An Inverse Problem for Toeplitz Matrices*

Bradley W. Dickinson

Department of Electrical Engineering and Computer Science
Princeton University
Princeton, New Jersey 08544

Submitted by Stephen Barnett

ABSTRACT

A simplified solution to an inverse problem for Toeplitz matrices using central mass sequences is presented. Some connections with discrete transmission lines are mentioned.

1. INTRODUCTION

In this note, we will give a simplified solution to an inverse problem for Toeplitz matrices which was solved by Caflisch [1] in a more complicated way. In [1], no use was made of the well-known Levinson-Durbin recursion associated with a positive definite Toeplitz matrix. In particular, the parameterization in terms of reflection coefficients may be directly employed to provide the desired results.

First, we give some notation and background. Let \( T_m \) be a positive definite, symmetric \( m \times m \) Toeplitz matrix with \((i, j)\)th element \( t_{ij} \). For \( 1 \leq n \leq m \), the principal \( n \times n \) submatrix of \( T_m \) is simply \( T_n \). The central mass sequence \( \{ \rho_1, \ldots, \rho_m \} \) associated with \( T_m \) is defined as

\[
\rho_n = \sup \{ \rho : T_n - \rho \Pi_n > 0 \} \tag{1}
\]

where \( \Pi_n \) is the \( n \times n \) matrix whose entries are all 1, and the inequality denotes that the matrix on the left-hand side is positive definite. We will give formulas for the elements of the central mass sequence in terms of the reflection coefficient sequence, or Szegö parameter sequence, \( \{ k_0, \ldots, k_{m-1} \} \), which is easily computed by applying the Levinson-Durbin algorithm to \( T_m \). The inverse problem of computing \( T_m \) from its central mass sequence is thus

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*This work was supported in part by the National Science Foundation through grant ECS82-05772.
reduced to a well-known procedure for obtaining $T_m$ from the corresponding reflection-coefficient sequence.

We will briefly review the Levinson-Durbin algorithm [2, 3]. For more elaboration, see [4]–[6]. A Cholesky factorization of the inverse of the Toeplitz matrix $T_m$ takes the form

$$T_m^{-1} = A'_m \Lambda_m A_m,$$

where $A'_m$ is a unit lower triangular matrix and $\Lambda_m$ is a diagonal matrix. The Levinson-Durbin recursions provide a way of computing the successive rows of the Cholesky factor $A_m$ and the diagonal elements of $\Lambda_m$. Let $(a_{n-1,n-1}, a_{n-1,n-2}, \ldots, a_{n-1,1}, 1, 0, \ldots, 0)$ denote the $n$th row of $A_m$ for $1 > n > m$. The first row of $A_m$ is $(1, 0, \ldots, 0)$. Also, let $\Lambda_m = \text{diag}(\lambda_0, \ldots, \lambda_{m-1})$. Then the Levinson-Durbin recursions are given by

$$k_{i+1} = a_{i+1,i+1} = -\left( t_{i+1} + \sum_{j=1}^{i} a_{i,j} t_{i+1-j} \right) \lambda_i,$$  \hfill (3.1)  

$$a_{i+1,j} = a_{i,j} + k_{i+1} a_{i,i+1-j}, \quad 1 \leq j \leq i,$$  \hfill (3.2)  

$$\lambda_{i+1} = \frac{\lambda_i}{1 - k_{i+1}^2}.$$  \hfill (3.3)  

The recursions are initialized with

$$k_0 = \frac{1}{\lambda_0} = t_0,$$  \hfill (3.4)  

$$k_1 = a_{1,1} = \frac{t_1}{t_0}.$$  \hfill (3.5)  

Because $T_m$ is assumed to be positive definite, it follows from (2) and the form of elements of $\Lambda_m$ in Equation (3.3) that $k_0 > 0$ and $|k_i| < 1$ for $1 \leq i \leq m - 1$. Thus, these recursions can be used to compute the reflection-coefficient sequence from the Toeplitz matrix elements, or the Toeplitz matrix elements may be computed from the reflection-coefficient sequence.

2. MAIN RESULT

To make the connection between the central mass sequence and the reflection-coefficient sequence, we start with an observation of Caflisch [1].
By the definition of $\rho_n$, it follows that

$$\det(T_n - \rho_n \Pi_n) = 0. \quad (4)$$

We use the standard formula for the determinant of a rank-one modification of a nonsingular matrix, since

$$\Pi_n = \pi_n \pi_n^\prime,$$  \quad (5)

where prime denotes transpose and $\pi_n$ is the $n$-dimensional column vector whose entries are all 1. Thus

$$\det(T_n - \rho_n \Pi_n) = \det T_n \cdot (1 - \rho_n \pi_n^\prime T_n^{-1} \pi_n),$$ \quad (6)

so that

$$\frac{1}{\rho_n} = \pi_n^\prime T_n^{-1} \pi_n.$$ \quad (7)

In words, $\rho_n$ is the reciprocal of the sum of the elements of the matrix $T_n^{-1}$.

In work on detection of discrete-time signals, the expression for $1/\rho_n$ arises as the signal-to-noise ratio of an optimum detector based on $n$ observations of a unit-magnitude signal received in additive Gaussian noise whose covariance matrix is $T_n$. In [7], we derived an explicit formula for the expression in terms of the reflection-coefficient sequence. The derivation involved using (2) and the following relation:

$$1 + \sum_{j=1}^{i} a_{i,j} = \prod_{j=1}^{i} (1 + k_j). \quad (8)$$

Equation (8) is due to Cybenko [8] and is proved by induction using the Levinson-Durbin recursions. The resulting formula is

$$\frac{1}{\rho_n} = \pi_n^\prime T_n^{-1} \pi_n = \frac{1}{k_0} \left[ 1 + \sum_{i=1}^{n-1} \prod_{j=1}^{i} \frac{1 + k_j}{1 - k_j} \right]. \quad (9)$$

From (9), and the conditions obeyed by reflection-coefficient sequences, namely $k_0 > 0$ and $|k_i| < 1$, $i > 1$, it follows that a central mass sequence
obey the conditions $\rho_1 > 0$ and $\rho_{i+1} < \rho_i$ for $i \geq 1$. The central mass sequence may be computed from the Toeplitz coefficients by adding equation (9) to the Levinson-Durbin algorithm; alternatively, the Toeplitz coefficients may be computed from the central mass sequence by using (9) to solve for the reflection-coefficient sequence.

3. DISCUSSION

The relation between the central mass sequence and discrete transmission lines is discussed by Caflisch [1]. We will give only a brief description here. The current and voltage in a transmission line are determined by the telegraph equations:

\begin{align}
   v(x, t) &= -L(x)i(x, t), \quad (10.1) \\
   i(x, t) &= -C(x)v(x, t), \quad (10.2)
\end{align}

where $L(x)$ and $C(x)$ are the inductance and capacitance tapers of the line. We consider only the case where $L(x) = \frac{1}{C(x)}$ and take the line to be half-infinite and piecewise constant. The capacitance taper is given by

\begin{equation}
   C(x) = C_n \quad \text{for } \frac{n-1}{2} \leq x < \frac{n}{2} \quad \text{and } n \geq 1. \quad (11)
\end{equation}

For an initially quiescent line, the transfer function between $i(0, t)$ and $v(0, t)$ corresponds to a sum of weighted delays:

\begin{equation}
   H(s) = \sum_{n=0}^{\infty} h_n e^{-ns}. \quad (12)
\end{equation}

Caflisch [1] showed that with the Toeplitz matrix elements defined by $t_0 = h_0$ and $t_j = h_j/2$, $j \geq 1$, the central mass sequence may be expressed as

\begin{equation}
   \frac{1}{\rho_n} = \sigma_n T_n = \sum_{j=1}^{n} C_j. \quad (13)
\end{equation}

From (9), it follows that

\begin{equation}
   \frac{C_{j+1}}{C_j} = \frac{1 + k_j}{1 - k_j}, \quad j \geq 1, \quad (14)
\end{equation}
with $C_1 = 1/k_0$. Solving for the reflection coefficients gives

$$k_j = \frac{C_{j+1} - C_j}{C_{j+1} + C_j},$$

which is the expression for the electromagnetic reflection coefficients of the transmission line [1].

There is an alternate method for computing the reflection coefficients from the Toeplitz matrix elements, the Schur algorithm [9], which also has interesting network-theoretic and stochastic interpretations. For a discussion, see Dewilde, Vieira, and Kailath [10].

REFERENCES


Received 11 January 1983