Existence of Periodic Solutions for Ginzburg–Landau Equations of Superconductivity

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In an earlier paper (B. Wang, 1999, *J. Math. Anal. Appl.* 232, 394–412) the author proved the existence of periodic solutions to the Ginzburg–Landau equations of superconductivity in two dimensions. Here we will show that the Ginzburg–Landau equations admit periodic solutions in three-dimensions as well. We also prove the existence of periodic solutions to the phase-lock equations introduced by the author (in press, *J. Nonlinear Anal.*).

Key Words: Ginzburg–Landau (TDGL) equations; periodic solutions; phase-lock equations.

INTRODUCTION

In a previous article [18], we introduced the following system of equations (phase-lock equations) to model the superconductivity phenomena,

\[
\begin{cases}
f_{t} + \kappa^{2}(|f|^{2} - 1)f - \Delta f + |\mathbf{q}|^{2}f = 0, \\
\eta \mathbf{q}_{t} + |f|^{2}\mathbf{q} + \text{curl}^{2} \mathbf{q} - \text{curl} H = 0, \\
\text{div} \mathbf{q} = 0.
\end{cases}
\]  

(0.1)

Here \(f\) is a real-valued function and \(\mathbf{q}\) is a vector-valued real function, with the initial conditions

\[
f(x, 0) = f_{0}(x), \quad \mathbf{q}(x, 0) = \mathbf{q}_{0}(x), \text{ in } \Omega
\]

(0.2)
and boundary conditions

\[ \nabla f \cdot \mathbf{n} = 0, \quad \text{curl} \mathbf{q} \times \mathbf{n} = H \times \mathbf{n}, \quad \mathbf{q} \cdot \mathbf{n} = 0, \text{ on } \partial \Omega. \quad (0.3) \]

The motivation of studying this system is because it is closely related to the time dependent Ginzberg–Landau equation of superconductivity with an applied force. To illustrate this point further, let \( \chi(t, x) \) be a real-valued function and set

\[
\begin{cases}
\psi = e^{i\chi} \\
A = Q + \nabla \chi \\
\phi = -\partial_t \chi.
\end{cases}
\]

Then formally, \( \psi, A, \phi \) satisfy the system of equations

\[
\begin{align*}
\psi_t + i\phi \psi + \kappa^2 (|\psi|^2 - 1) \psi + (i \nabla + A)^2 \psi &= 0, \\
\eta(A_t + \nabla \phi) + \frac{i}{2} (\psi^* \nabla \psi - \psi \nabla \psi^*) + |\psi|^2 A + \text{curl}^2 A - \text{curl} H &= 0,
\end{align*}
\]

with the initial conditions

\[
\psi(x, 0) = \psi_0(x), \quad A(x, 0) = A_0(x), \text{ in } \Omega \subseteq R^3, \quad (0.5)
\]

and boundary conditions

\[
(i \nabla + A \psi) \cdot \mathbf{n} = 0, \quad \text{curl} A \times \mathbf{n} = H \times \mathbf{n}, \quad A \cdot \mathbf{n} = 0, \text{ on } \partial \Omega,
\] 

which, as it is well known, are the time dependent Ginzberg–Landau equations of superconductivity with the applied field \( H \). Notice \( H \) is the dimensionless applied electromagnetic field in the sense that if \( \mathbf{H} \) is the applied electromagnetic field then \( \mathbf{H} = \sqrt{2} H_e \mathbf{H} \) where \( H_e \) is the thermodynamic critical field. (See [12, 17].) This model involves three unknown functions: a complex valued function \( \psi: \Omega \to \mathbb{C} \) for the order parameter, a vector valued function \( A: \Omega \to R^3 \) for the magnetic potential, and a scalar valued function \( \phi: \Omega \to R^3 \) for the electric potential. Here we may take \( \kappa \) as the Ginzburg–Landau parameter and \( \eta \) as the nondimensional diffusivity.

As it is well known, the square of modulus of \( \psi \) equals the number of superconducting charge carriers, so the behavior of the modulus of \( ||\psi|| \) plays an important role in understanding the superconductivity.
From the transformation described above, one can see that the unknown function \( f \) in the phase-lock equations is closely related to the modulus of \( \psi \). So by studying \( f \) we will gain a better understanding of \( ||\psi|| \) and hence the distribution and variation of the number of superconducting charge carriers, the essence of superconductivity.

Furthermore, the system (0.1) is simpler than (0.4), we can certainly hope that by studying the simpler system, we gain a better understanding of the original Ginzburg–Landau equations and hence the superconductivity ultimately. Indeed, this article supports such hope in the sense that, with the help of the phase-lock equations, we are able to prove the existence of at least one periodic solution to the Ginzburg–Landau equations in three dimensions.

Namely in this article we prove the following theorem for the Ginzburg–Landau equation, which asserts that the Ginzburg–Landau equations admit at least one \( T \)-periodic solution in three-dimensions for every positive value \( T > 0 \).

**THEOREM 0.1.** Let \( T > 0 \) be any given positive real number. Assume that \( H \in L^\infty(0, T; \mathbf{H}(\Omega)) \cap L^2(0, T; \mathbf{H}^1(\Omega)), \frac{\partial H}{\partial T} \in L^\infty(0, T; \mathbf{H}^2(\Omega)), \) and \( H \) is \( T \)-periodic, that is, \( H(t + T, x) = H(t, x), \forall (t, x) \in \mathbb{R} \times \Omega \). Then the system (0.4)–(0.6) possesses at least one time-periodic solution \((\psi(t, x), \mathbf{A}(t, x))\) that satisfies \( \psi(T + t, x) = \psi(t, x), \) and \( \mathbf{A}(T + t, x) = \mathbf{A}(t, x), \forall (t, x) \in \mathbb{R} \times \Omega \).

In [16], the author proved the existence of periodic solutions to the Ginzburg–Landau equations of superconductivity in two-dimensions. As pointed out by the author, the method used there can’t be directly extended to the 3-dimensional case. So one has to search for a different route.

In [18], we proved the existence and uniqueness of both strong solutions and weak solutions to the phase-block system (0.1). We also proved that for any given solutions to (0.1) one can construct a solution to (0.4). That is, we proved the following result concerning the connection between the solutions of the phase-lock system and Ginzburg–Landau equations.

**THEOREM 0.2.** For any \( T > 0 \), let \( f, Q \in C([0, T), \mathbf{H}^p(\Omega)) \cap L^2([0, T); \mathbf{V}(\Omega)) \) for \( p \geq 2 \), be a weak solution of (0.1). Then for any \( \chi \in C^4([0, T), \mathbf{H}^4(\Omega)) \cap C([0, T) \times \Omega) \) satisfying \( \nabla \chi \cdot \mathbf{n} = 0 \), the functions \( \psi, \phi, \mathbf{A} \) defined by the (TG) transformation

\[
\begin{align*}
\psi &= f e^{i\chi} \\
\mathbf{A} &= q + \nabla \chi \\
\phi &= -\partial_t \chi
\end{align*}
\]

are a weak solution of the Ginzburg–Landau equations (0.4)–(0.6).
In light of this theorem, Theorem 0.1 is a corollary of the following theorem concerning the existence of a periodic solution to the phase-lock system.

**Theorem 0.3.** Let $T > 0$ be any given positive real number. Assume that
$$
\frac{\partial H}{\partial t} \in L^\infty(0, T; H^2(\Omega)), \quad H \in L^\infty(0, T; H(\Omega)) \cap L^2(0, T; H^4(\Omega)),
$$
and $H$ is $T$-periodic, that is, $H(T + t, x) = H(t, x)$, $\forall (t, x) \in \mathbb{R} \times \Omega$. Then the system (0.1)–(0.3) possesses at least one time-periodic solution $(f(t, x), q(t, x))$ that satisfies
$$
f(T + t, x) = f(t, x) \quad \text{and} \quad q(T + t, x) = q(t, x), \quad \forall (t, x) \in \mathbb{R} \times \Omega.
$$

The paper is organized as follows: in Section 1, we first introduce some necessary mathematical notations. Then we recall the existence and uniqueness of a weak solution to the phase-lock equations.

In Section 2, we derive some a priori estimates for the weak solutions stated in Section 1. Then following the standard arguments, we prove Theorem 0.3. Throughout this paper, we always assume $\Omega \subset \mathbb{R}^3$. For $\Omega \subset \mathbb{R}^2$ the same result holds for the phase-lock equations, and the proof is the same.

### 1. Preliminary Results

In this section, we will recall the existence and uniqueness of a weak solution to the phase-lock systems. Following the convention of [17], we use $W^{s,p}(\Omega)$ for the standard Sobolev spaces for real functions defined on $\Omega$ and as usual, $W^{s,2}(\Omega)$ is denoted by $H^s(\Omega)$. We use $W^{k,p}$ and $H^s$ with bold-face letters to denote Sobolev spaces of the vector valued functions.

We use $u = (f, q)$ to represent the unknown functions:

$$
V_v = \{ q \in C^\infty(\Omega) | \text{div} \ q = 0, q \cdot n|_{\partial \Omega} = 0 \}.
$$

Then we define

$$
\begin{align*}
H &= H_1 \times H_2, \\
V &= V_1 \times V_2, \\
H_1 &= L^2(\Omega), \quad V_1 = H^1(\Omega), \\
H_2 &= \text{the closure of } V_v \text{ for the } L^2 \text{-norm}, \\
V_2 &= \text{the closure of } V_v \text{ for the } H^1 \text{-norm}.
\end{align*}
$$

(1.2)
The $L^2$-norms and inner products in $H_1$, $H_2$, and $H$ are denoted by $|\cdot|$ and $(\cdot, \cdot)$. The $H^1$-norm and inner product of $V_1$ are denoted by $\|\cdot\|$ and $(\cdot, \cdot)$, respectively. It is easy to see that

$$\|q\| = \left(\int_{\Omega} |\nabla q|^2 \, d\Omega \right)^{1/2}$$

is an equivalent norm of $V_2$; the corresponding inner product is also denoted by $(\cdot, \cdot)$. We also use the notations $\|\cdot\|$ and $(\cdot, \cdot)$ for the induced norm and inner product in $V$.

Moreover, it is classical that (see among others [14])

$$H_2 = \{ q \in L^2(\Omega) \mid \text{div } q = 0, \text{ } q \cdot n|_{\partial\Omega} = 0 \},$$

$$V_2 = \{ q \in H^1(\Omega) \mid \text{div } q = 0, \text{ } q \cdot n|_{\partial\Omega} = 0 \}.$$

Notice, since $\text{div } q = 0$, the phase-lock equations can be rewritten as

$$\begin{aligned}
f_t + \kappa^2(|f|^2 - 1)f - \Delta f + |q|^2f &= 0, \\
\eta q_t + |f|^2q - \Delta q - \nabla H &= 0, \\
\text{div } q &= 0.
\end{aligned}$$

We introduce an unbounded self-adjoint positive operator $L = -\Delta$ with domain

$$D(L) = \{ u = (f, q) \in H^2(\Omega) \times H^2(\Omega) : f \text{ and } q \text{ satisfy } \text{div } q = 0 \}.$$

Now let $u = (f, q)$ denote the unknown functions; we can rewrite the phase-lock equations in the operator form

$$\frac{d}{dt}u(t) + DLu(t) + Ru(u(t)) = g,$$

where

$$D = \begin{pmatrix}
1 & 0 \\
0 & \frac{1}{\eta}
\end{pmatrix}, \quad g = \begin{pmatrix}
0 \\
\frac{1}{\eta}, H
\end{pmatrix},$$

and $R(u) = (R_1(u), R_2(u))$ is a nonlinear operator on $D(L)$, with components defined by

$$\begin{aligned}
R_1(u) &= \kappa^2(|f|^2 - 1)f + |q|^2f, \\
R_2(u) &= |f|^2q.
\end{aligned}$$
As in [18], we introduce the weak formulation of the system (0.1) as follows:

**Problems 2.1 (Weak Formulation).** For \( u_0 = (f_0, q_0) \in H \) given, find a solution \( u = (f, q) \) of the system (0.1) in the sense

\[
\begin{aligned}
&u \in L^\infty(0, T; H) \cap L^2(0, T; V), \\
&\text{for all } T > 0, \\
&\frac{d}{dt} (f, g) + (L_1 f, g) + (R_1(u), g) = 0, \quad \forall g \in V_1, \\
&\frac{d}{dt} (q, \tilde{Q}) + \frac{1}{\eta} (L_2 q, \tilde{Q}) + (R_2(u), \tilde{Q}) = 0, \quad \forall \tilde{Q} \in V_2, \\
&u_{t=0} = u_0.
\end{aligned}
\]  

(1.9) (1.10) (1.11) (1.12)

Alternatively, (2.20) and (2.21) can be written as

\[
\begin{aligned}
&\frac{d}{dt} (u, \tilde{u}) + \langle Lu, \tilde{u} \rangle + (R(u), \tilde{u}) = 0, \quad \forall \tilde{u} \in V, \\
&u_{t=0} = u_0.
\end{aligned}
\]  

(1.13)

where

\[
\begin{aligned}
Lu &= \begin{pmatrix}
L_1 f, \\
-\frac{1}{\eta} L_2 q
\end{pmatrix}, \\
R(u) &= \begin{pmatrix}
R_1(u), \\
R_2(u)
\end{pmatrix}.
\end{aligned}
\]  

(1.14)

**Theorem 1.1.** For any initial data \( u_0 \in H \) and \( f_0 \in L^4(\Omega) \), Problem 2.1 has an unique solution \( u \in C([0, T]; H) \cap L^2(0, T; V(\Omega)) \) which satisfies

\[
f \in L^2(0, T; L^6(\Omega)) \cap C([0, T]; L^4(\Omega)).
\]

**Proof.** See [18].

2. PERIODIC SOLUTION TO THE PHASE-LOCK EQUATIONS

From the result in the previous section, we know that there exists a semigroup \( S(t) \) \((t \geq 0) : L^4 \times H \to L^4 \times H\) such that \( S(t)(f_0, q_0) = (f(t), q(t)), \) the solution of the phase-lock equations. In particular, the operator \( S(T) \) with \( T \) given in Theorem 0.3 is called the evolution operator. Following the standard argument, one can easily see that the existence of \( T\)-periodic solutions of phase-lock equations is equivalent to the existence of a fixed point of \( S(T) \). To this end, we will show that \( S(T) \) maps a suitable bounded set into itself and is a compact operator as was done in [16].
Lemma 2.1. There exists constants $R_1$ and $R_2$ depending on $\kappa, |\Omega|,$ and $T$ such that for $f_0 \in L^4, \mathbf{q}_0 \in H^2$, with $\|f_0\|_{L^4} \leq R_1$ and $|\mathbf{q}_0| \leq R_2$, the solution $(f(t), \mathbf{q}(t))$ of phase-lock equations satisfies

$$
\begin{align}
\|f\|_{L^4} &\leq R_1, \\
|\mathbf{q}| &\leq R_2.
\end{align}
$$

(2.15)

Proof. Multiplying the equation for $f$ by $f^3$ we have

$$
\frac{d}{dt} \int_{\Omega} |f|^4 \, dx + 4\kappa^2 \int_{\Omega} |f|^4
\begin{align}
+ 4 \int_{\Omega} \left[ \kappa^2 (|f|^2 - 2)|f|^4 + 3\text{grad} \, f \right] |f|^2 + |\mathbf{q}|^2 |f|^4 \right] \, dx = 0. \tag{2.16}
\end{align}
$$

From this and

$$
\int_{\Omega} |f|^4 \, dx \leq \frac{1}{4} \int_{\Omega} |f|^6 \, dx + 9|\Omega|, \tag{2.17}
$$

we have

$$
\frac{d}{dt} \int_{\Omega} |f|^4 \, dx + 4\kappa^2 \int_{\Omega} |f|^4 \leq 9\kappa^2 |\Omega|. \tag{2.18}
$$

Hence choose

$$
R_1 = \left( \frac{9|\Omega|}{1 - e^{-4\kappa^2 T}} \right)^{1/4}
$$

and we get (2.15).

Now multiplying the equations for $\mathbf{q}$ by $\mathbf{q}$ and integrating over $\Omega$, we have

$$
\eta \frac{d}{dt} \int_{\Omega} |\mathbf{q}|^2 \, dx + \int_{\Omega} \left[ |f|^2 |\mathbf{q}|^2 + |\text{curl} \, \mathbf{q}|^2 \right] \, dx = \int_{\Omega} \text{curl} \, H \mathbf{q} \, dx. \tag{2.19}
$$

Since

$$
\int_{\Omega} |\text{curl} \, \mathbf{q}| \, dx \geq c|\mathbf{q}|^2
$$

$$
\int_{\Omega} \text{curl} \, H \mathbf{q} \, dx \leq \frac{1}{2} \int_{\Omega} |\text{curl} \, \mathbf{q}|^2 \, dx + \frac{1}{2} \int_{\Omega} |H|^2 \, dx,
$$

we have

$$
\eta \frac{d}{dt} \int_{\Omega} |\mathbf{q}|^2 \, dx + c \int_{\Omega} |\mathbf{q}|^2 \leq \int_{\Omega} |H|^2 \, dx. \tag{2.20}
$$
Let

\[ R_2 = \left( \frac{2\int_0^T |H|^2 \, dx}{\eta(1 - e^{-(c/\eta)T})} \right)^{1/2}. \]

Then we have (2.16).

These prove the lemma. \( \blacksquare \)

Hereafter, we will use \( R_1 \) and \( R_2 \) to denote the constant obtained in the above proof.

From the proof of the previous lemma we can easily get the following useful results.

**Lemma 2.2.** Assume that \( \|f_0\|_{L^2} \leq R_1 \) and \( \|q_0\| \leq R_2 \), there exists a constant \( C \) that depends only on \( \kappa, |\Omega|, \) and \( T \) such that

\[
\int_0^T \int_\Omega \left[ |f|^6 + |\nabla f|^2 (|f|^2 + 1) + |q|^2 (|f|^4 + |f|^2) + |\nabla q|^2 \right] \, dx \, dt \leq C. \tag{2.21}
\]

**Proof.** Integrating the following three equations over \([0, T]\) with respect to \( t \),

\[
\frac{d}{dt} \int_\Omega |f|^2 \, dx + 2 \int_\Omega \left[ \kappa^2 (|f|^2 - 2)|f|^2 + 3|\nabla f|^2 + |q|^2 |f|^2 \right] \, dx = 0 \tag{2.22}
\]

\[
\frac{d}{dt} \int_\Omega |f|^4 \, dx + 4\kappa^2 \int_\Omega |f|^4
\]

\[
+ 4 \int_\Omega \left[ \kappa^2 (|f|^2 - 2)|f|^4 + 3|\nabla f|^2 |f|^2 + |q|^2 |f|^4 \right] \, dx = 0 \tag{2.23}
\]

\[
\eta \frac{d}{dt} \int_\Omega |q|^2 \, dx + \int_\Omega \left[ |f|^2 |q|^2 + |\nabla q|^2 \right] \, dx = \int_\Omega \nabla \cdot Hq \, dx. \tag{2.24}
\]

and including the fact that \( |f| \leq |f|_{L^\infty(\Omega)} \) we can easily obtain the result. \( \blacksquare \)

From [16, 14], one choose \( q_H \in H^4(\Omega) \) so that

\[
\Delta q_H + \nabla H = 0 \quad \text{in } \Omega
\]

\[
\nabla \times q_H \times n = H \times n, \quad q_H \cdot n = 0, \text{ on } \partial \Omega \tag{2.25}
\]
and this satisfies, for a constant $C$ independent of time $t$,

\[ \|\mathbf{q}_H\|_{H^1} \leq C|H(t)| \]

\[ \|\mathbf{q}\|_{H^2} \leq C\|H(t)\|_{H^1}. \]

Furthermore,

\[ \frac{\partial \mathbf{q}_H}{\partial t} = \mathbf{q}_{\delta H/\delta t}. \]

Now letting $\mathbf{Q} = \mathbf{q} - \mathbf{q}_H$, we have, under the condition of Lemma 2.1,

\[ \int_0^T (\|f\|_{H^1} + \|\mathbf{Q}\|_{H^1}) \, dt \leq C \]

with $C$ depending only on $\kappa, |\Omega|$, and $T$, and

\[ \begin{cases} 
  f_t + \kappa^2(|f|^2 - 1)f - \Delta f + |\mathbf{Q} + \mathbf{q}_H|^2 f = 0, \\
  \eta \mathbf{Q}_t + |f|^2(\mathbf{Q} + \mathbf{q}_H) + \Delta \mathbf{Q} + \frac{\partial \mathbf{q}_H}{\partial t} = 0,
\end{cases} \quad (2.26) \]

with the initial conditions

\[ f(x, 0) = f_0(x), \quad \mathbf{Q}(x, 0) = \mathbf{q}_0(x) - \mathbf{q}_H(0), \text{ in } \Omega \quad (2.27) \]

and the boundary conditions

\[ \text{grad } f \cdot \mathbf{n} = 0, \quad \text{curl } \mathbf{Q} \times \mathbf{n} = 0, \quad \mathbf{Q} \cdot \mathbf{n} = 0, \text{ on } \partial \Omega. \quad (2.28) \]

Now, we have,

**Lemma 2.3.** Assume that $\|f_0\|_{L^1} \leq R_1$ and $|\mathbf{q}| \leq R_2$, there exists a constant $C$ that depends only on $\kappa, |\Omega|$, and $T$ such that any solution $(f(t), \mathbf{Q}(t))$ of (2.28) satisfies

\[ \|f(t)\|_{H^1} + \|\mathbf{Q}\|_{H^1} \leq C, \quad \forall t \in \left[ \frac{T}{2}, T \right]. \quad (2.29) \]

**Proof.** Multiplying the equation for $f$ by $\Delta f$ and integrating over $\Omega$ we have

\[ \frac{d}{dt} \int_{\Omega} |\text{grad } f|^2 \, dx + 2\int_{\Omega} |\Delta f|^2 \, dx = 2\int_{\Omega} \left[ \kappa^2(|f|^2 - 2)f \Delta f + |\mathbf{q}_H|^2 f \Delta f \right] \, dx. \]

Now

\[ \int_{\Omega} (|f|^2 - 2)f \Delta f \, dx \leq C(\epsilon) \int_{\Omega} (|f|^6 + |f|^2) \, dx + \epsilon \int_{\Omega} |\Delta f|^2 \, dx \]
and
\[
\left| \int_{\Omega} |q|^2 f \Delta f \, dx \right| \leq \int_{\Omega} |q|^2 |f| |\Delta f| \, dx
\]
\[
\leq \left( \int_{\Omega} |q|^6 \, dx \right)^{1/4} \left( \int_{\Omega} |q|^2 |f|^4 \, dx \right)^{1/4} \left( \int_{\Omega} |\Delta f|^2 \, dx \right)^{1/2}
\]
\[
\leq C(\varepsilon) \|q\|_{L^2}^2 + \|q\|_{L^2}^2 \int_{\Omega} |q|^2 |f|^4 \, dx + \varepsilon \int_{\Omega} |\Delta f|^2 \, dx
\]
\[
\leq C(\varepsilon) \|q\|_{L^2}^2 + \left( \|q\|_{L^2}^2 + \int_{\Omega} |q|^2 |f|^4 \, dx \right) + \varepsilon \int_{\Omega} |\Delta f|^2 \, dx.
\]
Hence
\[
\frac{d}{dt} \int_{\Omega} |\text{grad} f|^2 \, dx + \int_{\Omega} |\Delta f|^2 \, dx
\]
\[
\leq C \|q\|_{L^2}^2 \left( \|q\|_{L^2}^2 + \int_{\Omega} |q|^2 |f|^4 \, dx \right) + C \int_{\Omega} |f|^6 \, dx + C.
\]
Multiplying the equation for $Q$ and by $\Delta Q$ and integrating over $\Omega$, we have, by the fact that $|\partial q_{tt}/\partial t|_H \leq |\partial H/\partial t|_H$.
\[
\eta \frac{d}{dt} \int_{\Omega} |\text{grad} Q|^2 \, dx + 2 \int_{\Omega} |\Delta Q|^2 \, dx
\]
\[
= 2 \int_{\Omega} \left[ |f|^4 q \Delta Q + \frac{\partial q_{tt}}{\partial t} \Delta Q \right] \, dx
\]
\[
\leq C \int_{\Omega} \left[ |f|^4 |q|^2 + \left| \frac{\partial H}{\partial t} \right|^2 \right] \, dx + \int_{\Omega} |\Delta Q|^2 \, dx.
\]
Hence
\[
\eta \frac{d}{dt} \int_{\Omega} |\text{grad} Q|^2 \, dx + \int_{\Omega} |\Delta Q|^2 \, dx \leq C \int_{\Omega} \left[ |f|^4 |q|^2 + \left| \frac{\partial H}{\partial t} \right|^2 \right] \, dx.
\]
So
\[
\frac{d}{dt} \left[ \int_{\Omega} |\text{grad} f|^2 \, dx + \int_{\Omega} |\text{grad} Q|^2 \, dx \right]
\]
\[
\leq C \int_{\Omega} \left[ |f|^4 |q|^2 + \left| \frac{\partial H}{\partial t} \right|^2 \right] \, dx
\]
\[
+ C \|q\|_{H^1} \left( \|q\|_{H^1}^2 + \int_{\Omega} |q|^2 |f|^4 \, dx \right) + C \int_{\Omega} |f|^6 \, dx + C. \tag{2.30}
\]
Note that \( \|q\|_{H^1} \leq \|Q\|_{H^1} + \|q_H(t)\|_{H^1} \leq \|Q\|_{H^1} + C \), and let

\[
y(t) = \int_{\Omega} |\nabla f|^2 \, dx + \int_{\Omega} |\nabla Q|^2 \, dx
\]

\[
g(t) = \|q\|_{H^1}^2 + \int_{\Omega} |q|^2 |f|^4 \, dx
\]

\[
h(t) = C \int_{\Omega} \left[ |f|^4 |q|^2 + \frac{\partial H}{\partial t} \right]^2 \, dx + C \int_{\Omega} |f|^6 \, dx + C.
\]

Then \( y(t), g(t), \) and \( h(t) \) satisfy the condition of [16, Lemma 3.5]; hence applying that lemma we have our result.

Again noting that \( \|q\|_{H^1} \leq \|Q\|_{H^1} + \|q_H(t)\|_{H^1} \leq \|Q\|_{H^1} + C \), we have

\[
\|f(t)\|_{H^1} + \|q\|_{H^1} \leq C, \quad \forall t \in \left[ \frac{T}{2}, T \right]. \quad (2.31)
\]

Finally, we can prove Theorem 0.3.

**Proof of Theorem 0.3.** As in [16], let \( B_1 = \{ f \in L^4(\Omega) : \|f\|_{L^4} \leq R_1 \} \) and \( B_2 = \{ q \in H(\Omega) : \|q\|_H \leq R_2 \} \), with \( R_1 \) and \( R_2 \) the constants obtained in Lemma 2.1. Notice that the embedding \( H^1(\Omega) \rightarrow L^4(\Omega) \) is compact for \( \Omega \subset \mathbb{R}^3 \). Applying the same argument of [16], we see that \( S(T) \) has a fixed point. Hence the phase-lock equations have periodic solutions. The proof is complete.

**Conclusion.** In this article we proved that Ginzburg–Landau equation admits at least one \( T \)-periodic solution for any given \( T > 0 \) in space dimension \( n = 3 \), which extends the results obtained in [16] where a similar result was proved for \( n = 2 \). From the physics point of view, this result indicates that when an external applied magnetic field has \( H \) as time periodic, the superconducting state of the sample will also vary periodically in time with a period coinciding with that of an applied magnetic field. From the mathematical point of view, this result may help us to further understand the long term dynamic of Ginzburg–Landau equations. In [12, 19], the authors investigated the long term behavior of Ginzburg–Landau and phase-lock equations. They proved the existence of global attractors and gave qualitative estimations of the upper bound on the Hausdorff dimension of the global attractors. A related interesting question would be to find the estimation of the lower bound on the Hausdorff dimension of the global attractors. Preliminary results indicate that, for given boundary conditions, Ginzburg–Landau equations admit a finite number of steady state solutions. This combined with the existence...
of periodic solutions might enable one to find a lower bound for the global attractors; see [11] for related discussions. Another interesting question is whether or not the time periodic solutions obtained in Theorem 0.1 and Theorem 0.3 are stable (asymptotically or exponentially).

REFERENCES