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# Curvatures of the Quadratic Rational Bézier Curves

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**Abstract**—We find necessary and sufficient conditions for the curvature of a quadratic rational Bézier curve to be monotone in $[0,1]$, to have a unique local minimum, to have a unique local maximum, and to have both extrema in $(0,1)$, and we also visualize them in figures. As an application, we present a necessary and sufficient condition for the offset curve to be regular and to have the same tangent direction with the given quadratic rational Bézier curve, and give a simple algorithm to find it. © 1998 Elsevier Science Ltd. All rights reserved.

**Keywords**—Quadratic rational Bézier curves, Monotone curvature, Local extrema of curvature, Fairing curves, Offset curves.

## 1. INTRODUCTION

The curvature is one of the most important geometric concepts of curves and surfaces. In particular, the curvature needs to be considered for the approximate curve with $GC^k$, $k \geq 2$ (see [1–3]), or for the interpolant curve with minimal fairness criterion [4–6], or in order to investigate the regularity of offset curves [7].

The quadratic rational Bézier curves which are usually called the *conic* section curves have been widely used in industry due to its well-known properties [8] and convenient implementations for the users. In CAD system, a circular arc and an ellipse can be expressed by quadratic rational Bézier curves, but not by any polynomial Bézier curves. In particular, the quadratic rational Bézier curves are used, for example, to design the bodies of aircraft or to design fonts [9,10]. Many papers have been published on the quadratic rational Bézier curves with topics such as curvature continuous interpolation problem [11,12], expression as conics [13], contact order [14], and high accuracy approximation [15], etc.

Sapidis and Frey [5] characterized a necessary and sufficient condition for the curvature of quadratic polynomial Bézier curve to be monotone. We found the corresponding condition for the case of quadratic rational Bézier curve, but we are informed that the same condition was found by Frey and Field [16]. Our characterization is more complete in the sense that it tells the extrema as well as the monotonicity of curvature of the quadratic rational Bézier curves. In the derivation of conditions, Frey and Field [16] used 'differentiation' of the curvature $\kappa(t)$ of conics, but we used the 'symmetry' of the conics. For better understanding of our characterizations, we present Figures 2–4 which describe all of our assertions. As an application, we also give an algorithm for finding the region of offset distance which make the offset curve to be regular and

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to have the same tangent direction with the given quadratic rational Bézier curve at both end points.

In Section 2, we give some basic facts of the quadratic rational Bézier curves and characterize the local extrema of curvature of a quadratic rational Bézier curve when it is symmetric. In Section 3, we extend the characterizations in Section 2 to the nonsymmetric case to cover all cases and we exhibit our findings in Figures 2–4. In Section 4, using our main results in Section 3, we present an application to the offset curve of the quadratic rational Bézier curves and an easy and simple algorithm to find the region of offset distance for the regular offset curve matching endpoint tangents. We summarize our results in Section 5.

## 2. PRELIMINARIES

Let $\mathbf{r}(t)$ be the quadratic rational Bézier curve with noncolinear control points $\mathbf{b}_i \in \mathbf{R}^2$ and weights $w_i > 0$, $i = 0, 1, 2$, which can be expressed as

$$\mathbf{r}(t) = \frac{\sum_{i=0}^{2} w_i B_i(t)\mathbf{b}_i}{\sum_{j=0}^{2} w_j B_j(t)}, \qquad t \in [0,1],$$

where $B_0(t) = (1-t)^2$, $B_1(t) = 2t(1-t)$, and $B_2(t) = t^2$. Any quadratic rational Bézier curve $\mathbf{r}(t)$ can be converted to the standard form, i.e., $w_0 = w_2 = 1$, without changing the shape of the curve $\mathbf{r}$. In this paper, we mainly treat the standard form for each quadratic rational Bézier curve. The new weights for the standard form are 1, $\mu$, and 1, in order, where $\mu := w_1/\sqrt{w_0 w_2}$ is called the *fullness factor* of the conic $\mathbf{r}$. It is easier to handle the quadratic rational Bézier curve in its standard form than in the general form. By the fullness factor, the quadratic rational Bézier curve is classified into one of the conics, as we see in the following.

PROPOSITION 2.1. *Let $\mathbf{r}(t)$ be a quadratic rational Bézier curve with its fullness factor $\mu$:*

(i) *$\mathbf{r}$ is a segment of ellipse if $\mu < 1$;*
(ii) *$\mathbf{r}$ is a segment of parabola if $\mu = 1$;*
(iii) *$\mathbf{r}$ is a segment of hyperbola if $\mu > 1$.*

PROOF. See [8].

Let $\kappa(t)$ be the curvature of $\mathbf{r}(t)$, i.e.,

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3},$$

where the notation '$\times$' denotes the vector product of two vectors and $\| \ \|$ means the Euclidean length of the vector. Using the above proposition, we can characterize the point $t$ such that $\kappa(t)$ is a local extremum in $(0,1)$.

PROPOSITION 2.2. *The following statements are equivalent:*

(i) *$\kappa$ has a local extremum at $t_0 \in (0,1)$;*
(ii) *$\mathbf{r}(t_0)$ lies on the symmetric axes of the conic containing the curve $\mathbf{r}$;*
(iii) *$\kappa'(t_0) = 0$.*

*Furthermore, $\kappa(t_0)$ is the unique maximum or unique minimum of $\kappa(t)$, $t \in [0,1]$, if $\mathbf{r}$ is not a circular arc.*

PROOF. See [13,17]. ∎

The next proposition is also easily derived from Proposition 2.1.

PROPOSITION 2.3. *The quadratic rational Bézier curve* $\mathbf{r}$ *has not any point at which its curvature* $\kappa(t)$ *is an inflection point in the open interval* $(0,1)$.

*For* $\mu \geq 1$, $\mathbf{r}(t)$ *has at most one point at which* $\kappa(t)$ *is a local maximum and has not any point at which* $\kappa(t)$ *is a local minimum in the open interval* $(0,1)$.

*For* $\mu < 1$, *if* $\mathbf{r}(t)$ *is not a circular arc, then it has at most one point at which* $\kappa(t)$ *is a local maximum and at most one point at which* $\kappa(t)$ *is a local minimum in the open interval* $(0,1)$.

PROOF. See [8,12,13]. ∎

For the standard quadratic rational Bézier curve $\mathbf{r}$, put $\mathbf{s} := \mathbf{r}(1/2)$ and $\mathbf{m} := (\mathbf{b}_0 + \mathbf{b}_2)/2$. We call $\mathbf{s}$ a *shoulder point* of $\mathbf{r}$. It is well known (refer to [8]) that $\mathbf{s}$ lies on the line $\overline{\mathbf{b}_1\mathbf{m}}$, and

$$\mathbf{s} = \frac{1}{1+\mu}\mathbf{m} + \frac{\mu}{1+\mu}\mathbf{b}_1.$$

Note that the tangent line of $\mathbf{r}(t)$ at $\mathbf{s}$ is parallel to the line $\overline{\mathbf{b}_0\mathbf{b}_2}$. Using this fact and the following proposition, we can prove Proposition 2.5.

PROPOSITION 2.4. *The quadratic rational Bézier curve* $\mathbf{r}$ *is symmetric if and only if* $\|\Delta\,\mathbf{b}_0\| = \|\Delta\,\mathbf{b}_1\|$, *where* $\Delta\,\mathbf{b}_i = \mathbf{b}_{i+1} - \mathbf{b}_i$, $i = 0, 1$. *Furthermore, a quadratic rational Bézier curve* $\mathbf{r}(t)$ *is a circular arc if and only if* $\mu < 1$ *and*

$$\mathbf{b}_1 = \mathbf{b}_\mu^{\pm} := \mathbf{m} \pm \frac{\sqrt{1-\mu^2}}{\mu}\,\|\mathbf{b}_0 - \mathbf{m}\|\,\mathbf{n}, \tag{2.1}$$

*where* $\mathbf{n}$ *is a unit normal vector to* $\Delta\,\mathbf{b}_0^2 := \mathbf{b}_2 - \mathbf{b}_0$.

PROOF. See [13,18,19]. ∎

For $\mu \geq 1$, $\mathbf{b}_\mu^{\pm}$ does not exist ($\mathbf{b}_i$'s are not collinear), and for $\mu < 1$, $\mathbf{b}_\mu^{\pm}$ satisfies that $\angle\mathbf{b}_\mu^{\pm}\mathbf{m}\mathbf{b}_0$ is a right angle and $\mu = \cos(\angle\mathbf{b}_\mu^{\pm}\mathbf{b}_0\mathbf{m})$.

PROPOSITION 2.5. *The curvature* $\kappa(t)$ *has a local extremum at* $t = 1/2$ *if and only if* $\|\Delta\,\mathbf{b}_0\| = \|\Delta\,\mathbf{b}_1\|$. *Furthermore, for* $\mathbf{b}_1 \neq \mathbf{b}_\mu^{\pm}$, $\kappa(t)$ *has a local extremum only at* $t = 1/2$ *in* $(0,1)$ *if and only if* $\|\Delta\,\mathbf{b}_0\| = \|\Delta\,\mathbf{b}_1\|$.

PROOF. If $\|\Delta\,\mathbf{b}_0\| = \|\Delta\,\mathbf{b}_1\|$, then $\mathbf{r}$ is symmetric, so is $\kappa(t)$. Since the curvature $\kappa(t)$ is differentiable, $k'(1/2) = 0$. By Proposition 2.3, $\kappa(1/2)$ is a local extremum.

Conversely, by Proposition 2.2, $\mathbf{r}(1/2)$ lies on a symmetric axis, say $\chi$, of the conic containing $\mathbf{r}$. Thus $\chi \perp \mathbf{r}'(1/2)$, and so $\chi \perp \overline{\mathbf{b}_0\mathbf{b}_2}$, as shown in Figure 1. Thus, the two points $\mathbf{b}_0$ and $\mathbf{b}_2$ are mutually symmetric with respect to $\chi$, and $\mathbf{m} = (\mathbf{b}_0 + \mathbf{b}_2)/2 \in \chi$ and $\mathbf{s} = \mathbf{r}(1/2) \in \chi$ imply $\mathbf{b}_1 \in \chi$. Hence, $\|\Delta\,\mathbf{b}_0\| = \|\Delta\,\mathbf{b}_1\|$.

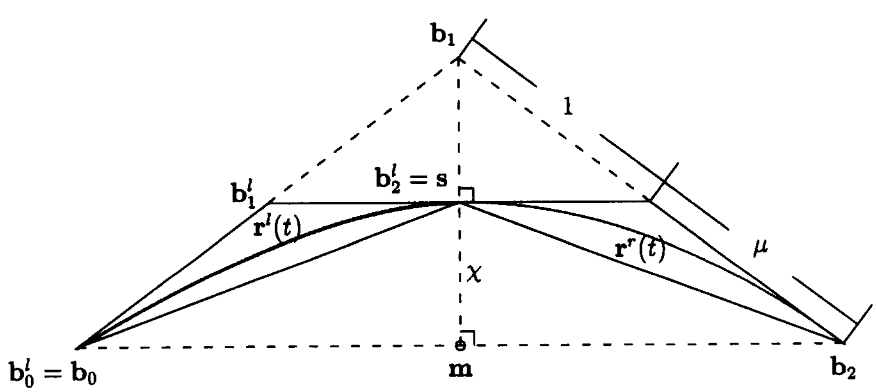


Figure 1. The left subdivision curve $\mathbf{r}^l$ plotted by a thick line and the right subdivision curve $\mathbf{r}^r$ plotted by a thin line are obtained by subdividing $\mathbf{r}(t)$ at $t = 1/2$.

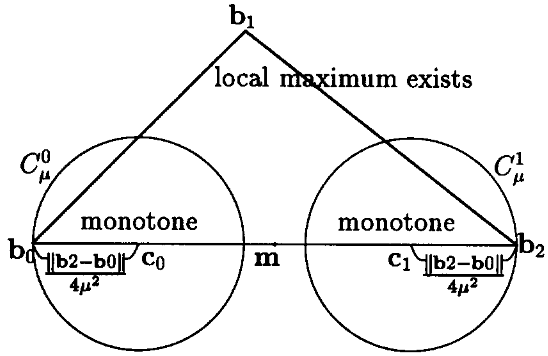


Figure 2. $1 \le \mu^2 < \infty$.

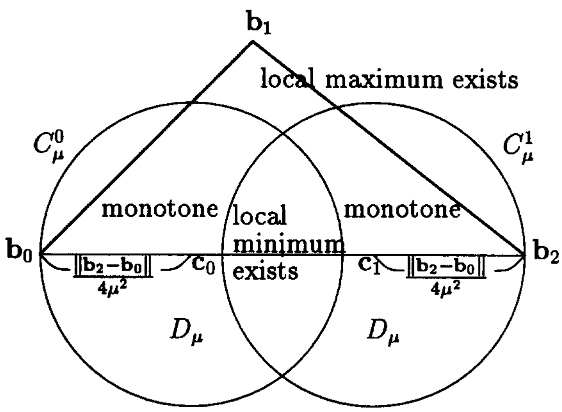


Figure 3. $1/2 \le \mu^2 < 1$.

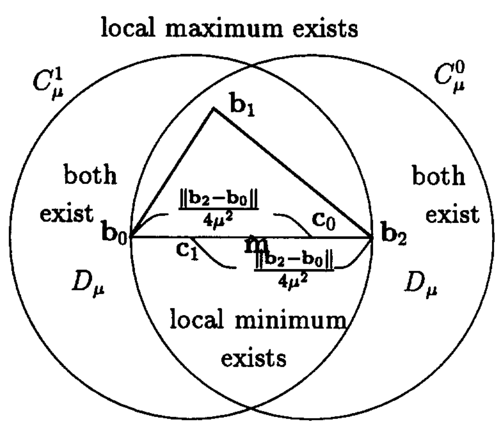


Figure 4. $0 < \mu^2 < 1/2$.

Let $\mathbf{r}$ be not a circular arc. Assume that $\kappa(t)$ has another local extremum at $t_1 \neq 1/2$ in the open interval $(0,1)$. Then $\kappa(1-t_1)$ is also a local extremum, and the curvature $\kappa(t)$ has local extrema at three distinct places, $t_1$, $1/2$, $1-t_1 \in (0,1)$, which contradicts Proposition 2.3. ∎

It is a well-known fact that (refer to [11,12])

$$\kappa(i) = \frac{\mathcal{A}(\triangle \mathbf{b}_0\mathbf{b}_1\mathbf{b}_2)}{\mu^2\|\Delta\,\mathbf{b}_i\|^3}, \qquad \text{for } i = 0,1, \tag{2.2}$$

where $\mathcal{A}(\triangle \mathbf{b}_0\mathbf{b}_1\mathbf{b}_2)$ is the area of the triangle $\triangle \mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$. Thus, for $i,j \in \{0,1\}$ and $i \neq j$,

$$\kappa(i) > \kappa(j), \qquad \text{if and only if } \|\Delta\,\mathbf{b}_i\| < \|\Delta\,\mathbf{b}_j\|. \tag{2.3}$$

To evaluate $\kappa(1/2)$, we subdivide $\mathbf{r}(t)$ at $t = 1/2$, then we have two quadratic Bézier curves, say $\mathbf{r}^l(t)$ and $\mathbf{r}^r(t)$, as shown in Figure 1 by a subdivision of $\mathbf{r}$ at $t = 1/2$, i.e.,

$$\mathbf{r}^l(t) = \mathbf{r}\left(\frac{t}{2}\right) \quad \text{and} \quad \mathbf{r}^r(t) = \mathbf{r}\left(\frac{1}{2}+\frac{t}{2}\right),$$

for $t \in [0,1]$. By equation (2.2), $\kappa(1/2) = \mathcal{A}(\triangle \mathbf{b}_0^l\mathbf{b}_1^l\mathbf{b}_2^l)/(\mu_l^2\|\Delta\,\mathbf{b}_1^l\|^3)$, where $\mathbf{b}_i^l$, $i = 0,1,2$, are the control points and $\mu_l$ is the fullness factor of left subdivision curve $\mathbf{r}^l(t)$. Since $\mathcal{A}(\triangle \mathbf{b}_0^l\mathbf{b}_1^l\mathbf{b}_2^l) = (\mu/2(1+\mu)^2)\mathcal{A}(\triangle \mathbf{b}_0\mathbf{b}_1\mathbf{b}_2)$, $\mu_l = \sqrt{1+\mu}/\sqrt{2}$ and $\|\Delta\,\mathbf{b}_1^l\| = \|\mathbf{b}_0 - \mathbf{m}\|/(1+\mu)$, (see Figure 1)

$$\kappa\left(\frac{1}{2}\right) = \frac{\mu\mathcal{A}\left(\triangle \mathbf{b}_0\mathbf{b}_1\mathbf{b}_2\right)}{\|\mathbf{b}_0 - \mathbf{m}\|^3}. \tag{2.4}$$

Using the above propositions and equation (2.4), we characterize whether the local extremum $\kappa(1/2)$ of the curvature of symmetric quadratic Bézier curve is its local maximum or not in the following proposition.

PROPOSITION 2.6. *Assume* $\|\Delta\,\mathbf{b}_0\| = \|\Delta\,\mathbf{b}_1\|$ *and* $t \in (0,1)$. *For* $\mu \geq 1$, $\kappa(t)$ *has its local maximum only at* $t = 1/2$ *and has not any local minimum. For* $\mu < 1$,

(i) $\|\mathbf{b}_1 - \mathbf{m}\| > \|\mathbf{b}_\mu^\pm - \mathbf{m}\|$ *if and only if* $\kappa(t)$ *has its local maximum only at* $t = 1/2$ *and has not any local minimum;*

(ii) $\|\mathbf{b}_1 - \mathbf{m}\| = \|\mathbf{b}_\mu^\pm - \mathbf{m}\|$ *if and only if* $\kappa(t) \equiv \sqrt{1-\mu^2}/\mu\|\Delta\,\mathbf{b}_0\|$;

(iii) $\|\mathbf{b}_1 - \mathbf{m}\| < \|\mathbf{b}_\mu^\pm - \mathbf{m}\|$ *if and only if* $\kappa(t)$ *has its local minimum only at* $t = 1/2$ *and has not any local maximum.*

PROOF. For $\mu \geq 1$, by Proposition 2.5, $\kappa(t)$, $t \in (0,1)$ has its local extremum only at $t = 1/2$, and by Proposition 2.3, the local extremum $\kappa(1/2)$ is its unique local maximum.

For $\mu < 1$, by Proposition 2.5, $\kappa(1/2)$ is a unique local extremum in $(0,1)$ unless $\mathbf{r}$ is a circular arc. By equations (2.1) and (2.4),

$$\kappa\left(\frac{1}{2}\right) = \frac{\mathcal{A}\left(\triangle \mathbf{b}_0\mathbf{b}_1\mathbf{b}_2\right)}{\mu^2\left\|\mathbf{b}_\mu^\pm - \mathbf{b}_0\right\|^3},$$

and by equation (2.4),

$$\kappa\left(\frac{1}{2}\right) - \kappa(0) = \frac{\mathcal{A}}{\mu^2}\left(\frac{1}{\left\|\mathbf{b}_\mu^\pm - \mathbf{b}_0\right\|^3} - \frac{1}{\|\Delta\,\mathbf{b}_0\|^3}\right).$$

Thus, $\kappa(1/2) - \kappa(0) > 0$ if and only if $\|\Delta\,\mathbf{b}_0\| - \|\mathbf{b}_\mu^\pm - \mathbf{b}_0\| > 0$, so that $\kappa(1/2)$ is the unique local maximum if and only if $\|\mathbf{b}_1 - \mathbf{m}\| > \|\mathbf{b}_\mu^\pm - \mathbf{m}\|$. Hence, we get (i).

Also, $\kappa(1/2) - \kappa(0) = 0$ if and only if $\kappa(t)$ is a constant. By equations (2.1),(2.2), we have that $\|\mathbf{b}_1 - \mathbf{m}\| = \|\mathbf{b}_\mu^\pm - \mathbf{m}\|$ if and only if $\kappa(t) \equiv \sqrt{1-\mu^2}/\mu\|\Delta\,\mathbf{b}_0\|$. Thus, we get (ii), and since the remaining case is unique, (iii) can be obtained similarly to (i). ∎

## 3. RELATION BETWEEN THE CONTROL POLYGON AND THE LOCAL EXTREMA OF THE CURVATURES

We first give our main theorems and present Figures 2–4 for easier understanding. As shown in Figure 2, we define $C_\mu^i$, $i = 0,1$, by the circle with center $\mathbf{c}_i = \mathbf{b}_i + (\mathbf{b}_{2-2i} - \mathbf{b}_i)/4\mu^2$ with radius $\|\Delta\,\mathbf{b}_0^2\|/4\mu^2$, and denote the interior of $C_\mu^i$ by $\operatorname{Int} C_\mu^i$ and the union of $\operatorname{Int} C_\mu^i$ and its boundary by $\overline{\operatorname{Int} C_\mu^i}$.

THEOREM 3.1. *Let* $\mu \geq 1$ *(refer to Figure 2). The curvature* $\kappa(t)$ *is monotone if and only if* $\mathbf{b}_1 \in \overline{\operatorname{Int} C_\mu^0} \cup \overline{\operatorname{Int} C_\mu^1}$. *If* $\mathbf{b}_1 \notin \overline{\operatorname{Int} C_\mu^0} \cup \overline{\operatorname{Int} C_\mu^1}$, *then there exists* $t_A \in (0,1)$ *such that* $\kappa(t_A)$ *is the unique local maximum in* $(0,1)$ *and* $\kappa(t)$ *is monotone in each of subintervals* $[0, t_A)$ *and* $(t_A, 1]$.

THEOREM 3.2. *Let $0 < \mu < 1$ (refer to Figures 3 and 4).*

(i) *If $\mathbf{b}_1 \notin \overline{\operatorname{Int} C_\mu^0} \cup \overline{\operatorname{Int} C_\mu^1}$, then $\kappa(t)$ has the unique local maximum at $t_A \in (0,1)$ and is monotone in $[0, t_A)$ and $(t_A, 1]$.*

(ii) *If $\mathbf{b}_1 \in \operatorname{Int} C_\mu^0 \cap \operatorname{Int} C_\mu^1$, then $\kappa(t)$ has the unique local minimum at $t_B \in (0,1)$ and is monotone in $[0, t_B)$ and $(t_B, 1]$.*

(iii) *If $1/\sqrt{2} \le \mu < 1$ and $\mathbf{b}_1 \in \overline{D_\mu}$, where $D_\mu = (\operatorname{Int} C_\mu^0 \cup \operatorname{Int} C_\mu^1) - (\overline{\operatorname{Int} C_\mu^0} \cap \overline{\operatorname{Int} C_\mu^1})$, then $\kappa(t)$ is monotone in $[0,1]$. For $0 < \mu < 1/\sqrt{2}$, if $\mathbf{b}_1 \in D_\mu$, then $\kappa(t)$ has a local maximum only at $t_A$ and local minimum only at $t_B$ in $(0,1)$, and if $\mathbf{b}_1$ lies on the boundary of $D_\mu$ and $\mathbf{b}_1 \notin \{\mathbf{b}_\mu^\pm\}$, then $\kappa(t)$ has a local maximum or a local minimum in $(0,1)$, but not both.*

(iv) *If $\mathbf{b}_1 \in \{\mathbf{b}_\mu^\pm\}$, then $\mathbf{r}$ is a circular arc.*

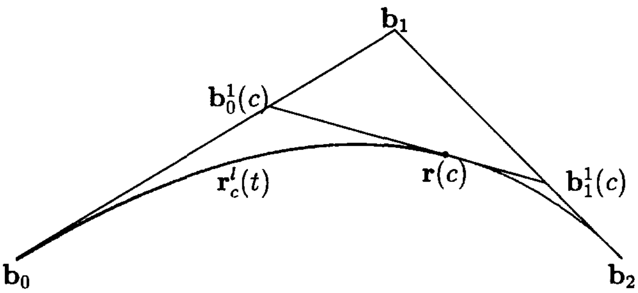


Figure 5. The left subdivision curve $\mathbf{r}_c^l(t)$ of $\mathbf{r}(t)$ at $c$ has the control points $\mathbf{b}_0$, $\mathbf{b}_0^1(c)$, and $\mathbf{r}(c)$, in order.

We prove the above theorems using the following lemmas and propositions. For each parameter $c \in [0,1]$, the left subdivision curve $\mathbf{r}_c^l : [0,1] \to \mathbf{R}^2$ defined by

$$\mathbf{r}_c^l(s) := \mathbf{r}(cs), \qquad 0 \le s \le 1,$$

whose endpoints are $\mathbf{r}(0)$ and $\mathbf{r}(c)$. As shown in Figure 5, the subdivision curve $\mathbf{r}_c^l(s)$ is also a quadratic rational Bézier curve with the control points $\mathbf{b}_0$, $\mathbf{b}_0^1(c)$, and $\mathbf{r}(c)$, where $\mathbf{b}_0^1(c) := ((1-c)\mathbf{b}_0 + c\mu\mathbf{b}_1)/((1-c)+c\mu)$, and the weights

$$1, \quad u(c) := (1-c) + c\mu, \quad \text{and} \quad w(c) := B_0(c) + \mu B_1(c) + B_2(c),$$

in order. (See [8] for a wealth of information about subdivision curves.) Since the quadratic rational Bézier curve $\mathbf{r}_c^l(s)$ is not a standard form, the shoulder point of $\mathbf{r}_c^l$ is not equal to $\mathbf{r}_c^l(1/2)$ but equal to $\mathbf{r}(t)$ with $t = c/(\sqrt{w(c)}+1)$. We define a map $\tau_0 : [0,1] \to [0,1/2]$ by

$$\tau_0(c) = \frac{c}{\sqrt{w(c)}+1}.$$

Then $\tau_0(0) = 0$ and $\tau_0(1) = 1/2$. Since

$$\tau_0'(c) = \frac{2w(c) + 2\sqrt{w(c)} - cw'(c)}{2\sqrt{w(c)}(\sqrt{w(c)}+1)^2} = \frac{(1-c) + c\mu + \sqrt{w(c)}}{\sqrt{w(c)}\left(\sqrt{w(c)}+1\right)^2} > 0,$$

the map $\tau_0$ is bijective and strictly increasing. For each $t \in [0,1/2]$, the map $\tau_0^{-1} : [0,1/2] \to [0,1]$ means that the point $\mathbf{r}(t)$ is the shoulder point of quadratic rational Bézier curve $\mathbf{r}^l_{\tau_0^{-1}(t)}$. Thus, the curve $\mathbf{r}_c^l(s)$, $c \in (0,1)$, is symmetric if and only if the curvature $\kappa(t)$ of $\mathbf{r}$ has a local extremum at $t = \tau_0(c) \in (0,1/2)$. We define a map $F_0 : [0,1] \to \mathbf{R}$ by

$$F_0(c) := \left\|\mathbf{b}_0 - \mathbf{b}_0^1(c)\right\|^2 - \left\|\mathbf{b}_0^1(c) - \mathbf{r}(c)\right\|^2. \tag{3.5}$$

For $c \in (0,1)$, $F_0(c) = 0$ if and only if the curvature $\kappa$ has a local extremum at $t = \tau_0(c) \in (0,1/2)$.

By the fact that $\mathbf{r}(c) = \mathbf{b}_1 + (-B_0(c)\Delta\,\mathbf{b}_0 + B_2(c)\Delta\,\mathbf{b}_1)/w(c)$, we have

$$\mathbf{r}(c) - \mathbf{b}_0^1(c) = \frac{c}{w(c)}\left[\mu\frac{2c-1}{u(c)}\Delta\,\mathbf{b}_0 - c\Delta\,\mathbf{b}_0^2\right].$$

Since $\|\mathbf{b}_0 - \mathbf{b}_0^1(c)\| = c\mu\|\Delta\,\mathbf{b}_0\|/u(c)$, using $w(c)^2 - (2c-1)^2 = 4c(1-c)u(c)u(1-c)$, we get

$$F_0(c) = \frac{c^3}{u(c)w(c)^2}h_0(c), \tag{3.6}$$

where

$$h_0(c) := 4\mu^2(1-c)u(1-c)\left\|\Delta\,\mathbf{b}_0\right\|^2 + 2\mu(2c-1)\Delta\,\mathbf{b}_0\cdot\Delta\,\mathbf{b}_0^2 - cu(c)\left\|\Delta\,\mathbf{b}_0^2\right\|^2. \tag{3.7}$$

LEMMA 3.3. *For $c \in (0,1)$, the following statements are equivalent:*
- (i) $h_0(c) = 0$;
- (ii) *the subdivision curve $\mathbf{r}_c^l$ is symmetric;*
- (iii) *the curvature $\kappa(t)$ of $\mathbf{r}$ has a local extremum at $t = \tau_0(c) \in (0,1/2)$.*

PROOF. By equation (3.6), $F_0(c) = 0$ if and only if $h_0(c) = 0$. Thus, equation (3.5) yields that (i) is equivalent to (ii). By Proposition 2.5, (ii) and (iii) are equivalent. ∎

In the above lemma, we characterized whether $\kappa(t)$ has a local extremum in $(0,1/2)$ or not by using a quadratic polynomial $h_0(c)$. By using the following lemma, we can also characterize when $\kappa(t)$ has a local extremum in $(1/2,0)$. Let $\hat{\mathbf{r}}$ be the quadratic rational Bézier curve with control points $\mathbf{b}_0$, $2\mathbf{m} - \mathbf{b}_1$, and $\mathbf{b}_2$, as shown in Figure 6.

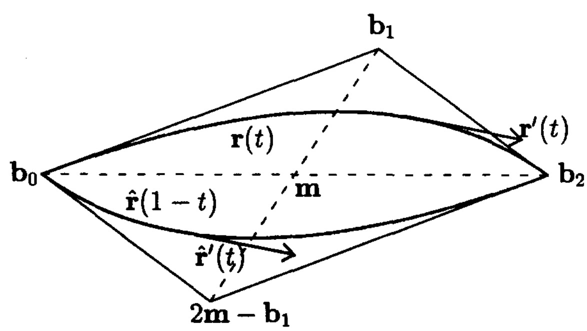


Figure 6. $\hat{\mathbf{b}}(t)$ having the control points $\mathbf{b}_0$, $2\mathbf{m} - \mathbf{b}_1$, and $\mathbf{b}_2$, in order.

LEMMA 3.4. *The curvature $\kappa(t)$ of $\mathbf{r}(t)$ has a local maximum (or minimum) at $t_0 \in (0,1)$ if and only if the curvature of $\hat{\mathbf{r}}(t)$ has a local maximum (or minimum, respectively) at $1 - t_0 \in (0,1)$.*

PROOF. Let $\hat{\kappa}(t)$ be the curvature of $\hat{\mathbf{r}}(t)$. It follows from $\hat{\mathbf{r}}'(1-t) = \mathbf{r}'(t)$ and $\hat{\mathbf{r}}''(1-t) = -\mathbf{r}''(t)$ that

$$\hat{\kappa}(1-t) = \frac{\|\hat{\mathbf{r}}'(1-t)\times\hat{\mathbf{r}}''(1-t)\|}{\|\hat{\mathbf{r}}'(1-t)\|^3} = \frac{\|-\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \kappa(t). \tag{3.8}$$ ∎

We can define a map $h_1 : [0,1] \to \mathbf{R}$ so that $h_1$ is to $\hat{\mathbf{r}}$ what $h_0$ is to $\mathbf{r}$. It is easily seen from equation (3.7) that

$$h_i(c) = 4\mu^2(1-c)u(1-c)\left\|\Delta\,\mathbf{b}_i\right\|^2 + 2\mu(2c-1)\Delta\,\mathbf{b}_i\cdot\Delta\,\mathbf{b}_0^2 - cu(c)\left\|\Delta\,\mathbf{b}_0^2\right\|, \tag{3.9}$$

for $i = 0,1$. By Lemma 3.4, we can see that the curvature $\kappa(t)$ has a local extremum at $1 - \tau_0(c)$ if $h_1(c) = 0$ for some $c \in (0,1)$. Thus, it is reasonable that we define a map $\tau_1 : [0,1] \to [1/2,1]$ by $\tau_1(c) = 1 - \tau_0(c)$. Hence, by Lemmas 3.3 and 3.4, we can find a quadratic polynomial $h_i(c)$, $i = 0,1$, such that $h_i(c)$ has a zero in $(0,1)$ if and only if $\kappa(t)$ has an extremum in $(0,1/2)\cup(1/2,1)$. Combining this fact and Proposition 2.5, we obtain the following proposition.

PROPOSITION 3.5. *$\kappa(1/2)$ is a local extremum if and only if $\|\Delta \mathbf{b}_0\| = \|\Delta \mathbf{b}_1\|$, and $\kappa(t)$ has a local extremum in $(0,1/2)$ or $(1/2,1)$ if and only if $h_0(c)$ or $h_1(c)$ has a zero in $(0,1)$, respectively.*

If $\|\Delta \mathbf{b}_0\| < \|\Delta \mathbf{b}_1\|$, then we can guess that $\kappa(t) > \kappa(1-t)$ for $t \in [0,1)$. The following proposition confirms our guess and will be used in the proof of Proposition 3.7.

PROPOSITION 3.6. *Let $\|\Delta \mathbf{b}_0\| \neq \|\Delta \mathbf{b}_1\|$. For $c \in [0,1/2)$,*

$$(\|\Delta \mathbf{b}_0\| - \|\Delta \mathbf{b}_1\|)\,(\kappa(c) - \kappa(1-c)) < 0.$$

PROOF. By equation (3.8), it suffices to show that if $\|\Delta \mathbf{b}_0\| < \|\Delta \mathbf{b}_1\|$, then $\kappa(c) > \kappa(1-c)$ for $c \in [0,1/2)$.

Assume that $\|\Delta \mathbf{b}_0\| < \|\Delta \mathbf{b}_1\|$ and $c \in [0,1/2)$. We define a subdivision curve $\mathbf{r}^c(s)$ of $\mathbf{r}$ such that $\mathbf{r}^c(0) = \mathbf{r}(c)$, $\mathbf{r}^c(1) = \mathbf{r}(1-c)$, i.e., $\mathbf{r}^c(s) = \mathbf{r}(c + (1-2c)s)$. Then $\mathbf{r}^c(s)$ is a quadratic rational Bézier curve, with control points $\mathbf{b}_0^c = \mathbf{r}(c)$, $\mathbf{b}_2^c = \mathbf{r}(1-c)$, and $\mathbf{b}_1^c$ is the intersection of two tangent lines of $\mathbf{r}$ at $\mathbf{r}(c)$ and $\mathbf{r}(1-c)$, as shown in Figure 7.

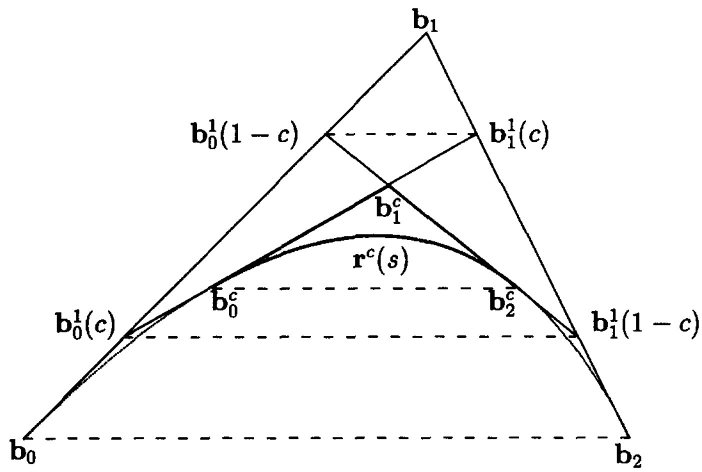


Figure 7. The subdivision curve $\mathbf{r}^c(s)$ of $\mathbf{r}(t)$ on $[c, 1-c]$ for some $c \in (0,1/2)$ has its control points $\mathbf{b}_0^c$, $\mathbf{b}_1^c$, and $\mathbf{b}_2^c$ in order.

For $i = 0,1$ and $c \in [0,1/2)$, let $\mathbf{b}_i^1(c)$ be the intersection of the line $\overline{\mathbf{b}_1\mathbf{b}_{2i}}$ and the tangent line of $\mathbf{r}$ at $\mathbf{r}(c)$. Then the four lines $\overline{\mathbf{b}_0\mathbf{b}_2}$, $\overline{\mathbf{b}_0^1(c)\mathbf{b}_1^1(1-c)}$, $\overline{\mathbf{b}_0^1(1-c)\mathbf{b}_1^1(c)}$, and $\overline{\mathbf{r}(c)\mathbf{r}(1-c)}$ are parallel as shown in Figure 7. We then have

$$\frac{\left\|\mathbf{b}_0^1(c) - \mathbf{b}_0^1(1-c)\right\|}{\left\|\mathbf{b}_1^1(c) - \mathbf{b}_1^1(1-c)\right\|} = \frac{\|\Delta \mathbf{b}_0\|}{\|\Delta \mathbf{b}_1\|} < 1,$$

so that

$$\left\|\mathbf{b}_1^1(c) - \mathbf{b}_0^1(c)\right\| < \left\|\mathbf{b}_1^1(1-c) - \mathbf{b}_0^1(1-c)\right\|.$$

It follows from (6) and

$$\frac{\|\mathbf{b}_1^c - \mathbf{b}_0^c\|}{\|\mathbf{b}_2^c - \mathbf{b}_1^c\|} = \frac{\left\|\mathbf{b}_1^c - \mathbf{b}_0^1(c)\right\|}{\left\|\mathbf{b}_1^1(1-c) - \mathbf{b}_1^c\right\|} = \frac{\left\|\mathbf{b}_1^1(c) - \mathbf{b}_0^1(c)\right\|}{\left\|\mathbf{b}_1^1(1-c) - \mathbf{b}_0^1(1-c)\right\|} < 1$$

that $\kappa(c) > \kappa(1-c)$. ∎

By Propositions 2.3 and 3.6, we can see that if $\|\Delta \mathbf{b}_0\| < \|\Delta \mathbf{b}_1\|$ the curvature $\kappa(t)$ can have neither any local minima in $(0,1/2)$ nor any local maxima in $(1/2,1)$. This fact can be explained in detail as in the next proposition.

PROPOSITION 3.7. *Let $i, j \in \{0,1\}$ and $i \neq j$, and assume that $\|\Delta \mathbf{b}_i\| < \|\Delta \mathbf{b}_j\|$. If $h_i(c) = 0$ and $c \in (0,1)$, then the curvature $\kappa(t)$ has a local maximum at $\tau_i(c)$. If $\mu \geq 1$, then $h_j(c)$ cannot have any zero in the open interval $(0,1)$. If $\mu < 1$, $h_j(c) = 0$, and $c \in (0,1)$, then $\kappa(t)$ has a local minimum at $\tau_j(c)$.*

PROOF. Since $\|\Delta \mathbf{b}_i\| < \|\Delta \mathbf{b}_j\|$, the quadratic rational Bézier curve is not a circular arc and $\kappa(1/2)$ is not a local extremum. Let $h_i(c) = 0$ for $c \in (0,1)$. By Lemma 3.3 and equation (2.3), we obtain

$$\kappa(i) = \kappa(c) > \kappa(j),$$

and the curvature $\kappa(t)$ has a local extremum at $\tau_i(c)$ which lies between $i$ and $1/2$. Assume $\kappa(\tau_i(c))$ is a local minimum. By Proposition 2.2, $\kappa(\tau_i(c))$ is the unique minimum in $[0,1]$, which contradicts to Proposition 3.6 which contains the fact $\kappa(1-\tau_i(c)) < \kappa(\tau_i(c))$. Hence, $\kappa(t)$ has a local maximum at $\tau_i(c)$.

By a similar method, we obtain that $\kappa(t)$ has a local minimum at $\tau_j(c)$ if $\mu < 1$, $h_j(c) = 0$, and $c \in (0,1)$. But if $\mu \geq 1$, then $h_j(c)$ cannot have any zero in $(0,1)$, since $\kappa(t)$ cannot have a local minimum. ∎

By Proposition 2.3, the quadratic polynomial $h_i(c)$, $i = 0,1$, has at most one zero in $(0,1)$. In the following lemma, we make clear the reason why the multiplicity of zero of $h_i(c)$ is at most one.

LEMMA 3.8. *Each function $h_i(c)$ has at most one zero including multiplicity in the open interval $(0,1)$.*

PROOF. By Propositions 2.3 and 3.7, the quadratic polynomial $h_i(c)$ cannot have two distinct zeros in $(0,1)$. Suppose that $h_i(c)$ has a zero, say at $c_0 \in (0,1)$, with multiplicity two. By Proposition 3.5, $\kappa(i) = \kappa(c_0)$, and by equation (2.3) and since for all sufficiently small $\varepsilon > 0$,

$$h(c_0+\varepsilon)\, h(c_0-\varepsilon) = \left(h(c_0+\varepsilon) - h(c_0)\right)\left(h(c_0-\varepsilon) - h(c_0)\right) > 0,$$

we have

$$\left(\kappa(c_0+\varepsilon) - \kappa(i)\right)\left(\kappa(c_0-\varepsilon) - \kappa(i)\right) = \left(\kappa(c_0+\varepsilon) - \kappa(c_0)\right)\left(\kappa(c_0-\varepsilon) - \kappa(c_0)\right) > 0,$$

so that the curvature $\kappa$ has a local extremum at $c_0$ as well as at $\tau_i(c_0)$. This is impossible for $\mu \geq 1$. If $\mathbf{r}$ lies on an ellipse, then $\mathbf{r}(i)$, $\mathbf{r}(c_0)$, and $\mathbf{r}(\tau_i(c_0))$ are the vertices of the ellipse, which is also a contradiction. ∎

By Lemma 3.8, we can see that, for each $i$, $h_i(c)$ has a zero in $(0,1)$ if and only if $h_i(0)h_i(1) < 0$. By equation (3.7), we have

$$h_0(0)h_0(1) = 16\mu^4 \left\{ \|\mathbf{b}_1 - \mathbf{c}_0\|^2 - \left\| \frac{1}{4\mu^2} \Delta\, \mathbf{b}_0^2 \right\|^2 \right\} \{(\mathbf{b}_1 - \mathbf{m}) \cdot (\mathbf{b}_2 - \mathbf{m})\}.$$

From equation (3.9), we can easily obtain

$$h_i(0)h_i(1) = 16\mu^4 \left\{ \|\mathbf{b}_1 - \mathbf{c}_i\|^2 - \left\| \frac{1}{4\mu^2} \Delta\, \mathbf{b}_0^2 \right\|^2 \right\} \{(\mathbf{b}_1 - \mathbf{m}) \cdot (\mathbf{b}_{2-2i} - \mathbf{m})\}, \tag{3.10}$$

for $i = 0,1$. Note that $\|\mathbf{b}_1 - \mathbf{c}_i\|^2 < (1/16\mu^2)\|\mathbf{b}_2 - \mathbf{b}_0\|^2$ if and only if $\mathbf{b}_1$ lies in the circle $C_\mu^i$.

REMARK.

(i)

$$C_\mu^0 \cap C_\mu^1 = \begin{cases} \phi, & \text{if } \mu > 1, \\ \{\mathbf{m}\}, & \text{if } \mu = 1, \\ \{\mathbf{b}_\mu^\pm\}, & \text{if } \mu < 1, \mu \neq \dfrac{1}{\sqrt{2}}, \\ C_\mu^0, & \text{if } \mu = \dfrac{1}{\sqrt{2}}. \end{cases}$$

(ii) $\mathrm{Int}\, C_\mu^0 \cap \mathrm{Int}\, C_\mu^1 \neq \phi$ if and only if $\mu < 1$, and $D_\mu = \phi$ if and only if $\mu = 1/\sqrt{2}$.
(iii) $\mathbf{b}_{2-2i} \in \overline{\mathrm{Int}\, C_\mu^i}$ if and only if $\mu \le 1/\sqrt{2}$.

Now, we characterize whether the quadratic rational Bézier curve $\mathbf{r}(t)$ has the local extremum of the curvature $\kappa(t)$, $t \in (0,1)$, or not in terms of the control points $\mathbf{b}_i$ of $\mathbf{r}$.

PROOF OF THEOREM 3.1. Let $\mu \ge 1$. If $\|\Delta\,\mathbf{b}_0\| = \|\Delta\,\mathbf{b}_1\|$, then $\kappa(t)$ has a unique local maximum at $t = 1/2$ in $(0,1)$ by Proposition 2.6. Let $\|\Delta\,\mathbf{b}_i\| < \|\Delta\,\mathbf{b}_j\|$, for $i, j \in \{0,1\}$ and $i \neq j$. By Propositions 3.6 and 3.7 and equation (3.10), $h_i(c)$ has a unique zero at a point $c_A \in (0,1)$ and $h_j(c)$ has no zeros in $(0,1)$ if and only if

$$\{(\mathbf{b}_1 - \mathbf{m}) \cdot (\mathbf{b}_{2-2i} - \mathbf{m})\} \left\{ \|\mathbf{b}_1 - \mathbf{c}_i\|^2 - \left\| \frac{1}{4\mu^2} \Delta\, \mathbf{b}_0^2 \right\|^2 \right\} < 0.$$

Since $(\mathbf{b}_1 - \mathbf{m}) \cdot (\mathbf{b}_{2-2i} - \mathbf{m}) < 0$, $\kappa(t)$ has a local maximum at $\tau_i(c_A)$ if and only if

$$\|\mathbf{b}_1 - \mathbf{c}_i\| > \left\| \frac{1}{4\mu^2} \Delta\, \mathbf{b}_0^2 \right\|,$$

i.e., $\mathbf{b}_1 \notin \overline{\mathrm{Int}\, C_\mu^i}$, as shown in Figure 2. By the symmetry, we can see that $\kappa(t)$ is monotone in $[0,1]$ if and only if $\mathbf{b}_1 \in \overline{\mathrm{Int}\, C_\mu^0} \cup \overline{\mathrm{Int}\, C_\mu^1}$. Hence, if $\mathbf{b}_1 \notin \overline{\mathrm{Int}\, C_\mu^0} \cup \overline{\mathrm{Int}\, C_\mu^1}$, we can define $t_A \in (0,1)$ as

$$t_A = \begin{cases} \dfrac{1}{2}, & \text{if } \|\Delta\,\mathbf{b}_0\| = \|\Delta\,\mathbf{b}_1\|, \\ \tau_i(c_A), & \text{otherwise}, \end{cases} \tag{3.11}$$

where $c_A \in (0,1)$ is the unique zero of the quadratic polynomial $h_i(c)$ so that we deduce that $\kappa(t)$ has the unique local maximum at $t_A \in (0,1)$ if and only if $\mathbf{b}_1 \notin \overline{\mathrm{Int}\, C_\mu^0} \cup \overline{\mathrm{Int}\, C_\mu^1}$. Thus, we obtain the assertions (refer to Figure 2). ∎

PROOF OF THEOREM 3.2. Let $0 < \mu < 1$. If $\|\Delta\,\mathbf{b}_0\| = \|\Delta\,\mathbf{b}_1\|$, by Proposition 2.6, we have
(i) $\kappa(1/2)$ is the unique local maximum, for $\mathbf{b}_1 \notin \overline{\mathrm{Int}\, C_\mu^0} \cup \overline{\mathrm{Int}\, C_\mu^1}$,
(ii) $\kappa(t)$ is a constant for $\mathbf{b}_1 = C_\mu^0 \cap C_\mu^1$,
(iii) $\kappa(1/2)$ is the unique local minimum for $\mathbf{b}_1 \in \mathrm{Int}\, C_\mu^0 \cap \mathrm{Int}\, C_\mu^1$.

Let $\|\Delta\,\mathbf{b}_i\| < \|\Delta\,\mathbf{b}_j\|$, for $i, j \in \{0,1\}$ and $i \neq j$. Then $h_i(c)$ has a unique zero at a point $c_A \in (0,1)$ if and only if

$$\{(\mathbf{b}_1 - \mathbf{m}) \cdot (\mathbf{b}_{2j} - \mathbf{m})\} \left\{ \|\mathbf{b}_1 - \mathbf{c}_i\|^2 - \left\| \frac{1}{4\mu^2} \Delta\, \mathbf{b}_0^2 \right\|^2 \right\} < 0,$$

and $h_j(c)$ has a unique zero at a point $c_B \in (0,1)$ if and only if

$$\{(\mathbf{b}_1 - \mathbf{m}) \cdot (\mathbf{b}_{2i} - \mathbf{m})\} \left\{ \|\mathbf{b}_1 - \mathbf{c}_j\|^2 - \left\| \frac{1}{4\mu^2} \Delta\, \mathbf{b}_0^2 \right\|^2 \right\} < 0.$$

Since $(\mathbf{b}_1 - \mathbf{m}) \cdot (\mathbf{b}_{2j} - \mathbf{m}) < 0$ and $(\mathbf{b}_1 - \mathbf{m}) \cdot (\mathbf{b}_{2i} - \mathbf{m}) > 0$, $\kappa(t)$ has a unique local maximum at $\tau_i(c_A)$ if and only if

$$\|\mathbf{b}_1 - \mathbf{c}_i\| > \left\| \frac{1}{4\mu^2} \Delta\, \mathbf{b}_0^2 \right\|, \tag{3.12}$$

i.e., $\mathbf{b}_1 \notin \overline{\mathrm{Int}\, C_\mu^i}$, and $\kappa(t)$ has a unique local minimum at $\tau_j(c_B)$ if and only if

$$\|\mathbf{b}_1 - \mathbf{c}_j\| < \left\| \frac{1}{4\mu^2} \Delta\, \mathbf{b}_0^2 \right\|, \tag{3.13}$$

i.e., $\mathbf{b}_1 \in \operatorname{Int} C_\mu^j$. The intersection of two regions of $\mathbf{b}_1$ bounded by equations (3.12) and (3.13) is empty for $1/\sqrt{2} \le \mu < 1$ and equals a component of $D_\mu$ for $0 < \mu < 1/\sqrt{2}$. By combining this with the case $\|\Delta \mathbf{b}_0\| = \|\Delta \mathbf{b}_1\|$, $\kappa(t)$ has a local maximum in $(0,1)$ if and only if

$$\begin{aligned} &\mathbf{b}_1 \notin \overline{\operatorname{Int} C_\mu^0} \cup \overline{\operatorname{Int} C_\mu^1}, && \text{for } \frac{1}{\sqrt{2}} \le \mu < 1 \quad \text{or} \\ &\mathbf{b}_1 \notin \overline{\operatorname{Int} C_\mu^0} \cap \overline{\operatorname{Int} C_\mu^1}, && \text{for } 0 < \mu < \frac{1}{\sqrt{2}}, \end{aligned} \tag{3.14}$$

and $\kappa(t)$ has a local minimum in $(0,1)$ if and only if

$$\begin{aligned} &\mathbf{b}_1 \in \operatorname{Int} C_\mu^0 \cap \operatorname{Int} C_\mu^1, && \text{for } \frac{1}{\sqrt{2}} \le \mu < 1 \quad \text{or} \\ &\mathbf{b}_1 \in \operatorname{Int} C_\mu^0 \cup \operatorname{Int} C_\mu^1, && \text{for } 0 < \mu < \frac{1}{\sqrt{2}}. \end{aligned} \tag{3.15}$$

Hence, if $\mathbf{b}_1$ satisfies equation (3.14) we can define $t_A$ as in equation (3.11), and if $\mathbf{b}_1$ satisfies equation (3.15) we can define $t_B$ as

$$t_B = \begin{cases} \dfrac{1}{2}, & \text{if } \|\Delta \mathbf{b}_0\| = \|\Delta \mathbf{b}_1\|, \\ \tau_j(c_B), & \text{otherwise}, \end{cases}$$

where $c_B \in (0,1)$ is the unique zero of $h_j(c)$.

Therefore, we can deduce the following facts. If $\mathbf{b}_1 \in \operatorname{Int} C_\mu^0 \cap \operatorname{Int} C_\mu^1$, then $\kappa(t)$ has a unique local minimum only at $t_B \in (0,1)$, and if $\mathbf{b}_1 \notin \overline{\operatorname{Int} C_\mu^0} \cup \overline{\operatorname{Int} C_\mu^1}$, then $\kappa(t)$ has a unique local maximum only at $t_A \in (0,1)$. If $\mathbf{b}_1 \in D_\mu$, then $\kappa(t)$ is monotone in $[0,1]$ for $1/\sqrt{2} \le \mu < 1$ and $\kappa(t)$ has a local maximum only at $t_A$ and local minimum only at $t_B$ in $(0,1)$ for $0 < \mu < 1/\sqrt{2}$, as shown in Figures 3 and 4. Thus, we obtain the assertions. ∎

## 4. APPLICATION

In a CAD/CAM system, to find the path of tool with radius $d \in \mathbf{R}$ for the quadratic rational Bézier curve $\mathbf{r}(t)$, we define an offset curve

$$\mathbf{r}^d(t) = \mathbf{r}(t) + d \cdot \mathbf{n}(t),$$

where $\mathbf{n}(t)$ is the unit normal vector of $\mathbf{r}$ at $\mathbf{r}(t)$ and its direction is outward from the curve $\mathbf{r}$, as shown in Figure 8. For practical use it is required that the path $\mathbf{r}^d(t)$ of tool must be regular and has the same tangent direction with $\mathbf{r}$ at both endpoints. Note that, since $\mathbf{r}(t)$ is convex, i.e., the signed curvature always has same sign, $\mathbf{r}^d(t)$ is regular in $[0,1]$ for $d \ge 0$. Thus, the necessary condition for $\mathbf{r}^d$ being regular is

$$d > -\frac{1}{\max\limits_{t\in[0,1]}\{\kappa(t)\}},$$

as shown in Figure 8. Using this fact, we get the following characterization.

PROPOSITION 4.1. *The offset curve $\mathbf{r}^d(t)$ is regular and $\mathbf{r}^d(t)$ has the same tangent direction with $\mathbf{r}(t)$ at both endpoints if and only if $d > -1/\kappa_0$, where*

$$\kappa_0 = \begin{cases} \kappa(i), & \text{if } \mathbf{b}_1 \in \overline{\operatorname{Int} C_\mu^i}, \\ \dfrac{4\mu c_A \|\Delta \mathbf{b}_0 \times (\mathbf{r}(c_A) - \mathbf{b}_0)\|}{\sqrt{w(c_A)}\,\|\mathbf{r}(c_A) - \mathbf{b}_0\|^3}, & \text{otherwise}, \end{cases} \tag{4.16}$$

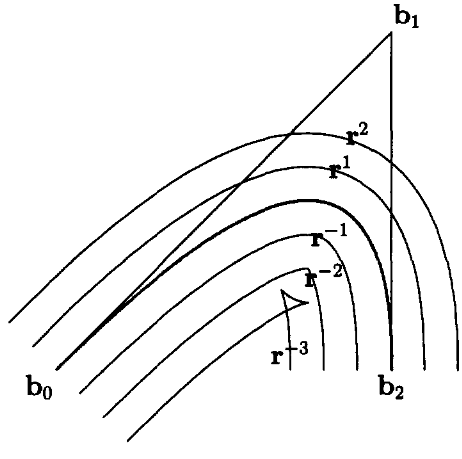


Figure 8. The offset curves $\mathbf{r}^d(t)$ of a quadratic rational Bézier curve $\mathbf{r}(t)$ having the control polygon $[\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2]$.

*where* $i \in \{0,1\}$ *such that* $\|\Delta\,\mathbf{b}_i\| = \min\{\|\Delta\,\mathbf{b}_0\|, \|\Delta\,\mathbf{b}_1\|\}$, *and* $c_A \in (0,1)$ *is the unique zero of* $h_i(c)$ *in equation (3.9).*

PROOF. It suffices to show that $\kappa_0 = \max_{t\in[0,1]} \kappa(t)$. The curvature $\kappa(t)$ of any quadratic rational Bézier curve is monotone in $[0,1]$ or has at most one point $t_A$ at which $\kappa(t_A)$ is a local maximum in $(0,1)$. By our main results in Section 4, the curvature $\kappa(t)$ has no local maxima in $(0,1)$ if and only if

$$\mathbf{b}_1 \in \overline{\operatorname{Int} C_\mu^0} \cup \overline{\operatorname{Int} C_\mu^1}, \quad \text{for } \mu > \frac{1}{\sqrt{2}} \qquad \text{or} \qquad \mathbf{b}_1 \in \overline{\operatorname{Int} C_\mu^0} \cap \overline{\operatorname{Int} C_\mu^1}, \quad \text{for } \mu \le \frac{1}{\sqrt{2}}.$$

Then $\mathbf{b}_1 \in \overline{\operatorname{Int} C_\mu^i}$ if and only if $\kappa(t)$ has no local maxima in $(0,1)$, i.e., the maximum of curvature is $\kappa(i)$. Otherwise, $\kappa$ has its maximum in $[0,1]$. Using (v) in the Remark in Section 3, and the facts that the weight of the subdivision curve at $c_A$ in standard form is $u(c_A)/\sqrt{w(c_A)}$ and that $\|\mathbf{b}_0^1(c_A) - \mathbf{b}_0\| = c_A\mu\|\Delta\,\mathbf{b}_0\|/u(c)$, we obtain that the maximum value of the curvature $\kappa(t)$ equals to equation (4.16). ∎

Using above theorem and the fact that $\mathbf{b}_1 \in \overline{\operatorname{Int} C_\mu^i}$ is equivalent to $\|\mathbf{b}_1 - \mathbf{c}_i\| \le \|\Delta\,\mathbf{b}_0^2\|/4\mu^2$, we present the following algorithm to find the region of $d$ in $\mathbf{R}$ such that the offset curve $\mathbf{r}^d(t)$ is regular and $\mathbf{r}^d$ has the same tangent direction with $\mathbf{r}$ at both endpoints.

ALGORITHM.

1. *Input* the control points $\mathbf{b}_j$ and weights $w_j$, for $j = 0,1,2$.
2. *Choose* $i \in \{0,1\}$ such that $\|\Delta\,\mathbf{b}_i\| = \min\{\|\Delta\,\mathbf{b}_0\|, \|\Delta\,\mathbf{b}_1\|\}$.
3. *Calculate* $\mu = (w_1/\sqrt{w_0 w_2})$, and $\mathbf{c}_i = (1 - (1/4\mu^2))\mathbf{b}_i + (1/4\mu^2)\mathbf{b}_{2-2i}$.
4. *Cases.*
   (a) If $\|\mathbf{b}_1 - \mathbf{c}_i\| \le \|\Delta\,\mathbf{b}_0^2\|/4\mu^2$, then
   $$\kappa_0 = \kappa(i) = \frac{\|\Delta\,\mathbf{b}_i \times \Delta\,\mathbf{b}_0^2\|}{2\mu^3\|\Delta\,\mathbf{b}_i\|^3}.$$
   (b) Otherwise, find the unique zero $c_A \in (0,1)$ of the quadratic polynomial $h_i(c)$ in equation (3.9), and calculate $t_A = \tau_i(c_A) \in (0,1)$ and
   $$\kappa_0 = \kappa(t_A) = \frac{4\mu c_A\,\|\Delta\,\mathbf{b}_0 \times (\mathbf{r}(c_A) - \mathbf{b}_0)\|}{\sqrt{w(c_A)}\,\|\mathbf{r}(c_A) - \mathbf{b}_0\|^3}.$$
5. *Output* the minimum value of $d : -1/\kappa_0$.

The merit of this algorithm is that it is easy not only to understand but also to implement.

## 5. COMMENTS

In this paper, we characterized necessary and sufficient conditions for the curvature of a quadratic rational Bézier curve to be monotone in $[0, 1]$, to have a unique local minimum, to have a unique local maximum, or to have both extrema in $(0, 1)$. We also visualized them in three figures.

As an application, we presented a necessary and sufficient condition for the offset curve to be regular and to have the same tangent direction with the quadratic rational Bézier curve, and gave a simple algorithm to find it.

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