Eigenvalues of the fractional Laplace operator
in the interval

Mateusz Kwaśnicki a,b,*

a Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-976 Warszawa, Poland
b Institute of Mathematics and Computer Science, Wrocław University of Technology, ul. Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

Received 6 December 2010; accepted 1 December 2011
Available online 12 December 2011
Communicated by Daniel W. Stroock

Abstract

Two-term Weyl-type asymptotic law for the eigenvalues of the one-dimensional fractional Laplace operator $(-\Delta)^{\alpha/2}$ ($\alpha \in (0, 2)$) in the interval $(-1, 1)$ is given: the $n$-th eigenvalue is equal to $(n\pi/2 - (2 - \alpha)\pi/8) + O(1/n)$. Simplicity of eigenvalues is proved for $\alpha \in [1, 2)$. $L^2$ and $L^\infty$ properties of eigenfunctions are studied. We also give precise numerical bounds for the first few eigenvalues.

© 2011 Elsevier Inc. All rights reserved.

Keywords: Fractional Laplacian; Stable process; Eigenvalues; Interval

1. Introduction and statement of the result

Let $D = (-1, 1)$ and $\alpha \in (0, 2)$. Below we study the asymptotic behavior of the eigenvalues of the spectral problem for the one-dimensional fractional Laplace operator in the interval $D$:

$$\left(-\frac{d^2}{dx^2}\right)^{\alpha/2} \varphi(x) = \lambda \varphi(x), \quad x \in D,$$

© Work supported by the Polish Ministry of Science and Higher Education grant No. N N201 373136 and by Foundation for Polish Science.
* Correspondence to: Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-976 Warszawa, Poland.
E-mail address: m.kwasnicki@impan.pl.

0022-1236/S – see front matter © 2011 Elsevier Inc. All rights reserved.
where \( \varphi \in L^2(D) \) is extended to \( \mathbb{R} \) by 0 (for details, see below). It is known that there exists an infinite sequence of eigenvalues \( \lambda_n, 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \), and the corresponding eigenfunctions \( \varphi_n \) form a complete orthonormal set in \( L^2(D) \). The following is the main result of this article.

**Theorem 1.** We have

\[
\lambda_n = \left( \frac{n\pi}{2} - \frac{(2 - \alpha)\pi}{8} \right)^\alpha + O \left( \frac{1}{n} \right). \tag{2}
\]

More precisely, there are absolute constants \( C, C' \) such that

\[
\left| \lambda_n - \left( \frac{n\pi}{2} - \frac{(2 - \alpha)\pi}{8} \right)^\alpha \right| \leq C \frac{(2 - \alpha)^\frac{1}{\alpha}}{\sqrt{\alpha}} \frac{1}{n}
\]

for \( n \geq (C'/\alpha)^{3/(2\alpha)} \).

The scaling property of the fractional Laplace operator \((-d^2/dx^2)^{\alpha/2}\) and its translation invariance imply that for a similar spectral problem in \( D' = (a, b) \), the corresponding eigenvalues \( \lambda'_n \) satisfy \( \lambda'_n = ((b - a)/2)^{-\alpha} \lambda_n(D) \). Hence, one easily finds the asymptotic formula for \( \lambda'_n \).

By following carefully the proof, one can take e.g. \( C = 30000 \) and \( C' = 4000 \) in Theorem 1. Note that the constant in the error term \( O(1/n) \) in (2) tends to zero as \( \alpha \) approaches 2, and in the limiting case \( \alpha = 2 \) (not considered below), we have \( \lambda_n = (n\pi/2)^2 \) without an error term. Theorem 1 for \( \alpha = 1 \) (with better numerical constants) was proved in [13].

The proof of Theorem 1 is modeled after [13], and the idea can be sketched as follows. In [14], an explicit formula for the solution of the spectral problem similar to (1) in half-line \((0, \infty)\) was given: for all \( \lambda > 0 \) there is an eigenfunction \( F_\lambda(x) \) such that \((-d^2/dx^2)^{\alpha/2} F_\lambda(x) = \lambda^\alpha F_\lambda(x) \) for \( x \in (0, \infty) \), and \( F_\lambda(x) = 0 \) for \( x \leq 0 \). Furthermore, \( F_\lambda(x) \approx \sin(\lambda x + (2-\alpha)\pi\alpha/8) \) when \( \lambda x \) is large enough. The fractional Laplace operator \((-d^2/dx^2)^{\alpha/2}\) is a non-local operator, so the eigenfunctions in half-line are not restrictions of eigenfunctions in the entire real line. Nevertheless, when \( \lambda \) is large enough and \( x \) is not too close to 0, then \( F_\lambda(x) \) behaves nearly as \( \sin(\lambda x + (2-\alpha)\pi\alpha/8) \), which is an eigenfunction of \((-d^2/dx^2)^{\alpha/2}\) in \( \mathbb{R} \). One may expect a similar approximate localization phenomenon for the solutions of the spectral problem (1) in the interval \( D \): locally near \(-1\) and \(1\), the eigenfunctions \( \varphi_n(x) \) on the interval \( D \) are expected to be close to the eigenfunctions in half-lines \((-1, \infty)\) and \((-\infty, 1)\) respectively. In other words, for \( n \) large enough, and with \( \mu_n \approx \lambda_n^{1/\alpha} \), we expect that

\[
\varphi_n(x) \approx \begin{cases} 
C_1 F_{\mu_n}(1 + x) & \text{for } x \text{ close to } -1, \\
C_2 F_{\mu_n}(1 + x) & \text{for } x \text{ close to } 1, \\
C_3 \sin(\mu_n x + \theta_n) & \text{for } x \in D \text{ away from the boundary},
\end{cases}
\]

for some constants \( C_1, C_2, C_3, \theta_n \).

The above observation is exploited as follows. We define the function \( \tilde{\varphi}_n(x) \) to be equal to \( F_{\mu_n}(1 + x) \) for \( x \) close to \(-1\), \( \tilde{\varphi}_n(x) = \pm F_{\mu_n}(1 - x) \) for \( x \) close to \(1\) (the sign depends on \(n\)), and so that \( \tilde{\varphi}_n(x) \) is approximately equal to \( \pm \cos(\mu_n x) \) or \( \pm \sin(\mu_n x) \) (again, depending on \(n\)) for \( x \in D \) away from the boundary. Such a construction is possible when \( \mu_n = \frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8} \).
Table 1

Comparison of the approximation \(\tilde{\lambda}_n = \left(\frac{n\pi}{\alpha} - \frac{(2-\alpha)n}{8}\right)^\alpha\) (roman font), and numerical approximations to \(\lambda_n\) obtained using the method of [15] with 5000 \(\times\) 5000 matrices (slanted font).

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>(\lambda_1)</th>
<th>(\lambda_2)</th>
<th>(\lambda_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.998</td>
<td>0.997</td>
<td>1.009</td>
</tr>
<tr>
<td>0.1</td>
<td>0.981</td>
<td>0.973</td>
<td>1.091</td>
</tr>
<tr>
<td>0.2</td>
<td>0.971</td>
<td>0.957</td>
<td>1.195</td>
</tr>
<tr>
<td>0.5</td>
<td>0.991</td>
<td>0.970</td>
<td>1.598</td>
</tr>
<tr>
<td>1</td>
<td>1.178</td>
<td>1.158</td>
<td>2.749</td>
</tr>
<tr>
<td>1.5</td>
<td>1.611</td>
<td>1.597</td>
<td>5.055</td>
</tr>
<tr>
<td>1.8</td>
<td>2.056</td>
<td>2.048</td>
<td>7.500</td>
</tr>
<tr>
<td>1.9</td>
<td>2.248</td>
<td>2.243</td>
<td>8.594</td>
</tr>
</tbody>
</table>

Then we are able to prove that \(A\tilde{\varphi}_n(x) \approx \mu_n^{\alpha} \tilde{\varphi}_n(x)\) for all \(x \in D\). This means that \(\tilde{\varphi}_n\) is an approximate eigenfunction. Using \(L^2(D)\) decomposition of \(\tilde{\varphi}_n\) in the orthonormal basis of (true) eigenfunctions \(\varphi_k\), we can show that \(\mu_n^{\alpha}\) must be close to some eigenvalue \(\lambda_k\). This proves that there is an infinite sequence of eigenvalues satisfying (2). It remains to prove that there are no other eigenfunctions. This is achieved using a trace estimate for the semigroup generated by \((-d^2/dx^2)^{\alpha/2}\) on \(D\) (with zero exterior condition).

The paper is organized as follows. First we briefly recall the history of the problem (1) and state it a more formal way. In Section 2, an auxiliary estimate for the fractional Laplace operator is given. The formula from [14] for the eigenfunctions \(F_{\lambda}(x)\) on the half-line is recalled in Section 3. An approximation \(\tilde{\varphi}_n\) to eigenfunctions is given in Section 4, and Theorem 1 is proved in Section 5. Further properties of eigenfunctions and eigenvalues are studied in Section 6; Sections 4–6 correspond to Sections 8–10 in [13]. Proposition 4 in Section 6 gives simplicity of the eigenvalues when \(\alpha \in [1, 2)\). This result follows relatively easily from the result for \(\alpha = 1\) in [13], and monotonicity in \(\alpha\) properties from [9]. In Propositions 1 and 2, also in Section 6, \(L^2(D)\) and \(L^\infty(D)\) bounds for the eigenfunctions are given. Finally, in Section 7, numerical estimates of \(\lambda_n\) in terms of eigenvalues of large dense matrices are obtained.

The spectral problem studied in this article has long history. First-term Weyl-type asymptotic law for \(\lambda_n\) was proved (in a much more general context) by Blumenthal and Getoor in 1959 [3]. The best known general estimate for \(\lambda_n\) is \(\lambda_n \approx \frac{1}{\alpha} \left(\frac{n\pi}{2}\right)^\alpha \leq \lambda_n \leq \frac{\pi}{\alpha} \left(\frac{n\pi}{2}\right)^\alpha\) due to DeBlassie [9] and Chen and Song [7], also known in a more general setting. The important case of \(\alpha = 1\) was studied in detail by several authors, see [1,13] and the references therein. It is known that \((\lambda_n)^{1/\alpha}\) is continuous and increasing in \(\alpha \in (0, 2]\), see [7–10]. For a discussion of related results and historical remarks, the reader is again referred to [1,13]. Theorem 1 is of interest in physics, the asymptotic formula (2) (without the information about the order of the error term) was stated, and supported by numerical experiments, in [15]. There is a considerable amount of related (mostly numerical) research in physics literature.

Noteworthy, although the values of \(C\) and \(C'\) given above are rather large, numerical evidence suggests that the error term in formula (2) is rather small also for small \(n\) in the full range of \(\alpha \in (0, 2]\), see Table 1 and the estimates in the last section of this article. It is an interesting open problem to prove Theorem 1 with \(C\) and \(C'\) non-exploding as \(\alpha\) approaches 0. This is related to simplicity of eigenvalues \(\lambda_n\), conjectured to hold for all \(\alpha \in (0, 2]\), proved for \(\alpha = 1\) in [13], and extended to \(\alpha \in [1, 2]\) in Proposition 3 in Section 6.

Motivated by the results of [13] and [14], as well as by Theorem 1 above, one can conjecture asymptotic law similar to (2) for eigenvalues on an interval for more general operators
$\mathcal{A} = \psi(-d^2/dx^2)$, studied in [14]. While such a result for each individual complete Bernstein function $\psi$ should present no difficulty (under some reasonable regularity and growth assumptions on $\psi$), it is an interesting (and much more difficult) problem to obtain estimates uniform also in $\psi$, for a given class of $\psi$. One important example here is the family of Klein–Gordon square-root operators $\mathcal{A} = \sqrt{m^2 - d^2/dx^2 - m}$, with mass $m$ ranging from 0 to $\infty$. This operator is close to $\sqrt{-d^2/dx^2}$ for small $m$, but when $m$ is large, it more similar to $-d^2/dx^2$.

To give a formal statement of the spectral problem (1), first we recall the definition of the one-dimensional fractional Laplace operator $\mathcal{A} = (-d^2/dx^2)^{\alpha/2}$. It is defined pointwise by the principal value integral, if convergent,

$$\mathcal{A}f(x) = c_\alpha \mathrm{pv} \int_{-\infty}^{\infty} \frac{f(x) - f(y)}{|x - y|^{1+\alpha}} dy, \quad x \in \mathbb{R},$$

where

$$c_\alpha = \frac{2^\alpha \Gamma(\frac{1+\alpha}{2})}{\sqrt{\pi} |\Gamma(-\frac{\alpha}{2})|} = \frac{\Gamma(1 + \alpha) \sin \frac{\alpha\pi}{2}}{\pi};$$

$\mathcal{A}f(x)$ is convergent if, for example, $f$ is $C^2$ in a neighborhood of $x$ and bounded on $\mathbb{R}$. Note that

$$\frac{1}{8} \alpha(2 - \alpha) \leq c_\alpha \leq \frac{1}{2} \alpha(2 - \alpha).$$

Indeed, for the lower bound simply use $\sin \frac{\alpha\pi}{2} \geq \frac{\pi}{2} \alpha(2 - \alpha)$ and $\Gamma(1 + \alpha) \geq \frac{1}{2}$, and for the upper bound, we have $\Gamma(1 + \alpha) \leq \max(1, \alpha)$ and $\max(1, \alpha) \sin \frac{\alpha\pi}{2} \leq \frac{\pi}{2} \alpha(2 - \alpha)$.

For $f \in C_\infty^\infty(\mathbb{R})$, the Fourier transform of $\mathcal{A}f$ is equal to $|\xi|^{\alpha} \hat{f}(\xi)$, and $\mathcal{A}$ extends to an unbounded self-adjoint operator on $L^2(\mathbb{R})$. We write $\mathcal{A}_D$ for the operator $\mathcal{A}$ on $D$ with zero exterior condition on $\mathbb{R} \setminus D$. More precisely, for $f \in C_\infty^\infty(D)$, $\mathcal{A}_D f$ is defined to be the restriction of $\mathcal{A}f$ to $D$. Again, the Friedrich’s extension of $\mathcal{A}_D$ is an unbounded self-adjoint operator on $L^2(D)$, denoted by the same symbol $\mathcal{A}_D$.

The operator $-\mathcal{A}$ (on an appropriate domain) is the generator of the one-dimensional symmetric $\alpha$-stable process $X_t$, and $-\mathcal{A}_D$ is the generator of $X_t$ killed upon leaving the interval $D$. This probabilistic interpretation is a primary source of our motivation, but will not be exploited in the sequel.

**Notation.** Throughout this article, $C, C', C''$ denote generic absolute constants (independent of $\alpha$), and their values may be different in each displayed equation. We will track the dependence of other constants employed below on $\alpha$ to catch their asymptotic behavior as $\alpha \searrow 0$ and $\alpha \nearrow 2$. For brevity, we denote $\beta = 2 - \alpha$.

### 2. Auxiliary estimates

Define, as in [13, Appendix C], an auxiliary function (see Fig. 1):
Fig. 1. Plot of $q(x)$ (dashed line), $f(x)$ (dotted line) and $g(x) = q(x)f(x)$ (solid line). With the notation of Lemma 1, plots correspond to $\alpha = \frac{1}{5}$ and (a) $n = 1$; (b) $n = 2$.

$$q(x) = \begin{cases} 0 & \text{for } x \in (-\infty, -\frac{1}{3}), \\ \frac{9}{2}(x + \frac{1}{3})^2 & \text{for } x \in (-\frac{1}{3}, 0), \\ 1 - \frac{9}{2}(x - \frac{1}{3})^2 & \text{for } x \in (0, \frac{1}{3}), \\ 1 & \text{for } x \in (\frac{1}{3}, \infty). \end{cases}$$ (5)

Note that $q$, $q'$ are continuous and bounded on $\mathbb{R}$, and $q''$ is continuous and bounded on $\mathbb{R} \setminus \{-\frac{1}{3}, 0, \frac{1}{3}\}$. Furthermore, $q(x) + q(-x) = 1$. Assume that $f$ is an integrable function on $\mathbb{R}$ such that $f$, $f'$ and $f''$ exist and are bounded in $[-\frac{1}{3}, \frac{1}{3}]$. We define $g(x) = q(x)f(x)$. Below we estimate $Ag$ on $(-1, 0)$ in a very similar way as in [13].

Let $M$ be the supremum of $\max(|f(x)|, |f'(x)|, |f''(x)|)$ over $x \in [-\frac{1}{3}, \frac{1}{3}]$, and let $I = \int_{0}^{\infty} |f(x)|\, dx$. Then $g''(x) = 0$ for $x < -\frac{1}{3}$ and

$$|g''(x)| \leq |f(x)q''(x)| + 2|f'(x)q'(x)| + |f''(x)q(x)| \leq CM, \quad x \in \left(-\frac{1}{3}, \frac{1}{3}\right) \setminus \{0\}.$$ 

Suppose first that $x \in (-1, -\frac{1}{3})$. Since $g$ vanishes in $(-1, -\frac{1}{3})$, we have

$$e_{\alpha}^{-1} |Ag(x)| \leq \int_{-\frac{1}{3}}^{\infty} \frac{q(y)|f(y)|}{|x-y|^{1+\alpha}} \, dy$$

$$\leq M \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{q(y)}{|y + \frac{1}{3}|^{1+\alpha}} \, dy + \int_{\frac{1}{3}}^{\infty} \frac{|f(y)|}{|x-y|^{1+\alpha}} \, dy$$

$$\leq 9M \int_{-\frac{1}{3}}^{\frac{1}{3}} \left| y + \frac{1}{3} \right|^{-\alpha} \, dy + \frac{1}{(\frac{2}{3})^{1+\alpha}} \int_{\frac{1}{3}}^{\infty} |f(y)| \, dy$$

$$\leq \frac{2^{1-\alpha}3^{\alpha}M}{2-\alpha} + \frac{3^{1+\alpha}I}{2^{1+\alpha}} \leq \frac{CM}{\beta} + CI.$$
In the third inequality we used the estimate $q(y) \leq \frac{9}{16}(y + \frac{1}{2})^2$ ($y \in \mathbb{R}$). For $x \in (-\frac{1}{2}, 0)$ the principal value integral in the definition of $Ag(x)$ can be estimated by splitting it into two parts.

By Taylor’s expansion of $g$, for $y \in (-\infty, \frac{1}{3}]$ we have

\[
|g(x) - g(y) - (y - x)g'(x)| = \left| \int_x^y g''(z)(x - z) \, dz \right|
\leq \left( \text{ess sup}_{z \in (-\infty, \frac{1}{3})} |g''(z)| \right) \left| \int_x^y (x - z) \, dz \right|
\leq CM(x - y)^2/2.
\]

Hence,

\[
\left| \text{pv} \int_{x - \frac{1}{3}}^{x + \frac{1}{3}} \frac{g(x) - g(y)}{|x - y|^{1+\alpha}} \, dy \right| = \left| \int_{x - \frac{1}{3}}^{x + \frac{1}{3}} \frac{g(x) - g(y) - (y - x)g'(x)}{|x - y|^{1+\alpha}} \, dy \right|
\leq \frac{CM}{2} \int_{x - \frac{1}{3}}^{x + \frac{1}{3}} \frac{(x - y)^2}{|x - y|^{1+\alpha}} \, dy = \frac{2^{1-\alpha}CM}{2^{2-\alpha}(2-\alpha)} \leq \frac{CM}{2^{\beta}}.
\]

Furthermore,

\[
\left| \left( \int_{-\infty}^{x - \frac{1}{3}} + \int_{x + \frac{1}{3}}^{\infty} \right) \frac{g(x) - g(y)}{|x - y|^{1+\alpha}} \, dy \right| \leq |g(x)| \left( \int_{-\infty}^{x - \frac{1}{3}} + \int_{x + \frac{1}{3}}^{\infty} \frac{1}{(x - y)^{1+\alpha}} \, dy \right) + 3^{1+\alpha} \int_{x + \frac{1}{3}}^{\infty} |f(y)| \, dy
\leq \frac{CM}{\alpha} + C1.
\]

We conclude that

\[
\epsilon^{-1}_{\alpha} |Ag(x)| \leq \frac{CM}{\alpha\beta} + C1, \quad x \in (-1, 0).
\]

3. Estimates for half-line

The main result of [14] is the formula for generalized eigenfunctions for a class of operators on $(0, \infty)$. The case of the fractional Laplace operator is studied in [14, Example 1]. In particular, the eigenfunction $F_\lambda$ of $A_{(0,\infty)}$ (defined pointwise, or as an operator on $L^\infty(0, \infty)$; see [14] for more details) corresponding to the eigenvalue $\lambda^\alpha$ ($\lambda > 0$) is given by $F_\lambda(x) = F(\lambda x) = \sin(\lambda x + \frac{\beta \pi}{8}) - G(\lambda x)$ (recall that $\beta = 2 - \alpha$), where $G$ is a completely monotone function.
More precisely, $G$ is the Laplace transform of a positive function $\gamma(s)$ ($s > 0$), given by the formula

$$
\gamma(s) = \frac{\sqrt{2\alpha} \sin\left(\frac{\alpha \pi}{2}\right)}{2\pi} \frac{s^\alpha}{1 + s^{2\alpha} - 2s^\alpha \cos\left(\frac{\alpha \pi}{2}\right)} \exp\left(\frac{1}{\pi} \int_0^\infty \frac{1}{1 + r^2} \log \frac{1 - r^\alpha s^\alpha}{1 - r^{2s^2}} \, dr\right).
$$

By [14, Lemma 16], for $x > 0$ we have

$$
G(x) \leq \sin\left(\frac{\beta \pi}{8}\right) \leq C\beta,
$$

and

$$
\int_0^\infty G(x) \, dx = \cos\left(\frac{\beta \pi}{8}\right) - \sqrt{\frac{\alpha}{2}} \leq C\beta.
$$

Note that the exponent in (7) is negative. Furthermore, for $\alpha \in (0, 1]$ we have

$$
1 + s^{2\alpha} - 2s^\alpha \cos\left(\frac{\alpha \pi}{2}\right) \geq \left(\sin\left(\frac{\alpha \pi}{2}\right)\right)^2 \geq \alpha^2,
$$

while for $\alpha \in (1, 2)$, the left-hand side is not less than one. Hence, for all $\alpha \in (0, 2]$,

$$
1 + s^{2\alpha} - 2s^\alpha \cos\left(\frac{\alpha \pi}{2}\right) \geq \min(\alpha^2, 1) \geq \frac{\alpha^2}{4}.
$$

Finally, $\sin\left(\frac{\alpha \pi}{2}\right) \leq \alpha(2 - \alpha) = \alpha\beta$. Therefore,

$$
\gamma(s) \leq \frac{2\sqrt{2\alpha\beta}}{\alpha \pi} s^\alpha.
$$

By direct integration, we find that for $x > 0$,

$$
G(x) = \int_0^\infty e^{-sx} \gamma(s) \, ds \leq \frac{2\sqrt{2\alpha\beta}}{\alpha \pi} \frac{\Gamma(1 + \alpha)}{x^{1-\alpha}} \leq \frac{C\beta}{\sqrt{\alpha}} x^{1-\alpha}.
$$

Furthermore, $-G'$ and $G''$ are the Laplace transforms of $s\gamma(s)$ and $s^2\gamma(s)$ respectively. Hence, (10) gives

$$
-G'(x) \leq \frac{C\beta}{\sqrt{\alpha}} x^{-2-\alpha}, \quad G''(x) \leq \frac{C\beta}{\sqrt{\alpha}} x^{-3-\alpha}
$$

for $x > 0$. For simplicity, we let $F(x) = 0$ and $G(x) = 0$ for $x \leq 0$. 

4. Approximation to eigenfunctions

Let $n$ be a fixed positive integer and $\mu_n = \frac{n\pi}{2} - \frac{\beta\pi}{8}$. Our goal is to show that $\mu_n^\alpha$ is close to $\lambda_n$.

Note that $\mu_n \geq \frac{\pi}{4}$ and $\frac{n\pi}{4} \leq \mu_n \leq \frac{n\pi}{2}$.

We construct approximations $\tilde{\varphi}_n$ to eigenfunctions $\varphi_n$ by combining shifted eigenfunctions for half-line, $F_{\mu_n}(1 + x)$ and $F_{\mu_n}(1 - x)$, and using the auxiliary function $q$ given above in (5) to join them in a sufficiently smooth way. We let (see Fig. 2)

$$\tilde{\varphi}_n(x) = q(-x)F_{\mu_n}(1 + x) - (-1)^n q(x)F_{\mu_n}(1 - x), \quad x \in \mathbb{R}. \quad (13)$$

Note that $\tilde{\varphi}_n(x) = 0$ for $x \notin (-1, 1)$. Suppose that $n$ is odd, $n = 2m + 1$. Then $\tilde{\varphi}_n$ is an even function. Furthermore,

$$\sin \left( \mu_n (1 - x) + \frac{\beta\pi}{8} \right) = \sin \left( \frac{n\pi}{2} - \mu_n x \right) = (-1)^m \cos(\mu_n x)$$

$$= \sin \left( \frac{n\pi}{2} + \mu_n x \right) = \sin \left( \mu_n (1 + x) + \frac{\beta\pi}{8} \right).$$

Recall that $F_{\lambda}(x) = \sin(\lambda x + \frac{\beta\pi}{8}) - G(\lambda x)$. Hence, for $x \in (-1, 1)$,
\[
\begin{align*}
\tilde{\varphi}_n(x) &= q(-x)F_{\mu_n}(1 + x) + q(x)F_{\mu_n}(1 - x) \\
&= (-1)^m \left(q(-x) + q(x)\right) \cos(\mu_n x) + q(-x)G(\mu_n(1 + x)) + q(x)G(\mu_n(1 - x)) \\
&= (-1)^m \cos(\mu_n x) + q(-x)G(\mu_n(1 + x)) + q(x)G(\mu_n(1 - x)).
\end{align*}
\] (14)

In a similar manner, when \(n\) is even, \(n = 2m\), then for \(x \in (-1, 1)\),

\[
\tilde{\varphi}_n(x) = (-1)^m \sin(\mu_n x) + q(-x)G(\mu_n(1 + x)) - q(x)G(\mu_n(1 - x)).
\] (15)

It follows that away from the boundary of \(D = (-1, 1)\), \(\tilde{\varphi}_n\) is close to \(\pm \cos(\mu_n x)\) or \(\pm \sin(\mu_n x)\), and it converges to zero near \(\pm 1\).

**Lemma 1.** We have

\[
\| A_D \tilde{\varphi}_n - \mu^n \tilde{\varphi}_n \|_2 \leq \frac{C\beta}{\sqrt{\alpha}} \frac{1}{n^\gamma}.
\] (16)

**Proof.** Note that for all \(x \in \mathbb{R}\) we have (see Fig. 2)

\[
\begin{align*}
\tilde{\varphi}_n(x) - F_{\mu_n}(1 + x) &= \left(q(-x) - 1\right)F_{\mu_n}(1 + x) - (-1)^n q(x)F_{\mu_n}(1 - x) \\
&= -q(x)\left(F_{\mu_n}(1 + x) + (-1)^n F_{\mu_n}(1 - x)\right).
\end{align*}
\]

Observe that

\[
\begin{align*}
\sin\left(\mu_n(1 + x) + \frac{\beta\pi}{8}\right) + (-1)^n \sin\left(\mu_n(1 - x) + \frac{\beta\pi}{8}\right) \\
= \sin\left(\frac{n\pi}{2} + \mu_n x\right) + (-1)^n \sin\left(\frac{n\pi}{2} - \mu_n x\right) = 0.
\end{align*}
\]

Since \(F_{\lambda}(x) = \sin(\lambda x + \frac{\beta\pi}{8})1_{[0, \infty)}(x) - G(\lambda x) (x \in \mathbb{R})\), it follows that for all \(x \in \mathbb{R}\) we have

\[
\begin{align*}
\tilde{\varphi}_n(x) - F_{\mu_n}(1 + x) &= q(x)\left(G(\mu_n(1 + x)) + (-1)^n G(\mu_n(1 - x))\right) \\
&- \sin\left(\mu_n(1 + x) + \frac{\beta\pi}{8}\right)1_{[1, \infty)}(x).
\end{align*}
\]

For \(x \in \mathbb{R}\), denote (see Fig. 1)

\[
\begin{align*}
h(x) &= \sin\left(\mu_n(1 + x) + \frac{\beta\pi}{8}\right)1_{[1, \infty)}(x), \\
f(x) &= G(\mu_n(1 + x)) + (-1)^n G(\mu_n(1 - x)), \\
g(x) &= q(x) f(x).
\end{align*}
\]

It follows that \(\tilde{\varphi}_n(x) = F_{\mu_n}(1 + x) + g(x) - h(x) (x \in \mathbb{R})\). For \(x \in (-1, 0)\), we have
\[ AF_\mu_n(1 + x) - \mu_\alpha^n F_\mu_n(1 + x) = 0 \text{ and } h(x) = 0. \] Hence,

\[ |A\tilde{\varphi}_n(x) - \mu_\alpha^n \tilde{\varphi}_n(x)| \leq |Ag(x)| + |\mu_\alpha^n g(x)| + |Ah(x)|, \quad x \in (-1, 0). \tag{17} \]

We will now estimate each summand on the right-hand side.

Recall that \( G \) is completely monotone, so that \( G, -G' \) and \( G'' \) are positive, convex functions on \((0, \infty)\). This fact and estimates (9), (11) and (12) give

\[
\sup_{x \in [-\frac{1}{3}, \frac{1}{3}]} |f(x)| \leq G\left(\frac{2}{3}\mu_n\right) + G\left(\frac{4}{3}\mu_n\right) \leq \frac{C\beta}{\sqrt{\alpha}} \mu_n^{-1-\alpha},
\]

\[
\sup_{x \in [-\frac{1}{3}, \frac{1}{3}]} |f'(x)| \leq -\mu_n G'\left(\frac{2}{3}\mu_n\right) - \mu_n G'\left(\frac{4}{3}\mu_n\right) \leq \frac{C\beta}{\sqrt{\alpha}} \mu_n^{-1-\alpha},
\]

\[
\sup_{x \in [-\frac{1}{3}, \frac{1}{3}]} |f''(x)| \leq \mu_n^2 G''\left(\frac{2}{3}\mu_n\right) + \mu_n^2 G''\left(\frac{4}{3}\mu_n\right) \leq \frac{CB}{\sqrt{\alpha}} \mu_n^{-1-\alpha},
\]

\[
\int_0^\infty |f(x)| \, dx \leq \int_0^{\infty} G_{\mu_n}(1 + x) \, dx + \int_0^1 G_{\mu_n}(1 - x) \, dx
\]

\[
= \frac{1}{\mu_n} \int_0^\infty G(y) \, dy \leq \frac{C\beta}{\mu_n}. \tag{18}
\]

By (6) and (4),

\[
|Ag(x)| \leq \frac{C\beta}{\sqrt{\alpha}} \mu_n^{-1-\alpha} + C\alpha\beta^2 \mu_n^{-1}, \quad x \in (-1, 0). \tag{19}
\]

For the second term in (17), we have \(|g(x)| = 0\) for \(x \in (-1, -\frac{1}{3})\). Furthermore, since \(q(x) \leq \frac{1}{2}\) for \(x < 0\), the estimate (18) gives

\[
|\mu_\alpha^n g(x)| = \mu_\alpha^n q(x) |f(x)| \leq \frac{\mu_\alpha^n |f(x)|}{2} \leq \frac{C\beta}{\sqrt{\alpha}} \mu_n^{-1}, \quad x \in \left(-\frac{1}{3}, 0\right). \tag{20}
\]

Finally, for the third term in (17), we use the following estimate: if \(u\) is a decreasing differentiable function such that \(\lim_{z \to \infty} u(z) = 0\), then, by integration by parts, for any \(a, \vartheta \in \mathbb{R}\) and \(\lambda > 0\) we have

\[
\left| \int_a^\infty u(z) \sin(\lambda z + \vartheta) \, dz \right| = \frac{1}{\lambda} \int_a^\infty u'(z) \left( \cos(\lambda a + \vartheta) - \cos(\lambda z + \vartheta) \right) \, dz
\]

\[
\leq \frac{2}{\lambda} \int_a^\infty |u'(z)| \, dz = \frac{2u(a)}{\lambda}.
\]
It follows that for all $x < 0$, we have
\[ |A_h(x)| = c_\alpha \left| \int_1^\infty \frac{\sin(\mu_n(y + y)) + \frac{\beta\pi}{8}}{|x-y|^{1+\alpha}} dy \right| \leq \frac{2c_\alpha}{\mu_n|x-1|^{1+\alpha}} \leq \alpha\beta\mu_n^{-1}. \] (21)

Estimates (19)–(21) applied to (17) yield that
\[ \left| A\tilde{\varphi}_n(z) - \mu_n^\alpha \tilde{\varphi}_n(z) \right| \leq C\beta \sqrt{\frac{1}{\alpha\mu_n^{-1}}}, \quad z \in (-1, 0). \] (22)

By symmetry, (22) also holds for $z \in (0, 1)$. Formula (16), with $A_D \tilde{\varphi}_n$ understood in the pointwise sense, follows. It remains to prove that $\tilde{\varphi}_n$ is in the domain of $A_D$. To this end, we will use the notion of the Green operator $G_D = A_D^{-1}$. The reader is referred e.g. to [6] for formal definition and properties of $G_D$; see also the last part of Section 7.

Since $A\tilde{\varphi}_n$ is bounded on $D$, the function $\tilde{\varphi}_n - G_D A\tilde{\varphi}_n$ is a bounded, continuous in $D$, weakly $\alpha$-harmonic function in $D = (-1, 1)$ with zero exterior condition. Such a function is necessarily zero (see [4,11]). It follows that $\tilde{\varphi}_n = G_D A\tilde{\varphi}_n$, and hence $\tilde{\varphi}_n$ is in the $L^\infty(D)$ domain of $A_D$. Since convergence in $L^\infty(D)$ is stronger than the one in $L^2(D)$, the proof is complete.

**Lemma 2.** We have
\[ 1 - \frac{C\beta}{n} \leq \|\tilde{\varphi}_n\|_2 \leq 1 + \frac{C\beta}{n}. \] (23)

In particular, there is an absolute constant $K$ such that $\|\tilde{\varphi}_n\|_2 \geq \frac{1}{2}$ for $n \geq K$.

**Proof.** By (13), we have (see also (14) and (15))
\[ \tilde{\varphi}_n(x) = \sin\left(\mu_n(1+x) + \frac{\beta\pi}{8}\right) + q(-x)G(\mu_n(1+x)) - (-1)^n q(x)G(\mu_n(1-x)). \]

Hence,
\[
\left| \|\tilde{\varphi}_n\|_2 - 1 \right| \\
= \left| \int_{-1}^{1} \left( (\tilde{\varphi}_n(x))^2 - \frac{1}{2} \right) dx \right| \\
\leq \left| \int_{-1}^{1} \left( \left( \sin\left(\mu_n(1+x) + \frac{\beta\pi}{8}\right) \right)^2 - \frac{1}{2} \right) dx \right| \\
+ 2 \left| \int_{-1}^{1} \left( q(-x)G(\mu_n(1+x)) - (-1)^n q(x)G(\mu_n(1-x)) \right) \sin\left(\mu_n(1+x) + \frac{\pi}{8}\right) dx \right| \\
+ \int_{-1}^{1} \left| q(-x)G(\mu_n(1+x)) - (-1)^n q(x)G(\mu_n(1-x)) \right|^2 dx.
\]

We estimate each term on the right-hand side. First, by direct integration,
\[
\begin{aligned}
&\left| \int_{-1}^{1} \left( \left( \sin \left( \mu_n (1 + x) + \frac{\beta \pi}{8} \right) \right)^2 - \frac{1}{2} \right) \, dx \right| = \frac{1}{2} \left| \int_{-1}^{1} \cos \left( 2 \mu_n (1 + x) + \frac{\beta \pi}{4} \right) \, dx \right| \\
&= \frac{1}{4 \mu_n} \left| \sin \left( 4 \mu_n + \frac{\beta \pi}{4} \right) - \sin \frac{\beta \pi}{4} \right| \leq \frac{C \beta}{\mu_n}.
\end{aligned}
\]

By (8) and (9),
\[
\int_{-1}^{1} \left( G\left( \mu_n (1 + x) \right) \right)^2 \, dx \leq C \beta \int_{-1}^{\infty} G\left( \mu_n (1 + x) \right) \leq \frac{C \beta^2}{\mu_n}.
\]

Hence,
\[
\int_{-1}^{1} \left( q(-x) G\left( \mu_n (1 + x) \right) - (-1)^n q(x) G\left( \mu_n (1 - x) \right) \right)^2 \, dx \\
\leq 2 \int_{-1}^{1} \left( G\left( \mu_n (1 + x) \right) \right)^2 \, dx + 2 \int_{-1}^{1} \left( G\left( \mu_n (1 - x) \right) \right)^2 \, dx \leq \frac{C \beta^2}{\mu_n}.
\]

Finally, again by (9),
\[
\left| \int_{-1}^{1} q(-x) G\left( \mu_n (1 + x) \right) \sin \left( \mu_n (1 + x) + \frac{\pi}{8} \right) \, dx \right| \leq \frac{1}{\mu_n} \int_{-1}^{1} G\left( \mu_n (1 + x) \right) \, dx \leq \frac{C \beta}{\mu_n},
\]

and we can replace \( q(-x) G\left( \mu_n (1 + x) \right) \) by \( q(x) G\left( \mu_n (1 - x) \right) \). Formula (23) follows.

### 5. Proof of Theorem 1

Since \( \hat{\phi}_n \in L^2(D) \), we have \( \hat{\phi}_n = \sum_j a_j \varphi_j \) for some \( a_j \in \mathbb{R} \). Moreover, \( \| \hat{\phi}_n \|_2^2 = \sum a_j^2 \) and \( A_D \hat{\phi}_n = \sum j \lambda_j a_j \varphi_j \). Let \( \lambda_k(n) \) be the eigenvalue nearest to \( \mu_n^\alpha \). Then
\[
\| A_D \hat{\phi}_n - \mu_n^\alpha \hat{\phi}_n \|_2^2 = \sum_{j=1}^{\infty} (\lambda_j - \mu_n^\alpha)^2 a_j^2 \geq (\lambda_k(n) - \mu_n^\alpha)^2 \sum_{j=1}^{\infty} a_j^2 = (\lambda_k(n) - \mu_n^\alpha)^2 \| \hat{\phi}_n \|_2^2.
\]

Let \( K \) be the constant defined in Lemma 2. By (16) and Lemma 2, it follows that for \( n \geq K \),
\[
|\lambda_k(n) - \mu_n^\alpha| \leq \frac{C \beta 1}{\sqrt{\alpha} n}.
\]

This will enable us to derive the two-term asymptotic formula for \( \lambda_j \).

Denote \( \varepsilon = \frac{1}{2} \frac{\beta \pi}{8} \). We claim that for each \( \alpha \in (0, 2) \), there is a positive integer \( L_\alpha \) such that \( \lambda_k(n) \in ((\mu_n - \varepsilon)^\alpha, (\mu_n + \varepsilon)^\alpha) \) for \( n \geq L_\alpha \). Namely, we take
\[ L_\alpha = \left\lceil \left( \frac{A\beta}{\alpha^{3/2} \varepsilon} \right)^{1/\alpha} \right\rceil = \left\lceil \left( \frac{CA}{\alpha^{3/2}} \right)^{1/\alpha} \right\rceil, \tag{25} \]

with the constant \( A \) large enough. In particular, we take \( A \geq 2^{3/4} K^2 \pi /16 \), so that \( L_\alpha \geq K \) for all \( \alpha \in (0, 2) \). By (24) and (25), for \( n \geq L_\alpha \) we have

\[
\left| \lambda_{k(n)} - \mu_n^{\alpha} \right| \leq \frac{C\beta}{\sqrt{\alpha}} n \leq \frac{C\beta}{\sqrt{\alpha}} \frac{\alpha^{3/2} \varepsilon n^{\alpha}}{A\beta} = \frac{C\alpha \varepsilon n^{\alpha-1} - 1}{A}. \tag{26} \]

On the other hand, we have \( \frac{n\pi}{8} \leq \mu_n - \varepsilon \leq \mu_n + \varepsilon \leq \frac{n\pi}{2} \). Hence, by the mean value theorem,

\[
\left| (\mu_n \pm \varepsilon)^{\alpha} - \mu_n^{\alpha} \right| \geq \alpha \varepsilon \min \left( (\mu_n - \varepsilon)^{\alpha-1}, (\mu_n + \varepsilon)^{\alpha-1} \right)
\geq \alpha \varepsilon n^{\alpha-1} \min \left( \frac{\pi}{7}^{\alpha-1}, \frac{\pi}{2}^{\alpha-1} \right) \geq C\alpha \varepsilon n^{\alpha-1}. \tag{27} \]

If \( A \) is large enough, then (26) and (27) give \( \lambda_{k(n)} \in ((\mu_n - \varepsilon)^{\alpha}, (\mu_n + \varepsilon)^{\alpha}) \). This proves our claim.

The intervals \( ((\mu_n - \varepsilon)^{\alpha}, (\mu_n + \varepsilon)^{\alpha}) \) are mutually disjoint. Thus, \( \lambda_{k(n)} \) for \( n \geq L_\alpha \) are all distinct. We claim that there are strictly less than \( L_\alpha \) eigenvalues not included in the above class. As in [13], the key step will be the trace estimate.

Let \( J \) be the set of those \( j > 0 \) for which \( j \neq k(n) \) for all \( n \geq L_\alpha \). We need to show that \#\( J < L_\alpha \). Denote by \( p_t(x - y) \) and \( p_t^D(x, y) \) the heat kernels for \( A \) and \( A_D \) respectively; we have \( \hat{p}_t(\xi) = \exp(-t|\xi|^\alpha) \). For \( t > 0 \), we have (see e.g. [2,12])

\[
\sum_{j=1}^{\infty} e^{-\lambda_j \cdot t} = \int_D \sum_{j=1}^{\infty} e^{-\lambda_j \cdot t} (\phi_j(x))^2 \, dx = \int_D p_t^D(x, x) \, dx \leq \int_D p_t(0) \, dx = 2p_t(0) = \frac{2}{\pi} \int_0^\infty e^{-ts^\alpha} \, ds.
\]

In the last step, Fourier inversion formula was used. We find that

\[
\frac{\pi}{2} \sum_{j \in J} e^{-\lambda_j \cdot t} = \frac{\pi}{2} \sum_{j=1}^{\infty} e^{-\lambda_j \cdot t} - \frac{\pi}{2} \sum_{n=L_\alpha}^{\infty} e^{-\lambda_k(n) \cdot t} \leq \int_0^\infty e^{-ts^{\alpha}} \, ds - \sum_{n=L_\alpha}^{\infty} \frac{\pi}{2} e^{-t(\mu_n + \varepsilon)^{\alpha}}.
\]

The series on the right-hand side is an upper Riemann sum for the integral of \( e^{-ts^{\alpha}} \) over \( (\mu_{L_\alpha} + \varepsilon, \infty) \). Hence,

\[
\frac{\pi}{2} \sum_{j \in J} e^{-\lambda_j \cdot t} \leq \int_0^{\mu_{L_\alpha} + \varepsilon} e^{-ts^{\alpha}} \, ds \leq \mu_{L_\alpha} + \varepsilon.
\]
As \( t \downarrow 0 \), the left-hand side converges to \((\pi/2)\#J\). It follows that
\[
\#J \leq \frac{2}{\pi}(\mu L_\alpha + \varepsilon) = L_\alpha - \frac{\beta}{4} + \frac{2\varepsilon}{\pi}.
\]
Since \( \varepsilon < \frac{\beta \pi}{8} \), the right-hand side is less than \( L_\alpha \), and our claim is proved.

By [8,9], for \( j < L_\alpha \) we have \( \lambda_j \leq (j\pi/2)^\alpha \leq ((L_\alpha - 1)\pi/2)^\alpha \). On the other hand, \( \lambda_{k(n)} \geq (\mu n - \varepsilon)^\alpha > ((L_\alpha - 1)^2)^{\alpha/2} \) for \( n \geq L_\alpha \). It follows that \( J \) contains \( \{1, 2, \ldots, L_\alpha - 1\} \). But since \( \#J \leq L_\alpha - 1 \), we must have \( J = \{1, 2, \ldots, L_\alpha - 1\} \). We conclude that \( k(n) = n \) for all \( n \geq L_\alpha \).

Theorem 1 follows now from (24).

6. Further properties of eigenvalues and eigenfunctions

In this section we study three additional properties of \( \varphi_n \) and \( \lambda_n \): the \( L^2(D) \) estimates of \( \tilde{\varphi}_n - \tilde{\varphi}_n \), the \( L^\infty(D) \) bound for \( \varphi_n \), and simplicity of \( \lambda_n \). This part is modeled after [13, Section 10]. A number of open problems is suggested at the end of the section.

Proposition 1. (Cf. Lemma 3 and Corollary 4 in [13].) We can choose the sign of the eigenfunctions \( \varphi_n \) in such a way that there are constants \( C, C' \) with the following property:
\[
\|\tilde{\varphi}_n - \tilde{\varphi}_n\|_2 \leq \frac{C'}{n} \quad \text{when } \alpha \geq 1,
\]
\[
\|\tilde{\varphi}_n - \tilde{\varphi}_n\|_2 \leq \frac{C'(2 - \alpha)}{\alpha^{3/2}n^\alpha} \quad \text{when } \alpha < 1.
\]
In particular, if \( \varphi_n^*(x) = (-1)^{(n-1)/2} \cos(\mu_n x) \) for odd \( n \) and \( \varphi_n^*(x) = (-1)^{n/2} \sin(\mu_n x) \) for even \( n \), then there is a constant \( C'' \) such that for \( n \geq (C/\alpha)^{3/(2\alpha)} \),
\[
\|\varphi_n^* - \varphi_n\|_2 \leq \frac{C''(2 - \alpha)}{\alpha^{3/2}n^\alpha} \quad \text{when } \alpha < 1/2.
\]

Proof. Fix \( n \geq L_\alpha + 1 \) and \( \varepsilon = \frac{\beta \pi}{8} \), and write, as in the previous section, \( \tilde{\varphi}_n = \sum_j a_j \varphi_j \). By changing the sign of \( \varphi_n \) if necessary, we may assume that \( a_n > 0 \). Recall that \( L_\alpha \) was chosen in such a way that \( |\lambda_j - \mu_n^\alpha| \geq C\alpha n^{\alpha - 1} \) for \( j \neq n \) (see (27) and following discussion). Hence,
\[
\|A_D\tilde{\varphi}_n - \mu_n^\alpha \tilde{\varphi}_n\|_2^2 = \sum_{j=1}^{\infty} (\lambda_j - \mu_n^\alpha)^2 a_j^2 \geq C(\alpha n^{\alpha - 1})^2 \sum_{j \neq n} a_j^2.
\]
By (16), we obtain that
\[
\|\varphi_n - a_n \varphi_n\|_2^2 \leq \sum_{j \neq n} a_j^2 \leq C \left( \frac{\beta}{\sqrt{\alpha}} \frac{1}{n} \right)^2 \frac{1}{(\alpha n^{\alpha - 1})^2} = C \left( \frac{\beta}{\alpha^{3/2}n^\alpha} \right)^2.
\]
Since $\|\varphi_n\|_2 = 1$, we have

$$\|\tilde{\varphi}_n - \|\tilde{\varphi}_n\|_2\|_2 \leq \|\tilde{\varphi}_n - a_n\varphi_n\|_2 + |a_n - \|\tilde{\varphi}_n\|_2|.$$ 

Furthermore,

$$|a_n - \|\tilde{\varphi}_n\|_2|^2 \leq (\|\tilde{\varphi}_n\|_2 - a_n)(\|\tilde{\varphi}_n\|_2 + a_n) = \|\tilde{\varphi}_n\|_2^2 - a_n^2 = \|\tilde{\varphi}_n - a_n\varphi_n\|_2^2.$$ 

Hence, by (28),

$$\|\tilde{\varphi}_n - \|\tilde{\varphi}_n\|_2\|_2 \leq 2\|\tilde{\varphi}_n - a_n\varphi_n\|_2 \leq \frac{C\beta}{\alpha^{3/2}n^{\alpha}}.$$ 

(29)

Finally, by (23) and (29),

$$\|\tilde{\varphi}_n - \varphi_n\|_2 \leq \|\tilde{\varphi}_n - \|\tilde{\varphi}_n\|_2\|_2 + \|\tilde{\varphi}_n\|_2 - 1 | \leq \frac{2C\beta}{\alpha^{3/2}n^{\alpha}} + \frac{C\beta}{n}.$$ 

The first part of the proposition is proved. The other part is a simple consequence of the first one and the definition of $\tilde{\varphi}_n$. Indeed, by (14) and (15),

$$\|\varphi_n - \varphi_n^\ast\|_2 \leq \left( \int_{-1}^{1} (G(\mu_n(1 + x)))^2 \, dx \right)^{1/2} + \left( \int_{-1}^{1} (G(\mu_n(1 - x)))^2 \, dx \right)^{1/2}$$

$$= \frac{2}{\sqrt{\mu_n}} \left( \int_{0}^{2\mu_n} (G(y))^2 \, dy \right)^{1/2} \leq \frac{C\beta}{\sqrt{n}};$$

the last step follows by (8), (9) and the inequality $\mu_n \geq Cn$. □

**Proposition 2.** (Cf. Corollary 5 in [13].) If $\alpha \geq \frac{1}{2}$, then the eigenfunctions $\varphi_n(x)$ are bounded uniformly in $n \geq 1$ and $x \in D$.

**Proof.** Let $P^D_t = \exp(-tA_D)$ ($t > 0$) be the heat semigroup for $-A_D$ (or the transition semigroup of the symmetric $\alpha$-stable process in $D$), and let $p^D_t(x, y)$ be the corresponding heat kernel (or transition density). We have $P^D_t \varphi_n(x) = e^{-t\lambda_n} \varphi_n(x)$ for $x \in D$. It is well known that $p^D_t(x, y) \leq p_t(y - x)$, where $p_t(x)$ is the heat kernel for $-A$, $\hat{p}_t(\xi) = \exp(-t|\xi|^\alpha)$; see e.g. [5].

By Cauchy–Schwarz inequality and Plancherel’s theorem, we obtain

$$e^{-\lambda_n t} |\varphi_n(x)| \leq |P^D_t(\varphi_n - \tilde{\varphi}_n)(x)| + |P^D_t \tilde{\varphi}_n(x)|$$

$$\leq \int_D p_t(x - y) |\varphi_n(y) - \tilde{\varphi}_n(y)| \, dy + \|P^D_t \tilde{\varphi}_n\|_\infty$$

$$\leq \left( \int_{-\infty}^{\infty} \left( p_t(x - y)^2 \, dy \right)^{1/2} \right)^{1/2} \|\varphi_n - \tilde{\varphi}_n\|_2 + \|\tilde{\varphi}_n\|_\infty$$
\[
\left( \frac{1}{2\pi} \int_{\infty}^{-\infty} e^{-2r|z|^\alpha} \, dz \right)^{1/2} \| \phi_n - \tilde{\phi}_n \|_2 + \sup_{x \in (0, \infty)} |F(x)| 
\leq 2\sqrt{\Gamma(1 + 1/\alpha)(2t)^{-1/(2\alpha)}} \left\| \phi_n - \tilde{\phi}_n \right\|_2 + 2.
\]

Let \( t = 1/\lambda_n \). Then \( e^{-\lambda_n t} = 1/e \) and \( t^{-1/(2\alpha)} = \lambda_n^{1/(2\alpha)} \leq (n\pi/2)^{1/2} \). If \( n \geq L_\alpha + 1 \) and \( \alpha \geq \frac{1}{2} \), then \( \| \phi_n - \tilde{\phi}_n \|_2 \leq C\beta/\sqrt{n} \), and finally \( |\phi_n(x)| \leq C \) (for all \( n \geq L_\alpha + 1 \), \( x \in D \)). Since \( \phi_n \in L^\infty(D) \) also for each \( n \leq L_\alpha \), the proof is complete. \( \square \)

**Proposition 3.** (Cf. Theorem 6 in [13].) If \( \alpha \geq 1 \), then the eigenvalues \( \lambda_n \) are simple.

**Proof.** Let us write \( \lambda_{n,\alpha} \) for \( \lambda_n \) in this proof. Since \( (\lambda_{n,\alpha})^{1/\alpha} \) is increasing in \( \alpha \), we have

\[
(\lambda_{n,\alpha})^{1/\alpha} \leq (\lambda_{n,2})^{1/2} = \frac{n\pi}{2}.
\]

By Theorem 6 in [13], for \( n \geq 3 \) we have

\[
\frac{(n+1)\pi}{2} - \frac{\pi}{8} - \frac{\pi}{10} < \lambda_{n+1,1} \leq (\lambda_{n+1,\alpha})^{1/\alpha}.
\]

Therefore, \( \lambda_{n,\alpha} < \lambda_{n+1,\alpha} \), except perhaps \( n = 1 \) or \( n = 2 \). But a similar argument works also for \( n = 1 \) and \( n = 2 \), since by [1] we have

\[
(\lambda_{1,\alpha})^{1/\alpha} \leq (\lambda_{1,2})^{1/2} = \frac{\pi}{2} < 2 \leq (\lambda_{2,\alpha})^{1/\alpha},
\]

\[
(\lambda_{2,\alpha})^{1/\alpha} \leq (\lambda_{2,2})^{1/2} = \pi < 3.83 \leq \lambda_{3,\alpha} \leq (\lambda_{3,\alpha})^{1/\alpha}.
\]

The proof is complete. \( \square \)

Numerical experiments suggest that \( \phi_n \) are uniformly bounded also for \( \alpha < \frac{1}{2} \). Furthermore, it would be interesting to obtain an upper estimate of \( \sup_n \| \phi_n \|_\infty \), and in particular, to find its behavior when \( \alpha \) approaches 0. Finally, as stated in the introduction, better bounds for \( \lambda_n \) may yield simplicity of eigenvalues also when \( \alpha < 1 \).

7. Numerical bounds for eigenvalues

No general efficient algorithm giving mathematically correct numerical bounds for \( \lambda_n \) is known to the author. For \( \alpha = 1 \), a satisfactory method (an application of Rayleigh–Ritz and Weinstein–Aronszajn methods) is described in [13]. For general \( \alpha \), even approximation of \( \lambda_n \) is difficult: all known methods converge rather slowly, and thus the computation of eigenvalues of very large matrices is required. In this section a version of finite element method for obtaining a lower bound for \( \lambda_n \) is described. It shares the main drawbacks of many related algorithms: compared to the technique applied in [15], it converges slowly, and it suffers large errors as \( \alpha \) approaches 2. On the other hand, the method presented below gives mathematically correct lower bounds, and there is no error estimate for the numerical scheme of [15]. At the end of the section,
a somewhat similar method for the upper bound for \( \lambda_1 \) is given. It gives satisfactory results for large \( \alpha \), but deteriorates as \( \alpha \) gets close to 0.

It should be pointed out that in some cases (e.g. \( \alpha \) close to 2 or \( n \) large), the bound \( \frac{1}{2}(\frac{n\pi}{2})^\alpha \leq \lambda_n \leq (\frac{n\pi}{2})^\alpha \) of [7,9] is sharper than the estimates obtained below, unless extremely large matrices are used. Also, good numerical estimates of \( \lambda_n \) are available for \( \alpha = 1 \) due to [13]. By the monotonicity of \( (\lambda_n)^{1/\alpha} \) in \( \alpha \), this gives a lower bound for \( \lambda_n \) when \( \alpha \in (1, 2) \) and an upper bound for \( \alpha \in (0, 1) \). Finally, a good estimate for \( \lambda_1 \) can be found in [1]. For a comparison of the above, see Table 2.

Our method for the lower bound works for the fractional Laplace operator in an arbitrary bounded open set \( D \subseteq \mathbb{R}^d \), for any \( d \geq 1 \); in fact, it can be easily extended to more general pseudo-differential operators (or Lévy processes). Roughly speaking, we use the following monotonicity property: the eigenvalues \( \lambda_n \) decrease when the kernel of \( \mathcal{A} \), i.e. the function \( c_{d,\alpha}|x-y|^{-d-\alpha} \), is replaced by a smaller one. This fact is a simple consequence of the Rayleigh–Ritz variational formula. We cover the set \( D \) with small cubes \( I_k \), and replace the kernel of \( \mathcal{A} \) by a smaller kernel, which is constant whenever \( x \in I_k \) and \( y \in I_l \). The eigenvalues of the integral operator corresponding to the latter kernel can be easily expressed in terms of eigenvalues of a certain matrix.

Fix \( \varepsilon > 0 \) and let \( \{I_k: k \in \mathbb{Z}^d\} \) be the partition of \( \mathbb{R}^d \) into cubes \( I_k = \prod_{j=1}^d(kj \varepsilon, (kj + 1)\varepsilon] \), \( k \in \mathbb{Z}^d \). Let \( K_\varepsilon \subseteq \mathbb{Z}^d \) be the set of those \( k \in \mathbb{Z}^d \) for which \( I_k \) intersects \( D \), and let \( D_\varepsilon \) be the interior of \( \bigcup_{k \in K_\varepsilon} I_k \). Note that \( D \subseteq D_\varepsilon \).

The definition of \( \mathcal{A} = (-\Delta)^{\alpha/2} \) in higher dimension is similar to (3): for smooth bounded functions we have

\[
\mathcal{A} f(x) = c_{d,\alpha} \text{pv} \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x-y|^{d+\alpha}} dy, \quad x \in \mathbb{R}^d,
\]

where \( c_{d,\alpha} = 2^\alpha \Gamma((d + \alpha)/2)/\pi^{d/2} \Gamma(-\frac{\alpha}{2}) \). Fractional Laplace operator in \( D \) with zero exterior condition, denoted \( \mathcal{A}_D \), is defined as in dimension one. Below we denote by \( \lambda_n \) the eigenvalues of \( \mathcal{A}_D \). By domain monotonicity of \( \lambda_n \), the eigenvalues for \( D \) are not less than the eigenvalues of its superset \( D_\varepsilon \). For notational convenience, we assume that \( D = D_\varepsilon \).

The Dirichlet form \( \mathcal{E}(f, f) \) corresponding to \( \mathcal{A}_D \) is given by

\[
\mathcal{E}(f, f) = \frac{c_{d,\alpha}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(f(x) - f(y))^2}{|x-y|^{d+\alpha}} dx dy, \quad f \in L^2(D).
\]

As usual, \( f \in L^2(D) \) is extended to \( \mathbb{R}^d \) so that \( f(x) = 0 \) for \( x \in \mathbb{R}^d \setminus D \). By Rayleigh–Ritz variational principle,

\[
\lambda_n = \inf \{ \sup \{ \mathcal{E}_U(f, f): f \in U, \|f\|_2 = 1 \}: U < L^2(D), \dim U = n \},
\]

where \( U < L^2(D) \) means that \( U \) is a linear subspace of \( L^2(D) \). For \( k \in \mathbb{Z}^d \), we denote

\[
\varrho(k) = \sqrt{\sum_{j=1}^d (|k_j| + 1)^2}.
\]
Table 2
Comparison of bounds and approximations to $\lambda_n$. Each cell contains six numbers: lower bound $\lambda_{n,\varepsilon}$ with $\varepsilon = \frac{1}{2500}$, the best lower bound known before, approximation $\left(\frac{4n}{2} - \frac{(2-\alpha)n}{k}\right)^{\frac{1}{\alpha}}$, numerical approximation of [15], upper bound $\lambda_{n,\varepsilon}^*$, the best upper bound known before. The better estimates are underlined.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
<th>$\lambda_4$</th>
<th>$\lambda_5$</th>
<th>$\lambda_6$</th>
<th>$\lambda_7$</th>
<th>$\lambda_8$</th>
<th>$\lambda_9$</th>
<th>$\lambda_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.9966</td>
<td>1.0086</td>
<td>1.0137</td>
<td>1.0171</td>
<td>1.0196</td>
<td>1.0217</td>
<td>1.0234</td>
<td>1.0248</td>
<td>1.0261</td>
<td>1.0273</td>
</tr>
<tr>
<td></td>
<td>0.9943</td>
<td>0.5057</td>
<td>0.5078</td>
<td>0.5092</td>
<td>0.5104</td>
<td>0.5113</td>
<td>0.5121</td>
<td>0.5128</td>
<td>0.5134</td>
<td>0.5139</td>
</tr>
<tr>
<td></td>
<td>0.9976</td>
<td>1.0086</td>
<td>1.0138</td>
<td>1.0172</td>
<td>1.0198</td>
<td>1.0218</td>
<td>1.0235</td>
<td>1.0250</td>
<td>1.0263</td>
<td>1.0274</td>
</tr>
<tr>
<td></td>
<td>0.9966</td>
<td>1.0087</td>
<td>1.0137</td>
<td>1.0172</td>
<td>1.0197</td>
<td>1.0218</td>
<td>1.0235</td>
<td>1.0250</td>
<td>1.0263</td>
<td>1.0274</td>
</tr>
<tr>
<td></td>
<td>13.5210</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td></td>
<td>0.9974</td>
<td>1.0102</td>
<td>1.0148</td>
<td>1.0179</td>
<td>1.0203</td>
<td>1.0223</td>
<td>1.0239</td>
<td>1.0254</td>
<td>1.0266</td>
<td>1.0277</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9724</td>
<td>1.0919</td>
<td>1.1469</td>
<td>1.1863</td>
<td>1.2159</td>
<td>1.2405</td>
<td>1.2611</td>
<td>1.2791</td>
<td>1.2950</td>
<td>1.3094</td>
</tr>
<tr>
<td></td>
<td>0.9513</td>
<td>0.5606</td>
<td>0.5838</td>
<td>0.6008</td>
<td>0.6144</td>
<td>0.6257</td>
<td>0.6354</td>
<td>0.6440</td>
<td>0.6516</td>
<td>0.6585</td>
</tr>
<tr>
<td></td>
<td>0.9809</td>
<td>1.0917</td>
<td>1.1477</td>
<td>1.1867</td>
<td>1.2167</td>
<td>1.2412</td>
<td>1.2620</td>
<td>1.2802</td>
<td>1.2962</td>
<td>1.3107</td>
</tr>
<tr>
<td></td>
<td>0.9726</td>
<td>1.0922</td>
<td>1.1473</td>
<td>1.1868</td>
<td>1.2165</td>
<td>1.2413</td>
<td>1.2620</td>
<td>1.2802</td>
<td>1.2962</td>
<td>1.3107</td>
</tr>
<tr>
<td></td>
<td>1.8351</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td></td>
<td>0.9786</td>
<td>1.1067</td>
<td>1.1575</td>
<td>1.1941</td>
<td>1.2226</td>
<td>1.2462</td>
<td>1.2664</td>
<td>1.2840</td>
<td>1.2997</td>
<td>1.3138</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9572</td>
<td>1.1960</td>
<td>1.3182</td>
<td>1.4093</td>
<td>1.4801</td>
<td>1.5402</td>
<td>1.5915</td>
<td>1.6373</td>
<td>1.6780</td>
<td>1.7154</td>
</tr>
<tr>
<td></td>
<td>0.9181</td>
<td>0.6286</td>
<td>0.6817</td>
<td>0.7221</td>
<td>0.7550</td>
<td>0.7831</td>
<td>0.8076</td>
<td>0.8294</td>
<td>0.8492</td>
<td>0.8673</td>
</tr>
<tr>
<td></td>
<td>0.9712</td>
<td>1.1948</td>
<td>1.3199</td>
<td>1.4102</td>
<td>1.4819</td>
<td>1.5420</td>
<td>1.5939</td>
<td>1.6399</td>
<td>1.6812</td>
<td>1.7188</td>
</tr>
<tr>
<td></td>
<td>0.9575</td>
<td>1.1965</td>
<td>1.3191</td>
<td>1.4105</td>
<td>1.4817</td>
<td>1.5421</td>
<td>1.5938</td>
<td>1.6404</td>
<td>1.6811</td>
<td>1.7188</td>
</tr>
<tr>
<td></td>
<td>1.2376</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td></td>
<td>0.9675</td>
<td>1.2247</td>
<td>1.3398</td>
<td>1.4258</td>
<td>1.4947</td>
<td>1.5530</td>
<td>1.6036</td>
<td>1.6485</td>
<td>1.6890</td>
<td>1.7260</td>
</tr>
<tr>
<td>0.5</td>
<td>0.9692</td>
<td>1.5991</td>
<td>2.0247</td>
<td>2.3809</td>
<td>2.6862</td>
<td>2.9618</td>
<td>3.2118</td>
<td>3.4443</td>
<td>3.6608</td>
<td>3.8654</td>
</tr>
<tr>
<td></td>
<td>0.8862</td>
<td>0.8862</td>
<td>1.0854</td>
<td>1.2533</td>
<td>1.4012</td>
<td>1.5349</td>
<td>1.6579</td>
<td>1.7724</td>
<td>1.8790</td>
<td>1.9816</td>
</tr>
<tr>
<td></td>
<td>0.9908</td>
<td>1.5977</td>
<td>2.0306</td>
<td>2.3862</td>
<td>2.6954</td>
<td>2.9725</td>
<td>3.2259</td>
<td>3.4608</td>
<td>3.6808</td>
<td>3.8883</td>
</tr>
<tr>
<td></td>
<td>0.9701</td>
<td>1.6015</td>
<td>2.0288</td>
<td>2.3871</td>
<td>2.6947</td>
<td>2.9728</td>
<td>3.2254</td>
<td>3.4610</td>
<td>3.6805</td>
<td>3.8884</td>
</tr>
<tr>
<td></td>
<td>1.0002</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td></td>
<td>0.9863</td>
<td>1.6598</td>
<td>2.0777</td>
<td>2.4274</td>
<td>2.7314</td>
<td>3.0055</td>
<td>3.2562</td>
<td>3.4892</td>
<td>3.7074</td>
<td>3.9136</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$\lambda_1$</td>
<td>$\lambda_2$</td>
<td>$\lambda_3$</td>
<td>$\lambda_4$</td>
<td>$\lambda_5$</td>
<td>$\lambda_6$</td>
<td>$\lambda_7$</td>
<td>$\lambda_8$</td>
<td>$\lambda_9$</td>
<td>$\lambda_{10}$</td>
</tr>
<tr>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>1.1578^1</td>
<td>2.7548^3</td>
<td>4.3169^3</td>
<td>5.8922^3</td>
<td>7.4602^3</td>
<td>9.0329^3</td>
<td>10.6023^3</td>
<td>12.1742^3</td>
<td>13.7442^3</td>
<td>15.3156^3</td>
<td></td>
</tr>
<tr>
<td>1.3293^1</td>
<td>4.5727^3</td>
<td>8.9689^3</td>
<td>14.3024^3</td>
<td>20.3762^3</td>
<td>27.1479^3</td>
<td>34.5223^3</td>
<td>42.4772^3</td>
<td>50.9536^3</td>
<td>59.9375^3</td>
<td></td>
</tr>
<tr>
<td>1.6114</td>
<td>5.0545</td>
<td>9.5970</td>
<td>15.0171</td>
<td>21.1905</td>
<td>28.0344</td>
<td>35.4886</td>
<td>43.5076</td>
<td>52.0514</td>
<td>61.0922</td>
<td></td>
</tr>
<tr>
<td>1.5971^4</td>
<td>5.0586^4</td>
<td>9.5921^4</td>
<td>15.0154^4</td>
<td>21.1846^4</td>
<td>28.0289^4</td>
<td>35.4800^4</td>
<td>43.4972^4</td>
<td>52.0392^4</td>
<td>61.0786^4</td>
<td></td>
</tr>
<tr>
<td>1.5989</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td></td>
</tr>
<tr>
<td>1.6224^1</td>
<td>5.5684^2</td>
<td>10.2297^2</td>
<td>15.7497^2</td>
<td>22.0108^2</td>
<td>28.9339^2</td>
<td>36.4609^2</td>
<td>44.5467^2</td>
<td>53.1550^2</td>
<td>62.2558^2</td>
<td></td>
</tr>
<tr>
<td>1.8</td>
<td>5.1149</td>
<td>10.4477</td>
<td>17.2231</td>
<td>25.2907</td>
<td>34.5448</td>
<td>44.8969</td>
<td>56.2813</td>
<td>68.6385</td>
<td>81.9210</td>
<td></td>
</tr>
<tr>
<td>1.6765^1</td>
<td>6.1963^3</td>
<td>13.9088^3</td>
<td>24.3496^3</td>
<td>37.2347^3</td>
<td>52.5393^3</td>
<td>70.1002^3</td>
<td>89.9057^3</td>
<td>111.8432^3</td>
<td>135.9060^3</td>
<td></td>
</tr>
<tr>
<td>2.0555</td>
<td>7.5003</td>
<td>15.8014</td>
<td>26.7233</td>
<td>40.1148</td>
<td>55.8658</td>
<td>73.8905</td>
<td>94.1188</td>
<td>116.4923</td>
<td>140.9605</td>
<td></td>
</tr>
<tr>
<td>2.0481^4</td>
<td>7.5007^4</td>
<td>15.7948^4</td>
<td>26.7156^4</td>
<td>40.1014^4</td>
<td>55.8481^4</td>
<td>73.8661^4</td>
<td>94.0884^4</td>
<td>116.5451^4</td>
<td>140.9145^4</td>
<td></td>
</tr>
<tr>
<td>2.0501</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td></td>
</tr>
<tr>
<td>2.0777^1</td>
<td>7.8501^2</td>
<td>16.2868^2</td>
<td>27.3552^2</td>
<td>40.8427^2</td>
<td>56.7138^2</td>
<td>74.8501^2</td>
<td>95.1871^2</td>
<td>117.6664^2</td>
<td>142.2381^2</td>
<td></td>
</tr>
<tr>
<td>1.9</td>
<td>3.0353</td>
<td>3.7704</td>
<td>7.8734</td>
<td>13.1989</td>
<td>19.6379</td>
<td>27.1159</td>
<td>35.5691</td>
<td>44.9481</td>
<td>55.2082</td>
<td>66.3127</td>
</tr>
<tr>
<td>1.8273^1</td>
<td>6.8573^3</td>
<td>16.0993^3</td>
<td>29.0750^3</td>
<td>45.5221^3</td>
<td>65.4733^3</td>
<td>88.7683^3</td>
<td>115.4333^3</td>
<td>145.3521^3</td>
<td>178.5468^3</td>
<td></td>
</tr>
<tr>
<td>2.2477</td>
<td>8.5942</td>
<td>18.7177</td>
<td>32.4615</td>
<td>49.7204</td>
<td>70.4157</td>
<td>94.4848</td>
<td>121.8754</td>
<td>152.5433</td>
<td>186.4500</td>
<td></td>
</tr>
<tr>
<td>2.2432^4</td>
<td>8.5926^4</td>
<td>18.7104^4</td>
<td>32.4503^4</td>
<td>49.7021^4</td>
<td>70.3904^4</td>
<td>94.4504^4</td>
<td>121.8314^4</td>
<td>152.4878^4</td>
<td>186.3824^4</td>
<td></td>
</tr>
<tr>
<td>2.2455</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td></td>
</tr>
<tr>
<td>2.2743^1</td>
<td>8.8021^2</td>
<td>19.0178^2</td>
<td>32.8505^2</td>
<td>50.1962^2</td>
<td>70.9769^2</td>
<td>95.1929^2</td>
<td>122.6024^2</td>
<td>153.3517^2</td>
<td>187.3389^2</td>
<td></td>
</tr>
<tr>
<td>1.99</td>
<td>0.4174</td>
<td>0.5494</td>
<td>1.1671</td>
<td>1.9816</td>
<td>2.9788</td>
<td>4.1482</td>
<td>5.4811</td>
<td>6.9705</td>
<td>8.6101</td>
<td>10.3944</td>
</tr>
<tr>
<td>1.9816^1</td>
<td>7.5121^3</td>
<td>18.3642^3</td>
<td>34.1070^3</td>
<td>54.5469^3</td>
<td>79.8163^3</td>
<td>109.7856^3</td>
<td>144.5508^3</td>
<td>184.0144^3</td>
<td>228.2517^3</td>
<td></td>
</tr>
<tr>
<td>2.4441</td>
<td>9.7330</td>
<td>21.8288</td>
<td>38.7113</td>
<td>60.3666</td>
<td>86.7839</td>
<td>117.9546</td>
<td>153.8713</td>
<td>194.5275</td>
<td>239.9178</td>
<td></td>
</tr>
<tr>
<td>2.4427^4</td>
<td>9.7292^4</td>
<td>21.8200^4</td>
<td>38.6960^4</td>
<td>60.3426^4</td>
<td>86.7494^4</td>
<td>117.9077^4</td>
<td>153.8100^4</td>
<td>194.4500^4</td>
<td>239.8220^4</td>
<td></td>
</tr>
<tr>
<td>2.4452</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td></td>
</tr>
<tr>
<td>2.4563^2</td>
<td>9.7573^2</td>
<td>21.8651^2</td>
<td>38.7595^2</td>
<td>60.4267^2</td>
<td>86.8560^2</td>
<td>118.0385^2</td>
<td>153.9670^2</td>
<td>194.6351^2</td>
<td>240.0373^2</td>
<td></td>
</tr>
</tbody>
</table>

1 See [1].
2 See [7].
3 Combination of [13] with monotonicity in $\alpha$.
4 See [15].
Hence, when \( x \in I_k, y \in I_l, k, l \in \mathbb{Z}^d \), we have \( |x - y| \leq \varepsilon d (k - l) \). We define

\[
v_k = (\varrho(k))^{-d - \alpha}, \quad \tilde{v} = \sum_{k \in \mathbb{Z}^d} v_k,
\]

and

\[
\mathcal{E}_\varepsilon(f, f) = \frac{c_{d, \alpha} \varepsilon^{-d - \alpha}}{2} \sum_{k, l \in \mathbb{Z}^d} v_{k-l} \int_{I_k} \int_{I_l} (f(x) - f(y))^2 \, dx \, dy.
\]

Clearly, \( \mathcal{E}_\varepsilon(f, f) \leq \mathcal{E}(f, f) \). By Rayleigh–Ritz variational principle, the eigenvalues \( \lambda_n \) are bounded below by the sequence \( \lambda_{n, \varepsilon} \) of eigenvalues of the operator corresponding to the Dirichlet form \( \mathcal{E}_\varepsilon \). Here \( \lambda_{n, \varepsilon} \) are defined in the usual way,

\[
\lambda_{n, \varepsilon} = \inf \{ \sup \mathcal{E}_\varepsilon(f, f) : f \in U, \| f \|_2 = 1 \} : U < L^2(D), \dim U = n \}.
\]

We now express \( \lambda_{n, \varepsilon} \) as eigenvalues of a matrix. For \( f \in L^2(D) \) and \( k \in \mathbb{Z}^d \), let \( f_k = \varepsilon^{-d} \int_{I_k} f(x) \, dx \) be the mean value of \( f \) on \( I_k \), and define \( f^* \) to be equal to \( f_k \) on each \( I_k \), \( k \in \mathbb{Z}^d \). Hence \( f^* \in L^2(D) \) is the orthogonal projection of \( f \) onto the space of functions constant on each \( I_k \), and \( \int_{I_k} f^*(x) \, dx = \int_{I_k} f(x) \, dx \). In particular, \( \| f^* \|^2 = \| f^* \|^2_2 + \| f - f^* \|^2_2 \).

Furthermore,

\[
\mathcal{E}_\varepsilon(f, f) = \frac{c_{d, \alpha} \varepsilon^{-d - \alpha}}{2} \sum_{k, l \in \mathbb{Z}^d} v_{k-l} \int_{I_k} \int_{I_l} ((f(x))^2 - 2 f(x) f(y) + (f(y))^2) \, dx \, dy
\]

\[
= c_{d, \alpha} \varepsilon^{-\alpha} \left( \varepsilon \| f \|^2_2 - \varepsilon^d \sum_{k, l \in \mathbb{Z}^d} v_{k-l} f_k f_l \right).
\]

Since \( (f^*)_k = f_k \) for all \( k \in \mathbb{Z}^d \), and \( \| f \|^2_2 - \| f^* \|^2_2 = \| f - f^* \|^2_2 \), we obtain that

\[
\mathcal{E}_\varepsilon(f, f) = \mathcal{E}_\varepsilon(f^*, f^*) + c_{d, \alpha} \varepsilon^{-\alpha} \varepsilon \| f - f^* \|^2_2. \tag{30}
\]

This proves that the two orthogonal subspaces, \( \{ f \in L^2(D) : f^* = 0 \} \) and \( \{ f \in L^2(D) : f^* \neq f \} \), are invariant under the action of the operator corresponding to \( \mathcal{E}_\varepsilon \). By (30), the former subspace is in fact the eigenspace corresponding to the eigenvalue \( c_{d, \alpha} \varepsilon^{-\alpha} \). The latter one is finite-dimensional, and when \( f^* = f \), we have

\[
f(x) = \sum_{k \in K_\varepsilon} f_k 1_{I_k}(x), \quad \mathcal{E}_\varepsilon(f, f) = c_{d, \alpha} \varepsilon^{-d - \alpha} \left( \varepsilon \sum_{k \in K_\varepsilon} f_k^2 - \sum_{k, l \in K_\varepsilon} v_{k-l} f_k f_l \right). \tag{31}
\]

The normalized indicator functions of \( I_k \), that is, the functions \( \varepsilon^{-d/2} 1_{I_k}, k \in K_\varepsilon \), form an orthonormal basis of the space \( \{ f \in L^2(D) : f^* = f \} \). By (31), in this basis, the action of \( \mathcal{E}_\varepsilon \) is given by the following \( |K_\varepsilon| \times |K_\varepsilon| \) matrix \( V \) if \( \kappa \) is an enumeration of the elements of \( K_\varepsilon \) (that is, a bijection between \( \{1, 2, \ldots, |K_\varepsilon|\} \) and \( K_\varepsilon \)), then \( V_{p, q} = c_{d, \alpha} \varepsilon^{-\alpha} (b_{p, q} \varepsilon - v_{\kappa(p) - \kappa(q)}) \).
We conclude that the sequence $\lambda_{n, \varepsilon}$ starts with all eigenvalues of the matrix $V$ which are less than $c_{d, \alpha} \varepsilon^{-\alpha} \bar{\nu}$ (there are at most $|K_\varepsilon|$ of them), and then it is a constant sequence $c_{d, \alpha} \varepsilon^{-\alpha} \bar{\nu}$. We have thus proved the following result.

**Proposition 4.** Let $D \subseteq \mathbb{R}^d$ be an open set in $\mathbb{R}^d$, and let $\varepsilon > 0$. Let $K_\varepsilon$ be the set of those $k \in \mathbb{Z}^d$ for which $D \cap \prod_{j=1}^d [k_j \varepsilon, (k_j + 1) \varepsilon]$ is nonempty, and let $\kappa : \{1, 2, \ldots, |K_\varepsilon|\} \rightarrow K_\varepsilon$ be the enumeration of the elements of $K_\varepsilon$. Finally, for $k \in \mathbb{Z}^d$ let

$$\varphi(k) = \sqrt{\sum_{j=1}^d (|k_j| + 1)^2}, \quad \text{and} \quad \bar{\nu} = \sum_{k \in \mathbb{Z}^d} (\varphi(k))^{-d-\alpha}.$$

Define a $|K_\varepsilon| \times |K_\varepsilon|$ matrix $V$ with entries

$$V_{p,q} = -\frac{c_{d,\alpha}}{\varepsilon^\alpha} \varphi(\kappa(p) - \kappa(q))^{-d-\alpha}, \quad p, q = 1, 2, \ldots, |K_\varepsilon|, \quad p \neq q;$$

$$V_{p,p} = \frac{c_{d,\alpha}}{\varepsilon^\alpha} (\bar{\nu} - d^{-(d+\alpha)/2}), \quad p = 1, 2, \ldots, |K_\varepsilon|.$$

If $n \leq |K_\varepsilon|$ and $n$-th smallest eigenvalue of $V$ does not exceed $c_{d,\alpha} \varepsilon^{-\alpha} \bar{\nu}$, then let $\lambda_n$ be this eigenvalue. Otherwise, let $\lambda_{n,\varepsilon} = c_{d,\alpha} \varepsilon^{-\alpha} \bar{\nu}$. Then the eigenvalues $\lambda_n$ of $A_D$ satisfy $\lambda_n \geq \lambda_{n,\varepsilon}$.

Note that if $\bar{\nu}$ is replaced by a smaller number in the definition of $V$, the eigenvalues $\lambda_{n,\varepsilon}$ decrease. Hence, when doing numerical computations using Proposition 4, one should approximate $\bar{\nu}$ from below.

In the one-dimensional case, we have $c_{1,\alpha} = c_{\alpha}$, and $\bar{\nu} = 2 \zeta(1 + \alpha) - 1$, where $\zeta$ is the Riemann zeta function. Consider now $D = (-1, 1) \subseteq \mathbb{R}$, and $\varepsilon = \frac{\pi}{2}$. For simplicity, assume that $N$ is an even positive integer. Then $K_\varepsilon = \{-\frac{N}{2}, -\frac{N}{2} + 1, \ldots, \frac{N}{2} - 1\}$, so it is natural to choose $\kappa(p) = p - \frac{N}{2} - 1, \ p \in \{1, 2, \ldots, N\}$. Furthermore, $V$ is a Toeplitz matrix, that is, $V_{p,q} = V_{p-q}$ depends only on $p - q$. In this case we can prove that all eigenvalues of the matrix $V$ are less than $c_{\alpha} \varepsilon^{-\alpha} \bar{\nu}$. Indeed, the symbol of the Toeplitz matrix $V$ is given by (we omit some technical details here)

$$\sum_{k=-\infty}^{\infty} V_{k} e^{ikx} = \frac{2c_{\alpha}}{\varepsilon^\alpha} \left( \zeta(1 + \alpha) - \sum_{k=0}^{\infty} \frac{\cos(kx)}{(1+k)^{1+\alpha}} \right)$$

$$= \frac{2c_{\alpha}}{\varepsilon^\alpha} \left( \zeta(1 + \alpha) - \text{Re} \left( \frac{\text{Li}_{1+\alpha}(e^{ix})}{e^{ix}} \right) \right)$$

$$= \frac{2c_{\alpha}}{\varepsilon^\alpha} \left( \zeta(1 + \alpha) - \frac{1}{1+\alpha} \int_{0}^{\infty} t^{\alpha} (e^t - \cos x) \frac{dt}{e^{2t} - 2e^t \cos x + 1} \right).$$

The right-hand side is easily checked to be symmetric, $2\pi$-periodic and increasing in $x \in [0, \pi]$, and so it attains its global maximum for $x = \pi$. The symbol of $V$ is therefore bounded above by $2c_{\alpha} \varepsilon^{-\alpha}(\zeta(1 + \alpha) - \text{Li}_{1+\alpha}(-1)) = 2^{1-\alpha} c_{\alpha} \varepsilon^{-\alpha} \zeta(1 + \alpha) \leq c_{\alpha} \varepsilon^{-\alpha} \bar{\nu}$. By a general result, the eigenvalues of $V$ are bounded above by the supremum of the symbol. It follows that all $N$ eigenvalues
Table 3
Comparison of estimates of $\lambda_n$ for a square $(-1,1)^2$. LB and UB mean lower bounds and upper bounds respectively. Estimates of this section are given in roman font, best numerical estimates known before are typeset in slanted font. Better estimates are underlined.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\lambda_1$ (LB)</th>
<th>$\lambda_1$ (UB)</th>
<th>$\lambda_2$ (LB)</th>
<th>$\lambda_2$ (UB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.0308</td>
<td>0.5230</td>
<td>1.0462</td>
<td>0.5415</td>
</tr>
<tr>
<td>0.2</td>
<td>1.0506</td>
<td>0.5472</td>
<td>1.0946</td>
<td>0.5865</td>
</tr>
<tr>
<td>0.5</td>
<td>1.1587</td>
<td>0.6266</td>
<td>1.2534</td>
<td>0.7452</td>
</tr>
<tr>
<td>1</td>
<td>1.3844</td>
<td>0.7853</td>
<td>1.5708</td>
<td>1.1107</td>
</tr>
<tr>
<td>1.5</td>
<td>1.4135</td>
<td>0.9843</td>
<td>1.9688</td>
<td>1.6554</td>
</tr>
<tr>
<td>1.8</td>
<td>0.9167</td>
<td>1.1271</td>
<td>2.2544</td>
<td>2.1033</td>
</tr>
<tr>
<td>1.9</td>
<td>0.5427</td>
<td>1.1792</td>
<td>2.3585</td>
<td>2.2781</td>
</tr>
</tbody>
</table>


Table 4
Comparison of estimates of $\lambda_n$ for a unit disk. LB and UB mean lower bounds and upper bounds respectively. Estimates of this section are given in roman font, best numerical estimates known before are typeset in slanted font. Better estimates are underlined.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\lambda_1$ (LB)</th>
<th>$\lambda_1$ (UB)</th>
<th>$\lambda_2$ (LB)</th>
<th>$\lambda_2$ (UB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.0381</td>
<td>1.0157</td>
<td>1.0953</td>
<td>0.5718</td>
</tr>
<tr>
<td>0.2</td>
<td>1.0655</td>
<td>1.0396</td>
<td>1.1342</td>
<td>0.6541</td>
</tr>
<tr>
<td>0.5</td>
<td>1.1986</td>
<td>1.1618</td>
<td>1.3943</td>
<td>0.9787</td>
</tr>
<tr>
<td>1</td>
<td>1.4734</td>
<td>1.5707</td>
<td>2.0944</td>
<td>1.9158</td>
</tr>
<tr>
<td>1.5</td>
<td>1.5387</td>
<td>2.3891</td>
<td>3.4131</td>
<td>3.7502</td>
</tr>
<tr>
<td>1.8</td>
<td>1.0087</td>
<td>2.2210</td>
<td>5.014</td>
<td>5.6114</td>
</tr>
<tr>
<td>1.9</td>
<td>0.5990</td>
<td>3.5834</td>
<td>6.1369</td>
<td>6.4182</td>
</tr>
</tbody>
</table>

1 See [7].

2 See [1].

of $V$ are included in the sequence $\lambda_{n,\varepsilon}$, as desired. Therefore, we have the following specialized version of Proposition 4 (the case of odd $N$ is very similar).

**Proposition 5.** Let $D = (-1,1)$, $N > 0$ and $\varepsilon = 2/N$. Let $V$ be an $N \times N$ Toeplitz matrix with entries

$$V_{p,q} = -\frac{c\alpha \varepsilon^\alpha}{(|p-q|+1)^{1+\alpha}}, \quad p, q = 1, 2, \ldots, N, \ p \neq q;$$

$$V_{p,p} = \frac{2c\alpha(\zeta(1+\alpha)-1)}{\varepsilon^\alpha}, \quad p = 1, 2, \ldots, N.$$

Define $\lambda_{n,\varepsilon}$ to be the $n$-th smallest eigenvalue of $V$ when $n \leq N$, and $\lambda_{n,\varepsilon} = c\alpha \varepsilon^{-\alpha} (2\zeta(1+\alpha) - 1)$ for $n > N$. Then the eigenvalues $\lambda_n$ of $A_D$ satisfy $\lambda_n \geq \lambda_{n,\varepsilon}$.

The lower bounds $\lambda_{n,\varepsilon}$ for the interval $D = (-1,1)$ are presented in Table 2 above. In higher dimensions, the complexity of computations increases dramatically. For example, a unit disk $B(0,1)$ or a square $(-1,1)^2$ with $\varepsilon = \frac{1}{25}$ require handling matrices larger than $2000 \times 2000$. Some results for these two cases are given in Tables 3 and 4.
In principle, the upper bound is much more difficult. The above approach can be modified to give an upper bound for the first eigenvalue $\lambda_1$ whenever the Green function for $D$ can be computed. For the fractional Laplace operator, this is the case when $D$ is a ball. By a scaling property, it is enough to consider $D = B(0, 1)$.

Let $G_D(x, y)$ be the Green function of $D$, $G_D(x, y) = \int_0^\infty p_D^t (x, y) dt$, where $p_D^t$ is the heat kernel for $A_D$ (see the proof of Proposition 2). The Green function is the kernel of $A_D^{-1}$. M. Riesz proved that

$$G_D(x, y) = \frac{\Gamma(\frac{d}{2})|x - y|^{\alpha/d}}{2^{\alpha/2}\pi^{d/2}(\Gamma(\frac{\alpha}{2}))^2} \int_0^{s^{\alpha/2-1}} \frac{r^{\alpha/2-1}}{(1 + r)^{d/2}} dr = \frac{\Gamma(\frac{d}{2})(1 - x^2)^{\alpha/2}(1 - y^2)^{\alpha/2}}{2^{\alpha/2}\pi^{d/2}\Gamma(\frac{\alpha}{2})(1 + \frac{\alpha}{2})|x - y|^{\alpha/2}} F_1\left(\frac{\alpha}{2}, \frac{d}{2}; 1 + \frac{\alpha}{2}; \frac{1 - x^2(1 - y^2)}{|x - y|^2}\right).$$

The eigenvalues of the Green operator $G_D = A_D^{-1}$ are $\lambda_n^{-1}$. Hence,

$$\frac{1}{\lambda_1} = \sup\left\{ \int_D \int_D G_D(x, y) f(x) f(y) dxdy : f \in L^2(D), \|f\|_2 = 1 \right\}.$$

Since $G_D(x, y)$ is nonnegative, we may restrict the supremum to nonnegative functions only. It follows that whenever $0 \leq g(x, y) \leq G_D(x, y)$, we have

$$\lambda_1 \leq \left( \sup\left\{ \int_D \int_D g(x, y) f(x) f(y) dxdy : f \in L^2(D), \|f\|_2 = 1 \right\} \right)^{-1}.$$ 

For $k, l \in \mathbb{Z}^d$, let $g_{k,l}$ be the infimum of $G_D(u, v)$ over $u \in I_k$ and $v \in I_l$. When $x \in I_k$, $y \in I_l$, we choose $g(x, y) = g_{k,l}$. Hence, by an argument similar to one used for the lower bounds, $\lambda_1$ is bounded above by $\lambda_{1,\epsilon}^*$, the reciprocal of the largest eigenvalue of the matrix $U$ with entries $U_{i,j} = \epsilon^d g_{k(i),k(j)}$.

The results for $D = (-1, 1)^2 \subseteq \mathbb{R}$ and some values of $\alpha$ are given in Table 2. Estimates for the unit disk and the square $(-1, 1)^2$ are given in Tables 3 and 4. Noteworthy, for the unit disk and $\epsilon = \frac{1}{125}$, the estimate $\lambda_{1,\epsilon}^*$ is worse than the one obtained in [1] using analytical methods.

Acknowledgments

I would like to thank Krzysztof Bogdan and Tadeusz Kulczycki for helpful discussion and valuable suggestions. I am very grateful to the referee for a number of helpful comments, which greatly helped improve the quality of the article. Tables and plots were prepared using Wolfram Mathematica 8.0.1 at the Wroclaw University of Technology.

References


