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Order Analogues and Betti Polynomials

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We exhibit an order-preserving surjection from the lattice of subgroups of a finite abelian p-group of type λ onto the product of chains of lengths the parts of the partition λ. Thereby, we establish the subgroup lattice as an order-theoretic, not just enumerative, p-analogue of the chain product. This insight underlies our study of the simplicial complexes $\Delta_S(p)$, whose simplices are chains of subgroups of orders p^k, some $k \in S$. Each of these subgroup complexes is homotopy equivalent to a wedge of spheres of dimension $|S|-1$. The number of spheres in the wedge, $\beta_S(p)$, is known to have nonnegative coefficients as a polynomial in p. Our main result provides a topological explanation of this enumerative result. We use our order-preserving surjection to find $\beta_S(p)$ maximal simplices in $\Delta_S(p)$ whose deletion leaves a contractible subcomplex. This work suggests a definition of order analogue; our main result holds for any semimodular lattices that are order analogues of a semimodular lattice.

0. PRELIMINARIES AND HISTORICAL CONTEXT

In the 1950s Philip Hall's study of polynomials that count subgroups of finite abelian p-groups led him to define what are now known as Hall–Littlewood symmetric functions. Each of the polynomials Hall studied, $g^{\lambda}_{\mu\nu}(p)$, evaluated at primes p gives the number of subgroups H of type μ in a finite abelian p-group G of type λ such that G/H is of type ν. (A finite abelian p-group is said to be of *type* the partition λ if it is isomorphic to $\mathbf{Z}/p^{\lambda_1}\mathbf{Z} \times \cdots \times \mathbf{Z}/p^{\lambda_l}\mathbf{Z}$.) He showed (see, e.g., [14]) that these *Hall polynomials* have integral coefficients, but to date there is no formula for them akin to the simple formula (see, e.g., [15] or [10]) for the number $\left[\begin{smallmatrix}\lambda'\\ \mu'\end{smallmatrix}\right]_p$ of subgroups of type μ in a finite abelian p-group of type λ.

$$\begin{bmatrix}\lambda'\\ \mu'\end{bmatrix}_p = \prod_{i \geqslant 1} p^{\mu'_{i+1}(\lambda'_i - \mu'_i)} \begin{bmatrix}\lambda'_i - \mu'_{i+1}\\ \mu'_i - \mu'_{i+1}\end{bmatrix}_p. \tag{1}$$

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(On the right hand side are *p-binomial coefficients* or *Gaussian polynomials* and λ' is the partition *conjugate* to λ.) Given this simple formula, it is surprising that the machinery of Hall-Littlewood symmetric functions was used in the first proof [6] that the number, $[{\lambda' \atop k}]_p = \sum_{\mu \vdash k} [{\lambda' \atop \mu'}]_p$, of subgroups of order p^k does not exceed the number of subgroups of order p^{k+1} if $k < |\lambda|/2$. The proof shows that the polynomial $[{\lambda' \atop k+1}]_p - [{\lambda' \atop k}]_p$ has nonnegative coefficients if $k < |\lambda|/2$. This *p-unimodality* result inspired subsequent work on q-log-concavity [7, 13, 17].

This unimodality result is one of three elegant enumerative results [6, 9] on subgroup lattices first obtained using Hall–Littlewood symmetric functions. The other two concern *Betti polynomials*, which we now define. Let G be a finite abelian p-group of type $\lambda \vdash n$. For each set $S = \{s_1, ..., s_{|S|}\}_< \subseteq [n-1]$, let $[{\lambda' \atop S}]_p$ be the number of chains of subgroups

$$e \subset H_1 \subset \cdots \subset H_{|S|} \subset G,$$

where the order of H_i is p^{s_i}. From Equation (1) we see that $[{\lambda' \atop S}]_p$ is a polynomial in p with nonnegative coefficients. Stanley conjectured that the polynomial

$$\beta_S(p) = \sum_{T \subseteq S} (-1)^{|S|-|T|} \begin{bmatrix} \lambda' \\ T \end{bmatrix}_p \tag{2}$$

has nonnegative coefficients. It is not surprising that $\beta_S(p)$ yields a nonnegative number when evaluated at primes p. (By the Euler–Poincaré formula of algebraic topology, $\beta_S(p) = \dim_{\mathbf{Q}} \tilde{H}_{|S|-1}(\Delta_S(p); \mathbf{Q})$, since the simplicial complex $\Delta_S(p)$ of chains of subgroups with orders p^k, some $k \in S$, is homotopy equivalent to a wedge of spheres of dimension $|S|-1$.) Yet the symmetric function theoretic proof given by Stanley [6] that $\beta_S(p)$ has nonnegative coefficients gives no topological insight into the subgroup complex $\Delta_S(p)$ whose top dimensional Betti number is $\beta_S(p)$. The main theorem of this paper provides a striking topological explanation of this enumerative result on Betti polynomials.

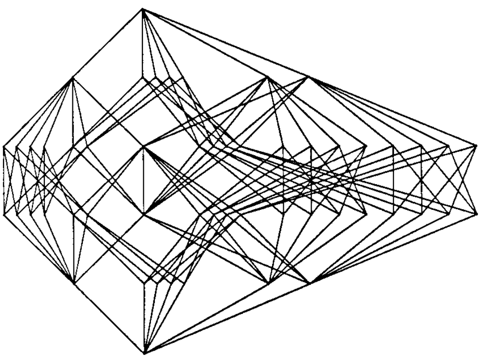
1. Illustration of the Main Result

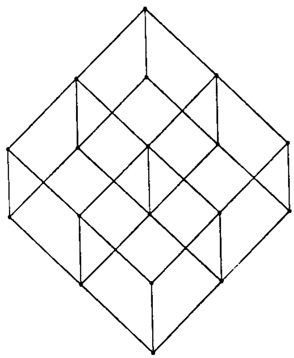
From general theory (see, e.g., [2] and [4]), we know that $\Delta_S(p)$ is homotopy equivalent to a wedge of $\beta_S(p) = \sum_{T \subseteq S} (-1)^{|S|-|T|} [{\lambda' \atop T}]_p$ spheres of dimension $|S|-1$. Our task is to find $\beta_S(p)$ maximal simplices in $\Delta_S(p)$ whose deletion leaves a contractible subcomplex. The way we find these simplices must make it obvious that $\beta_S(p)$ is a polynomial in p with nonnegative coefficients.

We begin with a lattice-theoretic analysis of an alternative proof [5] of Stanley's conjecture on Betti polynomials. In this proof we obtained a combinatorial formula for $[{\lambda' \atop S}]_p$, then applied inclusion-exclusion. The formula was inspired by an algorithm presented by Birkhoff [1] for obtaining a standard set of generators for each subgroup of $\mathbf{Z}/p^{\lambda_1}\mathbf{Z} \times \cdots \times \mathbf{Z}/p^{\lambda_l}\mathbf{Z}$. In [5] we use this algorithm to define a rank-preserving surjection of the lattice $L_\lambda(p)$ of subgroups onto the chain product $[0, \lambda] = [0, \lambda_1] \times \cdots \times [0, \lambda_l]$, such that the inverse image of each element in the chain product has cardinality a power of p.

To provide a topological explanation of the result on Betti polynomials we first show, in Section 2, that the surjection of $L_\lambda(p)$ onto $[0, \lambda]$ is order-preserving.

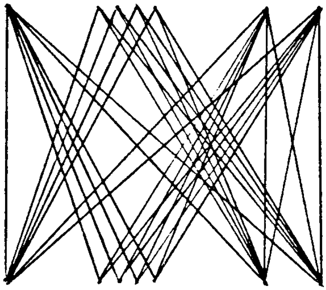
Example 1.1. Our surjection of the lattice of subgroups of $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, shown at the left, onto the chain product $[0, 2] \times [0, 2] \times [0, 1]$, shown at the right, is illustrated in the Appendix by six frames from a video produced by Toby Orloff of the Geometry Supercomputing Group.

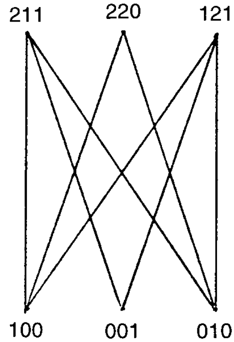




Since the surjection $\varphi: L_\lambda(p) \to [0, \lambda]$ is order preserving, it induces, for each set S, a simplicial map from the simplicial complex $\Delta_S(p)$ of chains of subgroups with orders p^k, $k \in S$, to the simplicial complex Δ_S of chains of elements in $[0, \lambda]$ with ranks in S. For simplicity, we also denote this induced simplicial map by φ.

Example 1.2. The order preserving surjection from $L_{221}(p)$ onto $[0, 2] \times [0, 2] \times [0, 1]$ illustrated at the end of this paper for $p = 2$ induces a simplicial map from $\Delta_{\{1, 4\}}(p)$ onto $\Delta_{\{1, 4\}}$. This simplicial map is illustrated below for $p = 2$.





The number of maximal simplices in $\Delta_S(p)$ mapped under the simplicial map φ to a given maximal simplex in Δ_S has cardinality a power of p. (See Section 2.) To show the Betti polynomial $\beta_S(p)$ has nonnegative coefficients we find a collection $\mathscr{C}$ of maximal simplices in Δ_S such that deletion of the $\beta_S(p)$ maximal simplices in $\varphi^{-1}(\mathscr{C})$ leaves a contractible subcomplex of $\Delta_S(p)$.

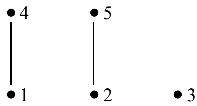
Example 1.3. In $\Delta_{\{1,4\}}(p)$, p^0 simplices map to $100 \subset 211$, p^2 simplices map to $100 \subset 220$, p^1 simplices map to $100 \subset 121$, p^2 simplices map to $001 \subset 211$, p^3 simplices map to $001 \subset 121$, p^1 simplices map to $010 \subset 211$, p^3 simplices map to $010 \subset 220$, p^2 simplices map to $010 \subset 121$. The powers are inversion numbers of tabloids corresponding to the chains. (See [5].) Entries in each row of a *tabloid* are nondecreasing. The tabloids are, respectively,

1 2	1 2	1 3	2 2	2 3	2 2	2 2	2 3
2 3,	2 2,	2 2,	2 3,	2 2,	1 3,	1 2,	1 2.
2	3	2	1	1	2	3	2

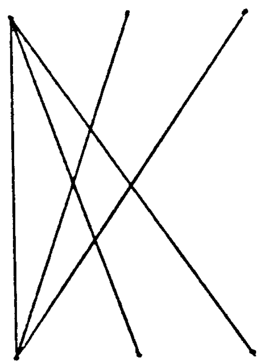
For each entry y in the tabloid, count the number of entries $x < y$ that lie below y in the same column or above y one column to the right. Sum these numbers over all entries y to obtain the inversion number of the tabloid.

From general theory it is known that the simpler simplicial complex Δ_S is homotopy equivalent to a wedge of spheres of dimension $|S|-1$; from Equations (1) and (2) we know that there are $\beta_S(1)$ spheres in this wedge. Moreover, Björner's theory ([2] and [4]) of lexicographic shellability can be used to find $\beta_S(1)$ maximal simplices whose deletion leaves a contractible subcomplex. Maximal simplices correspond to linear extensions of $[\lambda_1] + \cdots + [\lambda_l]$ with descent set contained in S; delete maximal simplices that correspond to linear extensions with descent set equal to S.

EXAMPLE 1.4. The chain product $[0,2]\times[0,2]\times[0,1]$ is isomorphic to the lattice of order ideals in the poset $[2]+[2]+[1]$. A labelling of the elements of this poset defines a 1–1 correspondence between linear extensions with descent set contained in S and maximal simplices of Δ_S. For $S=\{1,4\}$, the labelling

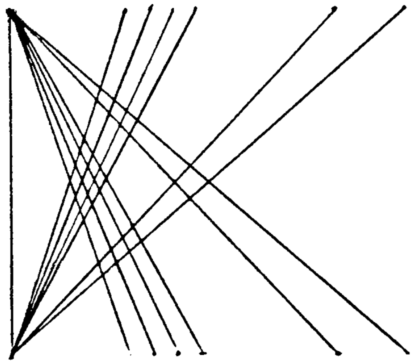
$[2]+[2]+[1]=$ 

pairs 12345 with $(100\subset 211)$, 12453 with $(100\subset 220)$, 12354 with $(100\subset 121)$, 31245 with $(001\subset 211)$, 31254 with $(001\subset 121)$, 21345 with $(010\subset 211)$, 21453 with $(010\subset 220)$, 21354 with $(010\subset 121)$. Of these linear extensions, three (21354, 21453, and 31254) have descent set $\{1,4\}$. Deletion of the corresponding maximal simplices $(010\subset 121,\ 010\subset 220$, and $001\subset 121)$ of $\Delta_{\{1,4\}}$ leaves a contractible subcomplex.



Let $\mathscr{C}$ denote the maximal simplices of Δ_S that correspond to linear extensions of $[\lambda_1]+\cdots+[\lambda_l]$ with descent set equal to S. (The collection $\mathscr{C}$ is computed using a certain labelling of elements in the disjoint sum of chains. The element of rank j in $[\lambda_i]$ is labelled $i+\sum_{k<j}\lambda'_k$.) Not only is $\beta_S(p)$ equal to the number of maximal simplices in $\Delta_S(p)$ which are mapped by φ to a member of $\mathscr{C}$, but also our main theorem states that deletion of these maximal simplices leaves a contractible subcomplex!

EXAMPLE 1.5. Deletion of the p^2 simplices that are mapped to $010\subset 121$, the p^3 simplices that are mapped to $010\subset 220$, and the p^3 simplices that are mapped to $001\subset 121$ leaves a contractible subcomplex of $\Delta_{\{1,4\}}(p)$.



THEOREM 1.6. *Let $\mathscr{C}$ be the above-described collection of maximal simplices of Δ_S whose deletion leaves a contractible subcomplex. Then*

$$\beta_S(p) = \sum_{c \in \mathscr{C}} |\varphi^{-1}(c)|$$

and deletion of the maximal simplices in $\varphi^{-1}(\mathscr{C})$ from $\Delta_S(p)$ leaves a contractible subcomplex.

2. ORDER ANALOGUES

Although there is no explicit definition in the literature of what is meant by calling posets $L(q)$ *q-analogues* of a poset L, there are numerous examples. Björner [4] discusses q-analogues of weak and strong Bruhat order on S_n; Dowling [11] is said to have defined q-analogues of partitions lattices; and Björner and Stanley [19, p. 164], inspired by Knuth [12], make a first attempt at construction of q-analogues for every finite distributive lattice. These examples have the following enumerative property: the number of elements of rank k in $L(q)$ is a polynomial in q with nonnegative coefficients that evaluates at $q=1$ to the number of elements of rank k in L.

This enumerative property prompted combinatorialists (see, e.g., [18]) to call the lattice $L_\lambda(p)$ of subgroups of $\mathbf{Z}/p^{\lambda_1}\mathbf{Z} \times \cdots \times \mathbf{Z}/p^{\lambda_l}\mathbf{Z}$ the p-analogue of the chain product $[0, \lambda] = [0, \lambda_1] \times \cdots \times [0, \lambda_l]$. One is left with a disquieting feeling that it might be just coincidence that $[{\lambda' \atop \mu'}]_p$ (see Equation (1)) evaluates at $p=1$ to the number of elements in $[0, \lambda]$ whose multiset of nonzero components is the multiset of parts of μ. The fact that the Gaussian coefficient $[{n \atop k}]_q$ evaluates at $q=1$ to the binomial coefficient $\binom{n}{k}$ is not viewed as a coincidence since Knuth [12] exhibited a rank-preserving surjection from the lattice $L_{1^n}(q)$ of subspaces of $(\mathbf{F}_q)^n$ to the Boolean algebra $B_n = [0, 1]^n$ such that the inverse image in $L_{1^n}(q)$ of a k-subset has cardinality a power of q. (The power depends on the k-subset but not on q.) In the examples cited above, the enumerative property which justified calling $L(q)$ a q-analogue of L is a consequence of the existence of a explicit rank-preserving surjection such that the inverse image in $L(q)$ of an element of rank k in L has cardinality a power of q. Henceforth we refer to ranked posets $L(q)$ as *enumerative q-analogues* of L only if such compatible rank-preserving surjections exist.

To provide a general context for the results of this paper we require a stronger notion of q-analogue. Knuth remarked that the rank-preserving surjections $L_{1^n}(q) \to [0, 1]^n$ are order-preserving. Our work on Betti polynomial suggested the following definition of *order-theoretic q-analogues.*

DEFINITION 2.7. The graded, rank n poset $L(q)$ in a family indexed by an infinite set of positive integers is called an *order-theoretic q-analogue* (or simply *order analogue*) of a graded, rank n poset L if there are surjections $\phi: L(q) \to L$ satisfying:

(i) If $H < K$, then $\varphi(H) < \varphi(K)$.

(ii) If $\alpha \leqslant \varphi(K)$, then the cardinality of $\{H \mid H \leqslant K \text{ and } \varphi(H) = \alpha\}$ is a power of q determined by α and $\varphi(K)$.

(iii) If $\{\alpha \mid \alpha \leqslant \varphi(K)\}$ is a chain in L, then $\{H \mid H \leqslant K\}$ is a chain in $L(q)$.

We now show that the lattice $L_\lambda(p)$ of subgroups of $\mathbf{Z}/p^{\lambda_1}\mathbf{Z} \times \cdots \times \mathbf{Z}/p^{\lambda_l}\mathbf{Z}$ is an order analogue of the chain product $[0, \lambda] = [0, \lambda_1] \times \cdots \times [0, \lambda_l]$. Rank-preserving surjections $\varphi: L_\lambda(p) \to [0, \lambda]$ were studied from an enumerative viewpoint in [5]. The theorem below states that these surjections are order-preserving. Property (ii) then follows from the enumerative study of φ in [5]. Property (iii) follows immediately from the definition of φ, which we now state.

DEFINITION 2.8. Let $\lambda = \{\lambda_1, ..., \lambda_l\}_\leqslant$ be a partition. Let H be a subgroup of type $\mu = \{\mu_1, ..., \mu_k\}_\leqslant$ in $\mathbf{Z}/p^{\lambda_1}\mathbf{Z} \times \cdots \mathbf{Z}/p^{\lambda_l}\mathbf{Z}$. An ordered set $\{g^{(1)}, ..., g^{(k)}\}$ of elements of H is called a set of *Hall generators* for H if it satisfies the following four conditions.

1. $H = \langle g^{(1)}, ..., g^{(k)} \rangle$.
2. The order of $g^{(i)} = (g_1^{(i)}, ..., g_l^{(i)})$ is p^{μ_i}.

Define a map $i \mapsto I$ so that I is largest with $p^{\mu_i} = \text{order}(g_I^{(i)})$.

3. If $j > i$, then $g_I^{(j)} = 0$.
4. If $j > i$ and $\mu_j = \mu_i$, then $J < I$.

Define $\varphi\colon L_\lambda(p) \to [0, \lambda]$ by $\varphi(H) = \bigvee_i \mu_i e_I$, where e_I has a 1 in the Ith component.

Think of the Ith component of $g^{(i)}$ as the rightmost component of the same order p^{μ_i} as $g^{(i)}$. Since the unit vectors e_I are distinct, the join above is actually a sum.

THEOREM 2.9. *The surjection* $\varphi\colon L_\lambda(p) \to [0, \lambda]$ *is order-preserving.*

Proof. Let $\{g^{(1)}, ..., g^{(k)}\}$ be a set of Hall generators for a subgroup of type $\mu = \{\mu_1, ..., \mu_k\}_\leqslant$ in the group $\mathbf{Z}/^{\lambda_1}\mathbf{Z} \times \cdots \times \mathbf{Z}/p^{\lambda_l}\mathbf{Z}$. For any element g in this group, define *type* $g = (\text{logorder}\, g) e_G$, where g has order $p^{\text{logorder}\, g}$ and the Gth component is the rightmost component of g of order $p^{\text{logorder}\, g}$. Notice

$$\varphi(\langle g^{(1)}, ..., g^{(k)} \rangle) = \bigvee_i \text{type}\, g^{(i)}.$$

It suffices to prove that: If $g \in \langle g^{(1)}, ..., g^{(k)} \rangle$, then type $g \leqslant \text{type}\, g^{(i)}$ for some i.

Since $g \in \langle g^{(1)}, ..., g^{(k)} \rangle$, there is an expression for g of the form

$$g = a_1 p^{\alpha_1} g^{(1)} + \cdots + a_k p^{\alpha_k} g^{(k)}$$

where a_j and p are relatively prime and $1 \leqslant \alpha_j \leqslant \mu_j$. Among all i such that $\mu_i - \alpha_i = \max_j \{\mu_j - \alpha_j\}$, pick i to maximize I. The logorder of $a_j p^{\alpha_j} g^{(j)}$ is $\mu_j - \alpha_j$, since the logorder of $g^{(j)}$ is μ_j. Furthermore the rightmost component of $a_j p^{\alpha_j} g^{(j)}$ of this order is the Jth component. Hence, inductive use of the following fact shows that type $g = (\mu_i - \alpha_i)\, e_I$.
Fact: Let type $a = (\text{logorder}\, a)\, e_A$ and type $b = (\text{logorder}\, b)\, e_B$. If $A \neq B$, then type $a + b = \max\{\text{logorder}\, a, \text{logorder}\, b\}\, e_C$, where

$$C = \begin{cases} A & \text{if logorder } a > \text{logorder } b, \\ \max\{A, B\} & \text{if logorder } a = \text{logorder } b, \\ B & \text{if logorder } a < \text{logorder } b. \end{cases} \qquad \blacksquare$$

Enumerative properties of φ are discussed in [5]. There are many Hall generating sets for a subgroup H, but exactly one is fixed once you require that $g_I^{(i)} = p^{\lambda_i - \mu_i}$ and that $g_I^{(j)} < g_I^{(i)}$ for $j < i$. These standard generating sets are used to establish the enumerative properties of φ encoded in tabloids weighted by inversion number. For example, the power in Property (ii) of

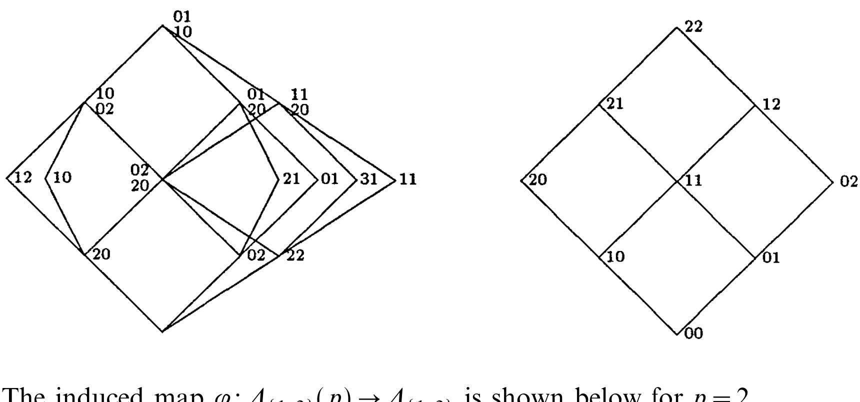
Definition 2.7 is easily calculated: Given a subgroup K such that $\varphi(K)=\beta$, the number of $H \subset K$ with $\varphi(H)=\alpha$ is p raised to the inversion number of the partial tabloid whose ith row has α_i 1's followed by β_i 2's. See Example 1.3.

COROLLARY 2.10. *$L_\lambda(p)$ is an order-analogue of* $[0, \lambda]$.

Our topological explanation of why the Betti polynomial of $\Delta_S(p)$ has nonnegative coefficients relies on the following corollary.

COROLLARY 2.11. *The order-preserving surjection $\varphi: L_\lambda(p) \to [0, \lambda]$ induces simplicial maps $\varphi: \Delta_S(p) \to \Delta_S$ that send maximal simplices to maximal simplices. The inverse image of any maximal simplex $c \in \Delta_S$ has cardinality a power of p.*

EXAMPLE 2.12. The surjection $\varphi: L_{22}(p) \to [0, 2] \times [0, 2]$ is shown below for $p=2$.



The induced map $\varphi: \Delta_{\{1, 3\}}(p) \to \Delta_{\{1, 3\}}$ is shown below for $p=2$.

3. Proof of the Main Result

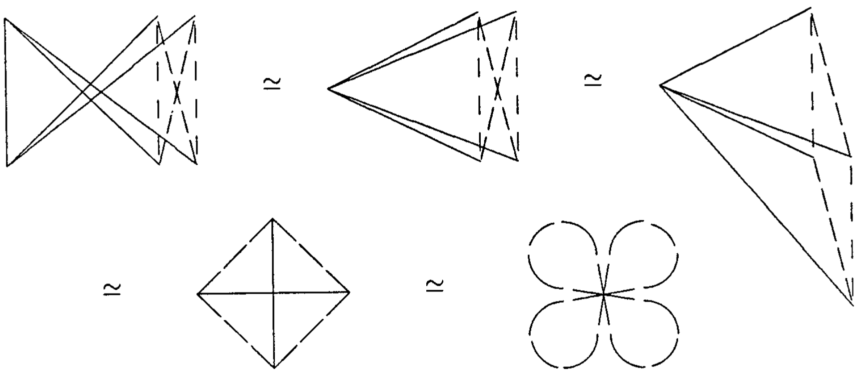
Our proof of Theorem 1.6 employs Björner's theory [2–4] of lexicographic shellability. Since $L_\lambda(p)$ is a modular lattice, the simplicial complex $\Delta_S(p)$ is *shellable* for all $S \subseteq [\,|\lambda| - 1\,]$. That is, there is an ordering

$$C_1, C_2, ..., C_{\left[{\lambda' \atop S}\right]_p}$$

of the maximal simplices of $\Delta_S(p)$ such that $\overline{C_k} \cap \{\overline{C_1} \cup \cdots \cup \overline{C_{k-1}}\}$ is pure of dimension $|S| - 2$. (Here $\overline{C_k}$ is the simplicial complex of faces of C_k, and a simplicial complex is *pure* if all its maximal simplices have the same dimension.) Since $\Delta_S(p)$ is shellable, it is homotopy equivalent to a wedge of spheres of dimension $|S| - 1$. The number of spheres in this wedge is

$$\beta_S(p) = \sum_{T \subseteq S} (-1)^{|S| - |T|} \begin{bmatrix} \lambda' \\ T \end{bmatrix}_p.$$

Example 3.13. The simplicial complex $\Delta_{\{1,3\}}(p)$ of Example 2.12 is homotopy equivalent to a wedge of $p^2 = (1 + 2p + p^2) - (1 + p) - (1 + p) + 1$ spheres of dimension 1. A homotopy equivalence is pictured for $p = 2$.



Given a shelling of the maximal simplices of $\Delta_S(p)$ it is easy (see, e.g., [4]) to find $\beta_S(p)$ maximal simplices whose deletion leaves a contractible subcomplex. Simply delete those maximal simplices C_k each of whose facets is also a facet of some C_i with $i<k$. The above well-known facts are summarized in the following lemma.

LEMMA 3.14. *For any shelling*

$$C_1, C_2, ..., C_{\left[{\lambda' \atop S}\right]_p}$$

of the maximal simplices of $\Delta_S(p)$, there are exactly

$$\beta_S(p) = \sum_{T \subseteq S} (-1)^{|S|-|T|} \begin{bmatrix} \lambda' \\ T \end{bmatrix}_p.$$

maximal simplices C_k each of whose facets is also a facet of some C_i with $i<k$. Furthermore, deletion of these $\beta_S(p)$ maximal simplices leaves a contractible subcomplex.

In [2, 3], Björner shows how to obtain a shelling of the maximal simplices of $\Delta_S(p)$ from any natural ordering of the join irreducibles (nontrivial cyclic subgroups) in the modular lattice $L_\lambda(p)$. Fix a natural ordering $E_1, E_2, ...$ of the join irreducibles. (The ordering is *natural* if $E_i < E_j \Rightarrow i<j$.) Label each edge $H<K$ of the Hasse diagram of $L_\lambda(p)$ by the smallest i such that $H \vee E_i = K$. Maximal chains are ordered lexicographically. That is, a maximal chain $\hat{0} = M_0 < M_1 < \cdots < M_{|\lambda|} = \hat{1}$ with edge labelling $(\pi_1, \pi_2, ..., \pi_{|\lambda|})$ (so π_i is smallest with $M_{i-1} \vee E_{\pi_i} = M_i$) comes earlier in the shelling order than any maximal chain whose edge labelling is lexicographically larger. For free [3] we get a shelling order of the maximal simplices in $\Delta_S(p)$, for each $S \subseteq [|\lambda|-1]$. Just notice that in every interval $[H, K]$ in $L_\lambda(p)$ there is a unique rising saturated chain. (A chain is *rising* if its edge labelling is nondecreasing.) So we associate to each maximal simplex $H_1 \subset H_2 \subset \cdots \subset H_{|S|}$ of $\Delta_S(p)$ the lexicographically smallest of edge labellings $(\pi_1, ..., \pi_{|\lambda|})$ of maximal chains that refine $H_1 \subset H_2 \subset \cdots \subset H_{|S|}$. Then order the maximal simplices of $\Delta_S(p)$ by lexicographic ordering of these associated edge labellings. We say that the shelling order so obtained of the maximal simplices of $\Delta_S(p)$ is *induced* by the fixed natural ordering of the join irreducibles of $L_\lambda(p)$.

To provide a topological proof that $\beta_S(p)$ is a polynomial in p with nonnegative coefficients, we find a natural ordering of the join irreducibles of $L_\lambda(p)$ such that the induced ordering

$$C_1, C_2, ..., C_{\left[{\lambda' \atop S}\right]_p}$$

of maximal simplices in $\Delta_S(p)$ has the following property.

Property $(*)$. There is a set $\mathscr{C}$ of $\beta_S(1)$ maximal simplices in Δ_S such that $\varphi^{-1}(\mathscr{C})$ consists of exactly those C_k each of whose facets is the facet of some C_i, with $i<k$.

Our topological proof is especially elegant because deletion of the $\beta_S(1)$ maximal simplices in our choice for $\mathscr{C}$ leaves a contractible subcomplex of Δ_S, just as deletion of the $\beta_S(p)$ maximal simplices in $\varphi^{-1}(\mathscr{C})$ leaves a contractible subcomplex of $\Delta_S(p)$. In fact, we construct $\mathscr{C}$ using a particular ordering of the join irreducibles in $[0, \lambda]$. (In the shelling $c_1, c_2, ..., c_{\binom{\lambda'}{S}}$ of the maximal simplices of Δ_S induced by this particular ordering, the collection $\mathscr{C}$ consists of exactly those c_k each of whose facets is the facet of some c_i, with $i<k$.) The distributive lattice $[0, \lambda_1] \times \cdots \times [0, \lambda_l]$ is the lattice of order ideals in the poset $[\lambda] = [\lambda_1] + \cdots + [\lambda_l]$. Join irreducibles in the lattice $[0, \lambda]$ correspond to elements of the poset $[\lambda]$. We label the element of rank j in $[\lambda_i]$ by the number $i + \sum_{k<j} \lambda'_k$. Our particular ordering of the join irreducibles of $[0, \lambda]$ is simply this standard numbering of the elements of $[\lambda]$.

Theorem 1.6 could be stated more generally. An identical theorem holds for any semimodular lattice $L(q)$ that is an order analogue of a semimodular lattice L, provided the surjection φ is *compatible* with some ordering of the join irreducibles of L. This compatibility condition is the following strengthening of Property (iii) in Definition 2.7 of order analogue.

Definition 3.15. Let $L(q)$ be an order analogue of a semimodular lattice L. The surjection $\varphi: L(q) \to L$ is *compatible* with a given ordering of the join irreducibles of L if

(iii)′ For every interval $[a, b]$ in L, with unique rising saturated chain

$$a = \alpha^{(0)} < \alpha^{(1)} < \cdots < \alpha^{(r)} = b,$$

if $H < K$ satisfies $\varphi(H) = a$ and $\varphi(K) = b$, then there is exactly one chain in $L(q)$ of the form

$$H = H_0 < H_1 < \cdots < H_r = K$$

where $\varphi(H_i) = \alpha^{(i)}$.

THEOREM 3.16. *The surjection* $\varphi\colon L_\lambda(p) \to [0, \lambda]$ *of Definition* 2.8 *is compatible with the ordering of join irreducibles in* $[0, \lambda]$ *obtained by labelling the element of rank* j *in* $[\lambda_i]$ *by the number* $i + \sum_{k<j} \lambda'_k$.

Proof. Fix K with $\varphi(K) = (b_1, ..., b_l)$. The number of chains $H \subset K$ such that $\varphi(H) = (a_1, ..., a_l)$ is p raised to the inversion number of the partial tabloid T whose ith row has a_i 1's followed by $b_i - a_i$ 2's. See Example 1.3. Let

$$a = \alpha^{(0)} < \alpha^{(1)} < \cdots < \alpha^{(r)} = b$$

be the unique rising saturated chain in $[a, b]$. The number of chains

$$H = H_0 \subset H_1 \subset \cdots \subset H_r = K$$

where $\varphi(H_i) = \alpha^{(i)}$ is p raised to the inversion number of a partial tabloid of the same shape b as T, but in the region of T occupied by 2's we find the entries 2, 3, ..., $1 + \sum b_i - a_i$. The entry $t+1$ occurs in the highest place (that was occupied by a 2 in T) below t in the same column, or if there is no such place, in the highest place (that was occupied by a 2 in T) in the next column to the right. Notice the inversion number of this new partial tabloid equals the inversion number of T. So for each $H \subset K$ with $\varphi(H) = a$ there is exactly one chain

$$H = H_0 \subset H_1 \subset \cdots \subset H_r = K$$

with $\varphi(H_i) = \alpha^{(i)}$. ∎

The following corollary was first established in [5]. Here we provide a proof that may be used whenever $L(q)$ is an order analogue of a semimodular lattice L, provided there is an ordering of the join irreducibles in L that is compatible with the surjection φ. Hence, it is easy to see that the Betti polynomials, defined as in Equation (2), of such order analogues have nonnegative coefficients.

COROLLARY 3.17. *Edge label maximal chains* m *in* $[0, \lambda]$ *using the ordering on join irreducibles of Theorem* 3.16. *Let* m_S *denote the subchain of* m *that is a maximal simplex in* Δ_S. *Then*

$$\beta_S(p) = \sum_{\substack{m \\ D(\pi(m)) = S}} |\varphi^{-1}(m_S)|.$$

Proof. Every maximal simplex in Δ_S refines uniquely to a maximal chain m in $[0, \lambda]$ with $D(\pi(m)) \subseteq S$. Hence,

$$\begin{bmatrix} \lambda' \\ S \end{bmatrix}_p = \sum_{\substack{m \\ D(\pi(m)) \subseteq S}} |\varphi^{-1}(m_S)| = \sum_{T \subseteq S} \sum_{\substack{m \\ D(\pi(m)) = T}} |\varphi^{-1}(m_S)|.$$

CLAIM. $|\varphi^{-1}(m_S)| = |\varphi^{-1}(m_T)|$, *whenever* $D(\pi(m)) = T \subseteq S$.

Proof. A bijection $\{C \mid \varphi(C) = m_T\} \leftrightarrow \{B \mid \varphi(B) = m_S\}$ is defined as follows: Since $T \subseteq S$, m_T is a subchain of m_S. Given B with $\varphi(B) = m_S$, let C be the subchain of B such that $\varphi(C) = m_T$. Given C with $\varphi(C) = m_T$, by Property (iii)′ of Definition 3.15 there is a unique refinement of C to a maximal chain M with $\varphi(M) = m$. Let B be the subchain of M with $\varphi(B) = m_S$.

The claim shows that

$$\begin{bmatrix} \lambda' \\ S \end{bmatrix}_p = \sum_{T \subseteq S} \sum_{\substack{m \\ D(\pi(m)) = T}} |\varphi^{-1}(m_T)|.$$

The desired formula now follows by inclusion-exclusion. ▮

Our main result in Theorem 1.6 is a consequence of Lemma 3.14, Corollary 3.17, and Theorem 3.19 below.

LEMMA 3.18. *Let the lattice $L(q)$ be an order analogue of a semimodular lattice L. Suppose $\varphi: L(q) \to L$ is compatible with some ordering of the join irreducibles of L. Given H in $L(q)$, if $e \leqslant \varphi(H)$ is join irreducible in L, then there is a join irreducible $E \leqslant H$ with $\varphi(E) = e$.*

Proof. Let $e \leqslant \varphi(H)$ be join irreducible. Since $e \leqslant \varphi(H)$, by Property (ii) of Definition 2.7 there is an $E \leqslant H$ with $\varphi(E) = e$. We use Property (iii)′ of Definition 3.15 to show that E is join irreducible.

Let X and Y be any elements of $L(q)$ covered by E. Since e is join irreducible, in the unique rising saturated chain

$$\hat{0} = \alpha^{(0)} < \cdots < \alpha^{(r-1)} < \alpha^{(r)} = e$$

from $\hat{0}$ to e, the element $\alpha^{(r-1)}$ is the only element in L covered by e. Hence $\varphi(X)=\varphi(Y)=\alpha^{(r-1)}$. By Property (ii) there are saturated chains

$$\hat{0}=H_0 < \cdots < H_{r-1}=X$$

and

$$\hat{0}=K_0 < \cdots < K_{r-1}=Y$$

where $\varphi(H_i)=\varphi(K_i)=\alpha^{(i)}$ for $i \leqslant r-2$. Take $H_r=K_r=E$ to produce saturated chains in $L(q)$ that φ maps to the unique rising saturated chain from $\hat{0}$ to e in L. By Property (iii)′ these chains must be identical. Hence $X=Y$. So E is join irreducible. ∎

THEOREM 3.19. *Order the join irreducibles of* $[0, \lambda]$ *as described in Theorem* 3.16. *Let* $\mathscr{C}$ *be the collection of maximal simplices of* Δ_S *of the form* m_S, *for maximal chains* m *in* $[0, \lambda]$ *whose edge labelling has descent set equal to* S. *Order the join irreducibles of* $L_\lambda(p)$ *so that: If* $\varphi(E)$ *precedes* $\varphi(F)$ *in the ordering of join irreducibles of* $[0, \lambda]$, *then* E *comes earlier than* F *in the ordering of join irreducibles of* $L_\lambda(p)$. *The induced shelling of maximal simplices* C *in* $\Delta_S(p)$ *has the following property: If* $\varphi(C) \in \mathscr{C}$, *then each facet of* C *is a facet of some maximal simplex that occurs earlier in the shelling order.*

Proof. Suppose C is the chain

$$\hat{0}=H_0 \subset H_1 \subset \cdots \subset H_{|S|} \subset H_{|S|+1}=\hat{1}$$

and $\varphi(C) \in \mathscr{C}$. Consider a facet

$$\hat{0}=H_0 \subset \cdots \subset H_{i-1} \subset H_{i+1} \subset \cdots \subset H_{|S|+1}=\hat{1}$$

of C. We construct a subgroup K such that the maximal simplex

$$\hat{0}=H_0 \subset \cdots \subset H_{i-1} \subset K \subset H_{i+1} \subset \cdots \subset H_{|S|+1}=\hat{1}$$

comes earlier than C in the shelling order.

Suppose H_{i-1} has order p^r, H_i has order p^s, and H_{i+1} has order p^t. Let

$$e_1, e_2, ..., e_{t-r}$$

be the join irreducibles e, in order, that satisfy $e \leqslant \varphi(H_{i+1})$ but $e \nleqslant \varphi(H_{i-1})$. For each $1 \leqslant j \leqslant k-r$, let E_j be the earliest join irreducible such that $E_j \leqslant H_{i+1}$ and $\varphi(E_j) = e_j$. The existence of such join irreducibles is guaranteed by Lemma 3.18. Choose

$$K = H_{i-1} \vee E_1 \vee \cdots \vee E_{s-r}.$$

To verify that the chain

$$\hat{0} = H_0 \subset \cdots \subset H_{i-1} \subset K \subset H_{i+1} \subset \cdots \subset H_{|S|+1} = \hat{1}$$

comes earlier than

$$\hat{0} = H_0 \subset H_1 \subset \cdots \subset H_{|S|} \subset H_{|S|+1} = \hat{1}$$

in the shelling order of maximal simplices of $\Delta_S(p)$, notice the following: Let

$$f_1, f_2, ..., f_{s-r}$$

be the join irreducibles f, in order, that satisfy $f \leqslant \varphi(H_i)$ but $f \nleqslant \varphi(H_{i-1})$. For $0 \leqslant j \leqslant s-r$, e_j occurs no later than f_j in the ordering of join irreducibles of $[0, \lambda]$. Furthermore

$$H_i = H_{i-1} \vee F_1 \vee \cdots \vee F_{s-r}$$

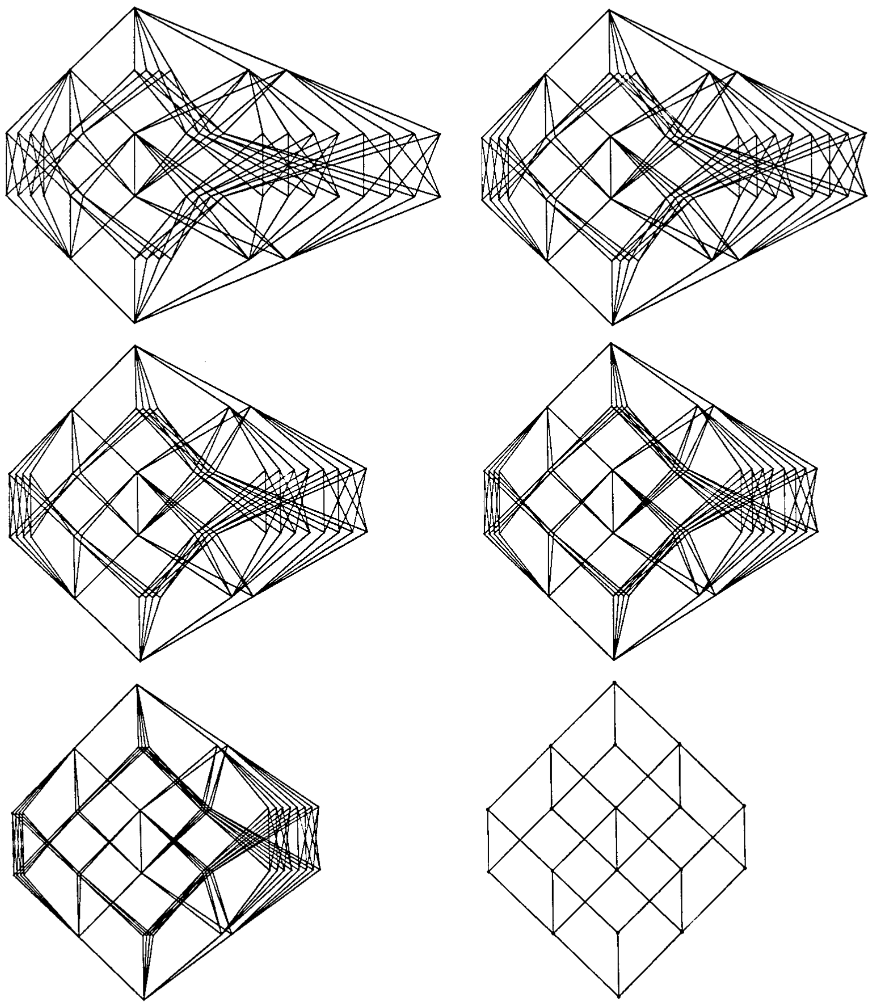
for some join irreducibles $F_j \leqslant H_{i+1}$ with $\varphi(F_j) = f_j$. For any such F_j, $1 \leqslant j \leqslant s-r$, E_j occurs no later than F_j in the list of join irreducibles of $L_\lambda(p)$. Furthermore it is impossible that $E_j = F_j$ for all $1 \leqslant j \leqslant s-r$. (If $E_j = F_j$ for all $1 \leqslant j \leqslant s-r$, then $e_j = f_j$ for all $1 \leqslant j \leqslant s-r$. But then $\varphi(C) \notin \mathscr{C}$.) ∎

EXAMPLE 3.20. The poset $[2]+[2] = |_1^3\,|_2^4$ has only one linear extension with descent set $\{1, 3\}$, namely 2143. This is the edge labelling of the maximal chain $00 \subset 01 \subset 11 \subset 12 \subset 22$ in $[0, 2] \times [0, 2]$. Hence the only maximal simplex in $\mathscr{C}$ is $01 \subset 12$. See Examples 2.12 and 3.13 to see what happens when you delete the p^2 maximal simplices in $\varphi^{-1}(\mathscr{C})$ from $\Delta_{\{1,3\}}(p)$.

Since Lemma 3.14 and the proofs of Corollary 3.17 and Theorem 3.19 are valid for any semimodular lattice that is an order analogue of a semimodular lattice, provided Property (iii)′ of Definition 3.15 is satisfied, our main result in Theorem 1.6 carries over to such order analogues.

Appendix

Surjection of the Lattice of Subgroups of $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ *onto* $[0, 2] \times [0, 2] \times [0, 1]$



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