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The approximate and exact solutions of the space- and time-fractional Burgers equations with initial conditions by variational iteration method

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A B S T R A C T

In this paper, a scheme is developed to study numerical solution of the space- and time-fractional Burgers equations with initial conditions by the variational iteration method (VIM). The exact and numerical solutions obtained by the variational iteration method are compared with that obtained by Adomian decomposition method (ADM). The results show that the variational iteration method is much easier, more convenient, and more stable and efficient than Adomian decomposition method. Numerical solutions are calculated for the fractional Burgers equation to show the nature of solution as the fractional derivative parameter is changed.

1. Introduction

In the last past decades, directly seeking for exact solutions of nonlinear partial differential equations has become one of the central theme of perpetual interest in Mathematical Physics. Nonlinear wave phenomena appear in many fields, such as fluid mechanics, plasma physics, biology, hydrodynamics, solid state physics and optical fibers. These nonlinear phenomena are often related to nonlinear wave equations. In order to better understand these phenomena as well as further apply them in the practical life, it is important to seek their more exact solutions. Many powerful methods had been developed such as Backlund transformation [1,2], Darboux transformation [3], the inverse scattering transformation [4], the bilinear method [5], the tanh method [6,7], the sine–cosine method [8,9], the homogeneous balance method [10], the Riccati method [11], the Jacobi elliptic function method [12] and the extended Jacobi elliptic function method [13], etc.

The fractional differential equations (FDE) appear more and more frequently in different research areas and engineering applications. The fractional derivative has been occurring in many physical problems such as frequency dependent damping behavior of materials, motion of a large thin plate in a Newtonian fluid, creep and relaxation functions for viscoelastic materials, the $PI^{\lambda}D^{\mu}$ controller for the control of dynamical systems, etc. Phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are also described by differential equations of fractional order. Bagley and Torvik [14–16] provided a review of work done in this area prior to 1980, and showed that half-order fractional differential models describe the frequency dependence of the damping materials very well. Other authors have demonstrated applications of fractional derivatives in the areas of electrochemical processes [17,18], dielectric polarization [19], colored noise [20], viscoelastic materials [21] and chaos [22]. Mainardi [23] and Rossikhin and Shitikova [24] presented survey of fractional derivatives, in general to solid mechanics, and in particular to modeling of visco elastic damping. Magin [25] presented a three part critical review of applications of fractional calculus in bioengineering. Application of fractional derivatives in other fields, related mathematical tools and techniques could be found in [26–30].

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Fractional calculus has been used to model physical and engineering processes that are found to be best described by FDEs. For that reason we need a reliable and efficient technique for the solution of FDEs. Atanackovic and Stankovic [31] have analyzed lateral motion of an elastic column fixed at one end loaded at the other, in terms of a system of FDE. Shawagfeh [32] has employed Adomian decomposition method in case of the nonlinear FDE. Daftardar-Gejji and Babakhani [33] have presented analysis of system of FDE. They have studied existence, uniqueness and stability of solutions of a system of FDE. Recently, Daftardar-Gejji and Babakhani [34] have used to obtain solutions of a system of FDEs by Adomian decomposition method. They have discussed convergence of the method with some illustrative examples. More recently, Daftardar-Gejji and Babakhani [35] have presented an iterative method for solving nonlinear functional equations. Most recently, Momani [36] has presented nonperturbative analytical solutions of the space- and time-fractional Burgers equations by Adomian decomposition method.

In this paper, we consider nonperturbation analytical solutions of the generalized Burgers equation with time- and space-fractional derivatives of the form [36]:

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\varepsilon u\frac{\partial u}{\partial x}-\nu\frac{\partial^2 u}{\partial x^2}+\eta\frac{\partial^\beta u}{\partial x^\beta}=0,\quad t>0,\ 0<\alpha,\ \beta\leqslant 1, \tag{1}$$

where ε, ν, η are parameters and, α and β are parameters describing the order of the fractional time- and space-derivatives, respectively. The function $u(x,t)$ is assumed to be a causal function of time and space, i.e., vanishing for $t<0$ and $x<0$. The fractional derivatives are considered in the Caputo sense. We refer to Eq. (1) as to the time-fractional Burgers and to the space-fractional Burgers equation in the cases $0<\alpha\leqslant 1,\ \eta=0$ and $0<\beta\leqslant 1,\ \alpha=1$, respectively. The space-fractional Burgers equation describes the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. The fractional derivative results from the memory effect of the wall friction through the boundary layer. The same form can be found in other systems such as shallow-water waves and waves in bubbly liquids [36]. Biler et al. [37] studied local and global in time solutions to a class of multidimensional generalized Burgers-type equations with a fractal power of the Laplacian in the principal part and with general algebraic nonlinearity. Biler et al. [38] presented the existence and uniqueness of the source solution to the fractal Burgers equation in the critical case. Mann and Woyczynski [39] presented asymptotic properties of solutions and the Monte Carlo type approximation algorithms of the fractal Burgers–KPZ equations. Recently, Stanescu et al. [40] worked a numerical method based on the interacting particles approximation for the solution of a large class of evolution problems involving the fractional Laplacian operator and a nonlocal quadratic-type nonlinearity.

The aim of this paper is to extend the variational iteration method by He [41–45] to derive the numerical and exact solutions of the space- and time-fractional Burgers equations and comparison with that obtained previously by the Adomian decomposition method [36].

2. Basic definitions

Definition 2.1. A real function $f(x),\ x>0$, is said to be in the space $C_\alpha,\ \alpha\in\Re$ if there exists a real number $p\ (>\alpha)$, such that $f(x)=x^p f_1(x)\in C[0,\infty]$. Clearly $C_\alpha\subset C_\beta$ if $\beta\leqslant\alpha$.

Definition 2.2. A function $f(x),\ x>0$, is said to be in the space $C_\alpha^m,\ m\in N\cup\{0\}$, if $f^{(m)}\in C_\alpha$.

Definition 2.3. The left sided Riemannian–Liouville fractional integral of order $\mu\geqslant 0$, of a function $f\in C_\alpha,\ \alpha\geqslant -1$, is defined as

$$I^\mu f(x)=\frac{1}{\Gamma(\mu)}\int_0^x\frac{f(t)}{(x-t)^{1-\mu}}\,dt,\quad \mu>0,\ x>0,\qquad I^0 f(x)=f(x). \tag{2}$$

Definition 2.4. Let $f\in C_{-1}^m,\ m\in N$. Then the Caputo fractional derivative of f is defined as [36,47]:

$$D^\mu f(x)=\begin{cases}[I^{m-\mu}f^{(m)}(x)], & m-1<\mu\leqslant m,\\ \frac{d^m}{dt^m}f(t), & \mu=m,\end{cases} \tag{3}$$

$$I^\mu I^\nu f=I^{\mu+\nu}f,\quad \mu,\nu\geqslant 0,\ f\in C_\alpha,\ \alpha\geqslant -1,$$

$$I^\mu x^y=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)}x^{\gamma+\mu},\quad \mu>0,\ \gamma>-1,\ x>0. \tag{4}$$

Lemma 2.1. *If $m-1<\alpha\leqslant m$, and $f\in L_1[a,b]$, then*

$$J_a^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt,\qquad D_a^\alpha J_a^\alpha f(x)=f(x), \tag{5}$$

and

$$J_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{(x-a)^k}{k!}, \quad x > 0. \tag{6}$$

Definition 2.5. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\,dt, \tag{7}$$

for $m-1 < \alpha \leqslant m,\ m \in \mathbb{N},\ x > 0$.

Definition 2.6. For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,t)}{\partial t^m}\,d\tau, & \text{for } m-1 < \alpha < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m \in \mathbb{N}, \end{cases} \tag{8}$$

and the space-fractional derivative operator of order $\beta > 0$ is defined as

$$D_x^\alpha u(x,t) = \frac{\partial^\beta u(x,t)}{\partial t^\beta} = \begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^x (t-\theta)^{m-\beta-1} \frac{\partial^m u(\theta,t)}{\partial t^m}\,d\theta, & \text{for } m-1 < \beta < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \beta = m \in \mathbb{N}. \end{cases} \tag{9}$$

3. Variational iteration method

Variational iteration method (VIM) was first proposed by the Chinese mathematician He [41–45]. This method has been employed to solve a large variety of linear and nonlinear problems with approximations converging rapidly to accurate solutions. This technique is used in [46] for solving nonlinear Jaulent–Miodek, coupled KdV and coupled MKdV equations. In [47] the applications of the present method to Shock-peakon and shock-compacton solutions for $K(p,q)$ equation are provided. The variational iteration technique is employed to solve the nonlinear dispersive equation, a nonlinear partial differential equation which arise in the process of understanding the role of nonlinear dispersion and in the forming of structures like liquid drops and exhibits compactons [48]. Also this method is applied in [49] for solving KdV and mKdV equations with initial conditions and in [50] for obtaining numerical doubly-periodic solution of the $(2+1)$-dimensional Boussinesq equation. Author of [51] employed VIM for determining exact and numerical solitons with compact support for the $K(m,p)$ equations. This technique is also employed in [52] to solve the Fokker–Planck equation. The linear and nonlinear cases are discussed in their work and several test examples are given to show the efficiency of this procedure. Draganescu [53] applied VIM to nonlinear oscillator with fractional damping and then [54] to nonlinear viscoelastic models with fractional derivatives. Odibat and Momani [55] applied the method to nonlinear differential equations of fractional order with great success. This method is successfully and effectively applied to delay differential equations [56], Cauchy reaction–diffusion problem [57], a differential equation arising in astrophysics [58], a biological population model [59], etc. The convergence of He's variational iteration method is investigated in [60]. Author of [61] employed VIM for solving the quadratic Riccati differential equation. A modified VIM is introduced to eliminate the shortcomings and in [62] the Padé technique was successfully linked with this modification. The idea of VIM is constructing a correction functional by a general Lagrange multiplier. The multiplier in the functional should be chosen such that its correction solution is superior to its initial approximation (trial function) and is the best within the flexibility of trial function, accordingly we can identify the multiplier by variational theory. The initial approximation can be freely chosen with possible unknowns, which can be determined by imposing the boundary/initial conditions.

To illustrate its basic concepts of the variational iteration method, we consider the following differential equation:

$$Lu + Nu = g(x), \tag{10}$$

where L is a linear operator, N a nonlinear operator and $g(x)$ an inhomogeneous term.

According to the variational iteration method, we can construct a correct functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\}\,d\tau, \tag{11}$$

where λ is a general Lagrangian multiplier [63] which can be identified optimally by the variational theory, the subscript n denotes the nth-order approximation, and $\tilde{u}_n$is considered as a restricted variation [64], i.e. $\delta\tilde{u}_n = 0$. For linear problems, the exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified. In nonlinear problems, in order to determine the Lagrange multiplier in a simple manner, the nonlinear terms have to be considered as restricted variations. Consequently, the exact solution may be obtained by using

$$u(x,t)=\lim_{n\to\infty} u_n(x,t).$$

The convergence of the VIM has been investigated in [65]. He obtained some results about the speed of convergence of this method.

To illustrate the above theory, two examples of special interest such as the space-fractional Burgers equation and the time-fractional Burgers equation are discussed in details and the obtained results are exactly the same with that found by the Adomian decomposition method [36].

4. Applications

4.1. The space-fractional Burgers equation

We consider the space-fractional Burgers equation with the following initial value problem [36]:

$$\begin{cases} \dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}-\nu\dfrac{\partial^2 u}{\partial x^2}+\eta\dfrac{\partial^\beta u}{\partial x^\beta}=0, \quad x,t>0,\ 0<\beta\leqslant 1,\\ u(0,t)=0, \quad u_x(0,t)=\dfrac{1}{t}-\dfrac{\pi^2}{2\nu t^2}. \end{cases} \tag{12}$$

To solve Eq. (12) by means of the variational iteration method, we construct a correction functional which reads

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\tau)\left\{\left(\frac{\partial u_n(x,\tau)}{\partial\tau}\right)_t+\big(\tilde u_n(x,\tau)\big)\left(\frac{\partial\tilde u_n(x,\tau)}{\partial\tau}\right)_x-\nu\left(\frac{\partial\tilde u_n(x,\tau)}{\partial\tau}\right)_{xx}+\eta\left(\frac{\partial\tilde u_n(x,\tau)}{\partial\tau}\right)_\beta d\tau\right\}, \tag{13}$$

where $\delta\tilde u_n$is considered as a restricted variation, $u_0(x,t)$ is its initial approximation or trial function. Making the above correction functional stationary and noticing that $\delta\tilde u_n=0$, we obtain

$$\begin{aligned}\delta u_{n+1}(x,t)&=\delta u_n(x,t)+\int_0^t \delta\lambda(\tau)\left\{\left(\frac{\partial u_n(x,\tau)}{\partial\tau}\right)_t+\big(\tilde u_n(x,\tau)\big)\left(\frac{\partial\tilde u_n(x,\tau)}{\partial\tau}\right)_x-\nu\left(\frac{\partial\tilde u_n(x,\tau)}{\partial\tau}\right)_{xx}+\eta\left(\frac{\partial\tilde u_n(x,\tau)}{\partial\tau}\right)_\beta d\tau\right\}\\ &=\delta u_n(x,t)+\lambda(\tau)\delta u_n(x,\tau)+\int_0^t \delta u_n(x,\tau)\lambda'(\tau)\,d\tau=\delta u_n(x,t)\big(1+\lambda(\tau)\big)+\int_0^t \delta u_n(x,\tau)\lambda'(\tau)\,d\tau=0\end{aligned}$$

which produces the stationary conditions:

$$\lambda'(\tau)=0, \tag{14a}$$

$$1+\lambda(\tau)|_{\tau=t}=0, \tag{14b}$$

where Eq. (14a) is called Lagrange–Euler equation and Eq. (14b) natural boundary condition.

The Lagrange multiplier, therefore, can be identified as $\lambda=-1$, and the following variational iteration formula can be obtained

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^x \big\{(u_n)_t+(\tilde u_n)(\tilde u_n)_x-\nu(\tilde u_n)_{xx}+\eta(\tilde u_n)_\beta\big\}\,d\tau. \tag{15}$$

We start with an initial approximation $u_0=u(0,t)+xu_x(0,t)$ given by Eq. (12), by the above iteration formula (15), we can obtain directly the other components as

$$u_0(x,t)=\left(\frac{1}{t}-\frac{\pi^2}{2\nu t^2}\right)x, \tag{16}$$

$$u_1(x,t)=\frac{\pi^4x^3}{24\nu^3t^4}-\frac{\eta x^{3-\beta}}{(3-\beta)(\beta-2)\Gamma(2-\beta)\nu}, \tag{17}$$

$$u_2(x,t)=-\frac{\pi^6x^5}{240\nu^5t^6}+\frac{\eta(-\pi^2+2\nu t)x^{5-\beta}}{2(5-\beta)(\beta-3)(\beta-2)\Gamma(\beta-2)\nu^3t^2}+\frac{\eta\Gamma(4-\beta)x^{5-2\beta}}{2(5-2\beta)(\beta-3)(\beta-2)^2\Gamma(\beta-2)\Gamma(3-2\beta)\nu}, \tag{18}$$

$$u_3(x,t)=\frac{\pi^8x^7}{40320\nu^7t^8}+\cdots, \tag{19}$$

and so on, in the same manner the rest of components of the iteration formula (15) were obtained the Mathematica Package.

Consequently, we have the solution of (12) in a series form

$$u(x,t)=\left(\frac{1}{t}-\frac{\pi^2}{2\nu t^2}\right)x+\frac{\pi^4x^3}{24\nu^3t^4}-\frac{\eta x^{3-\beta}}{(3-\beta)(\beta-2)\Gamma(2-\beta)\nu}-\frac{\pi^6x^5}{240\nu^5t^6}+\frac{\eta(-\pi^2+2\nu t)x^{5-\beta}}{2(5-\beta)(\beta-3)(\beta-2)\Gamma(\beta-2)\nu^3t^2}$$
$$+\frac{\eta\Gamma(4-\beta)x^{5-2\beta}}{2(5-2\beta)(\beta-3)(\beta-2)^2\Gamma(\beta-2)\Gamma(3-2\beta)\nu}+\frac{\pi^8x^7}{40320\nu^7t^8}+\cdots$$

which are exactly the same as obtained by Adomian decomposition method [36]. If we take $\beta=\frac{1}{2}$ in (19), then we get the solution of [66] as follows:

$$u(x,t)=\left(\frac{1}{t}-\frac{\pi^2}{2\nu t^2}\right)x+\frac{\pi^4x^3}{24\nu^3t^4}-\frac{\pi^6x^5}{240\nu^5t^6}+\frac{\pi^8x^7}{40320\nu^7t^8}+\cdots. \tag{20}$$

The exact solution, for the special case $\eta=0$, is given by

$$u(x,t)=\frac{x}{t}-\frac{\pi}{t}\tanh\left[\frac{\pi x}{2\nu}\right]. \tag{21}$$

4.2. The time-fractional Burgers equation

We now consider the one-dimensional time-fractional Burgers equation with the following initial value problem [36]:

$$\begin{cases}\dfrac{\partial^\alpha u}{\partial t^\alpha}+\varepsilon u\dfrac{\partial u}{\partial x}-\nu\dfrac{\partial^2u}{\partial x^2}=0, \quad t>0,\ 0<\alpha\leqslant 1,\\ u(x,0)=g(x).\end{cases} \tag{22}$$

To solve Eq. (22) by means of the variational iteration method, we construct a correction functional which reads

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t\lambda\left\{\left(\frac{\partial u_n(x,\tau)}{\partial\tau}\right)_\alpha+\left(\tilde{u}_n(x,\tau)\right)\left(\frac{\partial\tilde{u}_n(x,\tau)}{\partial\tau}\right)_x-\nu\left(\frac{\partial\tilde{u}_n(x,\tau)}{\partial\tau}\right)_{xx}\right\}d\tau, \tag{23}$$

where $\delta\tilde{u}_n$is considered as a restricted variation. $u_0(x,t)$ is its initial approximation or trial function. Making the above correction functional stationary and noticing that $\delta\tilde{u}_n=0$, we get

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\int_0^t\delta\lambda(\tau)\left\{\left(\frac{\partial u_n(x,\tau)}{\partial\tau}\right)_\alpha+\left(\tilde{u}_n(x,\tau)\right)\left(\frac{\partial\tilde{u}_n(x,\tau)}{\partial\tau}\right)_x-\nu\left(\frac{\partial\tilde{u}_n(x,\tau)}{\partial\tau}\right)_{xx}\right\}d\tau,$$

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\lambda(\tau)\delta u_n(x,\tau)+\int_0^t\delta u_n(x,\tau)\lambda'(\tau)\,d\tau=\delta u_n(x,t)\bigl(1+\lambda(\tau)\bigr)+\int_0^t\delta u_n(x,\tau)\lambda'(\tau)\,d\tau=0$$

which produces the stationary conditions:

$$\lambda'(\tau)=0, \tag{24a}$$

$$1+\lambda(\tau)|_{\tau=t}=0, \tag{24b}$$

where Eq. (24a) is called Lagrange–Euler equation and Eq. (24b) natural boundary condition.

The Lagrange multiplier, therefore, can be identified as $\lambda=-1$, and the following variational iteration formula can be obtained

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^x\left\{(u_n)_\alpha+(\tilde{u}_n)(\tilde{u}_n)_x-\nu(\tilde{u}_n)_{xx}\right\}d\tau. \tag{25}$$

We start with an initial approximation $u_0=u(x,0)$ given by Eq. (22), by the above iteration formula (25), we can obtain directly the other components as

$$u_0(x,t)=g(x)=\frac{\mu+\sigma+(\sigma-\mu)\exp(\gamma)}{1+\exp(\gamma)}, \tag{26}$$

$$u_1(x,t)=[\nu g''-\varepsilon gg']\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{27}$$

$$u_2(x,t)=\left[2\varepsilon^2gg'^2+\varepsilon^2g^2g''-4\varepsilon\nu g'g''-2\varepsilon\nu gg''+\nu^2g^{(4)}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{28}$$

and so on, in the same manner the rest of components of the iteration formula (25) were obtained the Mathematica Package.

Table 1
The numerical results for 10 iterations in comparison with the exact solution $u(x,t)=\frac{x}{t}-\frac{\pi}{t}\tanh[\frac{\pi x}{2\nu}]$ when $\nu=0.1$ for the space-fractional Burgers equation with initial conditions of Eq. (12)

t/x	0.1	0.2	0.3	0.4	0.5
0.1	2.9329E−15	2.8755E−14	2.8198E−14	2.7656E−13	2.7131E−12
0.2	7.3462E−14	7.2729E−13	7.2007E−13	7.1295E−13	7.0593E−12
0.3	3.2328E−14	3.2109E−13	3.1893E−13	3.1678E−12	3.1466E−11
0.4	1.7885E−14	1.7792E−13	1.7700E−13	1.7609E−12	1.7519E−11
0.5	1.1196E−14	1.1148E−13	1.1101E−13	1.1054E−12	1.1008E−11

Consequently, we have the solution of (22) in a series form

$$u(x,t)=\frac{\mu+\sigma+(\sigma-\mu)\exp(\gamma)}{1+\exp(\gamma)}+[\nu g''-\varepsilon gg']\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$
$$+\left[2\varepsilon^2 gg'^2+\varepsilon^2 g^2 g''-4\varepsilon\nu g'g''-2\varepsilon\nu gg''+\nu^2 g^{(4)}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots \tag{29}$$

which are exactly same as obtained by Adomian decomposition method [36] and where $\gamma=\frac{\mu}{\nu}(x-\lambda)$ and the parameters μ,σ,λ and ν are arbitrary constants.

If we take $\alpha=1$ in (29), then we get the solution of [67] as follows:

$$u_0(x,t)=\frac{\mu+\sigma+(\sigma-\mu)\exp(\gamma)}{1+\exp(\gamma)}, \tag{30}$$

$$u_1(x,t)=\frac{2\mu\sigma^2\exp(\gamma)}{[1+\exp(\gamma)]^2\nu}t, \tag{31}$$

$$u_2(x,t)=\frac{[\mu^3\sigma^2\exp(\gamma)][\exp(\gamma)-1]}{[1+\exp(\gamma)]^3\nu^2}t^2, \tag{32}$$

$$u_3(x,t)=\frac{[\mu^4\sigma^3\exp(\gamma)][1-4\exp(\gamma)+\exp(\gamma)]}{3[1+\exp(\gamma)]^4\nu^3}t^3, \tag{33}$$

and so on, in the same manner the rest of components of the iteration formula (25) were obtained the Mathematica Package. Thus, we have the solution of (22) in a series form for $\alpha=1$,

$$u(x,t)=\frac{\mu+\sigma+(\sigma-\mu)\exp(\gamma)}{1+\exp(\gamma)}+\frac{2\mu\sigma^2\exp(\gamma)}{[1+\exp(\gamma)]^2\nu}t+\frac{[\mu^3\sigma^2\exp(\gamma)][\exp(\gamma)-1]}{[1+\exp(\gamma)]^3\nu^2}t^2$$
$$+\frac{[\mu^4\sigma^3\exp(\gamma)][1-4\exp(\gamma)+\exp(\gamma)]}{3[1+\exp(\gamma)]^4\nu^3}t^3+\cdots, \tag{34}$$

where $\gamma=\frac{\mu}{\nu}(x-\lambda)$ and in a close form solution by

$$u(x,t)=\frac{\mu+\sigma+(\sigma-\mu)\exp[\frac{\mu}{\nu}(x-\sigma t-\lambda)]}{1+\exp[\frac{\mu}{\nu}(x-\sigma t-\lambda)]}, \tag{35}$$

which are exactly same as obtained by Adomian decomposition method [67].

5. Numerical results

In this section, we consider the space- and time-fractional Burgers equations for numerical comparisons. In order to verify numerically whether the proposed methodology lead to higher accuracy, we can evaluate the approximate solution using the n-term approximations (15) and (25). Tables 1–3 show the difference of the exact and numerical solutions of the absolute errors. The graphs of the exact and approximate solutions are depicted in Figs. 1 and 2. It is to be note that 10 terms only were used in evaluating the approximate solutions. We achieved a good approximation with the exact solutions of the equations by using 10 terms only of the variational iteration method. It is evident that the overall errors can be made smaller by adding new terms of the series (20) and (34).

6. Conclusions

In this paper, we have presented a scheme used to obtain numerical and exact solutions of the space- and time-fractional Burgers equations with initial conditions using by the variational iteration method. The results show that the present method is a powerful mathematical tool for finding other numerical and exact solutions of many nonlinear fractional differential equations with initial and boundary conditions. It may be concluded that the VIM and the ADM are very

Table 2
The numerical results for 5 iterations in comparison with the exact solution $u(x,t)=\frac{x}{t}-\frac{\pi}{t}\tanh[\frac{\pi x}{2\nu}]$ when $\nu=3$ for the space-fractional Burgers equation with initial conditions of Eq. (12)

t/x	0.1	0.2	0.3	0.4	0.5
0.1	3.5570E–08	3.5585E–08	3.5401E–08	3.5219E–08	3.5038E–08
0.2	7.9651E–08	7.9422E–08	7.9195E–08	7.8969E–08	7.8743E–08
0.3	2.9931E–07	2.9868E–07	2.9805E–07	2.9742E–07	2.9680E–07
0.4	1.4201E–07	1.4178E–07	1.4155E–07	1.4132E–07	1.4109E–07
0.5	8.2596E–07	8.2508E–07	8.2420E–07	8.2333E–07	8.2246E–07

Table 3
The numerical results for 5 iterations in comparison with the exact solution $u(x,t)=\frac{\mu+\sigma+(\sigma-\mu)\exp[\frac{\mu}{\nu}(x-\sigma t-\lambda)]}{1+\exp[\frac{\mu}{\nu}(x-\sigma t-\lambda)]}$ when $\nu=0.1$, $\mu=1$, $\sigma=0.9$ and $\lambda=0.4$ for the time-fractional Burgers equation with initial condition of Eq. (22)

t/x	0.1	0.2	0.3	0.4	0.5
0.1	7.4250E–11	6.2058E–13	4.6082E–13	2.6843E–11	5.2917E–07
0.2	1.4023E–09	2.9744E–11	1.1711E–11	7.0830E–10	1.1307E–06
0.3	1.0542E–08	1.4908E–10	7.8477E–11	4.4604E–09	9.2668E–06
0.4	5.4003E–08	6.2471E–10	3.7936E–10	1.9815E–08	4.2411E–05
0.5	1.3301E–07	2.8958E–09	1.1608E–09	6.1020E–08	1.4161E–05

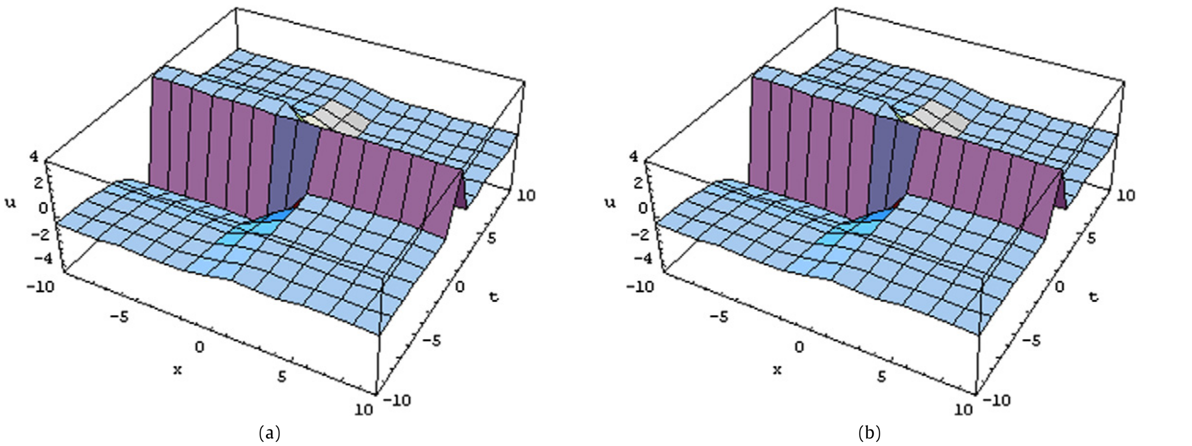

Fig. 1. The surface shows the solution $u(x,t)$ for Eq. (12): (a) exact solution; (b) approximate solution when $\nu=1$ for the space-fractional Burgers equation.

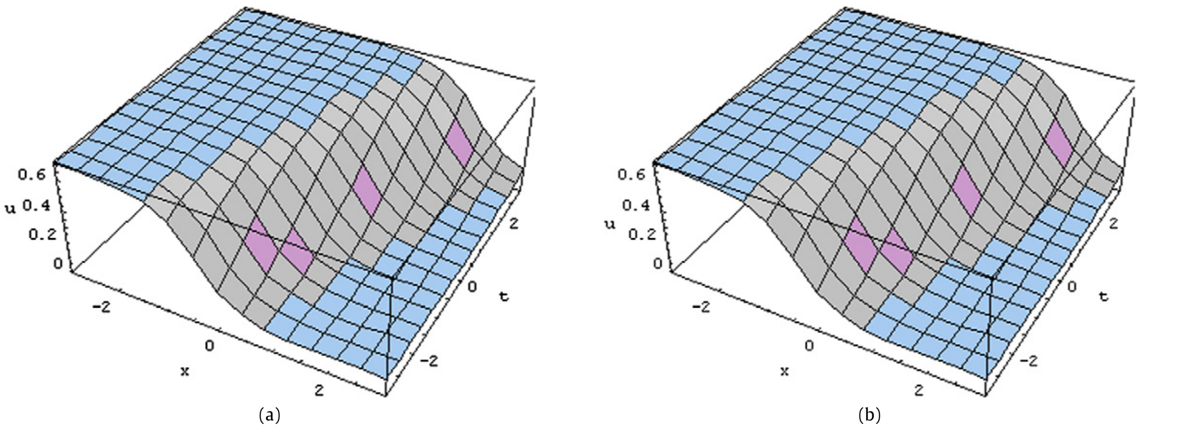

Fig. 2. The surface shows the solution $u(x,t)$ for Eq. (12): (a) exact solution; (b) approximate solution when $\nu=0.1$, $\mu=0.3$, $\sigma=0.4$ and $\lambda=0.8$ for the time-fractional Burgers equation.

powerful and efficient techniques in finding an acceptable solution for wide classes of nonlinear problems. The VIM requires the evaluation of the Lagrangian multiplier, λ. However, the ADM requires the evolution of the Adomian polynomials. Thus, the evaluation of the Adomian polynomials for every nonlinear terms mostly requires tedious algebraic calculations. The main advantage of VIM is to overcome the difficulty arising in calculating Adomian's polynomials in ADM. We can integration the equation directly without use calculation of Adomian polynomials by the VIM. This work show that the VIM is easier than the ADM. Furthermore, VIM needs relative less computational work than ADM. The behavior of the solution obtained by the variational iteration method is shown for different values of times in Figs. 1 and 2.

The numerical solutions of fractional differential equations obtained by the semi-analytic methods can be found in [68–72].

References

[1] M.J. Ablowitz, P.A. Clarkson, Solitons: Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, 1991.
[2] M.R. Miura, Backlund Transformation, Springer-Verlag, Berlin, 1978.
[3] C.H. Gu, et al., Darboux Transformation in Solitons Theory and Geometry Applications, Shangai Science Technology Press, Shanghai, 1999.
[4] C.S. Gardner, et al., Method for solving the Korteweg–de Vries equation, Phys. Rev. Lett. 19 (1967) 1095.
[5] R. Hirota, Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons, Phys. Rev. Lett. 27 (1971) 1192.
[6] W. Malfliet, Solitary wave solutions of nonlinear wave equations, Amer. J. Phys. 60 (1992) 659.
[7] M. Inc, E.G. Fan, Extended tanh-function method for finding travelling wave solutions of some nonlinear partial differential equations, Z. Naturforsch. A 60 (2005) 7.
[8] Z. Yan, H.Q. Zhang, New explicit and exact travelling wave solutions for a system of variant Boussinesq equations in mathematical physics, Phys. Lett. A 252 (1999) 291.
[9] M. Inc, D.J. Evans, A study for obtaining more solitary pattern solutions of fifth-order KdV-like equations, Int. J. Comput. Math. 81 (2004) 473.
[10] M.L. Wang, Exact solutions for a compound KdV–Burgers equation, Phys. Lett. A 215 (1996) 279.
[11] Z. Yan, H.Q. Zhang, New explicit solitary wave solutions and periodic wave solutions for Whitham–Broer–Kaup equation in shallow water, Phys. Lett. A 285 (2001) 355.
[12] Z. Fu, et al., New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations, Phys. Lett. A 290 (2001) 72.
[13] Z. Yan, The extended Jacobian elliptic function expansion method and its application in the generalized Hirota–Satsuma coupled KdV system, Chaos Solitons Fractals 15 (2003) 575.
[14] S.S. Ray, R.K. Bera, An approximate solutions of a nonlinear fractional differential equation by Adomian decomposition method, Appl. Math. Comput. 167 (2) (2005) 561–571.
[15] R.L. Bagley, P.J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, J. Rheol. 27 (1983) 201–210.
[16] R.L. Bagley, P.J. Torvik, Fractional calculus—A different approach to the analysis of viscoelastically damped structures, AIAA J. 21 (1983) 741–748.
[17] R.L. Bagley, P.J. Torvik, Fractional calculus in the transient analysis of viscoelastically damped structures, AIAA J. 23 (1985) 918–925.
[18] M. Ichise, Y. Nagayanagi, T. Kojima, An analog simulation of non integer order transfer functions for analysis of electrode processes, J. Electron. Chem. Interf. Electrochem. 33 (1971) 253–265.
[19] H.H. Sun, B. Onaral, Y. Tsao, Application of positive reality princible to metal electrode linear polarization phenomena, IEEE Trans. Bimed. Eng. 31 (1984) 664–674.
[20] H.H. Sun, A.A. Abdelvahab, B. Onaral, Linear approximation of transfer function with a pole of fractional order, IEEE Trans. Automat. Control 29 (1984) 441–444.
[21] B. Mandelbrot, Some noises with $1/f$ spectrum, a bridge between direct current and white noise, IEEE Trans. Inform. Theory 13 (1967) 289–298.
[22] T.T. Hartley, Chaos in a fractional order Chua system, IEEE Trans. Circuits Syst. 42 (1995) 485–490.
[23] F. Mainardi, Fractional calculus: Some basic problem in continuum and statistical mechanics, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum and Mechanics, Springer, New York, 1997, pp. 291–348.
[24] Y.A. Rossikhin, M.V. Shitikova, Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids, Appl. Mech. Rev. 50 (1997) 15–67.
[25] R.L. Magin, Fractional calculus in bioengineering, Crit. Rev. Biomed. Eng. 32 (2004) 1–104.
[26] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[27] A. Luchko, R. Groneflo, The initial value problem for some fractional differential equations with the Caputo derivative, preprint series, A08-98, Fachbereich Mathematik und Informatik, Freie Universität Berlin, 1998.
[28] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, New York, 1993.
[29] K.B. Oldham, J. Spainer, The Fractional Calculus, Academic Press, New York, 1974.
[30] M. Caputo, Linear model of dissipation whose q is almost frequency independent, J. Roy. Astr. Soc. 13 (1967) 529–539.
[31] T.M. Atanackovic, B. Stankovic, On a system of differential equations with fractional derivatives arising in rod theory, J. Phys. A 37 (2004) 1241–1250.
[32] N.T. Shawagfeh, Analytical approximate solutions for nonlinear fractional differential equations, Appl. Math. Comput. 131 (2002) 517–529.
[33] V. Daftardar-Gejji, A. Babakhani, Analysis of a system of fractional differential equations, J. Math. Anal. Appl. 293 (2004) 511–522.
[34] V. Daftardar-Gejji, A. Babakhani, Adomian decomposition: A tool for solving a system of fractional differential equations, J. Math. Anal. Appl. 301 (2005) 508–518.
[35] V. Daftardar-Gejji, A. Babakhani, An iterative method for solving nonlinear functional equations, J. Math. Anal. Appl. 316 (2006) 753–763.
[36] S. Momani, Non-perturbative analytical solutions of the space- and time-fractional Burgers equations, Chaos Solitons Fractals 28 (2006) 930–937.
[37] P. Biler, T. Funaki, W.A. Woyczynski, Fractal Burger equations, J. Differential Equations 148 (1998) 9–46.
[38] P. Biler, G. Karch, W.A. Woyczynski, Critical nonlinearity exponent and self-similar asyptotics for Lévy conservation laws, Ann. Henri Poincaré 5 (2001) 613–637.
[39] J.A. Mann, W.A. Woyczynski, Growing fractal interfaces in the presence of self-similar hopping surface diffusion, Phys. A 291 (2001) 159–183.
[40] D. Stanescu, D. Kim, W.A. Woyczynski, Numerical study of interacting particles approximation for integro-differential equations, J. Comput. Phys. 206 (2005) 706–726.
[41] J.H. He, A new approach to nonlinear partial differential equations, Commun. Nonlinear Sci. Numer. Simul. 2 (4) (1997) 230–235.
[42] J.H. He, Commun. Nonlinear Sci. Numer. Simul. 2 (4) (1997) 235–236.
[43] J.H. He, Approximate anlytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods Appl. Mech. Engrg. 167 (1–2) (1998) 57–68.
[44] J.H. He, Approximate solution of nonlinear differential equations with convolution product nonlinearities, Comput. Methods Appl. Mech. Engrg. 167 (1–2) (1998) 69–73.

[45] J.H. He, Variational iteration method a kind of non-linear analytical technique; some examples, Internat. J. Non-Linear Mech. 34 (1999) 699–708.
[46] D.D. Ganji, M. Jannatabadi, E. Mohseni, Application of He's variational iteration method to nonlinear Jaulent–Miodek equations and comparing it with ADM, J. Comput. Appl. Math. 207 (2007) 35–45.
[47] L. Tian, J. Yin, Shock–peakon and shock–compacton solutions for $K(p,q)$ equation by variational iteration method, J. Comput. Appl. Math. 207 (2007) 46–52.
[48] J.H. He, X.H. Wu, Construction of solitary solution and compacton-like solution by variational iteration method, Chaos Solitons Fractals 29 (2006) 108–113.
[49] M. Inc, Numerical simulation of KdV and mKdV equations with initial conditions by the variational iteration method, Chaos Solitons Fractals 34 (2007) 1071–1081.
[50] M. Inc, Numerical doubly-periodic solution of the $(2+1)$-dimensional Boussinesq equation with initial conditions by the variational iteration method, Phys. Lett. A 366 (2007) 20–24.
[51] M. Inc, Exact and numerical solitons with compact support for nonlinear dispersive $K(m,p)$ equations by the variational iteration method, Phys. A 375 (2007) 447–456.
[52] M. Dehghan, M. Tatari, The use of He's variational iterational method for solving a Fokker–Planck equation, Phys. Scr. 74 (2006) 310–316.
[53] G. Draganescu, N. Cofan, D. Rujan, Nonlinear vibrations of a nano sized sensor with fractional damping, J. Optoelectron. Adv. Mater. 7 (2005) 877–884.
[54] G. Draganescu, Application of a variational iteration method to linear and nonlinear viscoelastic models with fractional derivatives, J. Math. Phys. 47 (2006), Art. No. 082902.
[55] Z. Odibat, S. Momani, Application of variational iteration method to nonlinear differential equations of fractional order, Int. J. Nonlinear Sci. Numer. Simul. 7 (2006) 27–34.
[56] J.H. He, Some asymptotic methods for strongly nonlinear equations, Internat. J. Modern Phys. B 20 (2006) 1141–1199.
[57] M. Dehghan, F. Shakeri, Application of He's variational iteration method for solving the Cauchy reaction–diffusion problem, J. Comput. Appl. Math. 214 (2008) 435–446.
[58] M. Dehghan, F. Shakeri, Approximate solution of a differential equation arising in astrophysics using the variational iteration method, New Astron. 13 (2008) 53–59.
[59] F. Shakeri, M. Dehghan, Numerical solution of a biological population model using He's variational iteration method, Comput. Math. Appl. 54 (2007) 1197–1209.
[60] M. Tatari, M. Dehghan, On the convergence of He's variational iteration method, J. Comput. Appl. Math. 207 (2007) 121–128.
[61] S. Abbasbandy, A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials, J. Comput. Appl. Math. 207 (2007) 59–63.
[62] T.A. Abassy, M.A. El-Tawil, H. El Zoheiry, Solving nonlinear partial differential equations using the modified variational iteration Padé technique, J. Comput. Appl. Math. 207 (2007) 73–91.
[63] M. Inokvti, H. Sekine, T. Mura, General use of the Lagrange multiplier in nonlinear mathematical physics, in: S. Nemat-Nasser (Ed.), Variational Method in the Mechanics of Solids, Pergamon Press, Oxford, 1978.
[64] J.H. He, Generalized Variational Principles in Fluids, Science and Culture Publishing House of China, 2003 (in Chinese).
[65] J.H. He, Non-perturbative methods for strongly nonlinear problems, Disertation, de-Verlag in GmbH, Berlin, 2006.
[66] D. Kaya, A. Yokuş, A numerical comparison of partial solutions in the decomposition method for linear and nonlinear partial differential equations, Math. Comput. Simulation 60 (2002) 507–512.
[67] D. Kaya, An application of the decomposition method for the KdVB equation, Appl. Math. Comput. 152 (2004) 279–288.
[68] S. Momani, K. Al-Khaled, Numerical solutions for systems of fractional differential equations by the decomposition method, Appl. Math. Comput. 162 (2005) 1351–1365.
[69] S. Momani, Analytical approximate solution for fractional heat-like and wave-like equations with variable coefficients using the decomposition method, Appl. Math. Comput. 165 (2005) 459–472.
[70] S. Momani, An explicit and numerical solutions of the fractional KdV equations, Math. Comput. Simulation 70 (2005) 110–118.
[71] S. Momani, Analytic and approximate solutions of the space- and time- fractional telegraph equations, Appl. Math. Comput. 170 (2005) 1126–1134.
[72] S. Momani, Z. Odibat, Analytical approach to linear fractional partial differential equations arising in fluid mechanics, Phys. Lett. A 355 (2006) 271–279.