\Phi J_0, \Phi J) \geq \delta > 0$. This implies $(S^2_{\gamma_0} v, v) \leq (y'(l)/y^2(l)) - \delta$ and we compute $\langle \text{Hess}(Y \circ f^{(n)})_{\gamma_0(s)}, \gamma_0(s) \rangle = a^2 y(l)(S^2_{\gamma_0} v, v) + b^2 y'(l) \leq -a^2 y(l)\delta + [a^2 y(l)](y'(l)/y^2(l) + b^2 y'(l))$. The right hand side is less than $y'(l)$, provided $\eta$ is small. Hence this formula is a contradiction to inequality (15).

Remarks 5.3. (a) By inequality (2) $R_{\gamma(t)} \gamma_0(t) \gamma_0(t) \geq k(t)$. Hence $R_{\gamma} X = k \cdot X$ and $J(t) = y(t)/y(l) \cdot X(t)$ is a Jacobi field along $\gamma$ with $J(0) = 0$ and $J(l) = v$. Therefore, if we can construct a ruled surface consisting of minimizing geodesics, the corresponding Jacobi fields do not rotate.

(b) If there is a totally geodesically embedded ruled surface, Proposition 5.2 immediately leads to equation (6).
5.1. Construction of thin ruled surfaces

**Proposition 5.4.** There is a number \( \delta > 0 \) such that the following holds: If \( \gamma \) is a minimizing geodesic from \( \gamma_0(s_0) \) to \( p_0 \) and the angle between \( \gamma_0 \) and \( \gamma \) equals the corresponding angle in the model space, then there is a ruled surface consisting of minimizing geodesics from \( p_0 \) to \( \gamma_0([s_0, \min\{s_0 + \delta, l_0\}]) \). This surface can be constructed starting with \( \gamma \) and its interior is totally geodesically embedded.

The proof is divided into several parts (Section 5.1.1-5.1.3).

5.1.1. Choosing constants

I. \( R_0 := \min(\text{dist}_{p_0} \circ \gamma_0) \in (0, L), \ R_1 := \max(\text{dist}_{p_0} \circ \gamma_0) \in [R_0, L) \).

II. \( R := \begin{cases} L & \text{for } L < \infty, \\ R_1 + 1 & \text{for } L = \infty. \end{cases} \)

III. We assume \( \delta < \delta_1 := \min\left\{ \frac{1}{2}\left( R - R_1 \right), \frac{1}{2} R_0 \right\} \) and the following construction will be done in the compact balls \( B := B_{R_1 + 2\delta_1}(p_0) \) and \( \tilde{B} := B_{R_1 + 2\delta_1}(p_0) \).

IV. \( \kappa := \min\{\min K|_B, \min K|_{\tilde{B}}\} \).

V. \( \lambda := \max\{\max K|_B, \max K|_{\tilde{B}}\} \).

VI. We assume that \( \delta \) is sufficiently small. Then \( \frac{1}{2} \leq c_{s\kappa}(s) \leq c_{s\kappa}(s) \leq 2 \) for \( s \in [0, 2\delta] \), where \( c_{s\kappa} \) is the generalized Cosine Function in \( M_\kappa \). Suppose \( Y \) is a normal Jacobi field with \( Y'(0) = 0 \) along a geodesic contained in \( B \) or \( \tilde{B} \) of length less than \( 2\delta \). Then

\[
\frac{1}{2}\|Y(0)\| \leq \|Y\| \leq 2\|Y(0)\|. \tag{16}
\]

In particular this geodesic has neither focal nor conjugate points.

VII. Since \( k_{(0, L)} \) is locally Lipschitz continuous, there is a global Lipschitz constant \( A \) for \( k|_{[R_0/2 - 2\delta_1, R_1 + 2\delta_1]} \), i.e.,

\[
k(t + t') \leq k(t) + t'A \tag{17}
\]

for \( t \geq R_0/2 - 2\delta_1, t' \geq 0 \) with \( t + t' \leq R_1 + 2\delta_1 \).

VIII. Let \( C := 48\delta Re^2 \), where \( e > 1 \) is an arbitrary constant and

\[
\delta \leq \sqrt{\frac{1}{8Ce^2}} \leq \sqrt{\frac{1}{16\delta Re^2}}
\]

IX. Finally let \( \delta < \text{inj } |_{\gamma_0} \).

The claim of Proposition 5.4 is true for any \( \delta \) satisfying III, VI, VIII and IX. In the following proof we can assume that the angle \( \beta \) between \( \gamma_0 \) and \( \gamma \) equals the corresponding angle in the model space and \( \delta \in (0, l_0 - s_0] \).

5.1.2. Construction of the surface by an iterative process

Let \( c_0(s) := \gamma_0(s_0 + s) \) and \( c_t \) be the geodesic with \( c_t(0) = \gamma(t) \) and initial vector \( P_t c_0(0) \). In the model space let \( \tilde{c}_i \) denote the corresponding geodesic. The map \( \tilde{V}(s, t) := \tilde{c}_i(s) \) is of maximal
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By VI the geodesics $c_t$ and $\tilde{c}_t$ have no conjugate points on $[0, h(\delta, t)]$. By III $|c_t|_{[0, \tau]} \leq 2\delta < L - \text{dist}_{p_0}(c_t(0))$ for $s \in [0, h(\delta, t)]$. Hence the hinge version (corresponding to Theorem 2.8) holds for the hinges $(c_t|_{[0, \tau]}, \gamma^t|_{[\tau, h(\delta, t)]}, \beta)$, $t \in [0, t_0]$, $s \in [0, h(\delta, t)]$ and the function $G : \tilde{Q} := \{(s, t) \mid t \in [0, t_0], 0 \leq s \leq h(\delta, t)\} \rightarrow [0, R]$, $(s, t) \mapsto \text{dist}_{p_0} \circ \tilde{c}_t(s) - \text{dist}_{p_0} \circ c_t(s)$ is nonnegative. In particular

$$|\gamma^t_2| \leq |\tilde{\gamma}^t_2|.$$  

Let $J^t(s) = (\partial/\partial t)c_t(s)$. We can write $J^t(s) = Y^t(s) + \cos \beta \cdot \dot{c}_t(s)$, where $Y^t$ is a normal Jacobi field along $c_t$ with $(Y^t)'(0) = 0$ and $\|Y^t(0)\| = \sin \beta$. We compute

$$\gamma^t_2(t) = \tau \mapsto c_t(h(s, \tau))|_{\tau} = Y^t(h(s, t)) + \left[ \cos \beta + \frac{\partial h}{\partial t}(s, t) \right] \dot{c}_t(h(s, t)).$$

The corresponding formula holds for $\tilde{\gamma}^t_2(t)$. Since $R \geq |\gamma^t_2| \geq \int_0^t |\cos \beta + \partial h(s, t)/\partial t| \, dt$, we obtain the following rough estimate for the length of $\gamma^t_2$ (using formula (16)):

$$|\gamma^t_2| \leq \int_0^t \|Y^t(h(s, t))\| + \left| \cos \beta + \frac{\partial h}{\partial t}(s, t) \right| \, dt \leq 3R.$$  

By uniqueness $h$ is continuous and by the Implicit Function Theorem $h$ is differentiable. Moreover, $\partial h/\partial s$ is uniformly continuous and hence we can assume

$$s/e \leq h(s, t) \leq e \cdot s, \quad \text{for } (s, t) \in Q := [0, \delta] \times [0, t_0],$$  

provided $t_0$ is small enough. For $s \in [0, \delta]$ we set $\gamma^t_1(t) := c_t(h(s, t))$, $t \in [0, t_0]$. $\tilde{\gamma}^t_1$, the corresponding curve in the model space, is a length invariant reparametrization of the meridian, since $\tilde{V}$ is injective. Let $\gamma^t_2$ be a minimizing geodesic from $\gamma^t_1(t_0)$ to $p_0$ and $\tilde{\gamma}^t_2$ be the corresponding meridian in the model space, cf. Figure 3.

![Figure 3](image-url)
We will use the following modification of Rauch's second comparison theorem.

**Lemma 5.5.** Let $M, \tilde{M}$ be Riemannian manifolds with $\dim(\tilde{M}) \leq \dim(M)$ and $c, \tilde{c}$ geodesics in $M$ resp. $\tilde{M}$ each of them having no focal point in $(0, b]$. Suppose $K_{\tilde{c}(t)} \leq K_{c(t)} + g(t)$ for all planes, where $g$ is a continuous function. Let $Y, \tilde{Y}$ be normal Jacobi fields along $c$ resp. $\tilde{c}$ with $\|Y(0)\| = \|\tilde{Y}(0)\| > 0$, $Y'(0) = \tilde{Y}'(0) = 0$. Then for $t \in [0, b]$\[
\|\tilde{Y}(t)\|^2 \geq \|Y(t)\|^2 - 2\|Y(t)\|^2 \int_{0}^{t} \frac{1}{\|Y(u)\|^2} \int_{0}^{t} g(\tau) \|\tilde{Y}(\tau)\|^2 d\tau du.
\]
This lemma is proved similar to the classical theorem. A detailed proof can be found in [8].

**Lemma 5.6 (Iterative Improvement).** Let $G \leq \varepsilon \leq R$. Then \[
\Delta L(s) := |\gamma^s_t| - |\tilde{\gamma}^s_t| \leq \varepsilon Cs^2 \quad \text{for} \quad s \in [0, \delta], \quad G(s, t) \leq \varepsilon Ce^2 s^2 \quad \text{for} \quad (s, t) \in \tilde{Q}.
\]

**Proof.** Using inequality (17) we obtain \[
K \circ \tilde{c}(s) = k(\text{dist}_{p_0} \circ \tilde{c}(s)) \leq k(\text{dist}_{p_0} \circ c(s)) + A \cdot G(s, t) \leq K \circ c_t(s) + \varepsilon A.
\]

Lemma 5.5 and inequality (16) yield
\[
\|\tilde{\gamma}^s_t\|^2 \geq \|Y^t(s)\|^2 - 2\|Y^t(s)\|^2 \int_{0}^{t} \frac{1}{\|Y^t(u)\|^2} \int_{0}^{t} \varepsilon A \|\tilde{Y}(\tau)\|^2 d\tau du \geq \|Y^t(s)\|^2 - 16\varepsilon As^2\|Y^t(s)\|^2 \quad \text{for} \quad s \in [0, h(\delta, t)].
\]
Therefore, by inequality (18),
\[
\|\tilde{\gamma}^s_t\|^2 \geq (1 - 16\varepsilon A h(s, t)^2) \left(\|Y^t(h(s, t))\|^2 + \left[\cos \beta + \frac{\partial h}{\partial t}(s, t)\right]^2\right) \geq (1 - 16\varepsilon Ae^2 s^2)\|\tilde{\gamma}^s_t\|^2 \quad \text{for} \quad s \in [0, \delta], \quad t \in [0, t_0].
\]

Using VIII we conclude
\[
\|\tilde{\gamma}^s_t\| \geq \sqrt{1 - 16\varepsilon Ae^2 s^2}\|\tilde{\gamma}^s_t\| \geq (1 - 16\varepsilon Ae^2 s^2)\|\tilde{\gamma}^s_t\|. \quad (21)
\]
Combining this with inequality (20) we obtain the first claim of the lemma.

For $(s, t) \in \tilde{Q}$ there is $\delta \in [0, \delta]$ with $s = h(\delta, t)$. Then $G(s, t) \leq |\gamma^s_t|_{[0, \delta]} - |\tilde{\gamma}^s_t|_{[0, \delta]}$, since equality holds in the distance comparison. Combining this with inequality (21) and inequality (18) we obtain $G(s, t) \leq \int_{0}^{\delta} \|\tilde{\gamma}^s_t(\tau)\|^2 d\tau \leq 16\varepsilon Ae^2 s^2 |\gamma^s_t|_{[0, \delta]} \leq \varepsilon Cs^2 \leq \varepsilon Ce^2 s^2$. \quad \square

Now we apply Lemma 5.6 iteratively. Beginning with $\varepsilon = R$ and using VIII we obtain $\Delta L(s) \leq R \cdot Cs^2$ for $s \in [0, \delta]$ and $G(s, t) \leq R \cdot Ce^2 s^2 \leq \frac{1}{2} R$ for $(s, t) \in \tilde{Q}$. Then we apply Lemma 5.6 with $\varepsilon = \frac{1}{2} R \leq R$ and so on. The $m$th step yields\[
G(s, t) \leq \frac{R}{2^m} \quad \text{for} \quad (s, t) \in \tilde{Q} \quad \text{and} \quad \Delta L(s) \leq \frac{R}{2^m} \quad \text{for} \quad s \in [0, \delta].
\]
Therefore $G$ vanishes and $|\gamma^s_t| \leq |\tilde{\gamma}^s_t|$. Hence, by inequality (19), $\gamma^s_t \cup \tilde{\gamma}^s_t$ is a minimizing geodesic from $\gamma_0(s + s_0)$ to $p_0$. Thus $c_{s_0}$ does not meet the cut locus of $p_0$ and can be lifted by $\exp_{p_0}$ to
$T_{p_0}M^n$. Let $\hat{c}$ denotes this lift. Then we can define a ruled surface $F : \tilde{Q} := \{(s, t) \mid 0 \leq s \leq \delta, 0 \leq t \leq f(s + s_0)\} \rightarrow M$,

$$(s, t) \mapsto \exp_{p_0} \left[ t \frac{\hat{c}(h(s, t_0))}{\|\hat{c}(h(s, t_0))\|} \right].$$

This immersion is injective for $t > 0$ and thus an embedding:

Suppose $F(s, t) = F(s', t)$, $t > 0$ and, e.g., $s < s'$. Then $f(s) \neq f(s')$, since $\gamma|_{[0\delta\delta]}$ is injective by IX. Now $F(s, \cdot)$ is a minimizing geodesic from $\gamma_0(s)$ to $\gamma_0(s')$ different from $\gamma_0$, which is a contradiction to IX.

**Remark 5.7.** If $k$ is decreasing, e.g. in the classical case or in [4], Rauch's second comparison theorem immediately yields $G = \Delta L = 0$. In this case the iterative process is not necessary and $k$ does not have to be locally Lipschitz continuous.

5.1.3. The thin surfaces are totally geodesic

We merely give an outline of the proof. The details can be found in [8]. $W$, the interior of the image of $F$, is given in normal coordinates. Since the regularity properties of normal coordinates are not optimal, a priori, we do not have a differentiable frame for the normal bundle. But we will show that the field of orthogonal projectors onto the normal bundle is differentiable. For any point in $W$ one can construct a family of geodesics contained in $W$ and transversal to the geodesics $F(s, \cdot)$. This family is constructed like the geodesics $c_t$ in the previous section. A standard computation shows that the field of projectors onto the tangent bundle along the map $F$ is differentiable in the direction $\partial/\partial t$. Since we have a transversal family of geodesics, the same computation shows that $P$ is differentiable in a direction linearly independent to $\partial/\partial t$ too. Hence $P$ is differentiable in a neighbourhood of $p$ and so is the field $\text{Id} - P$ of normal projectors. Thus we obtain, locally, a differentiable frame for the normal bundle of $W$. In order to show that $W$ is totally geodesic, it is sufficient to show that an arbitrary second fundamental form $l$ for $W$ vanishes. A standard computation shows $l(F, F) = 0$. $J(s, \cdot)$ is a normal Jacobi field along $F(s, \cdot)$, which does not rotate, cf. Remark 5.3a. Therefore, another computation yields $l(F, J) = 0$. Since we have a transversal family of geodesics through any point $p \in W$, we have $l(v, v) = 0$ for a vector $v$ linearly independent to $F$. Thus $l$ vanishes in $p$, which completes the proof of Proposition 5.4.

**Remark 5.8.** We only require that $M^n$ is $C^2$-differentiable. If $M^n$ is even $C^3$-differentiable, $k$ is $C^1$-differentiable and thus locally Lipschitz continuous. In this case one can differentiate the Jacobi equation of the map $F$ to prove that the ruled surface is totally geodesic.

5.2. Construction of the global surface

In order to finish the proof of Theorem 2.6, we have to show that the thin ruled surfaces fit together to a global ruled surface. Let $F : \hat{Q} := \{(s, t) \mid 0 \leq s \leq \delta, 0 \leq t \leq f(s)\} \rightarrow M$ be a surface, constructed as above, from $p_0$ to $\gamma_0([0, \delta])$ and $\hat{F} : \hat{\hat{Q}} := \{(s, t) \mid \delta/2 \leq s \leq 3\delta/2, 0 \leq t \leq f(s)\} \rightarrow M$ be the surface, constructed starting with the geodesic $\hat{\gamma} = F(\delta/2, \cdot)^{-}$, from $p_0$
to $\gamma_0([\delta/2, 3\delta/2])$. It is sufficient to show that these two surfaces fit together, i.e.

$$F(s, t) = \hat{F}(s, t) \quad \text{for} \ s \in \left[\frac{1}{2}\delta, \delta\right]. \quad (22)$$

Let $t_0$, the geodesics $c_t$ and the function $h$ be defined as in Section 5.1.2 and let $W$ be the interior of the image of $F$. For the ruled surface $\hat{F}$ we use the corresponding notations but for convenience we write $c$ instead of $\hat{c}_{t_0}$. According to Remark 5.3a the Jacobi field $F_*(\partial/\partial s)_{|\delta/2, .}$ does not rotate. $\hat{c}(0)$ is obtained from $\gamma_0(\delta/2)$ by parallel transport along $\hat{y}$. Therefore $\hat{c}(0) \in TW$. Now let $\delta := \hat{h}(\delta/2, \hat{t}_0)$ be the parameter of $c$ corresponding to the parameter $\delta$ of $\gamma_0$ and $A := \{s \in [0, \delta] : c(s) \in F(Q)\}$. This set is closed and nonempty. Suppose $[0, s] \subset A$ and $s = \hat{h}(\delta - \delta/2, \hat{t}_0) < b$. Then $\hat{s} < \delta$ and hence $\hat{c}(s) \in TW$. Since $W$ is totally geodesic, we have $[0, s + \varepsilon] \subset A$ for some positive $\varepsilon$. Therefore $A = [0, b]$ and equation (22) follows from Lemma 5.9.

**Lemma 5.9.** Let $\delta \in [\delta/2, \delta]$ and $s$ be the parameter of $c$ corresponding to $\delta$, i.e., $s = \hat{h}(\delta - \delta/2, \hat{t}_0)$. If $s \in A$, then $F(\delta, t) = \hat{F}(\delta, t)$ for $0 \leq t \leq f(\delta)$. 

**Proof.** By assumption $c(s) \in F(Q)$, say $c(s) = F(s_1, t_1)$. Then $t \mapsto F(s_1, t), \ 0 \leq t \leq t_1$ is the unique minimizing geodesic from $p_0$ to $c(s)$. Therefore by construction

$$F(s_1, t) = \hat{F}(\delta, t) \quad \text{for} \ 0 \leq t \leq \min\{f(s_1), f(\delta)\}. \quad (23)$$

By $\gamma_0|_{[\delta, 1]}$ resp. $\gamma_0|_{[\delta, s]}$ is injective, because $|\delta_1 - \hat{s}| < \delta$. Therefore $\gamma_0(s_1) = \gamma_0(\delta)$, since, otherwise, the geodesic $F(s_1, \cdot)$ would be a minimizing connection of these points different from $\gamma_0$. We conclude $s_1 = \delta$ and equation (23) is the claim of the lemma.

It remains to show that the global ruled surface $F : Q := \{(s, t) \mid 0 < s < \delta, 0 < t < f(s)\} \to M$ is embedded. Suppose $F$ is not injective, e.g., $F(s, t) = F(s', t)$, where $s \neq s'$ and $t \in (0, f(s))$. Again we conclude $F(s, \tau) = F(s', \tau)$ for $0 \leq \tau \leq \min\{f(s), f(s')\}$. $\tilde{F} : Q \to M(k)$, the corresponding map into the model space, is an embedding. Let $\sigma^*$ be a minimizing geodesic from $\tilde{F}(s, \tau)$ to $\tilde{F}(s', \tau)$. Then $\tilde{F} \circ \tilde{F}^{-1} \circ \sigma^*$ is a nonconstant geodesic from $F(s, \tau)$ to $F(s', \tau)$, because the ruled surface is totally geodesic and $F \circ \tilde{F}^{-1}$ is a local isometry. We obtain arbitrary short nonconstant and noninjective geodesics near $p_0$, which is a contradiction. Therefore $F$ is an embedding and the proof of Theorem 2.6 is complete.

**Remarks 5.10.** (a) If $\alpha_1 = \hat{\alpha}_1$, we can construct the surface starting with $\gamma_1$. But in this case it possibly does not contain $\gamma_2$.

(b) In general, the ruled surface must be constructed starting with an inner geodesic.

(c) An example in Section 5.3 shows that the geodesic $\gamma_0$ does not have to be injective. Hence only the interior of the ruled surface is embedded.

(d) The argument above shows that $F$ is injective (for $t > 0$), if $\gamma_0$ is injective.

**5.3. Examples in the lens space $L(4, 1)$**

The lens space $L(4, 1)$ is defined as $S^3/\mathbb{Z}_4$, where $\mathbb{Z}_4 = \{i^k \mid k \in \mathbb{Z}\}$ operates isometrically on $S^3 \subset \mathbb{C}^2$ by $(i^k, (z_1, z_2)) \mapsto (i^kz_1, i^kz_2)$. 

Let $N$ denote the north pole $(1, 0, 0, 0) \in S^3$ and $[N] \in L(4, 1)$ its projection. The distance to the cut locus, $s : T_{[N]}L(4, 1) \to \mathbb{R}$, is given by
\[ s(v) = s(0, v_2, v_3, v_4) = \arccot(|v_2|). \tag{24} \]
We write $c_v(t) := \exp(tv)$ for $v \in T_{[N]}L(4, 1)$. For the first example we set $p_0 := [N]$. $\exp(C_{[N]} \cap \{v_4 = 0\})$ consists of the projection of two great circles, that intersect orthogonally in $(0, 0, 1, 0)$, cf. Figure 4. For any $\alpha \in (0, \pi/2)$ let $v_1 := (0, \sin \alpha, \cos \alpha, 0)$, $v_2 := (0, -\sin \alpha, \cos \alpha, 0) \in T_{p_0}L(4, 1)$. Then $s_1 := s(v_1) = s(v_2) \in (\pi/4, \pi/2)$. We set $\gamma_i := (c_{v_i}|_{[0,s_1]})^{-1}$ for $i = 1, 2$ and $v := (0, -\sin \alpha, 0, -\cos \alpha)$. $c_v$ is a closed geodesic of period $\pi$ with $c_v(s_1) = \gamma_1(0) = p_1$ and $c_v(\pi - s_1) = \gamma_2(0) = p_2$. The triangle $(\gamma_0, \gamma_1, \gamma_2)$, where $\gamma_0(t) = c_v(s_1 + t)$, $0 \leq t \leq \pi - 2s_1$, satisfies inequality (U) and the assumptions of Theorem 2.6. Since $\gamma_0|_{(0, \frac{\pi}{2})}$ does not meet the cut locus of $p_0$, the ruled surface is unique. But $\gamma_1 := (c_{v_1}|_{[0,s_1]})^{-1}$ is another minimizing geodesic from $p_1$ to $p_0$ and $\gamma_2 := c_{v_1}|_{[\pi - s_1, \pi]}$ is one from $p_2$ to $p_0$. The triangle $(\gamma_0, \gamma_1, \gamma_2)$ satisfies equation (5), but we have the strict inequality in the angle comparison for both angles. Hence equation (3) does not follow from equation (5). Neither $\gamma_1$ nor $\gamma_2$ is contained in the ruled surface.

For a second example we set $p_1 := [N]$ and $p_0 := [(0, 0, 1, 0)]$. According to formula (24) $\gamma_1 := c_{(0, 0, 1, 0)}|_{[0, \pi/2]}$ is a minimizing geodesic from $p_1$ to $p_0$. $\gamma_0 := c_{(0, 1, 0, 0)}$ is a closed geodesic of period $\pi/2$. For the distance to the cut locus of $p_0$ we obtain, similar to formula (24), $s(v_1, v_2, 0, v_4) = \arccot(|v_2|)$. Thus $\text{dist}_{p_0} \circ \gamma_0 = \frac{1}{2}\pi$. If $l_0 < \pi$, the triangle $(\gamma_0, \gamma_1, \gamma_2)$, where $\gamma_2$ is a minimizing geodesic from $p_2 = \gamma_0(l_0)$ to $p_0$, satisfies inequality (U) and the assumptions of Theorem 2.6. But $\gamma_0$ is not injective, provided $l_0 \geq \pi/2$, cf. Figure 5.
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References