Abstract

A general class of (finite dimensional) oscillatory integrals with polynomially growing phase functions is studied. A representation formula of the Parseval type is proven as well as a formula giving the integrals in terms of analytically continued absolutely convergent integrals. Their asymptotic expansion for "strong oscillations" is given. The expansion is in powers of $\bar{\hbar}^{1/2M}$, where $\bar{\hbar}$ is a small parameters and $2M$ is the order of growth of the phase function. Additional assumptions on the integrands are found which are sufficient to yield convergent, resp. Borel summable, expansions.

Résumé

On étudie une classe générale d’intégrales oscillatoires en dimension finie avec une fonction de phase à croissance polynomiale. Une formule de représentation du type Parseval est démontrée, ainsi qu’une formule donnant les intégrales au moyen de la continuation analytique d’intégrales absolument convergentes. On donne les développements asymptotiques de ces intégrales dans le cas d’"oscillations rapides". Ces développements sont en puissance de $\bar{\hbar}^{1/2M}$, où $\bar{\hbar}$ est un petit paramètre et $2M$ est l’ordre de croissance de la fonction de phase. Sous des conditions additionnelles sur les intégrands on obtient la convergence, resp. la sommabilité au sens de Borel, des développements asymptotiques.

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1. Introduction

The study of finite dimensional oscillatory integrals of the form

$$\int_{\mathbb{R}^N} e^{\hat{h}\Phi(x)} f(x) \, dx,$$  \hspace{1cm} (1)

where \(\hat{h}\) is a non vanishing real parameter, \(\Phi\) and \(f\) suitable real-valued smooth functions, is already a classical topic, largely developed in connections with various problems in mathematics and physics. Well known examples of simple integrals of the above form are the Fresnel integrals of the theory of wave diffraction and Airy’s integrals of the theory of rainbow. The theory of Fourier integral operators [25,26,34] also grew out of the investigation of oscillatory integrals. It allows the study of existence and regularity of a large class of elliptic and pseudoddelliptic operators and provides constructive tools for the solutions of the corresponding equations. In particular one is interested in discussing the asymptotic behavior of the above integrals when the parameter \(\hat{h}\) goes to 0. The method of stationary phase provides a tool for such investigations and has many applications, such as the study of the classical limit of quantum mechanics (see [2,3,8,22,33,44]). In the general case of degenerate critical points of the phase function \(\Phi\), the theory of unfoldings of singularities is applied, see [13,20].

The extensions of the definition of oscillatory integrals to an infinite dimensional Hilbert space \(\mathcal{H}\) and the implementation of a corresponding infinite-dimensional version of the stationary phase method has a particular interest in connection with the rigorous mathematical definition of the “Feynman path integrals”. Several methods has been discussed in literature, for instance by means of analytic continuation of Wiener integrals [16,17,29,30,32,35,36,42,43], or by “infinite dimensional distributions” in the framework of the Hida calculus [19,24], via “complex Poisson measures” [1,34], via a “Laplace transform method” [5,31], or via a “Fourier transform approach”, see [3,4,6–8,21,27,28]. The latter method is particularly interesting as it is the only one by which a development of an infinite dimensional stationary phase method has been performed. The phase functions that can be handled by this method are of the form “quadratic plus bounded perturbation”, that is \(\Phi(x) = \langle x, T x \rangle + V(x),\) where \(T\) is a self-adjoint operator and \(V\) is the Fourier transform of a complex bounded variation measure on \(\mathcal{H}\) [3,8,40,41].

We also mention that the problem of definition and study of integrals of the form (1) but with \(h \in \mathbb{C}, \text{Im}(h) < 0\) and \(\Phi\) lower bounded has also been discussed. The convergence of the integral in this case is a simple matter, so the analysis has concentrated on a “perturbation theoretical” computation of the integral, like in [14,15], resp. on a Laplace method for handling the \(h \to 0\) asymptotics, see, e.g. [3,9,11,12,38] (the latter method has some
relations with the stationary phase method). The aim of the present paper is the definition of a “generalized Fresnel integral” of the form

$$\int_{\mathbb{R}^N} e^{i\bar{h} \Phi(x)} f(x) \, dx,$$

where $\Phi$ is a smooth function bounded at infinity by a polynomial $P(x)$ on $\mathbb{R}^n$, $\text{Im}(\bar{h}) \leq 0$, $\bar{h} \neq 0$, and the study of the corresponding asymptotic expansion in powers of $\bar{h}$. The results we obtain will be generalized to the infinite dimensional case in a forthcoming paper [10] and applied to an extension of the class of phase functions for which the Feynman path integral had been defined before.

In Section 2 we introduce the notations, recall some known results and prove the existence of the oscillatory integral (2). In Section 3 we prove that when $f$ belongs to a suitable class of functions, this generalized Fresnel integral can be explicitly computed by means of an absolutely convergent Lebesgue integral. We prove a representation formula of the Parseval type (Theorem 3) (similar to the one which was exploited in [7] in the case of quadratic phase functions), as well as a formula (Corollary 1 to Theorem 3) giving the integral in terms of analytically continued absolutely convergent integrals. Even if our main interest came from the case $\bar{h} \in \mathbb{R} \setminus \{0\}$, both formulae are valid for all $\bar{h} \in \mathbb{C}$ with $\text{Im}(\bar{h}) \leq 0$, $\bar{h} \neq 0$. In the last section we consider the integral (2) in the particular case $P(x) = A_{2M}(x, \ldots, x)$, where $A_{2M}$ is a completely symmetric strictly positive covariant tensor of order $2M$ on $\mathbb{R}^N$, compute its detailed asymptotic power series expansion (in powers of $\bar{h}^{1/2M}$, for $\text{Im}(\bar{h}) \leq 0$, $\bar{h} \neq 0$) in the limit of “strong oscillations”, i.e. $\bar{h} \to 0$. We give assumptions on the integrand $f$ for having convergent, resp. Borel summable, expansions.

2. Definition of the generalized Fresnel integral

Let us consider a finite dimensional real Hilbert space $\mathcal{H}$, $\dim(\mathcal{H}) = N$, and let us identify it with $\mathbb{R}^N$. We will denote its elements by $x \in \mathbb{R}^N$, $x = (x_1, \ldots, x_N)$. We recall the definition of oscillatory integrals proposed by Hörmander [25,26].

**Definition 1.** Let $\Phi$ be a continuous real-valued function on $\mathbb{R}^N$. The oscillatory integral on $\mathbb{R}^N$, with $h \in \mathbb{R} \setminus \{0\}$,

$$\int_{\mathbb{R}^N} e^{i\Phi(x)} f(x) \, dx,$$

is well defined if for each test function $\phi \in \mathcal{S}(\mathbb{R}^N)$, such that $\phi(0) = 1$, the limit of the sequence of absolutely convergent integrals

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N} e^{i\Phi(x)} \phi(\varepsilon x) f(x) \, dx,$$
exists and is independent on \( \phi \). In this case the limit is denoted by
\[
\int_{\mathbb{R}^N} e^{i\Phi(x)} f(x) \, dx.
\]
If the same holds only for \( \phi \) such that \( \phi(0) = 1 \) and \( \phi \in \Sigma \), for some subset \( \Sigma \) of \( \mathcal{S}(\mathbb{R}^N) \), we say that the oscillatory integral exists in the \( \Sigma \)-sense and we shall denote it by the same symbol.

Let us consider the space \( \mathcal{M}(\mathbb{R}^N) \) of complex bounded variation measures on \( \mathbb{R}^N \) endowed with the total variation norm. \( \mathcal{M}(\mathbb{R}^N) \) is a Banach algebra, where the product of two measures \( \mu \ast \nu \) is by definition their convolution:
\[
\mu \ast \nu(E) = \int_{\mathbb{R}^N} \mu(E - x) \nu(dx), \quad \mu, \nu \in \mathcal{M}(\mathbb{R}^N)
\]
and the unit element is the Dirac measure \( \delta_0 \).

Let \( \mathcal{F}(\mathbb{R}^N) \) be the space of functions \( f: \mathbb{R}^N \to \mathbb{C} \) which are the Fourier transforms of complex bounded variation measures \( \mu_f \in \mathcal{M}(\mathbb{R}^N) \):
\[
f(x) = \int_{\mathbb{R}^N} e^{ik \cdot x} \mu_f(dk), \quad \mu_f \in \mathcal{M}(\mathbb{R}^N).
\]
If there exists a self-adjoint linear isomorphism \( T: \mathbb{R}^N \to \mathbb{R}^N \) such that the phase function \( \Phi \) is given by \( \Phi(x) = \langle x, Tx \rangle \) and \( f \in \mathcal{F}(\mathbb{R}^N) \), then the oscillatory integral
\[
\int_{\mathbb{R}^N} e^{i\bar{\hbar} \langle x, Tx \rangle} f(x) \, dx
\]
can be explicitly computed by means of the following Parseval-type formula [3,21]:
\[
\int_{\mathbb{R}^N} e^{\frac{i}{\hbar} \langle x, Tx \rangle} f(x) \, dx = (2\pi \hbar)^{N/2} e^{-\frac{\pi i}{4} \text{Ind}(T)} |\text{det}(T)|^{-1/2} \int_{\mathbb{R}^N} e^{-i\frac{\hbar}{\pi} \langle x, T^{-1} x \rangle} \mu_f(dx), \quad (3)
\]
where \( \text{Ind}(T) \) is the number of negative eigenvalues of the operator \( T \), counted with their multiplicity.

In the following we will generalize the latter result to more general phase functions \( \Phi \), in particular those given by an even polynomial \( P(x) \) in the variables \( x_1, \ldots, x_N \):
\[
P(x) = A_2 M(x, \ldots, x) + A_2 M - 1(x, \ldots, x) + \cdots + A_1(x) + A_0, \quad (4)
\]
where \( A_k \) are \( k \)th-order covariant tensors on \( \mathbb{R}^N \):
\[
A_k: \mathbb{R}^N \times \mathbb{R}^N \times \cdots \times \mathbb{R}^N \to \mathbb{R}, \quad \text{\( k \)-times}
\]
and the leading term, namely $A_{2M}(x, \ldots, x)$, is a $2M$th-order completely symmetric covariant tensor on $\mathbb{R}^N$. First of all, following the methods used by Hörmander [25, 26], we prove the existence of the following generalized Fresnel integral:

$$
\int_{\mathbb{R}^N} e^{i\Phi(x)} f(x) \, dx
$$

(5)

for suitable $\Phi$. We recall the definition of symbols (see [25]).

**Definition 2.** A $C^\infty$ map $f : \mathbb{R}^N \to \mathbb{C}$ belongs to the space of symbols $S^m_\lambda(\mathbb{R}^N)$, where $n, \lambda$ are two real numbers and $0 < \lambda \leq 1$, if for each $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}^N$ there exists a constant $C_\alpha \in \mathbb{R}$ such that

$$
\left| \frac{d^{\alpha_1}}{d x_1^{\alpha_1}} \cdots \frac{d^{\alpha_N}}{d x_N^{\alpha_N}} f \right| \leq C_\alpha (1 + |x|)^{n-\lambda|\alpha|}, \quad |x| \to \infty,
$$

(6)

where $|\alpha| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_N|$.

One can prove that $S^m_\lambda$ is a Fréchet space under the topology defined by taking as seminorms $|f|_{\alpha}$ the best constants $C_\alpha$ in (6) (see [25]). The space increases as $n$ increases and $\lambda$ decreases. If $f \in S^m_\lambda$ and $g \in S^m_\lambda$, then $fg \in S^{m+m}_\lambda$. We denote $\bigcup_\lambda S^m_\lambda$ by $S^m_\infty$. We shall see that $S^m_\infty$ is included in the class for which the generalized Fresnel integral (5) is well defined.

We say that a point $x = x_\epsilon \in \mathbb{R}^N$ is a critical point of the phase function $\Phi : \mathbb{R}^N \to \mathbb{R}$, $\Phi \in C^1$, if $\Phi'(x_\epsilon) = 0$. Let $\mathcal{C}(\Phi)$ be the set of critical points of $\Phi$. In fact we have:

**Theorem 1.** Let $\Phi$ be a real-valued $C^2$ function on $\mathbb{R}^N$ with the critical set $\mathcal{C}(\Phi)$ being finite. Let us assume that for each $N \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $\frac{1}{(\sqrt[N]{\Phi(x)})^k}$ is bounded for $|x| \to \infty$. Let $f \in S^0_N$, with $n, \lambda \in \mathbb{R}$, $0 < \lambda \leq 1$. Then the generalized Fresnel integral (5) exists for each $h \in \mathbb{R} \setminus \{0\}$.

**Proof.** We follow the method of Hörmander [25], see also [3,8,21].

Let us suppose that the phase function $\Phi(x)$ has $l$ stationary points $c_1, \ldots, c_l$, that is

$$
\nabla \Phi(c_i) = 0, \quad i = 1, \ldots, l.
$$

Let us choose a suitable partition of unity $1 = \sum_{i=0}^l \chi_i$, where $\chi_i$, $i = 1, \ldots, l$, are $C_0^\infty(\mathbb{R}^N)$ functions constant equal to 1 in an open ball centered in the stationary point $c_i$, respectively and $\chi_0 = 1 - \sum_{i=1}^l \chi_i$. Each of the integrals $I_i(f) \equiv \int_{\mathbb{R}^N} e^{i\Phi(x)} \chi_i(x) f(x) \, dx, \quad i = 1, \ldots, l$, is well defined in Lebesgue sense since $f \chi_i \in C_0(\mathbb{R}^N)$. Let $I_0 \equiv \int_{\mathbb{R}^N} e^{i\Phi(x)} \chi_0(x) f(x) \, dx$. To see that $I_0$ is a well defined oscillatory integral let us introduce the operator $L^+$ with domain $D(L^+)$ in $L^2(\mathbb{R}^N)$ given by

$$
L^+ g(x) = -ih \frac{\chi_0(x)}{\nabla \Phi(x)^2} \nabla \Phi(x) \nabla g(x),
$$

$$
g \in D(L^+) \equiv \left\{ g \in L^2(\mathbb{R}^N) \left| \frac{\chi_0(x)}{\nabla \Phi(x)^2} \nabla \Phi(x) \nabla g(x) \in L^2(\mathbb{R}^N) \right. \right\},
$$
while its adjoint in $L^2(\mathbb{R}^N)$ is given by

$$Lf(x) = i\hbar \frac{\chi_0(x)}{\lvert \nabla \Phi(x) \rvert^2} \nabla \Phi(x) \nabla f(x) + i\hbar \text{div} \left( \frac{\chi_0(x)}{\lvert \nabla \Phi(x) \rvert^2} \nabla \Phi(x) \right) f(x)$$

for $f \in L^2(\mathbb{R}^N) \cap C^\infty$ such that

$$\left\lvert \frac{f(x)g(x)\lvert x \rvert^N}{\lvert \nabla \Phi(x) \rvert^2} \nabla \Phi(x) \cdot x \right\rvert \to 0 \text{ as } \lvert x \rvert \to \infty, \forall g \in D(L^+) .$$

Let us choose $\psi \in S(\mathbb{R}^N)$, such that $\psi(0) = 1$. It is easy to see that if $f \in S^0_\lambda$ then $f_\varepsilon$, defined as $f_\varepsilon(x) := \psi(\varepsilon x) f(x)$, belongs to $S^{0+1}_\lambda \cap S(\mathbb{R}^N)$, for any $\varepsilon > 0$. By iterated application of the Stokes formula, we have:

$$\int_{\mathbb{R}^N} e^{i\Phi(x)} \psi(\varepsilon x) f(x) \chi_0(x) \, dx = \int_{\mathbb{R}^N} L^+(e^{i\Phi(x)}) \psi(\varepsilon x) f(x) \, dx$$

$$= \int_{\mathbb{R}^N} e^{i\Phi(x)} L f_\varepsilon(x) \, dx = \int_{\mathbb{R}^N} e^{i\Phi(x)} L^k f_\varepsilon(x) \, dx . \quad (7)$$

Now for $k$ sufficiently large the last integral is absolutely convergent and we can pass to the limit $\varepsilon \to 0$ by the Lebesgue dominated convergence theorem.

Considering $\sum_{i=0}^l I_i(f)$ we have, by the existence result proven for $I_0$ and the additivity property of oscillatory integrals, that $\int_{\mathbb{R}^N} e^{i\Phi(x)} f(x) \, dx$ is well defined and equal to $\sum_{i=0}^l I_i(f)$. □

**Remark 1.** If $C(\Phi)$ has countably many non accumulating points $\{x_i\}_{i \in \mathbb{N}}$, the same method yields $\int_{\mathbb{R}^N} e^{i\Phi(x)} f(x) \, dx = \sum_{i=0}^\infty I_i(f)$ provided this sum converges.

There are partial extensions of the above construction in the case of critical points which form a submanifold in $\mathbb{R}^N$ [20], or are degenerate [13], see also [18].

**Remark 2.** In particular we have proved the existence for $f \in S^0_\lambda$, $0 < \lambda \leq 1$, of the oscillatory integrals $\int_{\mathbb{R}^N} e^{iM(\Phi)} f(x) \, dx$, with $M$ arbitrary. For $M = 2$ one has the Fresnel integral of [8], for $M = 3$ one has Airy integrals [26].

**Remark 3.** If $\Phi$ is of the form (4), then the generalized Fresnel integral (5) also exists, even in Lebesgue sense, for $\hbar \in \mathbb{C}$ with $\text{Im}(\hbar) < 0$, as an analytic function in $\hbar$, as easily seen by the fact that the integrand is bounded by $\lvert f \rvert \exp(\text{Im}(\hbar) \Phi)$.

### 3. Generalized Parseval equality and analytic continuation

In this section we prove that, for a suitable class of functions $f : \mathbb{R}^N \to \mathbb{C}$ the generalized Fresnel integral (5) can be explicitly computed by means of a generalization of formula (3).
Lemma 1. Let $P : \mathbb{R}^N \to \mathbb{R}$ be given by (4). Then the Fourier transform of the distribution $e^{\frac{i}{\hbar} P(x)}$:
\[
\hat{F}(k) = \int_{\mathbb{R}^N} e^{ik \cdot x} e^{\frac{i}{\hbar} P(x)} \, dx, \quad \hbar \in \mathbb{R} \setminus \{0\}
\]  
(8)
is an entire bounded function and admits the following representation:
\[
\hat{F}(k) = e^{iN\pi/4M} \int_{\mathbb{R}^N} e^{i\varphi(x) k \cdot x} e^{\frac{i}{\hbar} P(x)} \, dx, \quad \hbar > 0
\]  
(9)
or
\[
\hat{F}(k) = e^{-iN\pi/4M} \int_{\mathbb{R}^N} e^{-i\varphi(x) k \cdot x} e^{\frac{i}{\hbar} P(x)} \, dx, \quad \hbar < 0.
\]  
(10)

Remark 4. The integral on the r.h.s. of (9) is absolutely convergent as
\[
e^{i\varphi(x) P(x)} = e^{-1/\hbar A_2 M(x, \ldots, x)} e^{i\varphi(x) (A_2 M - 1(x e^{i\pi/4M}, \ldots, x e^{i\pi/4M}) + \cdots + A_1 (x e^{i\pi/4M} + A_0))}.
\]
A similar calculation shows the absolute convergence of the integral on the r.h.s. of (10).

Proof of Lemma 1. Formulae (9) and (10) can be proved by using the analyticity of $e^{ikz + \frac{i}{\hbar} P(z)}, z \in \mathbb{C}$, and a change of integration contour (see Appendix A for more details).
Representations (9) and (10) show the analyticity properties of $\hat{F}(k), k \in \mathbb{C}$. By a study of the asymptotic behavior of $\hat{F}(k)$ as $|k| \to \infty$ we conclude that $\hat{F}$ is always bounded (see Appendix B for more details).

Remark 5. A representation similar to (9) holds also in the more general case $\hbar \in \mathbb{C}$, $\text{Im}(\hbar) < 0$, $\hbar \neq 0$. By setting $\hbar \equiv |\hbar| e^{i\phi}, \phi \in [-\pi, 0]$ one has:
\[
\hat{F}(k) = \int_{\mathbb{R}^N} e^{ik \cdot x} e^{\frac{i}{\hbar} P(x)} \, dx
\]  
\[
= e^{iN(\pi/4M + \phi/2M)} \int_{\mathbb{R}^N} e^{i\varphi(x) k \cdot x} e^{\frac{i}{\hbar} P(x)} \, dx
\]  
(11)
(see Appendix A for more details).

By mimicking the proof of Eq. (9) (Appendix A) one can prove in the case $\hbar > 0$ the following result (a similar one holds also in the case $\hbar < 0$):

Theorem 2. Let us denote by $\Lambda$ the subset of the complex plane
\[
\Lambda = \{\xi \in \mathbb{C} \mid 0 < \arg(\xi) < \pi/4M\} \subset \mathbb{C},
\]  
(12)
and let $\bar{\Lambda}$ be its closure. Let $f : \mathbb{R}^N \to \mathbb{C}$ be a Borel function defined for all $y$ of the form $y = \lambda x$, where $\lambda \in \bar{\Lambda}$ and $x \in \mathbb{R}^N$, with the following properties:
1. The function $\lambda \mapsto f(\lambda x)$ is analytic in $\Lambda$ and continuous in $\overline{\Lambda}$ for each $x \in \mathbb{R}^N$, $|x| = 1$.

2. For all $x \in \mathbb{R}^N$ and all $\theta \in (0, \pi/4M)$

   $$|f(e^{i\theta}x)| \leq AG(x).$$

where $A \in \mathbb{R}$ and $G : \mathbb{R}^N \rightarrow \mathbb{R}$ is a positive function satisfying bound (a) or (b) respectively:

(a) if $P$ is as in the general case defined by (4)

   $$G(x) \leq e^{B|x|^{2M-1}}, \quad B > 0;$$

(b) if $P$ is homogeneous, i.e. $P(x) = A_2M(x, \ldots, x)$

   $$G(x) \leq e^{-\frac{\bar{h}}{2}A_2M(x,x,\ldots,x)}g(|x|),$$

where $g(t) = O(t^{-(N+\delta)})$, $\delta > 0$, as $t \to \infty$.

Then the limit of regularized integrals:

$$\lim_{\varepsilon \to 0} \int e^{\bar{h}P(xe^{i\varepsilon})} f(xe^{i\varepsilon}) dx, \quad 0 < \varepsilon < \pi/4M, \quad \bar{h} > 0$$

is given by:

$$e^{iN\pi/4M} \int_{\mathbb{R}^N} e^{\bar{h}P(e^{i\pi/4M}x)} f(e^{i\pi/4M}x) dx.$$  \hspace{1cm} (13)

The latter integral is absolutely convergent and it is understood in Lebesgue sense.

The class of functions satisfying conditions (1) and (2) in Theorem 2 includes for instance the polynomials of any degree and the exponentials. In the case $f \in S^\omega_{\Lambda}$ for some $n, \lambda$, one is tempted to interpret expression (13) as an explicit formula for the evaluation of the generalized Fresnel integral $\int e^{\bar{h}P(x)} f(x) dx, \quad \bar{h} > 0$, whose existence is assured by Theorem 1. This is, however, not necessarily true for all $f \in S^\omega_{\Lambda}$ satisfying (1) and (2). Indeed the Definition 1 of oscillatory integral requires that the limit of the sequence of regularized integrals exists and is independent on the regularization. The identity

$$\lim_{\varepsilon \to 0} \int e^{\bar{h}P(x)} f(x) \psi(\varepsilon x) dx = e^{iN\pi/4M} \int_{\mathbb{R}^N} e^{\bar{h}P(e^{i\pi/4M}x)} f(e^{i\pi/4M}x) dx, \quad \bar{h} > 0$$

can be proven only by choosing regularizing functions $\psi$ with $\psi(0) = 1$ and $\psi$ in the class $\Sigma$ consisting of all $\psi \in S$ which satisfy (1) and are such that $|\psi(e^{i\theta}x)|$ is bounded as $|x| \to \infty$ for each $\theta \in (0, \pi/4M)$. In fact we will prove that expression (13) coincides with the oscillatory integral (5), i.e. one can take $\Sigma = S(\mathbb{R}^N)$, by imposing stronger assumptions on the function $f$. First of all we show that the representation (9) for the Fourier transform of $e^{\bar{h}P(x)}$ allows a generalization of Eq. (3). Let us denote by $\bar{D} \subset \mathbb{C}$ the lower semiplane in the complex plane

$$\bar{D} = \{z \in \mathbb{C} \mid \text{Im}(z) \leq 0\}. \hspace{1cm} (14)$$
Theorem 3. Let \( f \in \mathcal{F}(\mathbb{R}^N) \), \( f = \hat{\mu_f} \). Then the generalized Fresnel integral
\[
I(f) \equiv \int e^{ikP(x)} f(x) \, dx, \quad h \in \bar{D} \setminus \{0\}
\]
is well defined and it is given by the formula of Parseval’s type:
\[
\int e^{ikP(x)} f(x) \, dx = \int \tilde{F}(k) \mu_f(dk), \tag{15}
\]
where \( \tilde{F}(k) \) is given by (11) (see Lemma 1 and Remark 5)
\[
\tilde{F}(k) = \int e^{ikx} e^{i\bar{h}P(x)} \, dx.
\]
The integral on the r.h.s. of (15) is absolutely convergent (hence it can be understood in Lebesgue sense).

Proof. Let us choose a test function \( \psi \in \mathcal{S}(\mathbb{R}^N) \), such that \( \psi(0) = 1 \) and let us compute the limit
\[
I(f) \equiv \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N} e^{i\bar{h}P(x)} \psi(\varepsilon x) f(x) \, dx.
\]

By hypothesis \( f(x) = \int e^{ikx} \mu_f(dk) \) and substituting in the previous expression we get:
\[
I(f) = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N} e^{i\bar{h}P(x)} \psi(\varepsilon x) \left( \int_{\mathbb{R}^N} e^{ikx} \mu_f(dk) \right) \, dx.
\]

By Fubini theorem (which applies for any \( \varepsilon > 0 \) since the integrand is bounded by \( |\psi(\varepsilon x)| \) which is \( dx \)-integrable, and \( \mu_f \) is a bounded measure) the r.h.s. is
\[
= \lim_{\varepsilon \downarrow 0} \left( \int_{\mathbb{R}^N} e^{i\bar{h}P(x)} \psi(\varepsilon x) e^{ikx} \, dx \right) \mu_f(dk)
\]
\[
= \frac{1}{(2\pi)^N} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N} \tilde{F}(k - \alpha \varepsilon) \tilde{\psi}(\alpha) d\alpha \mu_f(dk) \tag{16}
\]
(here we have used the fact that the integral with respect to \( x \) is the Fourier transform of \( e^{i\bar{h}P(x)} \psi(\varepsilon x) \) and the inverse Fourier transform of a product is a convolution). Now we can pass to the limit using the Lebesgue bounded convergence theorem and get the desired result:
\[
\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N} e^{i\bar{h}P(x)} \psi(\varepsilon x) f(x) \, dx = \int_{\mathbb{R}^N} \tilde{F}(k) \mu_f(dk),
\]
where we have used that \( \int \tilde{\psi}(\alpha) d\alpha = (2\pi)^N \tilde{\psi}(0) \) and Lemma 1, which assures the boundedness of \( \tilde{F}(k) \). \( \square \)

Corollary 1. Let \( h = |h| e^{i\phi}, \phi \in [-\pi, 0], h \neq 0, f \in \mathcal{F}(\mathbb{R}^N), f = \hat{\mu_f} \) such that \( \forall x \in \mathbb{R}^N \)
\[
\int_{\mathbb{R}^N} e^{-kx \sin(\pi/4M + \phi/2M)} |\mu_f|(dk) \leq AG(x), \tag{17}
\]
where \( A \in \mathbb{R} \) and \( G : \mathbb{R}^N \to \mathbb{R} \) is a positive function satisfying bound (1) or (2) respectively:

1. if \( P \) is defined by (4),
   \[
   G(x) \leq e^{B|x|^{2M-1}}, \quad B > 0,
   \]
2. if \( P \) is homogeneous, i.e. \( P(x) = A_{2M}(x, \ldots, x) \):
   \[
   G(x) \leq e^{\bar{A}_{2M}(x, x, \ldots, x) g(|x|)},
   \]
   where \( g(t) = O(t^{-(N+\delta)}) \), \( \delta > 0 \), as \( t \to \infty \).

Then \( f \) extends to an analytic function on \( \mathbb{C}^N \) and its generalized Fresnel integral (5) is well defined and it is given by

\[
\int_{\mathbb{R}^N} e^{i\bar{P}(x) f(x)} dx = e^{iN(\pi/4M+\phi/2M)} \int_{\mathbb{R}^N} e^{i\tilde{P}(e^{i(\pi/4M+\phi/2M)x}) f(e^{i(\pi/4M+\phi/2M)x})} dx.
\]

**Proof.** By bound (17) it follows that the Laplace transform \( f^L : \mathbb{C}^N \to \mathbb{C} \), \( f^L(z) = \int_{\mathbb{R}^N} e^{ikx} f(dx) \), of \( f^L \) is a well defined entire function such that, for \( x \in \mathbb{R}^N \), \( f^L(ix) = f(x) \).

By Theorem 3 the generalized Fresnel integral can be computed by means of the Parseval type equality

\[
\int_{\mathbb{R}^N} e^{i\bar{P}(x) f(x)} dx = \int_{\mathbb{R}^N} \tilde{F}(k) \mu_f (dk) \\
= e^{iN(\pi/4M+\phi/2M)} \left( \int_{\mathbb{R}^N} e^{i k x e^{i(\pi/4M+\phi/2M)x}} e^{i\tilde{P}(e^{i(\pi/4M+\phi/2M)x})} \mu_f (dk) \right). 
\]

By Fubini theorem, which applies given the assumptions on the measure \( \mu_f \), this is equal to

\[
e^{iN(\pi/4M+\phi/2M)} \int_{\mathbb{R}^N} e^{i\tilde{P}(e^{i(\pi/4M+\phi/2M)x})} \mu_f (dk) dx \\
= e^{iN(\pi/4M+\phi/2M)} \int_{\mathbb{R}^N} e^{i\tilde{P}(e^{i(\pi/4M+\phi/2M)x})} f^L(i e^{i(\pi/4M+\phi/2M)x}) dx \\
= e^{iN(\pi/4M+\phi/2M)} \int_{\mathbb{R}^N} e^{i\tilde{P}(e^{i(\pi/4M+\phi/2M)x})} f( e^{i(\pi/4M+\phi/2M)x}) dx
\]

and the conclusion follows. \( \square \)
4. Asymptotic expansion

In this section we study the asymptotic expansion of the generalized Fresnel integrals (5) in the particular case where the phase function $\Phi(x)$ is homogeneous and strictly positive:

$$\phi(x) = A_{2M}(x, \ldots, x),$$

where $A_{2M} : \mathbb{R}^N \times \mathbb{R}^N \times \cdots \times \mathbb{R}^N \to \mathbb{R}$ is a completely symmetric strictly positive $2M$th-order covariant tensor on $\mathbb{R}^N$. Under suitable assumptions on the function $f$, we prove either the convergence or the Borel summability of the asymptotic expansion. In the general case one would have to consider the type of degeneracy of the phase function, cf. [3,8,13,20]. We leave the investigation of the corresponding expansions in our setting for a further publication.

Let us assume first of all $N = 1$ and study the asymptotic behavior of the integral:

$$\int_{-\infty}^{\infty} e^{i \bar{h} x} f(x) dx, \quad \bar{h} \in \bar{D} \setminus \{0\}.$$

Theorem 4. Let us consider a function $f \in \mathcal{F}(\mathbb{R})$, which is the Fourier transform of a bounded variation measure $\mu_f$ on the real line satisfying the following bounds for all $l \in \mathbb{N}, \rho \in \mathbb{R}^+$, $\bar{h} \in \bar{D} \setminus \{0\}$:

1. \[ \int |k|^{2l} |e^{ik \bar{h}^{1/2M} \rho^{i/2M}} + e^{-ik \bar{h}^{1/2M} \rho^{i/2M}}| \mu_f(dk) \leq F(l) g(\rho) e^{c|x|^{2M} - 1}, \]

2. \[ \left| \int k^{2l} (e^{ik \bar{h}^{1/2M} \rho^{i/2M}} + e^{-ik \bar{h}^{1/2M} \rho^{i/2M}}) \mu_f(dk) \right| \leq A c^l C(l, M), \]

where $A, c, C(l, M) \in \mathbb{R}$.

Then the generalized Fresnel integral

$$I(h) \equiv \int_{\mathbb{R}} e^{i \bar{h}^{1/2M} f(x)} dx, \quad \bar{h} \in \bar{D} \setminus \{0\}$$

(with $\bar{D}$ given by (14)) admits the following asymptotic expansion in powers of $h^{1/M}$:

$$I(h) = e^{i \frac{\pi}{4} h^{1/2M} \rho^{i/2M}} \sum_{j=0}^{n-1} e^{j \frac{i \pi}{2M}} h^{j/M} \Gamma \left( \frac{1 + 2j}{2M} \right) f(j)(0) + R_n(h)$$

(18)

with $|R_n(h)| \leq \frac{|h|^{1/2M} \rho^{i/2M} C(n, M)}{2M} \Gamma \left( \frac{1 + 2n}{2M} \right)$ (where $A, c, C(n, M)$ are the constants in (2)). If the constant $C(n, M)$ satisfies the bound

$$C(n, M) \leq (2n)! \Gamma \left( \frac{1 + 2n}{2M} \right)^{-1}, \quad \forall n \in \mathbb{N}$$

(19)
then the series given by (18) for \( n \to \infty \) has a positive radius of convergence, while if
\[
C(n, M) \leq (2n)!\Gamma\left(1 + \frac{n}{M}\right)\Gamma\left(1 + \frac{2n}{2M}\right)^{-1}, \quad \forall n \in \mathbb{N}
\] (20)
then the expansion (18) is Borel summable in the sense of, e.g., [23,37] and determines \( I(\bar{h}) \) uniquely.

Moreover if \( f \in \mathcal{F}(\mathbb{R}) \) instead of (1), (2) satisfies the following “moment condition”:
\[
\int |a|^l |\mu_f|(da) \leq C'(l, M)A^l, \quad A, c \in \mathbb{R}
\] (21)
for all \( l \in \mathbb{N} \), where \( C'(l, M) \sim \Gamma\left(l(1 - \frac{1}{2M})\right) \) as \( l \to \infty \) (where \( \sim \) means that the quotient of the two sides converges to 1 as \( l \to \infty \)), then the asymptotic expansion (18) has a finite radius of convergence.

**Proof.** First of all we recall that the integral \( \int e^{ix^2M/h} f(x) \, dx \) is a well defined convergent integral also for all \( \bar{h} \in \mathbb{C} \) with \( \text{Im}(\bar{h}) < 0 \), thanks to the exponential decay of \( e^{ix^2M/h} \) and to the boundedness of \( f \) (cf. Remark 3). Moreover it is an analytic function of the variable \( \bar{h} \in \mathbb{C} \) in the domain \( \text{Im}(\bar{h}) < 0 \) as one can directly verify the Cauchy–Riemann conditions.

Let us compute the asymptotic expansion of this integral, considered as a function of \( \bar{h} \in \mathbb{C} \), valid for \( \bar{h} \in \mathbb{D} \setminus \{0\} \).

By formula (15) we have
\[
\int_{\mathbb{R}} e^{\bar{x}x^2M} f(x) \, dx = h^{1/2M} \int \tilde{F}_{2M}(h^{1/2M}k) \mu_f(dk),
\] (22)
where, if \( h = |h|e^{i\phi} \), \( \phi \in [-\pi, 0] \), \( h^{1/2M} = |h|^{1/2M}e^{i\phi/2M} \) and \( \tilde{F}_{2M}(k) = \int_{\mathbb{R}} e^{ikx} e^{x^2M} \, dx \), which, for Lemma 1, is equal to \( \tilde{F}_{2M} = e^{i\pi} \int_{\mathbb{R}} e^{ikx} e^{ix^2M} \, dx \). Such a representation assures the analyticity of \( \tilde{F}_{2M}(k) \). We can now expand \( \tilde{F}_{2M}(h^{1/2M}k) \) in a convergent power series in \( h^{1/2M}k \) around \( h = 0 \):
\[
F_{2M}(h^{1/2M}k) = \sum_{n=0}^{\infty} \frac{F_{2M}^{(n)}(0)}{n!} h^{n/2M}k^n.
\]
The \( n \)th-derivative of \( F_{2M} \) can be explicitly evaluated by means of the representation (9):
\[
\tilde{F}_{2M}(0) = e^{i(n+1)\pi/4M}(i)^n(1+(-1)^n) \int_0^\infty \rho^n e^{-\rho^2M} \, d\rho
\]
that is \( F^{(n)}(0) = 0 \) if \( n \) is odd, while if \( n \) is even we have
\[
\tilde{F}_{2M}(0) = 2e^{i(2j+1)\pi/2M}(-1)^j \int_0^\infty \rho^{2j} e^{-\rho^2M} \, d\rho.
\]
By means of a change of variables one can compute the latter integral explicitly:
\[
\int_0^\infty \rho^{2j} e^{-\rho^2M} \, d\rho = \frac{1}{2M} \int_0^\infty e^{-\rho^2} \rho^{1+2j/2M} \, d\rho = \frac{1}{2M} \Gamma\left(1 + \frac{1+2j}{2M}\right).
\]
By substituting into (22) we get:

$$I(\bar{h}) = \frac{h^{1/2M} n!}{\Gamma(\frac{1+2j}{2M})} \frac{1}{(2j)!} \left( -1 \right) ^j \int \frac{\mu_f (dk)}{\int (k)^j} + R_n,$$

where

$$R_n = \frac{h^{1/2M} n!}{\Gamma(\frac{1+2j}{2M})} \frac{1}{(2j)!} \left( -1 \right) ^j \int \frac{\mu_f (dk)}{\int (k)^j}.$$

If assumption (21) is satisfied, one can verify by means of Stirling formula that the series (23) of powers of $\bar{h}^{1/2M}$ has a finite radius of convergence.

In the more general case in which assumptions (1), (2) are satisfied, we can nevertheless prove a suitable estimate for $R_n$, indeed:

$$R_n = 2h^{1/2M} e^{\pi/2M} \int \sum_{j \geq n} (-1)^j \frac{1}{2j!} e^{(2j+1)\pi i/2M} \int \frac{e^{2j\rho^2} k^{2j} \mu_f (dk)}{e^{2j\rho^2} k^{2j} \mu_f (dk)}.$$

By Fubini theorem and assumptions (1) and (2) we get the uniform estimate in $\bar{h}$:

$$|R_n| \leq \left| \frac{h^{1/2M} n!}{\Gamma(\frac{1+2j}{2M})} \frac{1}{(2j)!} \left( -1 \right) ^j \int \frac{\mu_f (dk)}{\int (k)^j} \right| \frac{h^n}{M}.$$

If assumption (19) is satisfied, then the latter becomes

$$|R_n| \leq \left| \frac{h^{1/2M} n!}{\Gamma(\frac{1+2j}{2M})} \frac{1}{(2j)!} \left( -1 \right) ^j \int \frac{\mu_f (dk)}{\int (k)^j} \right| \frac{h^n}{M},$$

and the series has a positive radius of convergence, while if assumption (20) holds, we get the estimate

$$|R_n| \leq \left| \frac{h^{1/2M} n!}{\Gamma(\frac{1+2j}{2M})} \frac{1}{(2j)!} \left( -1 \right) ^j \int \frac{\mu_f (dk)}{\int (k)^j} \right| \frac{h^n}{M}.$$

This and the analyticity of $I(h)$ in $\Im(h) < 0$ by Nevanlinna theorem [37] assure the Borel summability of the power series expansion (18). □

These results can be easily generalized to the study of $N$-dimensional oscillatory integrals:

$$I_N(h) = \int_{\mathbb{R}^N} e^{i \frac{A2M(x,...,x)}{2M}} f(x) \, dx, \quad h \in \mathbb{D} \setminus \{0\}$$

(25)
with $A_{2M}$ a completely symmetric $2M$th-order covariant tensor on $\mathbb{R}^N$ such that $A_{2M}(x, \ldots, x) > 0$ unless $x = 0$.

**Theorem 5.** Let $f \in \mathcal{F}(\mathbb{R}^N)$ be the Fourier transform of a bounded variation measure $\mu_f$ admitting moments of all orders.

Let us suppose $f$ satisfies the following conditions, for all $l \in \mathbb{N}$:

1. $\int_{\mathbb{R}^N} |kx|^{l} e^{-k|x|} |\mu_f|(dk) \leq F(l) g(|x|) e^{c|\frac{x}{2M}|^{2M-1}}$, $\forall x \in \mathbb{R}^N$,

where $c \in \mathbb{R}$, $F(l)$ is a positive constant depending on $l$, and $g : \mathbb{R}^+ \to \mathbb{R}$ is a positive function with polynomial growth;

2. $\int_{\mathbb{R}^N} (ku)^l e^{i\frac{1}{2M}h^{1/2M}e^{iM/4M} \mu_f (dk)} \leq A c^l C(l, M, N)$

for all $u \in S_{N-1}$, $\rho \in \mathbb{R}^+$, $h \in \overline{D} \setminus \{0\}$, where $A, c, C(l, M, N) \in \mathbb{R}$ (and $S_{N-1}$ is the $(N - 1)$-spherical hypersurface);

then the oscillatory integral (25) admits (for $h \in \overline{D} \setminus \{0\}$) the following asymptotic expansion in powers of $h^{1/2M}$:

$$I_N(h) = h^{\frac{N}{2M}} e^{i\frac{N\pi}{4M}} \frac{n-1}{2M!} \sum_{i=0}^{n-1} \frac{i^l}{i!} (e^{i\pi/4M})^i h^{1/2M} \Gamma\left(\frac{l + N}{2M}\right)$$

$$\times \int_{\mathbb{R}^N} \int_{S_{N-1}} (ku)^l P(u) \frac{d\Omega_{N-1}}{d\Omega} \mu_f (dk) + R_n,$$

(26)

with $|R_n| \leq A' |h|^n/2M (c')^n C(n, M, N) n^{-1} \Gamma(n+M/N)$ where $A', c' \in \mathbb{R}$ are suitable constants and $C(n, M, N)$ is the constant in (2). If $C(n, M, N)$ satisfies the following bound:

$$C(n, M, N) \leq n! \Gamma\left(\frac{n + N}{2M}\right)^{-1}$$

(27)

then the series has a positive radius of convergence, while if

$$C(n, M, N) \leq n! \Gamma\left(\frac{n + N}{2M}\right) \Gamma\left(\frac{n + N}{2M}\right)^{-1}$$

(28)

then the expansion is Borel summable in the sense of, e.g. [23,37] and determines $I(h)$ uniquely.

Moreover if $f \in \mathcal{F}(\mathbb{R}^N)$ instead of (1) and (2) satisfies the following moment condition:

$$\int_{\mathbb{R}^N} |a|^l |\mu_f|(d\alpha) \leq C'(l, M) A c^l,$$

(29)

for all $l \in \mathbb{N}$, where $C'(l, M) \sim \Gamma(l(1 - \frac{1}{2M}))$ as $l \to \infty$, then the asymptotic expansion has a finite radius of convergence.
Proof. Let \( \tilde{F}(k) \equiv \int_{\mathbb{R}^N} e^{ikx} e^{iA_2M(x,...,x)} \, dx \), then by Theorem 3 the oscillatory integral (25) is given by:

\[
\int_{\mathbb{R}^N} e^{iA_2M(x,...,x)} f(x) \, dx = h^{N/2M} \int_{\mathbb{R}^N} \tilde{F}(h^{1/2M} k) \, \mu_f (dk).
\]

(30)

By Lemma 1 \( \tilde{F} \) is given by

\[
\tilde{F}(\bar{h}^{1/2M} k) = e^{iN\pi/4M} \int_{\mathbb{R}^N} e^{i(h^{1/2M} k x) e^{i\phi_M}} e^{-\rho^2 A_2M(x,...,x)} \, dx
\]

where, if \( \bar{h} = |\bar{h}| e^{i\phi} \), \( \phi \in [-\pi, 0] \), \( h^{1/2M} = |h^{1/2M} e^{i\phi/2M} \). By representing the latter absolutely convergent integral using polar coordinates in \( \mathbb{R}^N \) we get:

\[
\tilde{F}(\bar{h}^{1/2M} k) = e^{iN\pi/4M} \int_{S_{N-1}} \int_{0}^{\infty} e^{i(h^{1/2M} e^{i\phi_M} \rho k u e^{-\rho^2 A_2M(u,...,u)} \rho^{N-1} d\rho d\Omega_{N-1}}
\]

where \( d\Omega_{N-1} \) is the measure on the \((N-1)\)-dimensional spherical hypersurface \( S_{N-1} \), \( x = \rho u, \rho = |x|, u \in S_{N-1} \) is a unitary vector. We can expand the latter integral in a power series of \( \bar{h}^{1/2M} \):

\[
\tilde{F}(\bar{h}^{1/2M} k) = e^{iN\pi/4M} \int_{S_{N-1}} \int_{0}^{\infty} \sum_{l=0}^{\infty} \frac{(i)^l}{l!} (e^{i\pi/4M})^l h^{l/2M} \rho^l (ku)^l
\]

\[
\times e^{-\rho^2 A_2M(u,...,u)} \rho^{N-1} d\rho d\Omega_{N-1}
\]

\[
= e^{iN\pi/4M} \sum_{l=0}^{\infty} \frac{(i)^l}{l!} (e^{i\pi/4M})^l h^{l/2M} \int_{S_{N-1}} (ku)^l
\]

\[
\times \int_{0}^{\infty} \rho^{l+N-1} e^{-\rho^2 A_2M(u,...,u)} d\rho d\Omega_{N-1}
\]

\[
= \frac{e^{iN\pi/4M}}{2M} \sum_{l=0}^{\infty} \frac{(i)^l}{l!} (e^{i\pi/4M})^l h^{l/2M} \Gamma\left(\frac{l + N}{2M}\right)
\]

\[
\times \int_{S_{N-1}} (ku)^l P(u) \frac{1}{4\pi^l} d\Omega_{N-1},
\]

(31)

where \( P(u) \equiv A_2M(u,...,u) \) is a strictly positive continuous function on the compact set \( S_{N-1} \), so that it admits an absolute minimum denoted by \( m \). This gives

\[
\left| \int_{S_{N-1}} (ku)^l P(u) \frac{1}{4\pi^l} d\Omega_{N-1} \right| \leq |k|^l m^{-\frac{lN}{2M}} \Omega_{N-1}(S_{N-1})
\]

\[
= |k|^l m^{-\frac{lN}{2M}} 2\pi^{N/2} \Gamma\left(\frac{N}{2}\right)^{-1}.
\]

(32)
The latter inequality and the Stirling formula assure the absolute convergence of the series (31). We can now insert this formula into (30) and get:

\[
\int_{\mathbb{R}^n} e^{x A_2 M(x, \ldots, x)} f(x) \, dx = h^{N/2} e^{k^n} \sum_{l=0}^{n-1} \frac{(i)^l}{l!} (e^{i\pi/4M} h^{l/2} \Gamma\left(\frac{l+N}{2M}\right)) \int \int (ku)^l P(u) \frac{j+1}{2M} d\Omega_{N-1} \mu_f(dk) + R_n.
\]

(33)

By estimate (32) and Stirling’s formula one can easily verify that if assumption (29) is satisfied, then the latter series in powers of \(h^{1/2M}\) has a strictly positive radius of convergence.

Eq. (33) can also be written in the following form:

\[
\int_{\mathbb{R}^n} e^{x A_2 M(x, \ldots, x)} f(x) \, dx = h^{N/2} e^{k^n} \sum_{l=0}^{n-1} \frac{1}{l!} (e^{i\pi/4M} h^{l/2} \Gamma\left(\frac{l+N}{2M}\right)) \int \int P(u) \frac{j+1}{2M} \frac{\partial^l}{\partial u^l} f(0) d\Omega_{N-1} + R_n,
\]

(34)

where \(\frac{\partial^l}{\partial u^l} f(0)\) denotes the \(l\)th partial derivative of \(f\) at 0 in the direction \(u\), and

\[
R_n = h^{N/2} e^{k^n} \int \int \int_{\mathbb{R}^n} \int_{S_{N-1}} \sum_{l=0}^{n-1} \frac{(i)^l}{l!} (e^{i\pi/4M} h^{l/2} \rho^l (ku)^l) \times e^{-\rho^{2M} A_2 M(u, \ldots, u)} \rho^{N-1} d\rho d\Omega_{N-1} \mu_f(dk).
\]

(35)

In the more general case in which assumptions (1) and (2) are satisfied we can prove the asymptoticity of the expansion (33), indeed

\[
R_n = h^{N/2} e^{k^n} \frac{(i)^n}{n!} (e^{i\pi/4M} h^{n/2} \Gamma\left(\frac{n+N}{2M}\right)) \int \int \int_{\mathbb{R}^n} \int_{S_{N-1}} (1-t)^{n-1} \times e^{i\kappa ut h^{1/2M} e^{n/4M}} e^{-\rho^{2M} A_2 M(u, \ldots, u)} (ku)^n \rho^{n+N-1} dt d\rho d\Omega_{N-1} \mu_f(dk).
\]

(36)

By assumptions (1), (2) and Fubini theorem the latter is bounded by

\[
|R_n| \leq \frac{A}{M} \pi^{N/2} \Gamma\left(\frac{N}{2}\right)^{-1} |h|^{(n+N)/2} e^{\rho^2} \frac{C(n, M, N)}{n!} \Gamma\left(\frac{n+N}{2M}\right).
\]

If assumption (27) is satisfied, then the latter becomes

\[
|R_n| \leq \frac{A}{M} \pi^{N/2} \Gamma\left(\frac{N}{2}\right)^{-1} |h|^{(n+N)/2} e^{\rho^2} \frac{n+N}{3M} \Gamma\left(\frac{n+N}{2M}\right)
\]

and the series has a positive radius of convergence, while if assumption (28) holds, we get the estimate

\[
|R_n| \leq \frac{A}{M} \pi^{N/2} \Gamma\left(\frac{N}{2}\right)^{-1} |h|^{(n+N)/2} e^{\rho^2} \frac{n+N}{3M} \Gamma\left(\frac{1+n}{2M}\right).
\]
This and the analyticity of the $I_N(h)$ in $\text{Im}(h) < 0$ (cf. Remark 3) by Nevanlinna theorem [37] (see also [39]) assure the Borel summability of the power series expansion (18).

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Appendix A. The Fourier transform of $e^{ihP(x)}$

Let us denote $D$ the region of the complex plane:

$$D \subset \mathbb{C}, \quad D \equiv \{ z \in \mathbb{C} \mid \text{Im}(z) < 0 \}.$$  

Let us assume $\bar{h}$ is a complex variable belonging to the region $\bar{D} \setminus \{0\}$. We are going to compute the Fourier transform of $e^{ihP(x)}$.

Let us introduce the polar coordinates in $\mathbb{R}^N$:

$$\int_{\mathbb{R}^N} e^{ik \cdot x} e^{ihP(x)} \, dx = \int_{S_{N-1}} \left( \int_{0}^{+\infty} e^{i|k|rf(\phi_1,\ldots,\phi_{N-1})} e^{i\bar{h}P(\phi_1,\ldots,\phi_{N-1})(r)} r^{N-1} \, dr \right) d\Omega_{N-1},$$  \tag{A.1}

where instead of $N$ cartesian coordinates we use $N - 1$ angular coordinates $(\phi_1, \ldots, \phi_{N-1})$ and the variable $r = |x|$. $S_{N-1}$ denotes the $(N-1)$-dimensional spherical surface, $d\Omega_{N-1}$ is the measure on it, $P(\phi_1,\ldots,\phi_{N-1})(r)$ is a $2M$th order polynomial in the variable $r$ with coefficients depending on the $N - 1$ angular variables $(\phi_1, \ldots, \phi_{N-1})$, namely:

$$P(x) = r^{2M} A_{2M} \left( \frac{x}{|x|}, \ldots, \frac{x}{|x|} \right) + r^{2M-1} A_{2M-1} \left( \frac{x}{|x|}, \ldots, \frac{x}{|x|} \right) + \cdots$$

$$+ r A_1 \left( \frac{x}{|x|} \right) + A_0$$

$$= a_{2M}(\phi_1, \ldots, \phi_{N-1}) r^{2M} + a_{2M-1}(\phi_1, \ldots, \phi_{N-1}) r^{2M-1} + \cdots$$

$$+ a_1(\phi_1, \ldots, \phi_{N-1}) r + a_0$$

$$= P(\phi_1, \ldots, \phi_{N-1})(r),$$  \tag{A.2}

where $a_{2M}(\phi_1, \ldots, \phi_{N-1}) > 0$ for all $(\phi_1, \ldots, \phi_{N-1}) \in S_{N-1}.$
Let us focus on the integral
\[ \int_{0}^{+\infty} e^{ikr} e^{i|k|^r} P(\phi_1, \ldots, \phi_{N-1}) r^{N-1} dr, \]  
(A.3)

which can be interpreted as the Fourier transform of the distribution on the real line
\[ F(r) = \Theta(r) r^{N-1} e^{i\bar{h}P(\phi_1, \ldots, \phi_{N-1})}, \]
with \( \Theta(r) = 1 \) for \( r \geq 0 \) and \( \Theta(r) = 0 \) for \( r < 0 \). Let us introduce the notation \( k' \equiv kf(\phi_1, \ldots, \phi_{N-1}), a_k \equiv a_k(\phi_1, \ldots, \phi_{N-1}), k = 0, \ldots, 2M, \ P'(r) = \sum_{k=0}^{2M} a_k r^k \) and \( h \in \mathbb{C}, \bar{h} = |h|e^{i\phi}, \) with \(-\pi \leq \phi \leq 0\).

Let us consider the complex plane and set \( z = \rho e^{i\theta} \). If \( \text{Im}(\bar{h}) < 0 \) the integral (A.3) is absolutely convergent, while if \( \bar{h} \in \mathbb{R} \setminus \{0\} \) it needs a regularization. If \( \bar{h} \in \mathbb{R}, \bar{h} > 0 \) we have
\[ \int_{0}^{+\infty} e^{ik'\rho} e^{i\bar{h}P'(\rho)} \rho^{N-1} d\rho = \lim_{\varepsilon \downarrow 0} \int_{\gamma_1(\varepsilon)} e^{ik'\rho} e^{i\bar{h}P'(\rho)} \rho^{N-1} d\rho, \]  
(A.4)
while if \( \bar{h} < 0 \)
\[ \int_{0}^{+\infty} e^{ik'\rho} e^{i\bar{h}P'(\rho)} \rho^{N-1} d\rho = \lim_{\varepsilon \downarrow 0} \int_{\gamma_2(\varepsilon)} e^{ik'\rho} e^{i\bar{h}P'(\rho)} \rho^{N-1} d\rho. \]  
(A.5)

We deal first of all with the case \( \bar{h} \in \mathbb{R}, \bar{h} > 0 \) (the case \( \bar{h} < 0 \) can be handled in a completely similar way). Let
\[ \gamma_1(R) = \{z \in \mathbb{C} | 0 \leq \rho \leq R, \theta = \varepsilon\}, \]
\[ \gamma_2(R) = \{z \in \mathbb{C} | \rho = R, \varepsilon \leq \theta \leq \pi/4M\}, \]
\[ \gamma_3(R) = \{z \in \mathbb{C} | 0 \leq \rho \leq R, \theta = \pi/4M\}. \]
From the analyticity of the integrand and the Cauchy theorem we have
\[ \int_{\gamma_1(R) \cup \gamma_2(R) \cup \gamma_3(R)} e^{ik'\rho} e^{i\bar{h}P'(\rho)} \rho^{N-1} d\rho = 0. \]
In particular:
\[ \left| \int_{\gamma_2(R)} e^{ik'\rho} e^{i\bar{h}P'(\rho)} \rho^{N-1} d\rho \right| = R^N \left| \int_{\varepsilon}^{\pi/4M} e^{ik'\rho e^{i\theta}} e^{i\bar{h}P'(Re^{i\theta})} e^{iN\theta} d\theta \right| \]
\[ \leq R^N \int_{\varepsilon}^{\pi/4M} e^{-k'\rho \sin(\theta)} e^{-\frac{1}{\pi} \sum_{k=0}^{2M} a_k R^k \sin(k\theta)} d\theta \]
\[ \leq R^N \int_{\varepsilon}^{\pi/4M} e^{-k''\rho \theta} e^{-a_{2M} R^{2M} \theta} e^{-\frac{1}{\pi} \sum_{k=0}^{2M-1} a_k R^k \theta} d\theta, \]  
(A.6)
where $k''$, $a'_k$, $k = 1, \ldots, 2M - 1$, are suitable constants. We have used the fact that if $\alpha \in [0, \pi/2]$ then $\frac{\alpha}{\pi} \leq \sin(\alpha) \leq \alpha$. The latter integral can be explicitly computed and gives:

$$R^N \left( e^{-\frac{(a'_2M)R^N + \sum_{k=1}^{2M-1} a'_k R_k)}{4M \pi^2} - e^{-\frac{(a'_2M)R^N + \sum_{k=1}^{2M-1} a'_k R_k)}{4M \pi^2}} \right),$$
which converges to 0 as $R \to \infty$. We get

$$\int_{z=\rho e^{i\varepsilon}}^{+\infty} e^{ikz} e^\bar{h} P'(z) z^{N-1} dz = \int_{z=\rho e^{i(\pi/4)M}}^{+\infty} e^{ikz} e^\bar{h} P'(z) z^{N-1} dz.$$

By taking the limit as $\varepsilon \downarrow 0$ of both sides one gets:

$$\int_0^{+\infty} e^{ikr} e^\bar{h} P'(r) r^{N-1} dr = e^{iN\pi/4M} \int_0^{+\infty} e^{ik\rho e^{i(\pi/4)M}} e^\bar{h} P'(\rho e^{i(\pi/4)M}) \rho^{N-1} d\rho.$$

By substituting into (A.1) we get the final result:

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ik \cdot x} e^\bar{h} P(x) + e^{iN\pi/4M} \int_{\mathbb{R}^N} e^{i\rho e^{i(\pi/4)M}} e^\bar{h} P'(\rho e^{i(\pi/4)M}) x dx. \quad (A.7)$$

In the case $\bar{h} < 0$ an analogous reasoning gives:

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ik \cdot x} e^\bar{h} P(x) + e^{-iN\pi/4M} \int_{\mathbb{R}^N} e^{i\rho e^{i(\pi/4)M}} e^\bar{h} P'(\rho e^{i(\pi/4)M}) x dx. \quad (A.8)$$

The analyticity of $\tilde{F}(k)$ is trivial in the case $\text{Im}(\bar{h}) < 0$, and follows from Eqs. (A.7) and (A.8) when $\bar{h} \in \mathbb{R} \setminus \{0\}$.

If $\text{Im}(\bar{h}) < 0$ a representation of type (A.7) still holds. By setting $\bar{h} = |\bar{h}| e^{i\phi}$, with $-\pi \leq \phi \leq 0$ and by deforming the integration contour in the complex $z$ plane, one gets

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ik \cdot x} e^\bar{h} P(x) + e^{iN(\pi/4M + \phi/2M)} \int_{\mathbb{R}^N} e^{i\rho e^{i(\pi/4M + \phi/2M)}} e^\bar{h} P'(\rho e^{i(\pi/4M + \phi/2M)}) x dx. \quad (A.9)$$

**Appendix B. The boundedness of $\tilde{F}(k)$ as $|k| \to \infty$**

Let us consider the distribution $e^\bar{h} P(x)$ and its Fourier transform

$$\tilde{F}(k) = \int_{\mathbb{R}^N} e^{ik \cdot x} e^\bar{h} P(x) dx.$$
Let us focus on the case $h \in \mathbb{R} \setminus \{0\}$ (in the case $\text{Im}(h) < 0 |\tilde{F}|$ is trivially bounded by $\int_{\mathbb{R}^N} |e^{ikx}e^{P(x)}| \, dx = \int_{\mathbb{R}^N} e^{\frac{ikx}{|h|}P(x)} \, dx < +\infty$). Let us assume for notation simplicity that $h = 1$, the general case can be handled in a completely similar way. In order to study $\int_{\mathbb{R}^N} e^{ikx}e^{P(x)} \, dx$ one has to introduce a suitable regularization. Chosen $\psi \in \mathcal{S}(\mathbb{R}^N)$, such that $\psi(0) = 1$ we have

$$e^{iP(x)} \psi(\varepsilon x) \to e^{iP(x)} , \quad \text{in } \mathcal{S}'(\mathbb{R}^N) \text{ as } \varepsilon \to 0,$$

$$\tilde{F}(k) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} e^{ikx}e^{iP(x)} \psi(\varepsilon x) \, dx .$$

Let us consider first of all the case $N = 1$ and $P(x) = x^2M/2m$. The unique real stationary point of the phase function $\Phi(x) = kx + x^2M$ is $c_k = -k \frac{2}{2M}$ . Let $\chi_1$ be a positive $C^\infty$ function such that $\chi_1(x) = 1$ if $|x - c_k| \leq 1/2$ , $\chi_1(x) = 0$ if $|x - c_k| \geq 1$ and $0 \leq \chi_1(x) \leq 1$ if $1/2 \leq |x - c_k| \leq 1$. Let $\chi_0 \equiv 1 - \chi_1$. Then $\tilde{F}(k) = I_1(k) + I_0(k)$, where $I_0(k) = \lim_{\varepsilon \to 0} \int e^{ikx}e^{i\chi_0 x^2M/2m} N \chi_0(x)\psi(\varepsilon x) \, dx$ and $I_1(k) = \int e^{ikx}e^{i\chi_1 x^2M/2m} N \chi_1(x) \, dx$. For the study of the boundedness of $|F(k)|$ as $|k| \to \infty$ it is enough to look at $I_0$, since one has, by the choice of $\chi_1$, that $|I_1| \leq 2$. By repeating the same reasoning used in the proof of Theorem 1 $I_0$ can be computed by means of Stokes formula:

$$\lim_{\varepsilon \to 0} \int e^{ikx}e^{i\chi_0 x^2M/2m} N \chi_0(x)\psi(\varepsilon x) \, dx \quad = \quad i \lim_{\varepsilon \to 0} \int e^{ikx}e^{i\chi_0 x^2M/2m} N \frac{\chi_0(x)}{k+x^2M-1} \, dx \quad + \quad i \lim_{\varepsilon \to 0} \int e^{ikx}e^{i\chi_1 x^2M/2m} N \frac{\chi_1(x)}{k+x^2M+1} \, dx .$$

(B.1)

Both integrals are absolutely convergent and, by dominated convergence, we can take the limit $\varepsilon \to 0$, so that

$$I_0(k) = i \int e^{ikx}e^{i\chi_0 x^2M/2m} N \frac{\chi_0(x)}{k+x^2M-1} \, dx \quad - \quad i \int e^{ikx}e^{i\chi_1 x^2M/2m} N \frac{\chi_1(x)}{k+x^2M+1} \, dx .$$

Thus:

$$|I_0(k)| \leq 2 \int_{c_k-1/2}^{c_k+1} \frac{1}{k+x^2M-1} \, dx + 2 \int_{c_k+1/2}^{c_k+1/2} \frac{1}{k+x^2M-1} \, dx$$

$$\quad + \quad (2M-1) \int_{-\infty}^{c_k-1/2} \frac{x^{2M-2}}{(k+x^2M-1)^2} \, dx$$
+ (2M − 1) \int_{c_1 + 1/2}^{+\infty} \left| \frac{x^{2M−2}}{(k + x^{2M−1})^2} \right| dx.

By a change of variables it is possible to see that both integrals remain bounded as |k| \to \infty. Let us consider for instance the first one:

\[
\int_{c_1 − 1}^{c_1−1/2} \frac{1}{k + x^{2M−1}} dx = \frac{k^{−1/2} \gamma\left(\frac{1}{2}, \frac{1}{k + x^{2M−1}}\right) \gamma\left(\frac{3}{2}, \frac{1}{k + x^{2M−1}}\right)}{k^{−1/2}} dy.
\]

The latter integral diverges logarithmically as |k| \to \infty, so that the r.h.s. goes to 0 as |k| \to \infty. Let us consider the integral \( \int_{−\infty}^{c_{1}−1/2} \frac{x^{2M−2}}{(k + x^{2M−1})^2} dx \). By a change of variables it is equal to

\[
\int_{−\infty}^{c_{1}−1/2} \frac{x^{2M−2}}{(k + x^{2M−1})^2} dx = \frac{1}{k^{−1/2}} \gamma\left(\frac{3}{2}, \frac{1}{k + x^{2M−1}}\right) \gamma\left(\frac{1}{2}, \frac{1}{k + x^{2M−1}}\right).
\]

The latter integral diverges as O(k) as |k| \to \infty, so that the r.h.s. remains bounded as |k| \to \infty. By such considerations we can deduce that |\tilde{F}(k)| is bounded as |k| \to \infty.

A similar reasoning holds also in the case \( N = 1 \) and \( P(x) = \sum_{i=1}^{2M} a_i x^i \) is a generic polynomial. Indeed for |k| sufficiently large the derivative of the phase function \( \Phi'(x) = k + P'(x) \) has only one simple real root, denoted by \( c_k \). One can repeat the same reasoning valid for the case \( P(x) = x^{2M}/2M \) and prove that for |k| \to \infty one has \(| \int e^{ikx_i + P(x)} dx | \leq C \) (where C is a function of the coefficients \( a_i \) of \( P \) at most with polynomial growth).

The general case \( \mathbb{R}^N \) can also be essentially reduced to the one-dimensional case. Indeed let us consider a generic vector \( k \in \mathbb{R}^N, k = |k|u_1 \), and study the behavior of \( \tilde{F}(k) \) as |k| \to \infty. By choosing as orthonormal base \( u_1, \ldots, u_N \) of \( \mathbb{R}^N \), where \( u_1 = k/|k| \), we have

\[
\tilde{F}(k) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{N−1}} e^{iQ(x_2, \ldots, x_N)} \psi(\varepsilon x_2) \cdots \psi(\varepsilon x_N) \times \left( \int_{\mathbb{R}} e^{ik_{x_1} x_1 + P_{x_2, \ldots, x_N}(x_1)} \psi(\varepsilon x_1) dx_1 \right) dx_2 \cdots dx_N,
\]

(B.2)

where \( \psi \in \mathcal{S}(\mathbb{R}), \psi(0) = 1; x_i = x \cdot u_i, P_{x_2, \ldots, x_N}(x_1) \) is the polynomial in the variable \( x_1 \) with coefficients depending on powers of the remaining \( N−1 \) variables \( x_2, \ldots, x_N \), obtained by considering in the initial polynomial \( P(x_1, x_2, \ldots, x_N) \) all the terms containing some power of \( x_1 \). The polynomial \( Q \) in the \( N−1 \) variables \( x_2, \ldots, x_N \) is given by \( P(x_1, x_2, \ldots, x_N) − P_{x_2, \ldots, x_N}(x_1) \).

Let us set \( I'(k, x_2, \ldots, x_N) = \int e^{ik_{x_1} x_1 + P_{x_2, \ldots, x_N}(x_1)} \psi(\varepsilon x_1) dx_1 \). By the previous considerations we know that, for each \( \varepsilon \geq 0, |I'(k, x_2, \ldots, x_N)| \) is bounded by a function of \( G(x_2, \ldots, x_N) \) of polynomial growth. By the same reasonings as in the proof of Theorem 1 we can deduce that the oscillatory integral (B.2) is a well defined bounded function of \( k \).
References


[34] V.P. Maslov, Méthodes Opérationelles, Mir, Moscow, 1987.


