Symplectic geometry, Poincaré algebra, and equal-global time commutators for \( p \)-branes in a curved background

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1. Introduction

In recent Letters [1,2] it has been proved that the covariant phase space for Dirac–Nambu–Goto \( p \)-branes propagating in a curved background spacetime corresponds to a symplectic manifold. Specifically, an exact (and, hence, an identically closed) two-form has been constructed on the corresponding reduced phase space \( \hat{Z} \) (the space of solutions of the classical equations of motion modulo gauge transformations), which preserves all relevant symmetries of the theory. Such a covariant and gauge invariant symplectic structure represents a starting point for the study of the symmetry aspects and also a covariant description of the canonical formulation of the theory for quantization. Hence, the purpose of this Letter is to give a step in such a direction and to generate, using the symplectic structure previously constructed, the Poincaré charges, the Poincaré algebra, and particularly, to construct covariantly the fundamental Poisson brackets-commutators of the theory, which show certain virtues that are absent in the descriptions of the known literature.

In the construction of our symplectic structure [1,2] a fully covariant scheme developed by Carter [3–5] for the general treatment of branes has been used, and particularly the covariant description of the geometry of deformations of such objects [4,5]. A great virtue of Carter scheme is, as pointed out by himself, that avoids the use of excess mathematical baggage that obscures the simplicity and generality of laws and results on the brane dynamics, which is also manifested in the study of covariant phase space of the theory in [1,2]. Moreover, the use of external background fields for describing the brane dynamics in the Carter scheme, and not the explicit use of the world surface fields employed in the traditional approaches, is not only a

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technical issue, but that represents a more objective description of the theory, in order to have a more direct interpretation of results [3,4].

With these ideas the presentation of this Letter is as follows. In the next section we summarize some basic results on standard symplectic geometry that will be useful in subsequent sections. In Section 3 the canonically conjugate variables and their Poisson brackets on a curved background are discussed. In Section 4, the Poincaré algebra, Poincaré charges, and the corresponding conservation properties are discussed. We finish in Section 5 with some remarks and prospects for the future.

2. Symplectic geometry on the covariant phase space

As mentioned previously, a (non-degenerate) differential form ω has been constructed on the physical phase space of the theory under consideration, which is covariant and invariant under spacetime diffeomorphisms and world surface reparametrizations [1,2]; explicitly

\[ \omega = m^{p+1} \int \delta ( - \sqrt{g} n^\alpha \delta X^\alpha ) d \Sigma, \]  

(1)

ω turns out to be independent on the choice of Σ, a spacelike section of the world surface corresponding to a Cauchy surface for the configuration of the p-brane. δ corresponds to the (covariant) deformation operator and is identified with the exterior derivative on the phase space (in particular, δX^\alpha = ξ^\alpha represents the deformations of the world surface geometry), γ is the determinant of the world surface metric, m is a fixed parameter having the dimensionality of a mass [4,5]. Additionally, n^\mu_\nu is the fundamental tensor of the world surface, that together with the complementary orthogonal projection 1^\mu_\nu, satisfy n^\mu_\nu + 1^\mu_\nu = g^\mu_\nu, being g^\mu_\nu the background metric. Moreover, d Σ_α = τ_α d Σ is the surface measure element of Σ, and is normal on Σ and tangent to the world surface; the timelike vector field τ_α can be considered as normalized,

\[ \tau_\alpha \tau^\alpha = -1. \]  

(2)

As well known in the standard symplectic geometry [6,7], if the symplectic structure ω of the theory is invariant under a group of transformations G (i.e., ω in Eq. (1) is invariant under spacetime diffeomorphisms in the embedding background), then the vector V tangent to a gauge orbit of G preserves ω in the sense that

\[ 0 = L_V \omega = V \delta \omega + \delta ( V \omega ), \]  

(3)

where L_V denotes the Lie derivative along V, and \( \delta \) the operation of contraction with V. The second equality in Eq. (3) represents a general relationship between the Lie derivative, the contraction with vector fields, and exterior derivative of differential forms within the scheme of exterior calculus. Since ω is closed, \( \delta \omega = 0 \) [1,2], Eq. (3) implies that

\[ V \omega = \delta \mathcal{H} = 0, \]  

(4)

where \( \mathcal{H} \), a function on the phase space, is the generator of the G transformations. Hence, our geometrical structure ω establishes a relationship between functions and vector fields on the phase space through Eq. (4), which is already known in the conventional Hamiltonian framework where, for example, the Hamiltonian function is related to time translations. Although the symplectic form ω will be worked out in its original form (1) in Section 4, we put it in a more manageable form for computations in the next section, without losing generality. Considering that the vector field e^α is tangent to the world surface, and that any deformation tangent to such a surface can be identified with the action of a world surface diffeomorphism (and then it is no physically relevant), ω can be rewritten as

\[ \omega = \int_\Sigma \delta X^\alpha \delta \hat{p}_\alpha d \Sigma, \]  

(5)

where \( \hat{p}_\alpha = \sqrt{-g} p_\alpha \), and \( p_\alpha = m^{p+1} \tau_\alpha \), which satisfies the mass shell condition for the p-brane:

\[ p_\alpha p^\alpha + m^{2p+2} = 0, \]  

(6)

where Eq. (2) has been considered. Since the present strongly covariant description is referred just to external background coordinates, quite independent of any choice of world surface coordinates, we can write \( X^\mu = X^\mu (\vec{y}, t) \), \( \hat{p}_\alpha = \hat{p}_\mu (\vec{y}, t) \) where \( \vec{y} \) represents the background spatial coordinates, and, particularly important, \( t \) denotes the background time coordinate; in
this sense, $\Sigma$ is the initial-value hypersurface at the
global time $t$ for the configuration of the $p$-brane.
In this manner, Eq. (5) allows us to identify to
$X^\alpha$ and $\hat{p}_\alpha$ as the canonically conjugate variables
in this covariant description of the phase space for
DNG $p$-branes in a curved background. Hence, we can consider to any function $F$
on the phase space as depending on $X^\alpha$ and $\hat{p}_\alpha$, $F = F(X^\alpha, \hat{p}_\alpha)$.
Similarly, if $V$ is any vector field on the phase space, then $V$ can be written as
\[ V = V_X \frac{\partial}{\partial X^\alpha} + V_p \frac{\partial}{\partial \hat{p}_\alpha}. \]  
Furthermore, if $f$ and $g$ are functions on the phase space, we can define using the symplectic structure $\omega$
a new function $[f, g]$, the Poisson bracket of $f$, and $g$, as [7]
\[ [f, g] = V_f(g) = -V_g(f) = -V_f(V_g)\omega, \]  
where $V_f$ and $V_g$ correspond to the Hamiltonian vector fields generated by $f$, and $g$, respectively, through Eq. (4). The Poisson bracket (8) is evidently antisymmetric, and the closeness of $\omega$ is equivalent to the corresponding Jacobi identity [7]. Therefore, $\omega$ represents a complete Hamiltonian description of the covariant phase space of the theory.
In the next section, we will employ these results for finding the relevant Poisson-brackets on the phase space, in the general case of a curved embedding background.

3. Covariant Poisson bracket on a curved background

From Eqs. (4), (5), and (7), it is very easy to find the Hamiltonian vector fields associated with the fundamental canonical variables on $Z$, $(X^\alpha, \hat{p}_\alpha)$:
\[ V_{X^\alpha} = \frac{\partial}{\partial X^\alpha}, \quad V_{\hat{p}_\alpha} = \frac{\partial}{\partial \hat{p}_\alpha}; \]  
hence, from Eqs. (8) and (9), we find the following equal-global time Poisson brackets:
\[ [X^\mu(\vec{y}, t), X^\nu(\vec{y}', t)] = 0 = [\hat{p}_\mu(\vec{y}, t), \hat{p}_\nu(\vec{y}', t)] \]
\[ [X^\mu(\vec{y}, t), \hat{p}_\nu(\vec{y}', t)] = \delta^\mu_{\nu}, \delta(\vec{y} - \vec{y}') \]
where $\delta^\alpha_{\nu}$ is the Kronecker symbol and $\delta(\vec{y} - \vec{y}')$ the Dirac delta function. Considering that the background metric depends only on $X^\mu$, i.e., $V_{X^\mu}(g^{\alpha\beta}) = 0$, from Eq. (10) we find that
\[ [X^\mu(\vec{y}, t), \tilde{\hat{p}}^\nu(\vec{y}', t)] = g^{\mu\nu}\delta(\vec{y} - \vec{y}'), \]  
which shows that the background metric is the measure of the non-commutativity of the canonical variables. As well known in a covariant quantization framework, the presence of an indefinite metric on the right-hand side in Eq. (11) introduces ghost states, which can destroy unitarity. This subject will be extend in subsequent works.
The expressions (10) and (11) deserve some remarks. They represent the first fully covariant description of Poisson brackets referred to an objective global time, unlike the conventional “covariant” descriptions (in a flat background) appearing in the known literature, where the reference time is a world surface time coordinate. Then, a conventional description necessarily implies a time gauge fixing condition, because of the invariance of the theory under world surface reparametrizations; such a gauge is, in passing, avoided in a natural way in the present setting. Furthermore, the commutators (10) and (11) have been obtained on the basis of a covariant description of the phase space of the theory, and the construction of the geometrical structure $\omega$ on it, and they are no assumed a priori, as a matter of fact, such as in the conventional schemes.

4. Flat embedding background, Poincaré charges, and Poincaré algebra

Let us use now the symplectic structure $\omega$ in its form (1), for determining the gauge-invariant energy–momentum tensor for the extendon embedded in a flat background.
In this manner, for $V = \epsilon^\mu \frac{\partial}{\partial X^\mu}$ along a gauge orbit on the phase space (being $\epsilon^\mu$ a constant spacetime vector), the contraction $V|\omega$ in (4) can be write as an exact form:
\[ V|\omega = \delta[\sqrt{-\gamma} \epsilon^\mu (m^{\mu+1} \eta_{\mu\nu})], \]  
From Eq. (12), one can identify the energy–momentum tensor
\[ \tilde{T}^{\mu\nu} = m^{\mu+1} \eta_{\mu\nu} = \tilde{T}_{\nu}^{\mu}, \]
which is essentially that given by Carter in [3]. Considering the equations of motion for the extendon [3]
\[ \nabla^\mu n^\nu \equiv 0 = \partial_\mu n^\nu, \tag{14} \]
where \( \nabla \) denotes the covariant differentiation with respect to the background metric, and \( \nabla^\mu = n^\mu, \nabla^\nu = n^\nu a \partial_{\nu} \equiv \partial_\nu \) in a flat background. \( \tilde{T}^{\mu\nu} \) is world surface covariantly conserved
\[ \partial_\mu \tilde{T}^{\mu\nu} = 0. \tag{15} \]

Once we know the energy–momentum tensor for the extendon, we can construct in the usual way the field variables describing the dynamics of the extendon as a whole is given by
\[ \tilde{P}^{\mu} \equiv \int_{\Sigma} \sqrt{-\gamma} \tilde{T}^{\mu\nu} d\tilde{\Sigma}_\nu = \int_{\Sigma} \tilde{P}^{\mu} d\Sigma; \tag{16} \]
moreover, Eq. (15) makes \( P^{\mu} \) independent of choice of \( \Sigma_t \),
\[ \int_{\Sigma_t} \sqrt{-\gamma} \tilde{T}^{\mu\nu} d\tilde{\Sigma}_\nu = \int_{\Sigma_t} \sqrt{-\gamma} \tilde{T}^{\mu\nu} d\tilde{\Sigma}_\nu, \tag{17} \]
and thus \( P^{\mu} \) is conserved. Note that the canonical momentum \( \tilde{P}^{\mu} \) in our symplectic structure (5), corresponds, in particular, to the linear momentum density in agreement with Eq. (16).

If we define the “mass center” of the extendon as
\[ \tilde{X}^{\mu} = \int_{\Sigma} X^{\mu} d\Sigma, \tag{18} \]
then from Eq. (11), the commutation relation for the field variables describing the dynamics of the extendon as a whole is given by
\[ [P^{\mu}, X^{\nu}] = N^{\mu\nu}, \tag{19} \]
where \( N^{\mu\nu} \) is the Minkowski background metric.

Similarly we define the (total) angular momentum tensor as
\[ M^{\mu\nu} \equiv \int_{\Sigma_t} \sqrt{-\gamma} \left( X^{\mu} \tilde{T}^{\nu\alpha} - X^{\nu} \tilde{T}^{\mu\alpha} \right) d\tilde{\Sigma}_\alpha \]
\[ = \int_{\Sigma} \left( X^{\mu} \tilde{p}^{\nu} - X^{\nu} \tilde{p}^{\mu} \right) d\Sigma; \tag{20} \]
and considering that \( \partial_\mu X^{\nu} = n^{\nu} a \), Eq. (15) implies that
\[ \partial_\mu (X^{\mu} \tilde{T}^{\nu\alpha} - X^{\nu} \tilde{T}^{\mu\alpha}) = n^{\mu} a \tilde{T}^{\nu\alpha} - n^{\nu} a \tilde{T}^{\mu\alpha} \]
\[ = \tilde{T}^{\mu\nu} - \tilde{T}^{\nu\mu} = 0, \tag{21} \]
where the symmetry of the energy–momentum tensor is considered. Hence, Eq. (21) makes \( M^{\mu\nu} \) independent of the choice of \( \Sigma_t \), and thus is conserved. The intrinsic momentum tensor or “spin” of the extendon is defined as usual,
\[ S^{\mu\nu} = \int_{\Sigma} \sqrt{-\gamma} \left( (X^{\mu} - \tilde{X}^{\mu}) \tilde{T}^{\nu\alpha} - (X^{\nu} - \tilde{X}^{\nu}) \tilde{T}^{\mu\alpha} \right) d\tilde{\Sigma}_\alpha \]
\[ = \int_{\Sigma} \left( (X^{\mu} - \tilde{X}^{\mu}) \tilde{p}^{\nu} - (X^{\nu} - \tilde{X}^{\nu}) \tilde{p}^{\mu} \right) d\Sigma, \tag{22} \]
which, considering Eq. (21) and that
\[ \partial_\alpha (X^{\mu} \tilde{T}^{\nu\alpha} - \tilde{X}^{\mu} \tilde{T}^{\nu\alpha}) = 0, \tag{23} \]
is also conserved. Note that the \( S^{\mu\nu} \) and the angular momentum tensor of the extendon as a whole, \( L \equiv \int (\tilde{X}^{\mu} \tilde{p}^{\nu} - \tilde{X}^{\nu} \tilde{p}^{\mu}) d\Sigma \), are conserved separately.

Finally, considering that, according to Eq. (4), the Hamiltonian vector field associated with the angular momentum tensor density \( X^{\mu} \tilde{p}^{\nu} - X^{\nu} \tilde{p}^{\mu} \) is given by
\[ (V_M)^{\mu\nu} = N^{\lambda\nu} \left( \frac{\partial X^{\mu}}{\partial X^\lambda} \tilde{p}^{\lambda} + \tilde{p}^{\mu} \frac{\partial \tilde{p}^{\nu}}{\partial \tilde{p}^{\lambda}} - N^{\nu\mu} \left( \frac{\partial \tilde{X}^{\nu}}{\partial \tilde{X}^\lambda} \tilde{p}^{\lambda} + \tilde{p}^{\nu} \frac{\partial \tilde{p}^{\mu}}{\partial \tilde{p}^{\lambda}} \right) \right); \tag{24} \]
is a straightforward matter to find that the following commutation relations are satisfied:
\[ [M^{\mu\nu}, P^{\alpha}] = N^{\alpha\nu} P^{\mu} - N^{\alpha\mu} P^{\nu}, \]
\[ [M^{\mu\nu}, M^{\rho\sigma}] = N^{\alpha\nu} M^{\mu\rho} + N^{\alpha\mu} M^{\nu\rho} + N^{\nu\rho} M^{\alpha\mu} + N^{\nu\mu} M^{\alpha\rho}, \tag{25} \]
which show explicitly that the Poincaré charges \( P \) and \( M \) indeed close correctly on the Poincaré algebra. Therefore, the Poisson brackets defined in terms of our symplectic structure \( \omega \) give us a Poincaré algebraic structure on the classical phase space, in order to provide a similar algebraic structure for the corresponding commutators in the quantum domain.
5. Remarks and prospects

It is important to emphasize that the results of Section 4 are not valid, in general, in a curved embedding background. For example, the expressions (16) and (20) for the Poincaré charges do not have a covariant meaning nor are conserved on a curved background. The existence of isometries in the background spacetime would represent an intermediate situation in which it may be possible to establish results analogous to those presented in Section 4, in a more general background. In this manner, the commutation relations in Section 3 in terms of distributional fields (and not their integral versions such as Eq. (19)), are the only ones that, preserving general covariance, are valid in an arbitrary background. These results and other aspects of the present covariant canonical formulation of the theory in a curved background will be considered in subsequent works.

We have presented a set of results that are common for a DNG $p$-brane, which covers a wide range of physically diverse systems: 0-brane for a point particle, 1-brane for a string, and the extreme case $(N-1)$-brane for a continuous medium. However, it is well known that the 1-brane case, the string, is very special among all these cases. In this manner, it is possible that a bifurcation point appears in subsequent analysis, where string theory would show its particularities. For example, whether we consider that our symplectic structure governs the transition between the classical and quantum domains of the theory, it may be interesting to study the conformal invariance properties of $\omega$, and its possible relation with the (critical) dimension of the background spacetime. This open question also will be the subject for forthcoming works.

In Ref. [8], the Poincaré charges are studied from a different point of view for higher order brane theories; however such a scheme is weakly covariant; it is possible that equivalent results can be obtained using the weakly covariant symplectic structure constructed in [9] following the ideas presented here.

Acknowledgements

This work was supported by the Sistema Nacional de Investigadores (México). The author wishes to thank Dr. G.F. Torres del Castillo for many conversations that have helped to clarify the results presented here. The author also thanks the referee for pointing out a blunder in the initial version, and for his constructive criticism.

References