JOURNAL OF FUNCTIONAL ANALYSIS 58, 20–52 (1984)

Polynomials of the Occupation Field and Related Random Fields*

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Communicated by L. Gross

Received January 23, 1984; revised January 30, 1984

To every symmetric Markov process there correspond two random fields over the state space: a Gaussian ("free") field ϕ_x and the occupation field T_x which describes the amount of time spent by a particle at each state. For the Brownian motion in $d \geqslant 2$ dimensions both fields are generalized. Using a relation between T_x and the field $\xi_x = :\phi_x^2:/2$ established in a previous publication, polynomials of the fields T and ξ are investigated. In particular, polynomials of T characterize self-intersections of the process.

1. INTRODUCTION

1.1. A typical example of a symmetric Markov process is the Brownian motion in $\mathbb{R}^d$ with killing rate k. This is a Markov process with the transition function

$$p_t(x, B) = \int_B p_t(x, y)\, m(dy), \tag{1.1}$$

where m is the Lebesgue measure and the transition density $p_t(x, y)$ is defined by the formulae

$$p_t(x, y) = t^{-d/2} e^{-kt} p\left(\frac{y-x}{\sqrt{t}}\right),$$
$$p(z) = (2\pi)^{-d/2}\, e^{-|z|^2/2}. \tag{1.2}$$

A path in $\mathbb{R}^d$ is a mapping w from an open interval (α, ζ) to $\mathbb{R}^d$ (α is the

* Research supported in part by NSF Grant MCS-8202286.

0022-1236/84 $3.00

birth-time and ζ is the death-time of w). There exists a measure P on the space W of all continuous paths such that

$$P(\alpha < t_1, w(t_1) \in B_1, \ldots, w(t_n) \in B_n, t_n < \zeta)$$
$$= \int_{B_1} \cdots \int_{B_n} m(dz_1) p_{t_2 - t_1}(z_1, dz_2) \cdots p_{t_n - t_{n-1}}(z_{n-1}, dz_n) \quad (1.3)$$

for all $t_1 < \cdots < t_n$ and all Borel sets $B_1, \ldots, B_n$. ($\alpha = -\infty$, $\zeta = +\infty$ P—a.s. if $k = 0$, and $-\infty < \alpha < \zeta < +\infty$ P—a.s. if $k > 0$.)

Along with P we introduce, for every x, y, a measure P_{xy}, concentrated on paths with $\alpha = 0$, $w(0+) = x$, $w(\zeta-) = y$, such that

$$P_{xy}(\alpha < t_1, w(t_1) \in B_1, \ldots, w(t_n) \in B_n, t_n < \zeta) \quad (1.4)$$
$$= \int_{B_1} \cdots \int_{B_n} p_{t_1}(x, dz_1) p_{t_2 - t_1}(z_1, dz_2) \cdots p_{t_n - t_{n-1}}(z_{n-1}, dz_n) g(z_n, y)$$

for all $0 < t_1 < t_2 \cdots < t_n$ and all Borel sets $B_1, \ldots, B_n$. Here

$$g(x, y) = \int_0^\infty p_t(x, y)\, dt \quad (1.5)$$

is the so-called Green's function. Heuristically, $P_{xy} = g(x, y) \times$ the probability law of the path conditioned to be born at time 0 at point x and to die at point y. To every pair of measures μ, ν there corresponds a measure

$$P_{\mu\nu} = \int P_{xy} \mu(dx)\, \nu(dy). \quad (1.6)$$

As usual, we put $X_t(w) = w(t)$.

1.2. In the case $d = 1$, there exists, for every z, a random measure A_z on $\mathbb{R}$ (local time at z) concentrated on (α, ζ) such that

$$\int_I p_s(z, X_u)\, du \to A_z(I) \qquad \text{in} \quad L^2(P) \text{ as } s \downarrow 0 \quad (1.7)$$

for every finite open interval I. Since $p_s(z, \cdot)$ tends to the delta function δ_z as $s \downarrow 0$, it is natural to write

$$A_z(I) = \int_I \delta_z(X_u)\, du. \quad (1.8)$$

The occupation field is defined by the formula

$$T_z = A_z(0, +\infty) = \int_0^\infty \delta_z(X_u)\, du. \quad (1.9)$$

For $d \geqslant 2$, T is a generalized random field. We denote by M_k the set of all σ-finite measures λ such that

$$\int \lambda(dx)\, g(x,y)^k\, \lambda(dy) < \infty. \tag{1.10}$$

If $\lambda \in M_1$, then there exists a random measure A_λ such that, for every finite open interval I,

$$A_\lambda(I) = \lim_{s \downarrow 0} \int_I p_s^\lambda(X_u)\, du \qquad \text{in} \quad L^2(P), \tag{1.11}$$

where

$$p_s^\lambda(x) = \int p_s(x,y)\, \lambda(dy). \tag{1.12}$$

(The integrand in formulae (1.8) and (1.11) is not defined for $t \notin (\alpha, \zeta)$ and we put it equal to 0.) We can get (1.11) by formally intergrating (1.8) with respect to λ, and we use the symbolism

$$T_\lambda(I) = \int_I du \int \lambda(dz)\, \delta_z(X_u).$$

The occupation field is defined by the formula

$$T_\lambda = \int_0^\infty du \int \lambda(dz)\, \delta_z(X_u). \tag{1.13}$$

We put

$$T_{t,z} = \int_0^\infty p_t(z, X_u)\, du. \tag{1.14}$$

It turns out (see Sect. 5) that

$$T_{t,z} = \int p_t(z, dx)\, T_x. \tag{1.15}$$

Formula (1.11) means that, in a certain sense, the generalized field T is the limit of $T_{t,z}$ as $t \downarrow 0$.

1.3. The powers of the occupation field are defined by a limit procedure starting from

$$\frac{:T_{t,z}^n:}{n!} = \sum_{r=0}^{n} B_{nr}(t,z)\, T_{t,z}^r. \tag{1.16}$$

To define the coefficients B_{nr}, we introduce the functions

$$C_1 = 1,$$
$$C_j(t, z) = \int p_t(z, dx_1) \cdots p_t(z, dx_j)\, g(x_1, x_2) \cdots g(x_{j-1}, x_j), \qquad j = 2, 3, \ldots, \tag{1.17}$$

and we consider the generating function

$$\mathscr{C}(u) = \sum_{j=1}^{\infty} C_j u^j \tag{1.18}$$

and its inverse $\mathscr{D}$. We define B_{nr} as the coefficients in the expansion

$$e^{\mathscr{D}(u)v} = \sum_{n,r=0}^{\infty} B_{nr}(t, z)\, u^n v^r. \tag{1.19}$$

Formula (1.16) can be rewritten in the form

$$\sum_{n=0}^{\infty} \frac{:T^n_{t,z}:}{n!} u^n = e^{\mathscr{D}(u) T_{t,z}}. \tag{1.20}$$

For $d = 1$, T_z is well defined for every z and we put

$$\frac{:T^n_z:}{n!} = Q_n(T_z),$$

where

$$Q_n(v) = \sum_{r=0}^{n} \beta_{nr} v^r, \qquad \beta_{nr} = \lim_{t \downarrow 0} B_{nr}(t, z).$$

By passing to the limit in (1.19), we get

$$\sum_{n,r=0}^{\infty} \beta_{nr} u^n v^r = \sum Q_n(v)\, u^n = \exp\{uv/1 + uh\}, \tag{1.21}$$

where $h = g(z, z)(k/2)^{1/2}$. By comparing this expression with the generating function of the Laguerre polynomials L^α_n, we arrive at the formula $Q_n(v) = (-h)^n L^{(-1)}_n(v/h)$. (Usually, the Laguerre polynomials are considered only for $\alpha > -1$.)

Passage to the limit in the case $d = 2$ is more sophisticated. We say that a measure λ is *admissible* if $\lambda(dz) = f(z)\, m(dz)$ and $\int f^k\, dm < \infty$ for all $k = 1, 2, \ldots$.

THEOREM 1.1. *Suppose that $d \leqslant 2$ and $k > 0$. There exists a random* *field* $:T^n:_\lambda$ *indexed by admissible measures* λ *such that, for all* $\mu, \nu \in N = M_1 \cap M_2$,

$$:T^n:_\lambda = \lim_{t \downarrow 0} \int \lambda(dz) :T^n_{t,z}: \qquad in \quad L^2(P_{\mu\nu}). \tag{1.22}$$

The identity

$$T^n_{t,z} = \int_0^\infty du_1 \cdots \int_0^\infty du_n \, p_t(z, X_{u_1}) \cdots p_t(z, X_{u_n}) \tag{1.23}$$

suggests the symbolism

$$:T^n_z: = \int_0^\infty du_1 \cdots \int_0^\infty du_n \, :\delta_z(X_{u_1}) \cdots \delta_z(X_{u_n}): . \tag{1.24}$$

Heuristically, $:T^n_z:$ describes the "size" of the random set $\{(u_1, ..., u_n): w(u_1) = \cdots = w(u_n) = z\}$. In other words, it characterizes the space location of self-intersections of order n.

1.4. To prove Theorem 1.1 we use a relation between the occupation field and the free field.

For $d = 1$, the free field can be defined as a Gaussian field ϕ_x such that $\langle \phi_x \rangle = 0$ and $\langle \phi_x \phi_y \rangle = g(x, y)$. (The random variables ϕ_x are defined on a probability space $(\Omega, \mathscr{F}, \Pi)$ which is totally unrelated to W and $\langle \ \rangle$ means the integral with respect to Π).[1]

If $d > 1$, then $g(x, x) = \infty$; however,

$$g_s(x, y) = \int_s^\infty p_t(x, y) \, dt, \qquad s > 0, \tag{1.25}$$

is finite for all x, y (if $k > 0$). There exists a Gaussian system $\phi_{s,x} s > 0$, $x \in \mathbb{R}^d$ such that $\langle \phi_{s,x} \rangle = 0$, $\langle \phi_{s,x} \phi_{t,y} \rangle = g_{s+t}(x, y)$. The function $\phi_{s,x}(\omega)$ can be chosen to be measurable in x, ω. The free field is a generalized field which is, in a sense, the limit of $\phi_{s,x}$ as $s \downarrow 0$. More precisely, for every $\lambda \in M_1$, there exists an $L^2(\Pi)$-limit[2]

$$\phi_\lambda = \lim_{s \downarrow 0} \int \phi_{s,x} \lambda(dx). \tag{1.26}$$

[1] Actually, $g(x, y) = (c/2) \, e^{-c|y-x|}$, where $c = \sqrt{2k}$ and ϕ_x is a stationary Ornstein–Uhlenbeck process.

[2] Integrals in formulae (1.26) and (1.28) are a certain type of improper integral described in Subsection 2.2.

Wick's powers $:Y^n:$ of a Gaussian random variable Y with mean 0 are defined by the generating function

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} :Y^n: = \exp(uY - u^2\langle Y^2\rangle/2). \tag{1.27}$$

In particular, $:Y^0: = 1$, $:Y^1: = Y$, $:Y^2: = Y^2 - \langle Y^2\rangle$. If $\lambda \in M_n$, then there exists an $L^2(\Pi)$-limit

$$:\phi^n:_\lambda = \lim_{s\downarrow 0} \int :\phi^n_{s,z}: \lambda(dz). \tag{1.28}$$

It is common to write $\int :\phi^n_z: \lambda(dz)$ for $:\phi^n:_\lambda$. The most important for us is the field

$$\xi_\lambda = :\phi^2:_\lambda/2, \qquad \lambda \in M_2.$$

The class M_2 contains all admissible measures if $d \leqslant 3$. If $d > 3$, then it contains only the null measure and the field ξ is trivial.

1.5. The functions ξ_λ and $:\varphi^n:_\lambda$ are defined up to Π-equivalence. So are the functions

$$Y = f(\xi_{\lambda_1}, ..., \xi_{\lambda_n}, ...), \tag{1.29}$$

where f is a Borel function on $\mathbb{R}^\infty$ and $\lambda_1, ..., \lambda_n, ..., \in N$. It has been proved in [1, see Theorem 6.1] that, if f is bounded, then for all $\mu, \nu \in N$,

$$\Pi_{\mu\nu} Y = P_{\mu\nu}\langle Y^*\rangle, \tag{1.30}$$

where

$$Y^* = f(\xi_{\lambda_1} + T_{\lambda_1}, ..., \xi_{\lambda_n} + T_{\lambda_n}, ...)$$

(for all versions of T_{λ_n}).

We extend (1.30) to all positive f by a monotone passage to the limit and then to all Borel f such that either $\Pi_{\mu\nu}|Y| < \infty$ or $P_{\mu\nu}\langle|Y^*|\rangle < \infty$, by linearity.

We say that two functions F_1, F_2 on $\Omega \times W$ are equivalent and we write $F_1 \approx F_2$ if $F_1 = F_2$ $\Pi \times P_{\mu\nu}$—almost everywhere for all $\mu, \nu \in N$. It follows from (1.30) that Y^* is defined up to equivalence. (In particular, T_λ is defined up to $P_{\mu\nu}$—equivalence for all $\mu, \nu \in N$.)

We denote by $\mathscr{F}_\xi$ the σ-algebra in Ω generated by the functions ξ_λ, $\lambda \in N$ and all sets of Π—measure 0. Every $\mathscr{F}_\xi$-measurable function Y is

Π-equivalent to a function of the form (1.29). Hence we have a map $Y \to Y^*$ from the space of classes of Π-equivalent $\mathscr{F}_\xi$-measurable functions on Ω to the space of classes of equivalent functions on $\Omega \times W$.

We note that if

$$Y = h(Y_1, \ldots, Y_n, \ldots) \qquad \Pi\text{—a.s.}, \tag{1.31}$$

then

$$Y^* \approx h(Y_1^*, \ldots, Y_n^*, \ldots). \tag{1.32}$$

In particular, if

$$Y = \lim Y_n \qquad \Pi\text{—a.s.} \tag{1.33}$$

then

$$Y^* \approx \lim Y_n^* \tag{1.34}$$

1.6. We write $Y = \operatorname{Lim} Y_n$ if $\langle (Y_n - Y)^p \rangle \to 0$ for every $p \geqslant 1$.

The proof of Theorem 1.1 is based on the identity (1.30) and the following result.

THEOREM 1.2. *Suppose $d \leqslant 2$ and $k > 0$. Then $:\varphi^n:_\lambda$ is defined for all n and all admissible measures λ. For every $t > 0$, there exist:*

(i) *a $\mathscr{B} \times \mathscr{F}_\xi$-measurable function $\xi_{t,z}(w)$ such that*

$$\xi_{t,z} = \int p_t(z, dx)\, \xi_x \qquad \Pi\text{—a.s.} \qquad \textit{for each} \quad z \in E; \tag{1.35}$$

(ii) *$\mathscr{B}$-measurable functions $b_{nr}(t, z)$, $0 \leqslant r \leqslant n$ such that, for all admissible measures λ,*

$$:\varphi^{2n}:_\lambda / 2^n = \operatorname*{Lim}_{t \downarrow 0} \int :\xi_{t,z}^n: \lambda(dz), \tag{1.36}$$

where

$$:\xi_{t,z}^n: = \sum_{r=0}^{n} b_{nr}(t, z)\, \xi_{t,z}^r. \tag{1.37}$$

We put

$$:\xi^n:_\lambda = :\varphi^{2n}:_\lambda / 2^n. \tag{1.38}$$

To describe b_{nr}, we introduce the functions

$$L_j(t, z) = \frac{1}{2j} \int p_t(z, dx_1) \cdots p_t(z, dx_j)\, g(x_1, x_2) \cdots g(x_{j-1}, x_j)\, g(x_j, x_1) \tag{1.39}$$

and the generating function

$$\mathscr{L}(u) = \sum_{j=2} L_j u^j. \tag{1.40}$$

The functions $b_{nr}(t, z)$ are defined by the formula

$$e^{v\mathscr{L}(u) - \mathscr{L}(\mathscr{L}(u))} = \sum_{n,r=0}^{\infty} b_{nr}(t, z) \frac{u^n}{n!} v^r. \tag{1.41}$$

In the case $d = 1$, $\xi_z = (\phi_z^2 - h)/2$ is defined for every z and we put

$$:\xi_z^n: = R_n(\xi_z), \tag{1.42}$$

where

$$R_n(v) = \sum_{r=0}^{n} \gamma_{nr} v^r, \qquad \gamma_{nr} = \lim_{t \downarrow 0} b_{ur}(t, z).$$

We get from (1.41) that

$$(1 + hu)^{-1/2} \exp\left\{\frac{v + h/2}{1 + hu} u\right\} = \sum_{n,r=0}^{\infty} \gamma_{nr} \frac{u^n}{n!} v^r = \sum R_n(v) \frac{u^n}{n!}. \tag{1.43}$$

Hence $R_n(v) = (-h)^n\, n!\, \mathscr{L}_n^{(-1/2)}((v/h) + (1/2))$, where $\mathscr{L}_n^{(-1/2)}$ are the Laguerre polynomials corresponding to the Γ-distribution with parameter $\frac{1}{2}$.

It is easy to check that $:T_z^n: = \langle :\xi_z^n: * \rangle$. We show in Section 5 that there is an analogous relation between $:T^n:_\lambda$ and $:\xi^n:_\lambda$ in the case $d = 2$.

1.7. Using the identity (1.30) we deduce from Theorem 1.2 the following result which contains Theorem 1.1 as a particular case.

THEOREM 1.3. *Under the conditions of Theorem* 1.1, *there exists, for every* $i, j \geqslant 0$, *a field* $:\xi^i: :T^j:$ *indexed by admissible measures* λ *such that, for all* $\mu, \nu \in N$,

$$(:\xi^i: :T^j:)_\lambda = \lim_{t \downarrow 0} \int \lambda(dz) :\xi^i_{t,z}: :T^j_{t,z}: \qquad \textit{in} \quad L^2(\Pi \times P_{\mu\nu}). \tag{1.44}$$

For every polynomial

$$f(u, v) = \sum c_{ij} u^i v^j,$$

we write

$$:f(\xi, T):_\lambda = \sum c_{ij}(:\xi^i: :T^j:)_\lambda. \tag{1.45}$$

If $f(u)$ is a polynomial, then

$$(:f(\xi):_\lambda)^* = :f(\xi + T):_\lambda. \tag{1.46}$$

We write, symbolically,

$$(:\xi^i: :T^j:)_\lambda = \int :\xi_z^i: :T_z^j: \lambda(dz).$$

1.8. Theorems 1.1, 1.2, and 1.3 will be proved for right Markov processes in a measurable state space $(E, \mathscr{B})$ with a symmetric transition density (see [1, Sect. 4]) which has the property: there exist positive constants, c, c_k, δ_k, δ such that:

1.8(A) For all $t > 0$, $x \in E$,

$$1 - p_t(x, E) \leqslant ct^\delta.$$

1.8(B) For every $k = 1, 2, \ldots$,

$$\int g(x, y)^k \, m(dy) \leqslant c_k \qquad \text{for all } x.$$

1.8(C) For every k,

$$\int p_t(z, dx) \, p_t(z, dy) \, g(x, y)^k \leqslant c_k \, |\log t|^{\delta_k}$$

for all z and all sufficiently small $t > 0$.

1.8(D) For all x and all sufficiently small t,

$$\int p_t(x, dy)(g(x, z) - g(y, z))^2 \, m(dz) < ct^\delta.$$

In addition, we assume that:

1.8(E) There exists a separable topology in the state space E such that $g_s(x, y)$ is continuous in x, y for every $s > 0$ and $\mathscr{B}$ is the Borel σ-algebra.

In the Appendix we check that conditions 1.8(A) through 1.8(E) are fulfilled for the Brownian motion in $\mathbb{R}^d$ for $d \leqslant 2$ and $k > 0$. They are satisfied also for a wide class of transient symmetric diffusions on two-dimensional manifolds.

1.9. This paper is a continuation of the article [1] to which we refer for the history of the subject.

2. Preliminaries

2.1. Let $\mathscr{M}$ be a family of finite measures on a measurable space $(E, \mathscr{B})$. We denote by $\mathscr{B}^\mu$ the completion of $\mathscr{B}$ with respect to a measure μ and by $\mathscr{B}^\mathscr{M}$ the intersection of $\mathscr{B}^\mu$ over all $\mu \in \mathscr{M}$. We say that a set $B \subset E$ is $\mathscr{M}$-negligible if $B \in \mathscr{B}^\mathscr{M}$ and if $\mu(B) = 0$ for all $\mu \in M$. If the set $\{f \neq g\}$ is $\mathscr{M}$-negligible, then we say that f and g are $\mathscr{M}$-equivalent and we write $f = g$ $\mathscr{M}$—a.s.

We do not distinguish $\mathscr{M}$-equivalent functions. We denote by $L^2(\mathscr{M})$ the set of all $\mathscr{B}^\mathscr{M}$-measurable functions f such that $\|f\|_\mu = (\mu(f^2))^{1/2} < \infty$ for all $\mu \in \mathscr{M}$. We introduce a topology into the linear space $L^2(\mathscr{M})$ using the family of norms $\|f\|_\mu$, $\mu \in \mathscr{M}$. It follows from ([2, Theorem 3]) that, if $f_m - f_n \to 0$ in $L^2(\mathscr{M})$ as $m, n \to \infty$, then there exists $f \in L^2(\mathscr{M})$ such that $f_n \to f$ in $L^2(\mathscr{M})$.

2.2. We denote by $L(\mathscr{M})$ the set of all $\mathscr{B}^\mathscr{M}$-measurable functions such that $\|f\|_{\mu,k} = (\mu(\|f\|^k))^{1/k} < \infty$ for all $k \geqslant 1$. By the Schwartz inequality,

$$\|fg\|_{\mu,k} \leqslant \|f\|_{\mu,2k} \|g\|_{\mu,2k} \tag{2.1}$$

and therefore $L(\mathscr{M})$ is an algebra. Of course, $L(\mathscr{M}) \subset L^2(\mathscr{M})$.

A real-valued function u_t defined for sufficiently small positive t is called a *logarithmic germ* if there exist constants c and $\delta > 0$ such that

$$|u_t| \leqslant c\,|\log t|^\delta \tag{2.2}$$

for all sufficiently small $t > 0$. It is called an *infinitesimal germ* if there exist c and $\delta > 0$ such that

$$|u_t| \leqslant ct^\delta \tag{2.3}$$

for all sufficiently small $t > 0$.

An $L(\mathscr{M})$-valued function f_t is called a *logarithmic germ in* $L(\mathscr{M})$ or a *logarithmic* $L(\mathscr{M})$-*germ* if $\|f_t\|_{\mu,k}$ is a logarithmic germ for all $\mu \in \mathscr{M}$ and all $k = 1, 2, \ldots$. We say that f_t is an *infinitesimal* $L(\mathscr{M})$-*germ* if, in addition, $\|f_t\|_{\mu,2}$ is an infinitesimal germ.

It follows from (2.1) that *the set of all logarithmic* $L(\mathscr{M})$-*germs is an* *algebra. We claim that the set of all infinitesimal $L(\mathscr{M})$-germs is its ideal.* Indeed, by the Schwartz inequality,

$$\mu(f_t^2 g_t^2) \leqslant \|f_t\|_\mu \|f_t g_t^2\|_\mu. \tag{2.4}$$

2.3. Let $\mathscr{M}_i$ be a family of finite measures on (E_i, B_i), $i = 1, 2$. We denote by $\mathscr{M}_1 \times \mathscr{M}_2$ the set of all product measures $\mu_1 \times \mu_2$, $\mu_1 \in \mathscr{M}_1$, $\mu_2 \in \mathscr{M}_2$. The spaces $L^2(\mathscr{M}_1)$ and $L^2(\mathscr{M}_2)$ are naturally imbedded into $L^2(\mathscr{M}_1 \times \mathscr{M}_2)$. On the other hand, to every $f \in L^2(\mathscr{M}_1 \times \mathscr{M}_2)$ there corresponds a family of functions $f_{x_1}(x_2) = f(x_1, x_2)$ which belong to $L^2(\mathscr{M}_2)$ for $\mathscr{M}_1$—almost all $x_1 \in E_1$. We note that

$$\int |f_{x_1}(x_2)| \, \mu_1(dx_1) < \infty \tag{2.5}$$

for μ_2—almost all x_2 and that, for every $\mu_1 \in \mathscr{M}_1$, the integral

$$f_{\mu_1}(x_2) = \int f_{x_1}(x_2) \, \mu_1(dx_1) \tag{2.6}$$

defines an element of $L^2(\mathscr{M}_2)$. Moreover

$$\|f_{\mu_1}\|_{\mu_2} \leqslant \mu_1(E_1)^{1/2} \|f\|_{\mu_1 \times \mu_2}. \tag{2.7}$$

Suppose that $f^{(n)} \to f$ in $L^2(\mathscr{M}_1 \times \mathscr{M}_2)$. It follows from (2.7) that $f^{(n)}_{\mu_1} \to f_{\mu_1}$ in $L^2(\mathscr{M}_2)$ for every $\mu_1 \in \mathscr{M}_1$.

Let A be a set of germs in $\mathscr{M}_1$ such that $\alpha_t(E_1)$ is logarithmic for each $\alpha \in A$. *We denote by $\mathscr{G}_A(\mathscr{M}_1 \times \mathscr{M}_2)$ the set of germs f_t in $L(\mathscr{M}_1 \times \mathscr{M}_2)$ such that*

$$I_k = \int |f_t(x_1, x_2)|^k \, \alpha_t(dx_1) \, \mu_2(dx_2)$$

is logarithmic for any $\alpha \in A$, $\mu_2 \in \mathscr{M}_2$, and $k = 1, 2, \ldots$. Obviously $\mathscr{G}_A(\mathscr{M}_1 \times \mathscr{M}_2)$ is an algebra. Let

$$J_k = \int \left| \int f_t(x_1, x_2) f_t(\tilde{x}_1, x_2) \, \mu_2(dx_2) \right|^k \alpha_t(dx_1) \, \alpha_t(d\tilde{x}_1).$$

We note that $J_k \leqslant J_l^{1/2} J_{2k-l}^{1/2}$ for $0 < l < 2k$ and

$$J_k \leqslant I_{2k} \mu_2(E_1)^{k-1} \, \alpha_t(E_1). \tag{2.8}$$

Hence if $f \in \mathscr{G}_A(\mathscr{M}_1 \times \mathscr{M}_2)$ and if J_1 is an infinitesimal germ, then J_k is infinitesimal for all $k = 1, 2, \ldots$. We denote by $\mathscr{G}_A^0(\mathscr{M}_1 \times \mathscr{M}_2)$ the set of all germs which satisfy these conditions.

We write $f_t \sim 0$ if $f_{t,\alpha_t} = \int f_t(x_1, x_2)\, \alpha_t(dx_1)$ is an infinitesimal germ in $L(\mathscr{M}_2)$ for every $\alpha \in A$. We note that

$$\mu_2(|f_{t,\alpha_t}|^k) \leqslant J_1^{1/2} I_{2k-2}^{1/2} \alpha_t(E_1)^{k-3/2} \qquad \text{for} \quad k = 2, 3, \ldots, \tag{2.9}$$

and therefore $f_t \sim 0$ for every $f \in \mathscr{S}_A^0(\mathscr{M}_1 \times \mathscr{M}_2)$. Writing $f_t \sim g_t$ means that $f_t - g_t \sim 0$.

2.4. We call S a *subspace* of $L^2(\mathscr{M})$ if S is closed in $L^2(\mathscr{M})$ and invariant under addition and multiplication by real numbers.

Let S be a subspace of $L^2(\mathscr{M}_2)$. We denote by $L^2(\mathscr{M}_1) \otimes S$ the minimal subspace of $L^2(\mathscr{M}_1 \times \mathscr{M}_2)$ which contains all functions

$$f(x_1, x_2) = f_1(x_1) f_2(x_2), \qquad f_1 \in L^2(\mathscr{M}_1), \quad f_2 \in S. \tag{2.10}$$

If f has the form (2.10), then, for every $\mu_1 \in \mathscr{M}_1$, $f_{\mu_1} = \mu_1(f_1) f_2$ belongs to S. Hence $f_{\mu_1} \in S$ *for all* $f \in L^2(\mathscr{M}_1) \otimes S$ *and all* $\mu_1 \in \mathscr{M}_1$.

2.5. Two elements f, g of $L^2(\mathscr{M})$ are *orthogonal* if $\mu(fg) = 0$ for all $\mu \in \mathscr{M}$. A function $\tilde{f}$ is the (*orthogonal*) *projection* of f *on a subspace* S if $\tilde{f} \in S$ and if $f - \tilde{f}$ is orthogonal to S. If the projection exists, it is determined uniquely up to $\mathscr{M}$-equivalence.

If $\tilde{f}$ is the projection of f on $L^2(\mathscr{M}_1) \otimes S$ then, for every $\mu_1 \in \mathscr{M}_1$, $\tilde{f}_{\mu_1}$ is the projection of f_{μ_1} on S, in other words, we have the following commutative diagram

$$\begin{array}{ccc} L^2(\mathscr{M}_1 \times \mathscr{M}_2) & \xrightarrow{\text{projection}} & L^2(\mathscr{M}_1) \otimes S \\ \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_1} \\ L^2(\mathscr{M}_2) & \xrightarrow{\text{projection}} & S \end{array} \tag{2.11}$$

The existence of the projection can be proved in one important particular case.

LEMMA 2.1. *Let* $\mathscr{M}$ *be defined on* (E_1, B_1) *and let* μ *be defined on* (E_2, B_2). *Let* S *be a subspace of* $L^2(\mu)$. *Then, for every* $f \in L^2(\mathscr{M} \times \mu)$, *the projection of* f *on* $L^2(\mathscr{M}_1) \otimes S$ *exists and it is given by the formula*

$$\tilde{f} = \sum g_n(x_1)\, h_n(x_2), \tag{2.12}$$

where h_n *is the orthonormal basis in* S *and*

$$g_n(x_1) = \int f(x_1, x_2)\, h_n(x_2)\, \mu(dx_2).$$

Proof. We have $\sum g_n^2 \leqslant f(x_1, x_2)^2 \mu(dx_2)$ and therefore $(\mu_1 \times \mu)(g_n^2 h_n^2) \leqslant (\mu_1 \times \mu)(f^2) < \infty$ for all $\mu_1 \in \mathscr{M}$. Elements $g_n h_n$ form an orthogonal system in $L^2(\mathscr{M}_1 \times \mu)$. Hence the series (2.12) converges in $L^2(\mathscr{M} \times \mu)$. Obviously the sum is the projection of f on $L^2(\mathscr{M}) \times S$.

2.6. The concluding part of this section is devoted to functions from a measurable space $(E, \mathscr{B})$ to a Hilbert space $L^2(\Pi)$, where Π is a probability measure on a measurable space $(\Omega, \mathscr{F})$. We note that $L^2(\Pi)$ is separable if $\mathscr{F}$ is countably generated (that is if it is generated by a countable family of sets and the sets of Π-measure 0).

We put $\langle F \rangle = \Pi(F)$ and $\|F\| = \langle F^2 \rangle^{1/2}$.

LEMMA 2.2. *Let an element U_x of $L^2(\Pi)$ be given for every $x \in E$ and let the function $\langle U_x U_y \rangle$ be $\mathscr{B} \times \mathscr{B}$-measurable. If $L^2(\Pi)$ is separable, then there exists an $\mathscr{B} \times \mathscr{F}^{\Pi}$-measurable function $V_x(\omega)$ such that $V_x = U_x$ Π-a.s. for every $x \in E$.*

Proof. The function $f_Z(x) = \|U_x - Z\|$ is $\mathscr{B}$-measurable for $Z = U_y$, $y \in E$. Therefore it is $\mathscr{B}$-measurable for all $Z \in L^2(\Pi)$. Let $Z_1, \ldots, Z_n, \ldots$, be a countable everywhere dense subset of $L^2(\Pi)$. For every $\varepsilon > 0$ and every n, the set $\{x: \|U_x - Z_n\| < \varepsilon\} = B_{n,\varepsilon}$ belongs to $\mathscr{B}$. We put $n(x, \varepsilon) = \min\{n: x \in B_{n,\varepsilon}\}$ and we note that $V_{\varepsilon,x}(\omega) = Z_{n(x,\varepsilon)}(\omega)$ is $\mathscr{B} \times \mathscr{F}$-measurable and that $\|V_{\varepsilon,x} - U_x\| < \varepsilon$ for all x. Using Chebyshev's inequality and the Borel–Cantelli lemma, we prove that

$$\lim V_{2^{-n},x} = U_x \qquad \Pi\text{—a.s.} \qquad \text{for every} \quad x \in E$$

(cf. proof of Lemma 1.1 in [1]). The function $\limsup_{n \to \infty} V_{2^{-n},x} = V_x$ is an $\mathscr{B} \times \mathscr{F}$-measurable version of U_x.

2.7. Let λ be a σ-finite measure on $(E, \mathscr{B})$ and let $X_z(\omega)$ be a $\mathscr{B} \times \mathscr{F}$-measurable function. If, for some $B \in \mathscr{B}$,

$$\int_B \|X_z\| \, \lambda(dz) < \infty, \tag{2.13}$$

then the Lebesgue integral

$$Y_B(\omega) = \int_B X_z(\omega) \, \lambda(dz)$$

exists for almost all ω, and Y_B belongs to the minimal subspace of $L^2(\Pi)$ which contains all X_z, $z \in B$. If $\|X_z\| < \infty$ for λ—almost all z, then there exists a sequence $B_n \uparrow E$ such that each B_n satisfies (2.13). We write $Y = \int_E X_z \lambda(dz)$ if $\|Y_{B_n} - Y\| \to 0$ for every such a sequence.

3. Random Fields Subordinate to the Free Field

3.1. In [1], we have defined a Gaussian algebra $G(x)$, $x=(x_1,\dots,x_n)$ as an algebra of polynomials of x_i, $i=1,\dots,n$ and $\langle x_i x_j\rangle=\langle x_j x_i\rangle$, $i,j=1,\dots,n$ with two operations $F\to\langle F\rangle$ and $F\to{:}F{:}$. For

$$F=\left(\prod_r \langle x_{i_r} x_{j_r}\rangle\right)\prod_\alpha {:}\prod_\beta x_{k_{\alpha\beta}}{:}, \tag{3.1}$$

we have

$$\langle F\rangle=\left(\prod_r \langle x_{i_r} x_{j_r}\rangle\right)\prod_\alpha \sum_{(a,b),(c,d)} \langle x_{k_{ab}} x_{k_{cd}}\rangle, \tag{3.2}$$

where the sum is taken over all pairings $((a,b),(c,d))$ of the pairs (α,β) subject to the condition $a\neq c$. (Formula (3.2) is convenient to describe in terms of Feynman's diagrams.) An example: in the algebra $G(x,y)$, $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$, we have

$$\langle {:}x_1\cdots x_n{:}\,{:}y_1\cdots y_n{:}\rangle=\sum \langle x_1 y_{k_1}\rangle\cdots\langle x_n y_{k_n}\rangle, \tag{3.3}$$

where $(k_1,\dots,k_n)$ runs over all permutations of $1,\dots,n$.

In the present paper we deal only with the monomials (3.1) subject to the conditions $i_r\neq j_r$ and $k_{ab}\neq k_{cd}$ for $a\neq c$. We denote by $G^*(x)$ the minimal algebra and by $G^*_+(x)$ the minimal cone which contains such monomials. By (3.2), if $F\in G^*_+$, then $\langle F\rangle$ is a linear combination with positive coefficients of the monomials

$$\sum_{i\neq j}\langle x_i x_j\rangle^{m_{ij}}. \tag{3.4}$$

If $X_1,\dots,X_n$ are random variables with a joint normal probability distribution, with mean 0, then to every element F of $G(x)$ there corresponds a random variable $F(X_1,\dots,X_n)$ which we get by substituting X_i for x_i and by interpreting $\langle X_i X_j\rangle$ as the covariance of X_i and X_j. For an arbitrary $F\in G(x)$, $\langle F\rangle(X_1,\dots,X_n)$ is equal to the expectation of $F(X_1,\dots,X_n)$. We write $\langle F\rangle_g$ for $\langle F\rangle(X_1,\dots,X_n)$ if $\langle X_i X_j\rangle=g(x_i,x_j)$.

The image of ${:}x_i^k{:}$ is Wick's power ${:}X_i^k{:}$ defined by (1.27).

3.2. Let $p_t(x,y)$ be a *symmetric transition density* in a measure space $(E,\mathscr{B},m)$ subject to conditions 1.8A through 1.8E. We consider the Gaussian family $\varphi_{s,x}$ described in Subsection 1.4 and we denote by $\mathscr{F}_\Omega$ the Π-completion of the σ-algebra generated by this family in the space Ω. We note that $\|\varphi_{r,x}-\varphi_{s,x}\|\to 0$ as $r\downarrow s$. Together with the condition 1.8E this fact implies that $\mathscr{F}_\Omega$ is countably generated.

The following theorem has been proved in [1]:

THEOREM 3.1. *Let $F \in G_+^*(x)$ and let*

$$F(\varphi_{s,x}) = F(\varphi_{s,x_1},\ldots,\varphi_{s,x_n}),$$
$$v_p(x^{(1)},\ldots,x^{(p)}) = \langle F(x^{(1)}) \cdots F(x^{(p)})\rangle_g.$$

If λ is a σ-finite measure on $(E^n, \mathscr{B}^n)$ and if

$$\int v_2(x,y)\,\lambda(dx)\,\lambda(dy) < \infty, \tag{3.5}$$

then the integral

$$\Phi_{s,\lambda} = \int F(\varphi_{s,x})\,\lambda(dx) \tag{3.6}$$

(*in the sense of Subsection* 2.6) *exists and it defines an element of $L(\Pi)$. Moreover there exists a limit*

$$\Phi_\lambda = \operatorname*{Lim}_{s\downarrow 0} \Phi_{s,\lambda} \tag{3.7}$$

and

$$\langle \Phi_{\lambda_1} \cdots \Phi_{\lambda_p}\rangle = \int v_p(x^{(1)},\ldots,x^{(p)})\,\lambda_1(dx^{(1)}) \cdots \lambda_p(dx^{(p)}) \tag{3.8}$$

for all measures $\lambda_1,\ldots,\lambda_p$ subject to condition (3.5).

To mention explicitly the element F of G_+^*, we write

$$\Phi_\lambda = \int F(\varphi_x)\,\lambda(dx) = F(\varphi)_\lambda.$$

The free field and its powers correspond to $F(x) = {:}x^n{:}$ and formula (1.28) is a particular case of (3.7). We put $F(\varphi)_\lambda = F_1(\varphi)_\lambda - F_2(\varphi)_\lambda$ if $F = F_1 - F_2$, $F_1, F_2 \in G_+^*(x)$.

3.3. We denote by Λ the set of all admissible measures λ on $(E, \mathscr{B})$ in the sense of Subsection 1.3 and by A the set of all Λ-germs $a_t(dz) = a_t(z)\,m(dz)$ such that $m(a_t^k)$ is a logarithmic germ for all $k = 1, 2, \ldots$. We consider sets $\mathscr{G}_A^0(\Lambda \times \Pi) \subset \mathscr{G}_A(\Lambda \times \Pi) \subset L(\Lambda \times \Pi)$ defined in Subsection 2.3.

We introduce the following measures on $(E^n, \mathscr{B}^n)$

$$\begin{aligned} p_t^n(z, dx) &= p_t(z, dx_1) \cdots p_t(z, dx_n), \\ \lambda_t^n(dx) &= p_t^n(\lambda, dx) = \int_E \lambda(dz)\,p_t^n(z, dx). \end{aligned} \tag{3.9}$$

The rest of Section 3 is devoted to proving the following theorems:

THEOREM 3.2. *Let* $F \in G^*(x)$, $x = (x_1,..., x_n)$. *For every sufficiently small t, the measures* $p_t^n(z, \cdot)$ *and* $\lambda_t^n = p_t^n(\lambda, \cdot)$, $\lambda \in \Lambda$ *satisfy condition* (3.5); *there exists a* $\mathscr{B} \times \mathscr{F}_\Omega$*-measurable function* $\Phi_t(z, w)$ *such that, for every z,*

$$\Phi_t(z, w) = \int F(\varphi_x)\, p_t^n(z, dx)\ \Pi\text{—a.s.} \tag{3.10}$$

We have

$$\Phi_{\lambda_t^n} = \int \Phi_t(z, w)\, \lambda(dz). \tag{3.11}$$

THEOREM 3.3. *Formula* (3.10) *defines an element of* $\mathscr{G}_\Lambda(\Lambda \times \Pi)$. *It belongs to* $\mathscr{G}_\Lambda^0(\Lambda \times \Pi)$ *if*

$$F(x) = :x_{i_1} \cdots x_{i_k}: - :x_{j_1} \cdots x_{j_k}:, \tag{3.12}$$

where k is an arbitrary positive integer and $i_1,..., i_k$, $j_1,..., j_k$ *take values in the set* $\{1, 2,..., n\}$.

3.4. We need the following lemma.

LEMMA 3.1. *Let* $\lambda(dz) = \rho(z)\, m(dz)$ *and let* $\|\rho\|_k = (m(\rho^k))^{1/k}$. *There exist constants c and* $\delta > 0$ (*depending on k but independent of* ρ) *such that*:

3.4(A) *For all t,*

$$\int p_t(\lambda, dx)\, p_t(\lambda, dy)\, g(x, y)^k \leqslant c\, \|\rho\|_1\, \|\rho\|_2.$$

3.4(B) *For all sufficiently small t,*

$$\int p_t^2(\lambda, dx, dy)\, g(x, y)^k \leqslant c\, \|\rho\|_1\, |\log t|^\delta.$$

3.4(C) *For all x and all sufficiently small t,*

$$\int p_t(x, dy)(g(x, z) - g(y, z))^2\, p_t(\lambda, dz) \leqslant c\, \|\rho\|_4\, t^\delta.$$

Proof. The integral in 3.4A is equal to $\int p_t(\lambda, dy)\, h(y)$, where

$$\begin{aligned} h(y) &= \int m(dx)\, p_t(x, dz)\, \rho(z)\, g(x, y)^k \\ &\leqslant \left(\int m(dx)\, p_t(x, dz)\, \rho(z)^2\right)^{1/2} \left(\int m(dx)\, p_t(x, dz)\, g(x, y)^{2k}\right)^{1/2} \\ &\leqslant \|\rho\|_2 \left(\int m(dx)\, g(x, y)^{2k}\right)^{1/2}, \end{aligned}$$

and 3.4A follows from 1.8B. The estimate 3.4B follows from 1.8C. Let

$$\gamma_x(dy, dz, du) = p_t(x, dy)\, m(dz)\, p_t(z, du).$$

By Hölder's inequality, the integral in 3.4C is not larger than $I_1^{1/2} I_2^{1/4} I_3^{1/4}$, where

$$I_1 = \int \gamma_x(dy, dz, du)(g(x, z) - g(y, z))^2,$$

$$I_2 = \int \gamma_x(dy, dz, du)(g(x, z) - g(y, z))^4,$$

$$I_3 = \int \gamma_x(dy, dz, du)\, \rho(u)^4.$$

By 1.8D,

$$I_1 \leqslant \int p_t(x, dy)(g(x, z) - g(y, z))^2\, m(dz) \leqslant c_1 t^\delta.$$

By 1.8B,

$$I_2 \leqslant \int p_t(x, dy)\, m(dz)(g(x, z) - g(y, z))^4 \leqslant c_2.$$

Since $I_3 \leqslant m(\rho^4)$, this proves 3.4C.

3.5. *Proof of Theorem* 3.2. It follows from (3.4) that $v_2(x, y)$ belongs to the algebra $Q(x, y)$ generated by the functions $g(x_i, x_j)$, $g(y_i, y_j)$, $i \neq j$ and $g(x_i, y_j)$. By Hölder's inequality, to prove that a measure λ satisfies the condition (3.5), it is sufficient to check that, for every k,

$$\int g(x_i, x_j)^k\, \lambda(dx) < \infty \qquad \text{for all } i \neq j, \tag{3.13}$$

$$\int g(x_i, y_j)^k\, \lambda(dx)\, \lambda(dy) < \infty \qquad \text{for all } i, j. \tag{3.14}$$

This follows from 1.8B,C for $p_t^n(z, \cdot)$ and from 3.4A,B for λ_t^n. The same arguments show that, for every $\lambda \in \Lambda$,

$$\int p_t^{2n}(\lambda, dx, dy)\, v_2(x, y) < \infty. \tag{3.15}$$

Since the Hilbert space $L^2(\Pi)$ is separable and since, by 1.8E, $g(x, y)$ is

$\mathscr{B} \times \mathscr{B}$-measurable, there exists, by Lemma 2.3, a $\mathscr{B} \times \mathscr{F}_\Omega$-measurable function Φ subject to condition (3.10).

We note that

$$\left(\int \lambda(dz\langle|\Phi_t(z)|\rangle)^2\right) < \lambda(E)\int \lambda(dz)\langle\Phi_t(z)^2\rangle.$$

By (3.8) and (3.15), the right side is finite. Using Fubini's theorem, we get that, for every μ which satisfies the condition (3.5),

$$\left\langle\Phi_\mu \int \Phi_t(z)\,\lambda(dz)\right\rangle = \int \langle\Phi_\mu\,\Phi_t(z)\rangle\,\lambda(dz)$$

$$= \int \mu(dx)\,p_t(z, dy)\,v_2(x, y)\,\lambda(dz)$$

$$= \int \mu(dx)\,\lambda_t^n(dy)\,v_2(x, y) = \langle\Phi_{\lambda_t^n}\Phi_\mu\rangle$$

which proves (3.11).

3.6. *Proof of Theorem* 3.3. For every integer k, Φ^k has again the form (3.10) (with n replaced by nk). Therefore, to prove the first statement of Theorem 3.3, we need only to check that, for every Φ of the form (3.10) and for every $\alpha \in A$,

$$\int \langle\Phi_t(z)\rangle\,\alpha_t(dz)$$

is a logarithmic germ. By (3.8),

$$\langle\Phi_t(z)\rangle = \int p_t^n(z, dx)\langle F(x)\rangle_g$$

and, by (3.4) and Hölder's inequality, it is sufficient to prove that

$$\int p_t^n(z, dx)\,g(x_i, x_j)^k\,\alpha_t(dz)$$

is logarithmic for any $i \neq j$, $k = 1, 2, \ldots$. This follows from 3.4B.

To prove the second statement of Theorem 3.3, we show that

$$J_2 = \int \langle\Phi_t(z)\,\Phi_t(\tilde{z})\rangle^2\,\alpha_t(dz)\,\alpha_t(d\tilde{z}) \tag{3.16}$$

is infinitesimal for every $\alpha \in A$. By (3.10) and (3.8),

$$J_2 = \int p_t^n(\alpha_t, dx)\, p_t^n(\alpha_t, dy)\, v_2(x, y).$$

It follows from (3.2) that, for F given by (3.12), $v_2(x, y) = \langle F(x)\, F(y) \rangle_g$ is the sum of terms

$$q_{ij}^l(x, y)(g(x_i, y_l) - g(x, y_l)),$$

where q_{ij}^l belong to the algebra $Q(x, y)$ defined in the proof of Theorem 3.2. Using 3.4A,B and Hölder's inequality, we show that

$$\int p_t^n(\alpha_t, dx)\, p_t^n(\alpha_t, dy)\, q(x, y)$$

is logarithmic for every $q \in Q(x, y)$. On the other hand,

$$\int p_t^n(\alpha_t, dx)\, p_t^n(\alpha_t, dy)(g(x_i, y_l) - g(x_j, y_l))^2 \leqslant \int p_t(\alpha_t, dy)\, q(y), \tag{3.17}$$

where

$$q(y) = \int p_t^2(\alpha_t, dx_1, dx_2)(g(x_1, y) - g(x_2 y))^2.$$

Using the inequality $(a + b)^2 \leqslant 2a^2 + 2b^2$, we get

$$q(y) \leqslant 4 \int \alpha_t(dz)\, p_t(z, dx)(g(z, y) - g(x, y))^2$$

and (3.17) is infinitesimal by 3.4C. So is (3.16).

3.7. We denote by $H(x)$ the set of all functions $h(x)$ such that $p_t^n(\alpha_t, h^r)$ is a logarithmic germ for every $\alpha \in A$ and every $r = 1, 2, \ldots$. By Hölder's inequality, $H(x)$ is an algebra and, by 3.4B, it contains all functions $g(x_i, x_j)$, $i \neq j$.

All the statements of Theorems 3.2 and 3.3 remain true if we replace measures $p_t^n(z, dx)$ with $p_t^n(z, dx)\, h(x)$. In particular,

$$\Psi_t(z, \omega) = \int F(\varphi_x)\, h(x)\, p_t^n(z, dx) \tag{3.18}$$

is an element of $\mathscr{S}_A(A \times \Pi)$ and Ψ_t belongs to $\mathscr{S}_A^0(A \times \Pi)$ if F is given by

(3.12). (The only change needed in proofs is the replacement of $v_2(x, y)$ by $h(x)\,h(y)\,v_2(x, y)$ and $\langle F\rangle_g$ by $h(x)\langle F\rangle_g$.)

4. Powers of the Field ξ

4.1. We note that, by 1.8A,

$$p_t(z, E) \sim 1. \tag{4.1}$$

LEMMA 4.1. *Let $h \in H(x)$, $x = (x_1, \ldots, x_n)$ and let $i_1, \ldots, i_k$ take values in the set $\{1, \ldots, n\}$. Put*

$$\eta^k_{t,z} = \int p_t(z, dy){:}\varphi^k_y{:}, \qquad a_h(t, z) = \int p^n_t(z, dx)\, h(x). \tag{4.2}$$

We have

$$\int p^n_t(z, dx)\, h(x){:}\varphi_{x_{i_1}} \cdots \varphi_{x_{i_k}}{:} \sim a_h(t, z)\, \eta^k_{t,z} \tag{4.3}$$

and

$${:}\varphi^n_{t,z}{:} = \int p^n_t(z, dx){:}\varphi_{x_1} \cdots \varphi_{x_n}{:} \sim \eta^n_{t,z}. \tag{4.4}$$

Proof. We have

$$\int p^{n+1}_t(z, dx, dy)\, h(x){:}\varphi^k_{s,y}{:} = a_h(t, z) \int p_t(z, dy){:}\varphi^k_{s,y}{:}.$$

By passing to the limit as $s \downarrow 0$ and by applying Theorem 3.1, we get

$$\int p^{n+1}_t(z, dx, dy)\, h(x){:}\varphi^k_y{:} = a_h(t, z)\, \eta^k_{t,z}. \tag{4.5}$$

Analogously, for $F \in G^*_+(x)$,

$$\int p^{n+1}_t(z, dx, dy)\, h(x)\, F(\varphi_x) = \int p^n_t(z, dx)\, h(x)\, F(\varphi_x)\, p_t(z, E). \tag{4.6}$$

By Subsection 3.7, $\mathscr{G}^0_A(\Lambda \times \Pi)$ contains the germ

$$\Phi_t(z, w) = \int p^{n+1}_t(z, dx, dy)\, h(x){:}\Phi_{x_{i_1}} \cdots \Phi_{x_{i_k}}{:} - \int p^{n+1}_t(z, dx, dy)\, h(x){:}\Phi^k_y{:} \tag{4.7}$$

and (4.3) follows from Subsection 2.3, (4.5), (4.6), and (4.1). To get the second part of (4.4), we put $h = 1$ in (4.3) and we use (4.1) once more.

To finish the proof, we note that $\tilde{\varphi}_{t,z} = \int p_t(z, dx)\, \varphi_x$ belongs to the minimal subspace of $L^2(\Pi)$ which contains $\varphi_{s,y}$, $s > 0$, $y \in E$, and that

$$\langle \tilde{\varphi}_{t,z} \varphi_{s,y} \rangle = \lim_{\varepsilon \downarrow 0} \left\langle \int p_t(z, dx)\, \varphi_{\varepsilon,x} \varphi_{s,y} \right\rangle$$

$$= \lim_{\varepsilon \downarrow 0} \int p_t(z, dx)\, g_{s+\varepsilon}(x, y) = \lim_{\varepsilon \downarrow 0} g_{s+t+\varepsilon}(z, y) = \langle \varphi_{t,z} \varphi_{s,y} \rangle.$$

Hence

$$\varphi_{t,z} = \int p_t(z, dx)\, \varphi_x = \operatorname*{Lim}_{s \downarrow 0} \int p_t(z, dx)\, \varphi_{s,x},$$

and we have

$$:\varphi^n_{t,z}: = \operatorname*{Lim}_{s \downarrow 0} \int p^n_t(z, dx) :\varphi_{s,x_1} \cdots \varphi_{s,x_n}:$$

$$= \int p^n_t(z, dx) :\varphi_{x_1} \cdots \varphi_{x_n}:.$$

4.2. We consider graphs Γ which consist of a finite number of vertices and a finite number of bonds: each bond connects two different vertices. We denote by $\Gamma(n)$ the set of all graphs with vertices $1, 2, \ldots, n$ such that each vertex has multiplicity 0, 1, or 2, i.e., it belongs to 0, 1, or 2 bonds. For every $\Gamma \in \Gamma(n)$, we denote by J_k the set of vertices of multiplicity k. Each connected component of Γ is either an m-chain

$$i_1 - i_2 - \cdots - i_{m+1}, \qquad m \geqslant 0$$

or an m-loop

$$\overbrace{i_1 - i_2 - \cdots - i_m}, \qquad m \geqslant 2$$

(in both cases m is the number of bonds, 0-chains are isolated points).

We denote the number of m-chains by c_m and the number of m-loops by l_m and we call the collection $c_1, c_2, \ldots, l_2, l_3, \ldots$, the characteristic of Γ. The total number of chains is equal to $r = n - \sum_{m \geqslant 1} mc_m - \sum_{m \geqslant 2} ml_m$. By permutations of the labels $1, \ldots, n$, we get from a graph Γ all the graphs with the same characteristics. We denote by S the set of permutations which do not change Γ (i.e., which map every bond into another bond). If $|S|$ is the order of S, then the number of different graphs with the same characteristic as Γ is equal to $n!/|S|$.

To evaluate $|S|$, we consider the subgroup H of S which preserves each connected component of Γ. Its order is equal to

$$2^{c-l_2} \prod_{m \geq 2} (2m)^{l_m}.$$

The cosets S/H are in a 1–1 correspondence with the transformations in the space of the connected components which map m-chains to m-chains and m-loops to m-loops. The number of such transformations is equal to $\prod_{m \geq 0} c_m! \prod_{m \geq 2} l_m!$. Hence

$$|S| = 2^{c-l_2} \prod_{m \geq 0} c_m! \prod_{m \geq 2} (2m)^{l_m} l_m!. \tag{4.7a}$$

4.3. The following identity has been proved in [1, see (2.16)]:

$$\prod_{i=1}^{n} (:x_i^2:/2) = \sum_{\Gamma \in \Gamma(n)} 2^{-l_2} \prod_{\text{bonds}} \langle x_p x_q \rangle : \prod_{i \in J_0} (x_i^2/2) \prod_{j \in J_1} x_j:, \tag{4.8}$$

where the first product is taken over all bonds of Γ. Since

$$\prod_{i=1}^{n} \operatorname*{Lim}_{s \downarrow 0} \int p_t(z, dx_i) :\varphi_{s,x_i}^2: = \operatorname*{Lim}_{s \downarrow 0} \int p_t^n(z, dx) \prod_{i=1}^{n} :\varphi_{s,x_i}^2:,$$

it follows from Theorem 3.1 that

$$\xi_{t,z}^n = \int p_t^n(z, dx) \prod_{i=1}^{n} :\varphi_x^2/2:,$$

and, by (4.8),

$$\xi_{t,z}^n = \sum_{\Gamma \in \Gamma(n)} \int p_t^n(z, dx)\, V_\Gamma(x) : \prod_{i \in J_0} (\varphi_{x_i}^2/2) \prod_{j \in J_1} \varphi_{x_j}: 2^{-l/2}, \tag{4.9}$$

where

$$V_\Gamma(x) = \prod_{\text{bonds}} g(x_p, x_q). \tag{4.10}$$

We note that

$$f_\Gamma(t, z) = \int p_t^n(z, dx)\, V_\Gamma(x) = \prod_{m \geq 1} C_m^{c_{m-1}} \prod_{m \geq 2} (2mL_m)^{l_m}, \tag{4.11}$$

where the functions C_m are defined by (1.17) and the functions L_m by (1.39).

We conclude from (4.9) and (4.3) that

$$\xi^n_{t,z} \sim \sum_{r=0}^{n} a_{nr}(t,z)\,\eta^{2r}_{t,z}/2^r, \tag{4.12}$$

where

$$a_{nr}(t,z) = \sum_{\Gamma} 2^{c-l_2} f_{\Gamma}(t,z), \tag{4.13}$$

Γ runs over all graphs in $\Gamma(n)$ with r chains. Obviously, $f_\Gamma(t,z)$ depends only on the characteristic of Γ. By (4.7a) and (4.11),

$$a_{nr} = n! \sum_{c_0, c_1, \ldots, l_2, \ldots,} \prod_{m \geqslant 1} \frac{\mathscr{C}_m^{c_{m-1}}}{c_{m-1}!} \prod_{m \geqslant 2} \frac{\mathscr{L}_m^{l_m}}{l_m!}, \tag{4.14}$$

where the sum is taken over all collections of nonnegative integers $c_0, c_1, c_2, \ldots, l_2, l_3, \ldots$, such that

$$\sum_{m \geqslant 1} c_{m-1} = r, \qquad \sum_{m \geqslant 1} m c_{m-1} + \sum_{m \geqslant 2} m l_m = n.$$

It follows from (4.14), that

$$e^{\mathscr{L}(u) + v\mathscr{C}(u)} = \sum_{n,r=0}^{\infty} \frac{a_{nr}}{n!} u^n v^r. \tag{4.15}$$

We note that (a_{nr}) is a triangular matrix with diagonal entries equal to 1. Such a matrix is invertible and the entries of the inverse matrix (b_{nr}) can be obtained from a_{nr} by addition, subtraction, and multiplication.

Formula (4.15) is equivalent to (1.41). To prove this, we rewrite (4.15) as the system of equations

$$\sum_n a_{nr} \frac{u^n}{n!} = e^{\mathscr{L}(u)} \frac{\mathscr{C}(u)^r}{r!}, \qquad r = 0, 1, \ldots, \tag{4.16}$$

and we rewrite (1.41) as the system

$$e^{-\mathscr{L}(\mathscr{D}(u))} \frac{\mathscr{D}(u)^r}{r!} = \sum_n \frac{u^n}{n!} b_{nr}, \qquad r = 0, 1, \ldots. \tag{4.17}$$

It is easy to see that (4.16) and (4.17) are equivalent.

Suppose that $f \in \mathscr{G}_A(A)$, i.e., $\int |f_t(z)|^k \alpha_t(dz)$ is logarithmic for every $\alpha \in A$ and every $k = 1, 2, \ldots$. Then $\tilde{\alpha}_t(dz) = f_t(z)\, \alpha_t(dz)$ belongs to A for every $\alpha \in A$ and therefore, *it is legitimate to multiply any equivalence relation by f.* By

Subsection 3.7, the algebra $\mathscr{G}_A(\Lambda)$ contains all functions a_h of the form (4.2) and, therefore, it contains the coefficients a_{nr} in (4.12) as well as the entries b_{nr} of the inverse matrix.

It follows from (4.4), (4.12), and (1.37) that

$$\frac{1}{2^n} :\varphi_{t,z}^{2n}: \sim \frac{\eta_{t,z}^{2n}}{2^n} \sim \sum_{n=0}^{r} b_{nr}(t, z)\, \xi_{t,z}^{r} = :\xi_{t,z}^{n}:. \tag{4.18}$$

By (4.18) and (1.28),

$$\int \frac{:\varphi_z^{2n}:}{2^n} \lambda(dz) = \lim_{t \downarrow 0} \int :\xi_{t,z}^{n}: \lambda(dz) \qquad \text{in} \quad L^2(\Pi). \tag{4.19}$$

4.4. We denote by Γ_p the subspace of the Hilbert space $L^2(\Pi)$ generated by $:\varphi_{\lambda_1} \cdots \varphi_{\lambda_p}:$, $\lambda_1,\dots,\lambda_p \in M_1$ (Γ_0 consists of constants). It is well known that $L^2(\Pi)$ is the orthogonal sum of Γ_p, $p = 0, 1, 2,\dots$. We put

$$\Gamma_{\leqslant p} = \sum_{k=0}^{p} \Gamma_k.$$

If $X \in \Gamma_p$, $Y \in \Gamma_q$, then $XY \in \Gamma_{\leqslant p+q}$. It follows form (1.28) that $:\varphi^p:_\lambda \in \Gamma_p$. In particular, $\xi_{t,z} \in \Gamma_2$. Hence $\xi_{t,z}^n$ and $:\xi_{t,z}^n:$ belong to $\Gamma_{\leqslant 2n}$.

By Nelson's hypercontractivity estimate,

$$\langle |Y|^k \rangle \leqslant (k-1)^{kp/2} \, \|Y\|^k \tag{4.20}$$

for all $k > 1$, $Y \in \Gamma_{\leqslant p}$ (see references in Sect. 1 of [1]).

Hence (4.19) implies (1.36).

4.5. We consider formal series

$$F(u) = \sum_{n=0}^{\infty} u^n \Phi_{t,z}^{(n)},$$

where $\Phi_{t,z}^{(n)}$ are random variables, and we write $F(u) \sim 0$ if $\Phi_{t,z}^{(n)} \sim 0$ for $n = 0, 1, 2,\dots$. If

$$F(u) = \sum_{n=0}^{\infty} u^n f_n(t, z)\, \xi_{t,z}^n,$$

then we put

$$:F(u): = \sum_{n=0}^{\infty} u^n f_n(t, z) :\xi_{t,z}^n:.$$

In this notation, formula (4.12) can be rewritten as

$$e^{u\xi_{t,z}} \sim :e^{\mathcal{L}(u)+\mathcal{C}(u)\xi_{t,z}}:. \tag{4.21}$$

Since $\langle \eta^n_{t,z} \rangle = 0$ for $n \geqslant 1$, we have from (4.1) that $\langle \xi^n_{t,z} \rangle \sim a_{n0}$ and therefore

$$\langle e^{u\xi_{t,z}} \rangle \sim e^{L(u)}. \tag{4.22}$$

5. Polynomials of ξ and T

5.1. We assume that there exists a right process $X_t(w)$ with the transition density $p_t(x, y)$ (see the definition in Sect. 4 of [1]).

First, we prove formula (1.15). If $\lambda(dx) = p_t(z, dx)$, then, by (1.11) and (1.12),

$$A_\lambda(0, v) = \lim_{s \downarrow 0} \int_0^v p_{t+s}(z, X_u)\, du \qquad \text{in} \quad L^2(P). \tag{5.1}$$

To prove (1.15), it is sufficient to show that

$$A_\lambda(0, v) = \int_0^v p_t(z, X_u)\, du \tag{5.2}$$

for all $v < \infty$. We fix v and we denote the right side in (5.2) by Y_t. Because of (5.1), formula (5.2) will be proved if we show that

$$Y_t = \lim_{s \downarrow 0} Y_{t+s} \qquad \text{in} \quad L^2(P). \tag{5.3}$$

It follows from (1.3) that, for $u_1 < u_2$,

$$Pp_{t_1}(z, X_{u_1})\, p_{t_2}(z, X_{u_2}) = \int p_{t_1}(z, x)\, m(dx)\, p_{u_2 - u_1}(x, y)\, m(dy)\, p_{t_2}(z, y)$$
$$= p_{t_1 + t_2 + u_2 - u_1}(z, z)$$

and therefore

$$PY_{t_1} Y_{t_2} = 2 \int_0^v du_1 \int_{t_1 + t_2}^{t_1 + t_2 + u_1} p_{u_2}(z, z)\, du_2$$
$$\downarrow 2 \int_0^v du_1 \int_{2t}^{2t + u_1} p_{u_2}(z, z)\, du_2 \qquad \text{as} \quad t_1, t_2 \downarrow t.$$

Since $g_t(z, z) < \infty$ for $t > 0$, this implies (5.3).

5.2. We denote by $\mathscr{N}$ the set of all measures $P_{\mu\nu}$, $\mu, \nu \in N$. All these measures are defined on the minimal σ-algebra $\mathscr{F}_W$ which contains all the sets $\{\alpha = 0, X_t(w) \in B\}$, $t > 0$, $B \in \mathscr{B}$. We consider the mapping $Y \to Y^*$ described in Subsection 1.5 and we note that, by (1.30) and the Schwartz inequality,

$$P_{\mu\nu}\langle (Y^*)^k \rangle = P_{\mu\nu}\langle (Y^k)^* \rangle = \langle \varphi_\mu \varphi_\nu Y^k \rangle \leqslant \langle \varphi_\mu^2 \varphi_\nu^2 \rangle^{1/2} \langle Y^{2k} \rangle^{1/2}. \tag{5.4}$$

We denote by $L_\xi(\Pi)$ the set of all elements of $L(\Pi)$ which are $\mathscr{F}_\xi$-measurable. By (5.4), $Y^* \in L(\Pi \times \mathscr{N})$ for all $Y \in L_\xi(\Pi)$ and

$$Y^* = \lim Y_n^* \text{ in } L(\Pi \times \mathscr{N}) \qquad \text{if} \quad Y = \operatorname{Lim} Y_n. \tag{5.5}$$

The algebra $L(\Lambda \times \Pi)$ contains all functions $\xi_t(z, \omega) = \xi_{t,z}(\omega)$. We denote by $L_\xi(\Lambda \times \Pi)$ the minimal subalgebra which contains ξ_t and $L(\Lambda)$. If $\Phi \in L_\xi(\Lambda \times \Pi)$, then, for every $z \in E$, the function Φ_z belongs to $L_\xi(\Pi)$, and the formula

$$\Phi^*(z, \omega, w) = \Phi_z^*(\omega, w)$$

defines a $\mathscr{B} \times \mathscr{F}_\xi \times \mathscr{F}_W$-measurable function. We claim that, for every $\lambda \in \Lambda$,

$$(\Phi^*)_\lambda \approx (\Phi_\lambda)^*. \tag{5.6}$$

This is an immediate implication of

LEMMA 5.1. *Let λ be a finite measure on $(E, \mathscr{B})$. Let $Y_z(\omega)$ be a $\mathscr{B} \times \mathscr{F}_\xi$-measurable function such that*

$$\int \| Y_z \| \lambda(dz) < \infty$$

and let

$$Y_\lambda = \int Y_z \lambda(dz).$$

If $U_z(w, \omega)$ is an $\mathscr{B} \times \mathscr{F}_\xi \times \mathscr{F}_W$-measurable function and if $U_z = Y_z^*$ for each z, then

$$Y_\lambda^* \approx \int U_z \lambda(dz).$$

Proof. We put

$$\|Y\|' = \langle |Y| \rangle, \qquad \|U\|'' = P_{\mu\nu}\langle |U| \rangle$$

and we note that

$$\|Y\|' \leqslant \lambda(E)^{1/2} \|Y\|,$$
$$\|U_z\|'' \leqslant \langle \varphi_\mu^2 \varphi_\nu^2 \rangle^{1/2} \|Y_z\|.$$

The functions $\|Y_z\|'$ and $\|U_z\|''$ are $\mathscr{B}$-measurable and finite λ—a.e. Hence (see, e.g., [4, p. 131]) for every $\varepsilon > 0$, there exists a partition of E into disjoint sets $A_1, \dots, A_n, \dots$, such that

$$\|Y_z - Y_x\|' \leqslant \varepsilon, \qquad \|U_z - U_x\|'' \leqslant \varepsilon \qquad \text{for all } z, x \in A_n, n = 1, 2, \dots. \tag{5.7}$$

We choose a point $z_n \in A_n$ and we put

$$Z = \sum_n Y_{z_n} \lambda(A_n), \qquad V = \sum_n U_{z_n} \lambda(A_n).$$

By (1.32), $Z^* \approx V$. Using (5.7) and Fubini's theorem, we get $\|Y - Z\|' \leqslant \varepsilon\lambda(E)$, $\|\int U_z \lambda(dz) - V\|'' \leqslant \varepsilon\lambda(E)$. Now we consider a sequence of partitions and we get two sequences Z_k and V_k such that $Z_k^* \approx V_k$, $\lim Z_k = Y$ Π—a.s., $\lim V_k = \int U_z \lambda(dz)$ $P_{\mu\nu} \times \Pi$—a.s. Lemma 5.1 follows from (1.34).

5.3. According to the definition in Subsection 2.3, $\mathscr{G}_A(\Lambda \times \Pi \times \mathscr{N})$ means the set of all germs $\Psi_t(z, \omega, w)$ in $L(\Lambda \times \Pi \times \mathscr{N})$ such that, for every $\alpha \in A$, $P \in \mathscr{N}$, and $k = 1, 2, \dots$,

$$\int P\langle \Psi_t(z) \rangle \, \alpha_t(dz)$$

is a logarithmic germ. The algebra $\mathscr{G}_A(\Lambda \times \Pi \times \mathscr{N})$ contains all elements of the form (3.10) and (4.2). Indeed, elements (3.10) and, in particular, $\eta_{t,z}^k(\omega)$ belong to $\mathscr{G}_A(\Lambda \times \Pi)$ by Theorem 3.3, and elements $a_h(t, z)$ belong to $\mathscr{G}_A(\Lambda)$.

Let us show that $\mathscr{G}_A(\Lambda \times \Pi \times \mathscr{N})$ contains the germs $T_{t,z}(w)$. We have

$$(\xi_{t,z}^n)^* = (\xi_{t,z} + T_{t,z})^n = \sum_{k=0}^n \binom{n}{k} \xi_{t,z}^k T_{t,z}^{n-k}.$$

Since $\langle \xi_{t,z}^k \rangle \geqslant 0$, we have $\langle (\xi_{t,z}^n)^* \rangle \geqslant T_{t,z}^n$ and, by (1.30) and (5.4),

$$P_{\mu\nu} T_{t,z}^n \leqslant P_{\mu\nu} \langle (\xi_{t,z}^n)^* \rangle \leqslant \|\varphi_\mu \varphi_\nu\| \langle \xi_{t,z}^{2n} \rangle^{1/2}.$$

Hence $\int P_{\mu\nu} T_{t,z}^n \alpha_t(dz)$ is logarithmic for each $\alpha \in A$.

We observe that the algebra $\mathscr{G}_A(\Lambda \times \Pi \times \mathscr{N})$ contains the functions $a_{nr}(t, z)$ and $b_{nr}(t, z)$ as well as the elements $:\xi_t^n:$ and $:T_t^n:$ defined by formulae (1.37) and (1.16).

LEMMA 5.2. *Let* $\Phi_t \in \mathscr{G}_A^0(\Lambda \times \Pi)$, $V_t \in \mathscr{G}_A(\Lambda \times V)$. *Then* $\Psi_t = \Phi_t V_t$ *belongs to* $\mathscr{G}_A^0(\Lambda \times \Pi \times \mathscr{N})$.

Proof. For every $P \in \mathscr{N}$

$$P\langle \Psi_t(z)\, \Psi_t(\tilde{z})\rangle = \langle \Phi_t(z)\, \Phi_t(\tilde{z})\rangle\, PV_t(z)\, V_t(\tilde{z}).$$

Hence, for every $\alpha \in A$,

$$\left(\int |P\langle \Psi_t(z)\, \Psi_t(\tilde{z})\rangle|\, \alpha_t(dz)\, \alpha_t(d\tilde{z})\right)^2$$

$$\leqslant \int \langle \Phi_t(z)\, \Phi_t(\tilde{z})\rangle^2\, \alpha_t(dz)\, \alpha_t(d\tilde{z}) \int (PV_t(z)\, V_t(\tilde{z}))^2\, \alpha_t(dz)\, \alpha_t(d\tilde{z}).$$

The first factor is infinitesimal, and the second factor is logarithmic.

5.4. If $\Psi_t \in \mathscr{G}_A(\Lambda \times \Pi \times \mathscr{N})$, then writing $\Psi_t \sim 0$ means that Ψ_{t,α_t}^2 is an infinitesimal germ in $L(\Pi \times \mathscr{N})$ for each $\alpha \in A$ (again this is a particular case of a general concept introduced in Subsection 2.3).

LEMMA 5.3. *It is legitimate to multiply the equivalence relations* (4.1), (4.3), (4.12), *and* (4.18) *by any germ* $V_t \in \mathscr{G}_A(\Lambda \times \mathscr{N})$.

Proof. This is obvious in the case of (4.1). Let Φ_t be defined by (4.7). Then $\Phi_t V_t$ belongs to $\mathscr{G}_A^0(\Lambda \times \Pi \times \mathscr{N})$ by Lemma 5.2, and therefore $\Phi_t V_t \sim 0$. This justifies the multiplication of (4.3) by V_t. Formula (4.4) follows from (4.1) and (4.3) and the proof of (4.12) and (4.18) uses only the equivalence relation (4.4) and certain equations. We denote by $\{\Psi\}_p$ the projection of $\Psi \in L^2(\Lambda \times \Pi \times \mathscr{N})$ on the subspace $L^2(\Lambda) \otimes \Gamma_p \otimes L^2(\mathscr{N})$ (such a projection exists by Lemma 2.1). By Subsection 2.5, $\{\Psi_\lambda\}_p = (\{\Psi\}_p)_\lambda$ for all $\lambda \in \Lambda$. Therefore $P(\{\Psi_\lambda\}_p)_\lambda^2 \leqslant P\Psi_\lambda^2$ for every $P \in \Pi \times \mathscr{N}$, and

$$\{\Psi_t\}_p \sim 0 \qquad \text{if} \quad \Psi_t \sim 0. \tag{5.8}$$

Suppose that Ψ_t is a germ in $\mathscr{G}_A(\Lambda \times \Pi)$ with values in $L_\xi(\Lambda \times \Pi)$. Then, by (5.7), (5.4) and the Schwartz inequality,

$$P_{\mu\nu}((\Psi_t^*)_\lambda)^2 \leqslant \langle \varphi_\mu^2 \varphi_\nu^2\rangle^{1/2} \langle \Psi_{t,\lambda}^4\rangle^{1/2}$$
$$\leqslant \langle \varphi_\mu^2 \varphi_\nu^2\rangle^{1/2} \langle \Psi_{t,\lambda}^2\rangle^{1/4} \langle \Psi_{t,\lambda}^6\rangle^{1/4}.$$

Hence

$$\Psi_t^* \sim 0 \qquad \text{if} \quad \Psi_t \sim 0. \tag{5.9}$$

5.5. THEOREM 5.1. *If p is odd, then $\{:\xi_t^n:*\}_p = 0$. If $p = 2i$ and $n = i + j$, then*

$$\frac{\{:\xi_t^n:*\}_p}{n!} \sim \frac{:\xi_t^i:}{i!} \frac{:T_t^j:}{j!}. \tag{5.10}$$

Proof. For the sake of brevity, we omit the subscript t in our computations. By (4.12)

$$\xi^i \sim \sum_{r=0}^{i} a_{ir} \eta^{2r}/2^r. \tag{5.11}$$

We note that

$$\{\eta^{2r}\}_p = \eta^{2r} \qquad \text{for} \quad p = 2r, \qquad = 0 \quad \text{otherwise}. \tag{5.12}$$

It follows from Lemma 5.3, (5.11), (5.8), and (5.12) that

$$\{\xi^i T^j\}_p \sim 0 \qquad \text{for} \quad p \text{ odd}, \tag{5.13}$$

and, taking into account (4.18), that

$$\{\xi^i T^j\}_{2l} \sim a_{il} \eta^{2l} T^j / 2^l \sim a_{il} :\xi^l:\, T^j. \tag{5.14}$$

By (4.16)

$$\sum_{i=l}^{\infty} a_{il} a^i / i! = e^{\mathscr{L}(u)} \mathscr{C}(u)^l / l!. \tag{5.15}$$

By virtue of (5.14) and (5.15),

$$\begin{aligned} \{e^{u(\xi+T)}\}_{2l} &= \sum_{i,j=0}^{\infty} \{\xi^i T^j\}_{2l} \frac{u^i}{i!} \frac{u^j}{j!} \\ &\sim \sum_{i,j=0}^{\infty} a_{il} \frac{u^i}{i!} \frac{u^j T^j}{j!} :\xi^l: = e^{\mathscr{L}(u)} \frac{\mathscr{C}(u)}{l!} e^{uT} :\xi^l:. \end{aligned} \tag{5.16}$$

On the other hand, by (5.11) and (4.18)

$$\xi^n \sim \sum_{r=0}^{n} a_{nr} :\xi^r:.$$

Hence, by (5.9),

$$(\xi + T)^n = (\xi^n)^* \sim \sum_{r=0}^{n} a_{nr} :\xi^r:^*$$

and, by (5.8),

$$\{(\xi + T)^n\}_{2l} \sim \sum_{r=0}^{n} a_{nr} \{:\xi^r:^*\}_{2l}.$$

In combination with (5.15), this implies

$$\begin{aligned}\{e^{u(\xi+T)}\}_{2l} &\sim \sum_{r=0}^{\infty} \frac{\mathscr{C}(u)^r}{r!} e^{\mathscr{L}(u)} \{:\xi^r:^*\}_{2l} \\ &= e^{\mathscr{L}(u)} \{:e^{\xi\mathscr{C}(u)}:^*\}_{2l}.\end{aligned} \tag{5.17}$$

By comparing (5.16) and (5.17), we get

$$\{:e^{\xi\mathscr{C}(u)}:^*\}_{2l} \sim \frac{\mathscr{C}(u)}{l!} e^{uT} :\xi^l:$$

or, by substituting $\mathscr{D}(u)$ for u,

$$\{:e^{\xi u}:^*\}_{2l} \sim \frac{u^l}{l!} e^{\mathscr{D}(u)T} :\xi^l:. \tag{5.18}$$

It follows from (1.20) that

$$e^{\mathscr{D}(u)T} = :e^{uT}: \tag{5.19}$$

and we can rewrite (5.18) in the form

$$\{:e^{\xi u}:^*\}_{2l} \sim \frac{u^l}{l!} :\xi^l: :e^{uT}: \tag{5.20}$$

which is equivalent to (5.10).

5.6. *Proof of Theorem* 1.3. By (1.36) and (1.38), for every $\lambda \in \Lambda$,

$$:\xi^n:_\lambda = \operatorname*{Lim}_{t \downarrow 0} :\xi_t^n:_\lambda.$$

By (2.11), (5.6), and (5.5), this implies

$$(\{:\xi_t^n:^*\}_p)_\lambda \to \{(:\xi^n:_\lambda)^*\}_p \qquad \text{in} \quad L^2(\Pi \times \mathscr{N}). \tag{5.21}$$

Formula (1.44) follows from (5.21) and (5.10). Moreover we have

$$(:\xi^i::T^j:)_\lambda = \frac{i!\,j!}{(i+j)!}\{(:\xi^{i+j}:_\lambda)^*\}_{2i}. \tag{5.22}$$

We conclude from (5.22) and (1.45) that

$$\begin{aligned}(:\xi^n:_\lambda)^* &= \sum_{l=0}^{n}\{(:\xi^n:_\lambda)^*\}_{2l} = \sum_{l=0}^{n}\binom{n}{l}(:\xi^l::T^{n-l}:)_\lambda \\ &= :(\xi+T)^n:_\lambda\end{aligned}$$

which implies (1.46).

Appendix

THEOREM. *The Brownian density* (1.2) *satisfies conditions* 1.8A *through* 1.8E *if* $d \leqslant 2$, $k>0$.

Proof. It is easy to see that, if this statement is true for a density $p_t(x,y)$, it is true also for the density $\tilde{p}_t(x,y) = \lambda^d p_{\lambda^2 t}(\lambda x, \lambda y)$, where $\lambda > 0$. Therefore without any loss of generality we can assume that $k=\frac{1}{2}$. If $d=1$, then $g(x,y)=\frac{1}{2}e^{-|y-x|}$ and the theorem is trivial. For $d=2$, $g(x,y)=\pi^{-1}K(|x-y|)$, where $K(r)$ is the positive solution of the Bessel equation

$$r^2K''(r)+rK'(r)-r^2K(r)=0$$

which has the following asymptotic:

$$K(r)\sim\left(\frac{\pi}{2r}\right)^{1/2}e^{-r} \qquad \text{as} \quad r\to\infty,$$

$$K(r)\sim\left|\ln\frac{r}{2}\right| \qquad \text{as} \quad r\to 0.$$

Besides $K_1 = -K'$ is positive monotone decreasing and

$$K_1(r)\sim\left(\frac{\pi}{2r}\right)^{1/2}e^{-r} \qquad \text{as} \quad r\to\infty,$$

$$K_1(r)\sim r^{-1} \qquad \text{as} \quad r\to 0$$

(see [3, Sects. 3.71 and 7.23]).

Condition 1.8A holds with $\delta = 1$. Changing variables by the formula $y - x = z$, we get that

$$\int g(x, y)^k \, m(dy) = \text{const} \int K(|z|)^k \, m(dz)$$

$$= \text{const} \int K(r)^k r \, dr < \infty.$$

Hence 1.8B is satisfied. Since $g(x, y) \leqslant \text{const}$ for all $|x - y| > 1$, we get

$$\int_{|x-y|>1} p_t(z, dx) \, p_t(z, dy) \, g(x, y)^k \leqslant \text{const for all } t.$$

On the other hand, for all sufficiently small t,

$$\int_{|x-y| \leqslant 1} p_t(z, dx) \, p_t(z, dy) \, g(x, y)^k$$

$$\leqslant \text{const} \int_{|x-y| \leqslant 1} p_t(z, x) \, p_t(z, y) \, \big|\ln |x - y|\big|^k \, m(dx) \, m(dy).$$

Changing variables by the formula $\tilde{x} = (x - z) \, t^{-1/2}$, $\tilde{y} = (y - z) \, t^{-1/2}$, we see that this expression does not exceed

$$\text{const} \int p(\tilde{x}) \, p(\tilde{y}) \, \big|\ln |\tilde{x} - \tilde{y}| + \ln \sqrt{t}\big|^k \, m(d\tilde{x}) \, m(d\tilde{y}).$$

Since

$$\int p(x) \, p(y) \, \big|\log |x - y|\big|^k \, m(dx) \, m(dy) < \infty \qquad \text{for all } k,$$

we get 1.8C.

The integral in 1.8D is equal to

$$\text{const} \int p(y) \, m(dy) \, m(dz) (K(|z - y|) - K(|z - x - y \sqrt{t}|))^2. \tag{1}$$

We have $|K(r_1) - K(r_2)| \leqslant |r_1 - r_2| \, K_1(r_1)$ for $r_1 < r_2$. Let r_1 be the smallest and r_2 be the largest of two quantities $|z - x|$, $|z - x - y \sqrt{t}|$. We note that $|r_1 - r_2| \leqslant \sqrt{t} \, y$, hence $|K(r_1) - K(r_2)|^2 \leqslant \sqrt{t} \, y K_1(r_1) \, K(r_1)$ and the expression (1) does not exceed

$$\text{const} \sqrt{t} \int p(y) \, m(dy) \, m(dz) \, |y| \, K_1(r_1) \, K(r_1). \tag{2}$$

To prove 1.8D, it is sufficient to show that the integrals of the form (2) with $r_1 = |z - x|$ and with $r_1 = |z - x - y\sqrt{t}|$ are finite. Changing variables by the formula $\tilde{z} = z - x$ in the first case and by the formula $\tilde{z} = z - x - y\sqrt{t}$ in the second case, we arrive at the same expression

$$\text{const}\sqrt{t}\int m(d\tilde{z})\, K_1(|\tilde{z}|)\, K(|\tilde{z}|) \int m(dy)\, p(y)\, |y|$$

which is finite.

Condition 1.8E holds in the topology of $\mathbb{R}^2$.

Acknowledgments

This work was done, in part, during the author's visits to the Zentrum für Interdisziplinäre Forschung, Bielefeld University, Germany and to the Mortimer and Raymond Sackler Institute of Advanced Studies, Tel Aviv University, Israel. It was completed at the Sackler Institute. The author is grateful to Professor Ludvig Streit and to Professor Yuval Ne'eman whose hospitality and support have made the visits both productive and enjoyable. The author is also indebted to D. Brydges and L. Gross for stimulating discussions and to R. Adler and W. S. Yang for suggested corrections to the manuscript.

References

1. E. B. Dynkin, Gaussian and non-Gaussian random fields associated with Markov processes, *J. Funct. Anal.* **55**, 1984.
2. P.-A. Meyer, Limites médiales d'après Mokobodzki, *in* "Séminaire de Probabilités VII, Université de Strasbourg," Lecture Notes in Mathematics, Springer-Verlag, Berlin/Heidelberg/New York, 1973.
3. G. N. Watson, "A Treatise on the Theory of Bessel Functions," 2nd ed., Cambridge Univ. Press, Cambridge, 1952.
4. K. Yosida, "Functional Analysis," 5th ed., Springer-Verlag, Berlin/Heidelberg/New York, 1978.