The approximate and exact solutions of the space- and time-fractional Burgers equations with initial conditions by variational iteration method

Mustafa Inc*

Department of Mathematics, Fırat University, 23119 Elazıg, Turkey

Article info

Article history:
Received 24 January 2006
Available online 9 April 2008
Submitted by I. Podlubny

Keywords:
Time- and space-fractional Burgers equations
Fractional derivative
Variational iteration method
Lagrange multiplier

1. Introduction

In the last past decades, directly seeking for exact solutions of nonlinear partial differential equations has become one of the central theme of perpetual interest in Mathematical Physics. Nonlinear wave phenomena appear in many fields, such as fluid mechanics, plasma physics, biology, hydrodynamics, solid state physics and optical fibers. These nonlinear phenomena are often related to nonlinear wave equations. In order to better understand these phenomena as well as further apply them in the practical life, it is important to seek their more exact solutions. Many powerful methods had been developed such as Backlund transformation [1,2], Darboux transformation [3], the inverse scattering transformation [4], the bilinear method [5], the tanh method [6,7], the sine–cosine method [8,9], the homogeneous balance method [10], the Riccati method [11], the Jacobi elliptic function method [12] and the extended Jacobi elliptic function method [13], etc.

The fractional differential equations (FDE) appear more and more frequently in different research areas and engineering applications. The fractional derivative has been occurring in many physical problems such as frequency dependent damping behavior of materials, motion of a large thin plate in a Newtonian fluid, creep and relaxation functions for viscoelastic materials, the $\Pi^D D^\mu$ controller for the control of dynamical systems, etc. Phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are also described by differential equations of fractional order. Bagley and Torvik [14–16] provided a review of work done in this area prior to 1980, and showed that half-order fractional differential models describe the frequency dependence of the damping materials very well. Other authors have demonstrated applications of fractional derivatives in the areas of electrochemical processes [17,18], dielectric polarization [19], colored noise [20], viscoelastic materials [21] and chaos [22]. Mainardi [23] and Rossikhin and Shitikova [24] presented survey of fractional derivatives, in general to solid mechanics, and in particular to modeling of visco elastic damping. Magin [25] presented a three part critical review of applications of fractional calculus in bioengineering. Application of fractional derivatives in other fields, related mathematical tools and techniques could be found in [26–30].

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doi:10.1016/j.jmaa.2008.04.007
Fractional calculus has been used to model physical and engineering processes that are found to be best described by FDEs. For that reason we need a reliable and efficient technique for the solution of FDEs. Atanackovic and Stankovic [31] have analyzed lateral motion of an elastic column fixed at one end loaded at the other, in terms of a system of FDE. Shawagfeh [32] has employed Adomian decomposition method in case of the nonlinear FDE. DaftarDar-Gejji and Babakhani [33] have presented analysis of system of FDE. They have studied existence, uniqueness and stability of solutions of a system of FDE. Recently, DaftarDar-Gejji and Babakhani [34] have used to obtain solutions of a system of FDEs by Adomian decomposition method. They have discussed convergence of the method with some illustrative examples. More recently, DaftarDar-Gejji and Babakhani [35] have presented an iterative method for solving nonlinear functional equations. Most recently, Momani [36] has presented nonperturbative analytical solutions of the space- and time-fractional Burgers equations by Adomian decomposition method.

In this paper, we consider nonperturbation analytical solutions of the generalized Burgers equation with time- and space-fractional derivatives of the form [36]:

\[
\frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} + \eta \frac{\partial^\beta u}{\partial x^\beta} = 0, \quad t > 0, \quad 0 < \alpha, \beta \leq 1, \tag{1}
\]

where \(\varepsilon, \nu, \eta\) are parameters and, \(\alpha\) and \(\beta\) are parameters describing the order of the fractional time- and space-derivatives, respectively. The function \(u(x,t)\) is assumed to be a causal function of time and space, i.e., vanishing for \(t < 0\) and \(x < 0\). The fractional derivatives are considered in the Caputo sense. We refer to Eq. (1) as to the time-fractional Burgers and to the space-fractional Burgers equation in the cases \(0 < \alpha \leq 1, \quad \eta = 0\) and \(0 < \beta \leq 1, \quad \alpha = 1\), respectively. The space-fractional Burgers equation describes the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. The fractional derivative results from the memory effect of the wall friction through the boundary layer. The same form can be found in other systems such as shallow-water waves and waves in bubbly liquids [36]. Biler et al. [37] studied local and global in time solutions to a class of multidimensional generalized Burgers-type equations with a fractal power of the Laplacian in the principal part and with general algebraic nonlinearity. Biler et al. [38] presented the existence and uniqueness of the source solution to the fractal Burgers equation in the critical case. Mann and Woyczynski [39] presented asymptotic properties of solutions and the Monte Carlo type approximation algorithms of the fractal Burgers–KPZ equations. Recently, Stanescu et al. [40] worked a numerical method based on the interacting particles approximation for the solution of a large class of evolution problems involving the fractional Laplacian operator and a nonlocal quadratic-type nonlinearity.

The aim of this paper is to extend the variational iteration method by He [41–45] to derive the numerical and exact solutions of the space- and time-fractional Burgers equations and comparison with that obtained previously by the Adomian decomposition method [36].

2. Basic definitions

**Definition 2.1.** A real function \(f(x), \ x > 0\), is said to be in the space \(C_\alpha, \ \alpha \in \mathbb{N}\) if there exists a real number \(p \ (> \alpha)\), such that \(f(x) = x^p f_1(x) \in C[0, \infty]\). Clearly \(C_\alpha \subset C_\beta\) if \(\beta \leq \alpha\).

**Definition 2.2.** A function \(f(x), \ x > 0\), is said to be in the space \(C^m_\alpha, \ m \in \mathbb{N} \cup \{0\}\), if \(f^{(m)} \in C_\alpha\).

**Definition 2.3.** The left sided Riemann–Liouville fractional integral of order \(\mu \geq 0\), of a function \(f \in C_\alpha, \ \alpha \geq -1\), is defined as

\[
I^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_0^x \frac{f(t)}{(x-t)^{1-\mu}} \, dt, \quad \mu > 0, \ x > 0, \quad I^0 f(x) = f(x). \tag{2}
\]

**Definition 2.4.** Let \(f \in C^m_{-1}, \ m \in \mathbb{N}\). Then the Caputo fractional derivative of \(f\) is defined as [36,47]:

\[
D^\mu f(x) = \begin{cases} \left\{f^{(m-\mu)} f^{(m)}(x)\right\}, & m - 1 < \mu \leq m, \\ \frac{d^m}{dx^m} f(t), & \mu = m, \end{cases} \tag{3}
\]

\[
I^\mu I^\nu f = I^{\mu + \nu} f, \quad \mu, \nu \geq 0, \quad f \in C_\alpha, \ \alpha \geq -1, \tag{4}
\]

\[
I^\mu x^\nu = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \mu + 1)} x^{\nu + \mu}, \quad \mu > 0, \ \gamma > -1, \ x > 0. \tag{5}
\]

**Lemma 2.1.** If \(m - 1 < \alpha \leq m, \) and \(f \in L_1[a, b]\), then

\[
J^\alpha_a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha - 1} f(t) \, dt, \quad D^\alpha_a J^\alpha_a f(x) = f(x). \tag{5}
\]
and
\[ J_0^\alpha D_0^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{(x-a)^k}{k!}, \quad x > 0. \]  

Definition 2.5. The fractional derivative of \( f(x) \) in the Caputo sense is defined as
\[ D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) \, dt, \]  

for \( m-1 < \alpha \leq m, \ m \in \mathbb{N}, \ x > 0. \)

Definition 2.6. For \( m \) to be the smallest integer that exceeds \( \alpha \), the Caputo time-fractional derivative operator of order \( \alpha > 0 \) is defined as
\[ D_t^\alpha u(x, t) = \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(\tau, t)}{\partial \tau^m} \, d\tau, & \text{for } m-1 < \alpha < m, \\
\frac{\partial^m u(x, t)}{\partial x^m}, & \text{for } \alpha = m \in \mathbb{N},
\end{array} \right. \]  

and the space-fractional derivative operator of order \( \beta > 0 \) is defined as
\[ D_x^\beta u(x, t) = \frac{\partial^\beta u(x, t)}{\partial x^\beta} = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(m-\beta)} \int_0^x (x-\theta)^{m-\beta-1} \frac{\partial^m u(\theta, t)}{\partial \theta^m} \, d\theta, & \text{for } m-1 < \beta < m, \\
\frac{\partial^m u(x, t)}{\partial x^m}, & \text{for } \beta = m \in \mathbb{N}.\]  

3. Variational iteration method

Variational iteration method (VIM) was first proposed by the Chinese mathematician He [41–45]. This method has been employed to solve a large variety of linear and nonlinear problems with approximations converging rapidly to accurate solutions. This technique is used in [46] for solving nonlinear Jaulent–Miodek, coupled KdV and coupled MKdV equations. In [47] the applications of the present method to Shock-peakon and shock-compacton solutions for \( K(p, q) \) equation are provided. The variational iteration technique is employed to solve the nonlinear dispersive equation, a nonlinear partial differential equation which arise in the process of understanding the role of nonlinear dispersion and in the forming of solitons. This technique is also employed in [52] to solve the Fokker–Planck equation. The linear and nonlinear cases are discussed in their work and several test examples are given to show the efficiency of this procedure.

To illustrate its basic concepts of the variational iteration method, we consider the following differential equation:
\[ Lu + Nu = g(x), \]  

where \( L \) is a linear operator, \( N \) a nonlinear operator and \( g(x) \) an inhomogeneous term.

According to the variational iteration method, we can construct a correct functional as follows
\[ u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left\{ Lu_n(\tau) + Nu_n(\tau) - g(\tau) \right\} \, d\tau, \]  

where \( \lambda \) is a general Lagrangian multiplier [63] which can be identified optimally by the variational theory, the subscript \( n \) denotes the \( n \)-th-order approximation, and \( \delta u_n = 0 \) for linear problems.
the exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified. In nonlinear problems, in order to determine the Lagrange multiplier in a simple manner, the nonlinear terms have to be considered as restricted variations. Consequently, the exact solution may be obtained by using

\[ u(x, t) = \lim_{n \to \infty} u_n(x, t). \]

The convergence of the VIM has been investigated in [65]. He obtained some results about the speed of convergence of this method.

To illustrate the above theory, two examples of special interest such as the space-fractional Burgers equation and the time-fractional Burgers equation are discussed in details and the obtained results are exactly the same with that found by the Adomian decomposition method [36].

4. Applications

4.1. The space-fractional Burgers equation

We consider the space-fractional Burgers equation with the following initial value problem [36]:

\[
\begin{aligned}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} + \eta \frac{\partial^\beta u}{\partial x^\beta} &= 0, \quad x, t > 0, \quad 0 < \beta \leq 1, \\
u(0, t) = 0, \quad u_x(0, t) = \frac{1}{\bar{r}} - \frac{\pi^2}{2vt^2}.
\end{aligned}
\]

To solve Eq. (12) by means of the variational iteration method, we construct a correction functional which reads

\[
\delta u_{n+1}(x, t) = \delta u_n(x, t) + \int_0^t \delta \lambda(t) \left\{ \left( \frac{\partial u_n(x, \tau)}{\partial \tau} \right)_t + \left( \tilde{u}_n(x, \tau) \right) \left( \frac{\partial \tilde{u}_n(x, \tau)}{\partial \tau} \right)_x - \nu \left( \frac{\partial \tilde{u}_n(x, \tau)}{\partial \tau} \right)_{xx} + \eta \left( \frac{\partial \tilde{u}_n(x, \tau)}{\partial \tau} \right)_\beta \right\} d\tau,
\]

where \( \delta u_n \) is considered as a restricted variation, \( u_0(x, t) \) is its initial approximation or trial function. Making the above correction functional stationary and noticing that \( \delta \tilde{u}_n = 0 \), we obtain

\[
\delta u_{n+1}(x, t) = \delta u_n(x, t) + \int_0^t \delta \lambda(t) \left\{ \left( \frac{\partial u_n(x, \tau)}{\partial \tau} \right)_t + \left( \tilde{u}_n(x, \tau) \right) \left( \frac{\partial \tilde{u}_n(x, \tau)}{\partial \tau} \right)_x \right\} d\tau = \delta u_n(x, t) \lambda'(t) + \int_0^t \delta u_n(x, \tau) \lambda'(t) d\tau = 0
\]

which produces the stationary conditions:

\[
\begin{aligned}
\lambda'(t) &= 0, \\
1 + \lambda(t)|_{t=0} &= 0,
\end{aligned}
\]

where Eq. (14a) is called Lagrange–Euler equation and Eq. (14b) natural boundary condition.

The Lagrange multiplier, therefore, can be identified as \( \lambda = -1 \), and the following variational iteration formula can be obtained

\[
u_{n+1}(x, t) = u_n(x, t) - \int_0^x \left[ (u_n)_t + (\tilde{u}_n)(\tilde{u}_n)_x - \nu((\tilde{u}_n))_{xx} + \eta((\tilde{u}_n))_\beta \right] d\tau.
\]

We start with an initial approximation \( u_0 = u(0, t) + xu_0(0, t) \) given by Eq. (12), by the above iteration formula (15), we can obtain directly the other components as

\[
\begin{aligned}
u_0(x, t) &= \left( \frac{1}{t} - \frac{\pi^2}{2vt^2} \right) x, \\
u_1(x, t) &= \frac{\pi^4x^3}{24\nu^4} \left( \frac{\eta x^{3-\beta}}{(3-\beta)(\beta-2)\Gamma(2-\beta)} \right), \\
u_2(x, t) &= -\frac{\pi^6x^5}{240\nu^6} \left( \frac{\eta(-\pi^2+2vt)x^{3-\beta}}{2(5-\beta)(\beta-3)(\beta-2)\Gamma(2-\beta)\nu^2x^2} + \frac{\eta\Gamma(4-\beta)x^{5-2\beta}}{2(5-2\beta)(\beta-3)(\beta-2)\Gamma(\beta-2)\Gamma(3-2\beta)} \right), \\
u_3(x, t) &= \frac{\pi^8x^7}{40320\nu^8} \left( \cdots \right).
\end{aligned}
\]

and so on, in the same manner the rest of components of the iteration formula (15) were obtained the Mathematica Package.
Consequently, we have the solution of (12) in a series form

\[
\begin{align*}
\frac{\pi^2}{2i^2} x + \frac{\pi^4 x^3}{244^3} - \frac{\pi^4 x^3}{244^3} - \frac{\pi^5 x^5}{245^5} x + \frac{\pi^5 x^5}{245^5} x - \frac{\pi^6 x^6}{246^6} x + \frac{\pi^6 x^6}{246^6} x - \frac{\pi^7 x^7}{247^7} x + \frac{\pi^7 x^7}{247^7} x - \ldots
\end{align*}
\]

which are exactly the same as obtained by Adomian decomposition method [36]. If we take \( \beta = \frac{1}{2} \) in (19), then we get the solution of [66] as follows:

\[
\frac{\pi^2}{2i^2} x + \frac{\pi^4 x^3}{244^3} - \frac{\pi^4 x^3}{244^3} - \frac{\pi^5 x^5}{245^5} x + \frac{\pi^5 x^5}{245^5} x - \frac{\pi^6 x^6}{246^6} x + \frac{\pi^6 x^6}{246^6} x - \frac{\pi^7 x^7}{247^7} x + \frac{\pi^7 x^7}{247^7} x - \ldots
\]

The exact solution, for the special case \( \eta = 0 \), is given by

\[
\frac{\pi^2}{2i^2} x + \frac{\pi^4 x^3}{244^3} - \frac{\pi^4 x^3}{244^3} - \frac{\pi^5 x^5}{245^5} x + \frac{\pi^5 x^5}{245^5} x - \frac{\pi^6 x^6}{246^6} x + \frac{\pi^6 x^6}{246^6} x - \frac{\pi^7 x^7}{247^7} x + \frac{\pi^7 x^7}{247^7} x - \ldots
\]

4.2. The time-fractional Burgers equation

We now consider the one-dimensional time-fractional Burgers equation with the following initial value problem [36]:

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\varepsilon}{\partial x} \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} &= 0, \quad t > 0, \quad 0 < \alpha \leq 1, \\
u(x, 0) = g(x),
\end{align*}
\]

To solve Eq. (22) by means of the variational iteration method, we construct a correction functional which reads

\[
\begin{align*}
\delta u_{n+1}(x, t) &= \delta u_{n}(x, t) + \int_0^t \lambda(\tau) \left( \frac{\partial u_n(x, \tau)}{\partial \tau} \right)_\alpha + \left( \tilde{u}_n(x, \tau) \right) \left( \frac{\partial \tilde{u}_n(x, \tau)}{\partial \tau} \right)_\alpha - \nu \left( \frac{\partial \tilde{u}_n(x, \tau)}{\partial \tau} \right)_x \right) d\tau,
\end{align*}
\]

which produces the stationary conditions:

\[
\begin{align*}
\lambda'(\tau) &= 0, \\
1 + \lambda(\tau)\bigg|_{t=0} &= 0,
\end{align*}
\]

where Eq. (24a) is called Lagrange–Euler equation and Eq. (24b) natural boundary condition.

The Lagrange multiplier, therefore, can be identified as \( \lambda = -1 \), and the following variational iteration formula can be obtained

\[
\begin{align*}
u_{n+1}(x, t) &= \nu_n(x, t) - \int_0^x \left( \nu_n(x, \tau) + (\tilde{u}_n)(\tilde{u}_n)_\alpha - \nu (\tilde{u}_n)_x \right) d\tau.
\end{align*}
\]

We start with an initial approximation \( u_0 = u(x, 0) \) given by Eq. (22), by the above iteration formula (25), we can obtain directly the other components as

\[
\begin{align*}
u_0(x, t) &= g(x) = \frac{\mu + \sigma + (\sigma - \mu)\exp(\gamma)}{1 + \exp(\gamma)}, \\
u_1(x, t) &= \frac{[\nu g'' - \nu g'']}{\Gamma(\alpha + 1)}, \\
u_2(x, t) &= \frac{[2\nu^2 g''^2 + \nu^2 g'']}{\Gamma(2\alpha + 1)},
\end{align*}
\]

and so on, in the same manner the rest of components of the iteration formula (25) were obtained the Mathematica Package.
where which are exactly same as obtained by Adomian decomposition method \[36\] and where additional differential equations with initial and boundary conditions. It may be concluded that the VIM and the ADM are very present method is a powerful mathematical tool for finding other numerical and exact solutions of many nonlinear fractional Burgers equations with initial conditions using by the variational iteration method. The results show that the overall errors can be made smaller by adding new terms of the series (20) and (34).

### 6. Conclusions

In this paper, we have presented a scheme used to obtain numerical and exact solutions of the space- and time-fractional Burgers equations with initial conditions using by the variational iteration method. The results show that the present method is a powerful mathematical tool for finding other numerical and exact solutions of many nonlinear fractional differential equations with initial and boundary conditions. It may be concluded that the VIM and the ADM are very

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<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
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<td>1.1054E–12</td>
<td>1.1008E–11</td>
</tr>
</tbody>
</table>

Consequently, we have the solution of (22) in a series form

\[
\begin{align*}
\frac{\mu + \sigma + (\sigma - \mu) \exp(\gamma)}{1 + \exp(\gamma)} + [\nu g'' - \epsilon gg] + \frac{t^\alpha}{\Gamma(\alpha + 1)} & \\
+ \left[2\epsilon^2 gg'' + \epsilon^2 g^2 g'' - 4\epsilon v gg'' - 2\epsilon vv g'' + v^2 g^4\right] & \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \cdots
\end{align*}
\]

which are exactly same as obtained by Adomian decomposition method \[36\] and where $\gamma = \frac{\mu}{\nu} (x - \lambda)$ and the parameters $\mu$, $\sigma$, $\lambda$, and $\nu$ are arbitrary constants.

If we take $\alpha = 1$ in (29), then we get the solution of \[67\] as follows:

\[
\begin{align*}
u_0(x, t) &= \frac{\mu + \sigma + (\sigma - \mu) \exp(\gamma)}{1 + \exp(\gamma)}, \\
u_1(x, t) &= \frac{2\mu \sigma^2 \exp(\gamma)}{[1 + \exp(\gamma)]^3 v t}, \\
u_2(x, t) &= \frac{[\mu^3 \sigma \exp(\gamma)] [\exp(\gamma) - 1] t^2}{[1 + \exp(\gamma)]^3 v^2}, \\
u_3(x, t) &= \frac{[\mu^4 \sigma^3 \exp(\gamma)] [1 - 4 \exp(\gamma) + \exp(\gamma)] t^3}{3[1 + \exp(\gamma)]^4 v^3},
\end{align*}
\]

and so on, in the same manner the rest of components of the iteration formula (25) were obtained the Mathematica Package. Thus, we have the solution of (22) in a series form for $\alpha = 1$,

\[
\begin{align*}
u(x, t) &= \frac{\mu + \sigma + (\sigma - \mu) \exp(\gamma)}{1 + \exp(\gamma)} + \frac{2\mu \sigma^2 \exp(\gamma)}{[1 + \exp(\gamma)]^2 v t} + \frac{[\mu^3 \sigma \exp(\gamma)] [\exp(\gamma) - 1] t^2}{[1 + \exp(\gamma)]^3 v^2} \\
+ \frac{[\mu^4 \sigma^3 \exp(\gamma)] [1 - 4 \exp(\gamma) + \exp(\gamma)] t^3}{3[1 + \exp(\gamma)]^4 v^3} + \cdots
\end{align*}
\]

where $\gamma = \frac{\mu}{\nu} (x - \lambda)$ and in a close form solution by

\[
u(x, t) = \exp\left[\frac{\mu}{\nu} (x - \sigma t - \lambda)\right],
\]

which are exactly same as obtained by Adomian decomposition method \[67\].

### 5. Numerical results

In this section, we consider the space- and time-fractional Burgers equations for numerical comparisons. In order to verify numerically whether the proposed methodology lead to higher accuracy, we can evaluate the approximate solution using the $n$-term approximations (15) and (25). Tables 1–3 show the difference of the exact and numerical solutions of the absolute errors. The graphs of the exact and approximate solutions are depicted in Figs. 1 and 2. It is to be note that 10 terms only were used in evaluating the approximate solutions. We achieved a good approximation with the exact solutions of the equations by using 10 terms only of the variational iteration method. It is evident that the overall errors can be made smaller by adding new terms of the series (20) and (34).
Table 2
The numerical results for 5 iterations in comparison with the exact solution \( u(x,t) = \frac{1}{4} - \frac{1}{4} \tanh\left( \frac{\pi t}{2x} \right) \) when \( \nu = 3 \) for the space-fractional Burgers equation with initial conditions of Eq. (12)

<table>
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<th>0.3</th>
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<th>0.5</th>
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</tr>
<tr>
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</tr>
<tr>
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</tr>
</tbody>
</table>

Table 3
The numerical results for 5 iterations in comparison with the exact solution \( u(x,t) = \mu + \sigma + (\sigma - \mu) \exp\left( \frac{\mu - \mu^2}{2(\sigma - \lambda)} \right) \) when \( \nu = 0.1 \), \( \mu = 1 \), \( \sigma = 0.9 \) and \( \lambda = 0.4 \) for the time-fractional Burgers equation with initial condition of Eq. (22)

<table>
<thead>
<tr>
<th>( t/x )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
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<tr>
<td>0.1</td>
<td>7.4250E−11</td>
<td>6.2058E−13</td>
<td>4.6082E−13</td>
<td>2.6843E−11</td>
<td>5.2917E−07</td>
</tr>
<tr>
<td>0.2</td>
<td>1.4023E−09</td>
<td>2.9744E−11</td>
<td>1.1711E−11</td>
<td>7.0830E−10</td>
<td>1.3076E−06</td>
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<tr>
<td>0.3</td>
<td>1.0542E−08</td>
<td>1.4908E−10</td>
<td>7.8477E−11</td>
<td>4.4604E−09</td>
<td>9.2668E−06</td>
</tr>
<tr>
<td>0.4</td>
<td>5.4003E−08</td>
<td>6.2471E−10</td>
<td>3.7936E−10</td>
<td>1.9815E−08</td>
<td>4.2411E−05</td>
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<tr>
<td>0.5</td>
<td>1.3301E−07</td>
<td>2.8958E−09</td>
<td>1.1608E−09</td>
<td>6.1020E−08</td>
<td>1.4161E−05</td>
</tr>
</tbody>
</table>

Fig. 1. The surface shows the solution \( u(x,t) \) for Eq. (12): (a) exact solution; (b) approximate solution when \( \nu = 1 \) for the space-fractional Burgers equation.

Fig. 2. The surface shows the solution \( u(x,t) \) for Eq. (12): (a) exact solution; (b) approximate solution when \( \nu = 0.1 \), \( \mu = 0.3 \), \( \sigma = 0.4 \) and \( \lambda = 0.8 \) for the time-fractional Burgers equation.
powerful and efficient techniques in finding an acceptable solution for wide classes of nonlinear problems. The VIM requires the evaluation of the Lagrangian multiplier, $\lambda$. However, the ADM requires the evolution of the Adomian polynomials. Thus, the evaluation of the Adomian polynomials for every nonlinear term mostly requires tedious algebraic calculations. The main advantage of VIM is to overcome the difficulty arising in calculating Adomian’s polynomials in ADM. We can integrate the equation directly without calculating Adomian polynomials by the VIM. This work shows that the VIM is easier than the ADM. Furthermore, VIM needs relatively less computational work than ADM. The behavior of the solution obtained by the variational iteration method is shown for different values of times in Figs. 1 and 2.

The numerical solutions of fractional differential equations obtained by the semi-analytic methods can be found in [68–72].

References


