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Irreducibility of some orthogonal polynomials

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Abstract

We give an explicit upper bound for the degree of reducible generalized Hermite–Laguerre polynomials in some particular cases.

Keywords: Irreducibility; Newton polygons; Hermite–Laguerre polynomials; Thue equations; Pell's equations

1. Introduction

For any integer $n \geq 1$, we denote by $P(n)$, the greatest prime factor of n with $P(1) = 1$, $\omega(n)$, the number of distinct prime divisors of n with $\omega(1) = 0$ and $\pi(n)$, the number of primes $\leq n$ with $\pi(1) = 0$. Let $u \geq -1$, $v > 0$, $m > 0$ be integers with $\gcd(u, v) = 1$. Let $a_j \in \mathbb{Z}$ and $b_j \in \mathbb{Z}$ for $0 \leq j \leq m$. We consider

$$f(x) = f(x, u, v) = \sum_{j=0}^{m} (vm + u) \ldots (v(j + 1) + u)\, x^j \tag{1}$$

and

$$g(x) = g(x, u, v, \{a_j\}) = \sum_{j=0}^{m} a_j (vm + u) \ldots (v(j + 1) + u)\, x^j. \tag{2}$$

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Two special families of orthogonal polynomials are Hermite and Laguerre polynomials denoted by $H_m(x)$ and $L_m(x)$. These are related to $g(x)$ as follows:

$$H_{2m}(x) = g\left(x^2, -1, 2, \left\{\binom{m}{j} 2^j\right\}\right),$$
$$H_{2m+1}(x) = xg\left(x^2, 1, 2, \left\{\binom{m}{j} 2^j\right\}\right),$$
$$L_m(x) = g\left(x, 0, 1, \left\{\binom{m}{j} j!\right\}\right),$$
$$L_m^*(x) = g\left(x, u, v, \left\{\binom{m}{j} j!\right\}\right), \quad u \geq 0, \tag{3}$$

the last being called generalized Laguerre polynomial. Thus we see that irreducibility questions concerning these classical polynomials can be answered by considering the polynomial $g(x)$. Note that $f(x)$ is a special case of $g(x)$ with $a_0 = \cdots = a_m = 1$. As it turns out, (see Lemma 2.1), information on factors of $g(x)$ can be obtained from the factors of $f(x)$. Schur [16,17] proved the irreducibility of some of these polynomials using algebraic tools. Following a paper by Coleman [4], Newton polygons became an integral part of the method to study the irreducibility of these polynomials as shown by Filaseta and his co-authors, see [1,2,8,9] to cite a few. Another important tool used is results on greatest prime factor of

$$\Delta_k = \Delta_k(u, v) = (vm + u) \ldots (v(k+1) + u) .$$

This has been extensively studied and applied in a sequence of papers recently, for example see [11–13,18]. In particular, these papers address the question of large factors of $g(x)$.

In [10], Filaseta and Lam showed that for all but finitely many positive integers m, the polynomial $L_m^*(x)$ is irreducible over the rationals. Our aim in this paper is to make the result explicit, i.e., to give an explicit lower bound for m in some particular cases. We say that a polynomial of degree m is *almost irreducible* if it may be written as a linear factor times an irreducible polynomial of degree $m - 1$.

Theorem 1.1. *The polynomial $g(x)$ with $|a_0| = |a_m| = 1$ and $v \in \{1, 2, 4\}$ is almost irreducible for all $m \geq m_0$ where*

$$m_0 = \exp\{10(v\mu + u)^2(1.65)^{(v\mu+u)}\} \tag{4}$$

with

$$\mu = \begin{cases} u & \text{if } v = 1, 2 \\ \max(600, 10u) & \text{if } v = 4. \end{cases}$$

Further the number of integers m for which $g(x)$ may fail to be almost irreducible is at most

$$n_0 = 2(v\mu + u)2^{\pi(v\mu+u)} + \theta, \tag{5}$$

where

$$\theta = \begin{cases} 11 & \text{if } v = 1 \\ 1 & \text{if } v = 2 \\ 0 & \text{if } v = 4. \end{cases}$$

In the next theorem, we restrict to $a_j = \binom{m}{j} b_j$, i.e., the case of generalized Hermite–Laguerre polynomials. Here linear factors can also be included. We show

Theorem 1.2. *The polynomial $g(x)$ with $a_j = \binom{m}{j} b_j$ for $0 \le j \le m$, $|b_0| = |b_m| = 1$ and $v \in \{1, 2, 4\}$ is irreducible for all $m \ge m_0$ and the number of m for which it fails to be irreducible is at most n_0, where m_0 and n_0 are as given in (4) and (5).*

In the case $v = 3$, we are able to give only ineffective result as follows.

Theorem 1.3. *The polynomial $g(x)$ with $|a_0| = |a_m| = 1$ and $v = 3$ is almost irreducible except for n_1 number of values of m where*

$$n_1 = 8 \times 5^{\pi(3\mu_1+u)}$$

with

$$\mu_1 = \max(500, 4u).$$

Similar assertion is true for irreducible generalized Hermite–Laguerre polynomials.

Remark 1. The particular cases $v = 1, u \le 30$; $v = 2, u < 29, u$ odd; $v = 3, u = 1, 2$ and $v = 4, u = 1, 3$ have been treated in [18,11–13], respectively.

The following theorem is on small factors of $g(x)$ for arbitrary u and v.

Theorem 1.4. *Let $2 \le k \le \frac{m}{e^{2.5(v+1)}}$. Let $\alpha = \max(e^{2.5(v+1)}, u)$. Then $g(x)$ with $|a_0| = |a_m| = 1$ has no factor of degree k except perhaps for*

$$M = 2 \times (3^{\omega(v)} + 1)5^{\pi(v\alpha+v)} \tag{6}$$

number of values of m. This assertion is true for $1 = k$ in the case of generalized Hermite–Laguerre polynomials.

Apart from the usage of Newton polygon, the proof in [10] depends on three ingredients, viz.,

(i) Prime Number Theorem for Arithmetic Progressions.
(ii) A combinatorial argument due to Erdős.
(iii) Finiteness of the number of solutions to Thue equations.

Explicit results in (i) are known only for restricted values of v, by the works of Ramaré and Rumley [15] and Dusart [5,6]. The second ingredient (ii) works well when the degree k of a factor of $g(x)$ is large compared to u and v. A Theorem of Evertse [7] gives an upper bound for the number of solutions of Thue equations. Although effective methods are known for finding the solutions of Thue equations, they give generally very large bounds which are complicated for explicit computations.

Let $v \in \{1, 2, 4\}$. A lower bound $\delta(u)$ for k in terms of u is obtained in Lemmas 4.1 and 4.2. Then we consider $k \le \delta(u)$. We resort to Pell's equations of the form

$$X^2 - DY^2 = E$$

and we need to find solutions in (X, Y) where both X and Y belong to a set of integers composed of primes from a given set. In 1960, Cassels [3] had shown that these solutions can be obtained in a finite number of steps. In 1962, Lehmer [14] considered the particular cases $E = 1$ and

4 in connection with a problem of Störmer. Using divisibility properties of Lucas sequences he showed that all the required solutions can be obtained in finite number of steps and further these solutions can be explicitly bounded from the bounds for the fundamental solution. Thus for $v = 1, 2, 4$ we get explicit results as given in Theorems 1.1 and 1.2.

2. Lemmas

2.1. Lemmas based on Newton polygons

We begin with the following useful lemma from [8].

Lemma 2.1. *Let k and l be integers with $k > l \geq 0$. Suppose $F(x) = \sum_{j=0}^{m} f_j x^j \in \mathbb{Z}[x]$ and p is a prime such that $p \nmid f_m$, $p | f_j$ for all $j \in \{0, 1, \ldots, m - l - 1\}$, and the right most edge of the Newton polygon for $F(x)$ with respect to p has slope less than $\frac{1}{k}$. Then for any integers $g_0, g_1, \ldots, g_m$ with $|g_0| = |g_m| = 1$, the polynomial $G(x) = \sum_{j=0}^{m} f_j g_j x^j$ cannot have a factor with degree in the interval $[l + 1, k]$. In fact, the assertion is even true if only p does not divide $g_0 g_m$.*

Remark 2. Let

$$\Delta_j = (vm + u) \cdots (v(j + 1) + u). \tag{7}$$

By Lemma 2.1, it is enough to study factors of $f(x) = \sum_{j=0}^{m} \Delta_j x^j$ given by (1) in order to know the factors of (2). Further we can restrict to factors of degree k with $1 \leq k \leq \frac{m}{2}$. We shall always assume this restriction on a generic factor of degree k for $f(x)$ without any mention.

The following lemma is an application of Lemma 2.1.

Lemma 2.2. *Suppose there exists a prime $p > vk + u$ such that p divides Δ_{m-k}. Then $f(x)$ has no factor of degree k.*

Proof. Let $\nu_p(n)$ denote the order of p in any positive integer n. The last slope of the Newton polygon of $f(x)$ with respect to p equals

$$L = \max_{1 \leq j \leq m} \frac{\nu_p(\Delta_0) - \nu_p(\Delta_j)}{j}. \tag{8}$$

Suppose

$$L < \frac{1}{k}. \tag{9}$$

Since $p | \Delta_{m-k}$, we see that $p | \Delta_j$ for $0 \leq j \leq m - k$. Then by (9) and Lemma 2.1 with $l + 1 = k$, we see that $f(x)$ has no factor of degree k, by taking $F(x) = f(x)$ and $g_j = 1$ for $0 \leq j \leq m$. Now we show inequality (9). If $1 \leq j < k$, then $\nu_p((u + v) \ldots (u + vj)) = 0$. Hence

$$\frac{\nu_p(\Delta_0) - \nu_p(\Delta_j)}{j} = 0 < \frac{1}{k}.$$

Now suppose $j \geq k$. We observe that

$$\frac{\nu_p(\Delta_0) - \nu_p(\Delta_j)}{j} = \frac{\nu_p((u + v) \ldots (u + vj))}{j}$$

$$\leq \frac{\nu_p((u+vj)!)}{j}$$
$$< \frac{u+vj}{j(p-1)} \leq \frac{u+vj}{j(u+vk)} \leq \frac{1}{k}.$$

This completes the proof of the lemma. □

2.2. A lemma on estimates

Lemma 2.3. *We have*

(i) $\pi(n) \leq \frac{n}{\log n}\left(1+\frac{1.2762}{\log n}\right)$, *for* $n > 1$,

(ii) $\prod_{p\leq x} p \leq (2.72)^x$,

(iii) $\nu_p(k!) \geq \frac{k-p}{p-1} - \frac{\log(k-1)}{\log p}$, *for* $k > 1$ *and* $p \leq k$,

(iv) $k! < \sqrt{(2\pi k)}\mathrm{e}^{-k}k^k\mathrm{e}^{\frac{1}{12k}}$, *for* $k > 1$.

The first estimate (i) and (ii) are due to Dusart [5,6] and (iii) is a classical result due to Legendre. The estimate (iv) is obtained from the well known Stirling's approximation.

2.3. Lemmas based on a method of Erdős

The following lemma is fundamental in finding large prime factors of Δ_{m-k}. This is available in various forms in many of the papers on this problem mentioned in References. From the proof of Lemma 4 in [11] the following lemma can be easily derived. Let $\pi_v(n)$ denote the number of primes not exceeding n and co-prime to v.

Lemma 2.4. *Let* $k \geq 2, c_0 > 1, c_1 > 0$ *and* $k - \pi_v(c_0k) \geq 1$. *Let*

$$\Lambda = n(n+v)\dots(n+(k-1)v)$$

with $\gcd(n, v) = 1$ *and* $n \geq c_1kv$ *and* $P(\Lambda) \leq c_0k$. *Then*

$$(c_1vk)^{k-\pi_v(c_0k)} \leq (k-1)!\prod_{p|v} p^{-\nu_p((k-1)!)}. \tag{10}$$

Now we apply the estimates in Lemma 2.3 together with Lemma 2.4 to derive the following result.

Corollary 2.5. *Let* $k \geq 2, c_0 > 1, c_1 > 0$. *Suppose* p *is a prime dividing* v *and* $p < c_0k$. *Assume that* $k - \pi_p(c_0k) \geq 1$ *and* $n \geq c_1kv$. *If* $P(\Lambda) \leq c_0k$, *then*

$$\left(\frac{1}{p-1} - \frac{p+1}{k(p-1)}\right)\log p \leq \left(1+\frac{1.2762}{\log k}\right)c_0 - \left(1 - \frac{c_0}{\log k} - \frac{1.2762c_0}{(\log k)^2}\right)\log c_1.$$

Proof. Since the conditions of Lemma 2.3 are satisfied and $\pi_v(c_0k) \le \pi(c_0k) - 1$ we get by (10) and Lemma 2.3 (iii) that

$$\begin{aligned}(k - \pi(c_0k))\log(c_1k) &\le (k - \pi(c_0k))\log(c_1vk)\\ &\le (k-1)\log k - \left(\frac{k-1-p}{p-1} - \frac{\log(k-2)}{\log p}\right)\log p\\ &\le k\log k - \left(\frac{k}{p-1} - \frac{p+1}{p-1}\right)\log p.\end{aligned}$$

Now we apply Lemma 2.3(i), to get

$$\begin{aligned}(\log p)\left(\frac{k}{p-1} - \frac{p+1}{p-1}\right) &\le \pi(c_0k)\log\ k - (k - \pi(c_0k))\log c_1\\ &\le c_0k\left(1 + \frac{1.2762}{\log k}\right) - k\left(1 - \frac{c_0}{\log k} - \frac{1.2762c_0}{(\log k)^2}\right)\log c_1.\end{aligned}$$

Dividing by k, we get the inequality in the corollary. □

Let c_0 and c_1 be fixed. It is easy to see that the function on the right hand side of the inequality in Corollary 2.5 is a decreasing function of k while the left hand side is an increasing function of k. Thus if for some value of $k = k_0$, the right hand side function becomes ≤ 0, while the left hand side function is >0, then we get a contradiction for all $k \ge k_0$. Then we conclude that

$$P(\Delta) > c_0k \quad \text{for } k \ge k_0.$$

Now we set $c_0 = v + 1$ and $c_1 = \mathrm{e}^{2.4(v+1)}$. Let $k = k_0 = \mathrm{e}^{2.5(v+1)}$. We see that for $v \ge 2$ the right hand side expression does not exceed

$$(1.171)(v+1) - (1.27)(v+1) = -(0.9)(v+1) < 0$$

and the left hand side remains positive since $p|v$. Thus for the chosen values of c_0 and c_1, we have $P(\Delta) > c_0k$ for $k \ge k_0$. Let us now consider Δ_{m-k}. The smallest term of this product is

$$u + (m - k + 1)v.$$

This exceeds $c_1kv = \mathrm{e}^{2.4(v+1)}kv$ provided

$$m - k + 1 > \mathrm{e}^{2.4(v+1)}k.$$

It suffices that

$$m > \mathrm{e}^{2.5(v+1)}k.$$

From the above discussion we conclude the following proposition.

Proposition 2.6. *Let* $m > \mathrm{e}^{2.5(v+1)}k \ge \mathrm{e}^{2.5(v+1)}\max(\mathrm{e}^{2.5(v+1)}, u)$. *Then*

$$P(\Delta_{m-k}) > vk + u.$$

Remark 3. In [10], c_1 is taken as $(\log k)^2$ for large k and hence $m > k(\log\ k)^2$. In the above proposition this condition is weakened to $m > \mathrm{e}^{2.5(v+1)}k$.

For application to small values of v, we need a more precise version of Corollary 2.5. We use the inequality (iv) of Lemma 2.3 in (10) to get the following result. As the derivation is similar to Corollary 2.5, we omit the proof.

Corollary 2.7. *Let $k \geq 2$, $c_0 > 1$, $c_1 > 0$. Suppose p is a prime dividing v and $p < c_0 k$. Assume that $k - \pi_p(c_0 k) \geq 1$ and $n \geq c_1 k v$. If $P(\Lambda) \leq c_0 k$, then*

$$\log\left(c_1 v e p^{\frac{1}{p-1}}\right) \leq c_0\left(1 + \frac{1.2762}{\log c_0 k}\right)\left(\frac{\log c_1 v k}{\log c_0 k}\right) + \frac{1}{k}\left(1 + \frac{1}{2}\log 2\pi + \frac{1}{12(k-1)} + \frac{p+1}{p-1}\log p\right).$$

2.4. *A theorem of Evertse*

A result due to Evertse [7] gives an upper bound for the number of solutions of Thue equations. The following lemma is a particular case of his result for cubic Thue equations.

Lemma 2.8. *Let*

$$\left|Nx^3 - My^3\right| = v$$

with N and M given positive integers. Then the above equation has at most

$$4 \times 3^{\omega(v)} + 3$$

solutions in positive integers (x, y).

3. Proof of Theorem 1.4

By Proposition 2.6 and Lemma 2.2, we conclude that $f(x)$ has no factor of degree $k \geq \alpha$, where $\alpha = \max(e^{2.5(v+1)}, u)$. Let us now consider the case $2 \leq k < \alpha$. We take

$$\Delta_2 = (mv + u)\left((m-1)v + u\right).$$

We may assume by Lemma 2.2 that

$$P(\Delta_2) \leq vk + u < v\alpha + u =: \beta.$$

Let $2 = p_1 < p_2 < \cdots < p_{\pi(\beta)}$ be the sequence of all the primes $\leq \beta$. Then

$$mv + u = p_1^{a_1} \ldots p_{\pi(\beta)}^{a_{\pi(\beta)}} \quad \text{and} \quad (m-1)v + u = p_1^{b_1} \ldots p_{\pi(\beta)}^{b_{\pi(\beta)}}, \tag{11}$$

with $a_i \geq 0, b_i \geq 0$ for $1 \leq i \leq \pi(\beta)$. Since $\gcd(u, v) = 1$, these two integers have no common factors. We reduce the powers a_i and b_i modulo 3. Then we get a cubic equation

$$\left|Ax^3 - By^3\right| = v \tag{12}$$

with $\gcd(Ax^3, By^3) = 1$. For any prime $p \leq p_{\pi(\beta)}$, we have $\left(\nu_p(A), \nu_p(B)\right) \in \{(0,0), (1,0), (2,0), (0,1), (0,2)\}$. Thus totally we have $5^{\pi(\beta)}$ choices for A and B. Since we consider only absolute value on the left hand side of (12), we get $(5^{\pi(\beta)} - 1)/2$ number of distinct cubic equations in (12). Now by Lemma 2.8, Eq. (12) has at most $4 \times 3^{\omega(v)} + 3$ solutions in (x, y). Thus the number of m for which (11) holds is at most

$$\left(4 \times 3^{\omega(v)} + 3\right)\left(5^{\pi(v\alpha+u)} - 1\right)/2 < 2(3^{\omega(v)} + 1)5^{\pi(v\alpha+u)}.$$

This proves the first assertion of Theorem 1.4. □

3.1. A lemma on generalized Laguerre polynomials

We include possible linear factors of $g(x)$ with $a_j = \binom{m}{j} b_j$. For this we need the following lemma.

Lemma 3.1. *Let $k \geq 1$. Suppose there exists a prime $p > vk + u$ dividing m. Then $g(x)$ with $a_j = \binom{m}{j} b_j$ has no factor of degree k.*

Proof. Consider

$$\binom{m}{j} = \frac{(j+1)\ldots m}{(m-j)!}, \quad 0 \leq j \leq m-1.$$

Hence $p | \binom{m}{j}$ if $m - j \leq p - 1$, i.e., if $j \geq m - p + 1$. Now

$$\Delta_j = (vm + u)\ldots(v(j+1) + u).$$

Since this is a product of $m - j$ terms in arithmetic progression with $\gcd(v, u) = 1$, it is divisible by p if $m - j \geq p$, i.e., if $j \leq m - p$. Thus

$$p \mid \binom{m}{j} \Delta_j \quad \text{for } 0 \leq j \leq m-1.$$

Now we follow the proof of Lemma 2.2, to get the assertion of the lemma. □

Proof of the second assertion of Theorem 1.4. By Lemma 2.1, it is enough to consider $g(x)$ with $a_j = \binom{m}{j}$. By Proposition 2.6, Lemma 2.2, we conclude that $g(x)$ has no factor of degree $k \geq \alpha$ where $\alpha = \max(e^{2.5(v+1)}, u)$. So we assume that $1 \leq k < \alpha$. We take

$$a_{m-1}\Delta_1 = m(mv + u).$$

By Lemmas 2.2 and 3.1, we may assume that

$$P(a_{m-1}\Delta_1) \leq vk + u < v\alpha + u = \beta.$$

Now we follow the argument in the proof of the first assertion of Theorem 1.4 to get the second assertion. □

4. The cases of $v \in \{1, 2, 3, 4\}$

The generalized Hermite–Laguerre polynomials for small values of v has been dealt with considerably in the papers [11–13,18]. For the purpose of this paper we use Theorem 1.1 of [18] when $v = 1$ and Theorem 2 of [13] when $v = 2$. We state the results as a lemma below in our notation. For the cases $v = 3$ and 4, a similar result is not given explicitly in [12,13]. So we state this case in Lemma 4.2 and give the necessary details.

4.1. Lemmas for large k

Lemma 4.1.

(i) *Let $v = 1$. Suppose $k \geq 2$ and $0 \leq u \leq \frac{3}{2}k$. If $g(x)$ has a factor of degree k, then*

$$(m, k, u) \in \{(6, 2, 3), (7, 2, 2), (7, 2, 3), (7, 3, 3), (8, 2, 1), (8, 3, 2), (12, 3, 4), (13, 2, 3), (22, 2, 3), (46, 3, 4), (78, 2, 3)\}.$$

(ii) *Let* $v = 2, k \geq 1, 0 \leq u \leq k$ *and* $a_0 a_n \in \{\pm 2^t : t \geq 0,\ t \in \mathbb{Z}\}$. *Then* $g(x)$ *is irreducible for* $u \neq 1$ *and when* $u = 1$ *it is almost irreducible.*

Lemma 4.2. *Let* $v = 3, 4$. *Suppose*

$$k \geq \begin{cases} \max(500, 4u) & \text{if } v = 3 \\ \max(600, 10u) & \text{if } v = 4. \end{cases}$$

Then $g(x)$ *cannot have a factor of degree* k.

Proof. Let $v = 3$. Assume that $k \geq \max(500, 4u)$. Suppose further we have the smallest term of Δ_{m-k} viz., $X := (m-k+1)v + u \leq 10.6 \times 3k$ and $X \geq 6450$. Then from Corollary 2.3 [13] we get

$$P(\Delta_{m-k}) > vk + u.$$

Note that the proof of this part depends on the explicit bounds from the results on primes in arithmetic progression when the common difference is 3. Now we assume that $X > 10.6 \times 3k$. We apply Corollary 2.7 with $c_0 = 3.25$ and $c_1 = 10.6$, $p = 3$ to find that the inequality in the corollary is not valid. This means that $P(\Delta_{m-k}) > vk + u$ in this case also. Lastly we consider $3.25k < X < 6450$. Then $500 \leq k \leq 1984$. We directly check that in this range $P(\Delta_{m-k}) > vk + u$. Now we apply Lemma 2.2 to get the assertion of the lemma.

Let $v = 4$. Assume that $k \geq \max(600, 10u)$. Suppose further we have $X = (m-k+1)v+u \leq 138 \times 4k$. Then using Corollaries 4.5 and 4.3 of [12] we get

$$P(\Delta_{m-k}) \geq vk + u.$$

The proof of this part uses results on primes in arithmetic progression when the common difference is 4. Now we assume that $X > 138 \times 4k$. We apply Corollary 2.7 with $c_0 = 4.1$ and $c_1 = 138$, $p = 2$ to find that the inequality in the corollary is not valid. This means that $P(\Delta_{m-k}) > vk + u$ in this case also. Now we apply Lemma 2.2 to get the assertion of the lemma. □

4.2. Lemma of Lehmer

In order to make the small cases $v \in \{1, 2, 4\}$, effective, we avoid cubic Thue equations as done in the proof of Theorem 1.4. We use results of Lehmer on the problem of finding all pairs $(s, s+v)$ both belonging to a given set. We refer to Theorems 4, 5, 7, 8 and 9 of [14] for the following lemma.

Lemma 4.3. *Let* $N_d(t)$ *be the number of pairs of integers* $(S, S+d)$ *whose product has its prime factors restricted to a given set of* t *primes* $q_1 < \cdots < q_t$ *and let* $L_d(t)$ *be the largest* S *for which such a product exists. Further let*

$$M = \max\left(3, \frac{q_t + 1}{2}\right) \quad \text{and} \quad P = q_1 \dots q_t.$$

Then

(a) $N_d(t) \leq M(2^t - 1)$ *if* $d = 1, 2$

(b) $N_4(t) \leq (M+1)2^t/3$

(c) $\log L_1(t) < M\,(2 + \log(8P))\sqrt{2P}$

(d) $\log L_d(t) < M\,(2 + \log(4P))\sqrt{P}$ *for* $d = 2, 4$.

Proof of Theorems 1.1, 1.2 and 1.3. By Lemma 4.1 for $v = 1, 2$ and Lemma 4.2 for $v = 3, 4$, we may assume that

$$1 \le k < k_0$$

where

$$k_0 = \begin{cases} \frac{2u}{3} & \text{if } v = 1 \\ u & \text{if } v = 2 \\ \max(500, 4u) & \text{if } v = 3 \\ \max(600, 10u) & \text{if } v = 4. \end{cases}$$

We follow the proof of Theorem 1.4. We may assume that

$$P(\Delta_2) \le vk + u < vk_0 + u = \beta.$$

In Lemma 4.3, we take $(m-1)v + u = S$, $t \le \pi(\beta)$ and $q_t \le \beta$. We have

$$P = q_1 \ldots q_t \le \prod_{p \le \beta} p \le (2.72)^{\beta},$$

by Lemma 2.3(ii). Also

$$M = \max\left(3, \frac{q_t + 1}{2}\right) \le 2\beta.$$

Applying Lemma 4.3, we get

$$N_v(t) \le 2(vk_0 + u)2^{\pi(vk_0+u)}.$$

Further

$$\begin{aligned} \log L_v(t) &\le 2\beta\left(2 + \log(8 \times (2.72)^{\beta})\right)\sqrt{2(2.72)^{\beta}} \\ &\le 10(1.65)^{\beta}\beta^2. \end{aligned}$$

Thus

$$\log L_v(t) \le 10(vk_0 + u)^2(1.65)^{vk_0+u}. \tag{13}$$

Combining with Lemmas 4.1 and 4.2 we find that the number of integers m for which $g(x)$ is *not* almost irreducible is bounded by

$$2(vk_0 + u)2^{\pi(vk_0+u)} + \theta,$$

where

$$\theta = \begin{cases} 11 & \text{if } v = 1 \\ 1 & \text{if } v = 2 \\ 0 & \text{if } v = 4. \end{cases}$$

By (13) when $v \in \{1, 2, 4\}$ the maximum of such m is bounded by $\exp\left(10(v\mu + u)^2(1.65)^{v\mu+u}\right)$.

In the case $v = 3$, we combine Lemmas 2.8 and 4.2 and the argument in the proof of Theorem 1.4 to get that the number of m for which $g(x)$ is not almost irreducible is bounded by $8 \times 5^{\pi(3\mu_1+u)}$.

The assertions in Theorems 1.2 and 1.3 regarding generalized Hermite–Laguerre polynomials are similar to the second half of the proof of Theorem 1.4. □

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