Blow-Up of Solutions for a Class of Semilinear Reaction Diffusion Equations with Mixed Boundary Conditions

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(Received March 2001; accepted May 2001)
Communicated by B. Fuchssteiner

Abstract—The Hopf's maximum principles are utilized to deal with the problem on blow-up of the solutions for a class of semilinear reaction diffusion equations \( u_t = \Delta u + f(x, u, q, t) \) \( (q = |\nabla u|^2) \), subject to mixed boundary conditions. Some nonexistence theorems of global solutions and the bounds of "blow-up time" are obtained. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Nonlinear, Reaction diffusion equations, Maximum principles.

1. INTRODUCTION

During the past decades, the blow-up phenomenon has been investigated by a large number of people. However, most of the previous research deal with the following the boundary-value problem with Dirichlet boundary conditions (see [1,2]):

\[
\begin{align*}
    u_t &= \Delta u + f(u), & \text{in } D \times (0, T), \\
    u &= 0, & \text{on } \partial D \times (0, T), \\
    u(x, 0) &= u_0(x), & \text{in } \bar{D}.
\end{align*}
\]

In this paper, we consider the semilinear reaction diffusion equations with mixed boundary conditions

\[
\begin{align*}
    u_t &= \Delta u + f(x, u, q, t), & \text{in } D \times (0, T), \\
    u &= 0, & \text{on } \Gamma_1 \times (0, T), \\
    \frac{\partial u}{\partial n} &= 0, & \text{on } \Gamma_2 \times (0, T), \\
    \Gamma_1 \cup \Gamma_2 &= \partial D, \\
    u(x, 0) &= u_0(x) \geq 0, & \text{in } \bar{D},
\end{align*}
\]

I would like to thank Professor Yu Mingqi for his useful help. This research is supported by Natural Science Fund of Shanxi Province for the Returned Workers.
where $D$ is a bounded domain in $R^N$ with a smooth boundary $\partial D$, $0 < T < +\infty$, $q = |\nabla u|^2$, denotes the outward normal derivative and the function $f$ is $C^1$ function of its arguments, and obtain some nonexistence theorems of global solution $u$ and bounds for the "blow-up time" $T^*$, which extend those obtained in [1,2]. Our approach depends heavily upon Hopf's maximum principles.

In what follows, for simplicity we use a bar to denote partial differentiation with respect to the explicit variable $t$, e.g.,

$$\bar{f} = \frac{\partial f(x,u,q,t)}{\partial t}.$$

The content of this paper is organized as follows. In Section 2, we shall give the main result and its proof. In Section 3, we shall give an example to which the theorems obtained in this paper may be applied.

2. THE MAIN RESULT AND ITS PROOF

Since one does not know the range of the dependent variables, the first infimum must be taken with respect to all $q$ and nonnegative $u$ as well as over $(x,t)$. Theorem 1. Let $u$ be a solution of (1.1)–(1.3). Suppose the following.

(i) The function $f$ has the property

$$f > 0, \quad f_u \geq 0, \quad f_q \geq 0, \quad \bar{f} > 0. \quad (2.1)$$

(ii) The initial function $u_0$ satisfies that when $u_0 = 0$,

$$\Delta u_0 + f(x,u_0,q_0,0) \geq 0; \quad (2.2)$$

(iii) The positive constant $\beta$ satisfies

$$\beta \leq \min \left\{ \frac{\bar{f}}{\Delta u_0 + f(x,u_0,q_0,0)} \right\} \quad (2.3)$$

where $q_0 = |\nabla u_0|^2$, $D_1 = \{ x \mid x \in \bar{D}, u_0(x) \neq 0 \}$, and $\alpha > 1$ is some positive constant.

Then $u(x,t)$ must blow-up in finite time $T^*$ and

$$T^* \leq \frac{M_{\alpha}^{1-\alpha}}{\beta(\alpha - 1)} \quad (2.4)$$

where $M_\alpha = \max_{\bar{D}} u_0(x) < +\infty$.

Proof. Because $f > 0$, it is easy to see that the solution $u(x,t)$ is nonnegative. Let

$$v = -u_{tt} + \beta u^\alpha, \quad (2.5)$$

then we have

$$\nabla v = -\nabla u_{tt} + \alpha \beta u^{\alpha - 1}\nabla u, \quad (2.6)$$

$$\Delta v = -\Delta u_{tt} + \beta \alpha (\alpha - 1) u^{\alpha - 2} |\nabla u|^2 + \beta \alpha u^{\alpha - 1} \Delta u,$$

and

$$v_{tt} = -\Delta u_{tt} - f_{uu} u_{tt} - 2f_{uq} \nabla u \cdot \nabla u_{tt} - \bar{f} + \beta \alpha u^{\alpha - 1} u_t. \quad (2.7)$$

Hence,

$$\Delta v - v_{tt} - \beta \alpha (\alpha - 1) u^\alpha - 2q + \beta \alpha u^{\alpha - 1} \Delta u + (f_{uu} - \beta \alpha u^{\alpha - 1}) u_{tt} + 2f_{uq} \nabla u \cdot \nabla u_{tt} + \bar{f}$$

$$= \beta \alpha (\alpha - 1) u^{\alpha - 2} q + \beta \alpha u^{\alpha - 1} (-v + \beta u^\alpha - f) + (f_{uu} - \beta \alpha u^{\alpha - 1}) (-v + \beta u^\alpha) + 2f_{uq} \nabla u \cdot (\nabla v + \beta \alpha u^{\alpha - 1} \nabla u) + \bar{f}$$

$$= -2f_{uq} \nabla u \cdot \nabla v - f_{uu} v + \beta \alpha (\alpha - 1) u^{\alpha - 2} q + 2\beta \alpha u^{\alpha - 1} f_{uq} q$$

$$+ \beta f_{uu} u^\alpha - \alpha f^{\alpha - 1} \left( \beta - \frac{\bar{f}}{\alpha f^{\alpha - 1}} \right). \quad (2.8)$$
Therefore, from (2.1) and (2.3) we know
\[
\Delta v + 2f_{uq} \nabla u \cdot \nabla v + f_{uu} v - v_{tt} = \beta \alpha (\alpha - 1) u^{\alpha - 2} q + 2 \beta \alpha u^{\alpha - 1} f_{uq} q + \beta f_{uu} u^\alpha
\]
\[
- \alpha f u^\alpha \left( \beta - \frac{f}{\alpha f u^{\alpha - 1}} \right) \geq 0.
\]
\[ \tag{2.9} \]

With (2.2) and (2.3), we have
\[
v_0 = v(x, 0) = -\Delta u_0 - f(x, u_0, q_0, 0) + \beta u_0^\alpha \leq 0.
\]

We note that on \( \Gamma_1 \times (0, T) \),
\[
u_t = 0
\]
and
\[
v = -u_t + \beta u^\alpha = 0.
\]

On \( \Gamma_2 \times (0, T) \), we have
\[
\frac{\partial u_{tt}}{\partial n} = 0,
\]
and therefore,
\[
\frac{\partial v}{\partial n} = -\frac{\partial u_t}{\partial n} + \beta \alpha u^{\alpha - 1} \frac{\partial u}{\partial n} = 0.
\]

By Hopf’s maximum principle [3], we find that in \( \bar{D} \times [0, T) \), the maximum of \( v \) is 0. Hence, we have in \( \bar{D} \times [0, T) \),
\[
v \leq 0
\]
and
\[
u_t \geq \beta u^\alpha.
\]

At the point \( x_0 \) at which \( u_0(x_0) = M_o \), we get by integration
\[
\frac{1}{\beta} \int_{M_o}^{u_0(x_0, t)} \frac{1}{s^\alpha} \, ds \geq t.
\]

By using
\[
\frac{1}{\beta} \int_{M_o}^{+\infty} \frac{1}{s^\alpha} \, ds = \frac{M_o^{1-\alpha}}{\beta(\alpha - 1)} < +\infty
\]
for some finite constant \( \alpha > 1 \), it follows that \( u(x, t) \) must blow-up for a finite value \( t = T^* \).

Further, the following inequality must hold:
\[
T^* \leq \frac{1}{\beta} \int_{M_o}^{+\infty} \frac{1}{s^\alpha} \, ds = \frac{M_o^{1-\alpha}}{\beta(\alpha - 1)}.
\]

The proof of Theorem 1 is complete.

Remark. In (1.2), \( \Gamma_1 \) or \( \Gamma_2 \) may be empty. If \( \Gamma_2 = \emptyset \) or \( \Gamma_1 = \emptyset \), then we refer to resulting problem as the Dirichlet problem or Neumann problem. For the Dirichlet problem or Neumann problem, the results of Theorem 1 are valid.
3. CONCLUDING REMARKS AND APPLICATIONS

One may give extensions of the blow-up problem for a uniformly parabolic equation \( u_{t} = Lu + f(x, u, q, t) \) under suitable assumptions, as shown in [4,5].

In the following, we present an example of a problem to which Theorem 1 maybe applied.

**Example.** Let \( u(x, t) \) be a solution of

\[
\begin{aligned}
  u_{t} &= \Delta u + 2N \exp \left\{ (g + t + 1)u + \sum_{i=1}^{N} x_{i}^{2} \right\}, & \text{in } D \times (0, T), \\
  u &= 0, & \text{on } \partial D \times (0, T), \\
  u(x, 0) &= u_{0}(x) = 1 - \sum_{i=1}^{N} x_{i}^{2}, & \text{in } \bar{D},
\end{aligned}
\]

where \( D = \{ x = (x_{1}, \ldots, x_{N}) \mid \sum_{i=1}^{N} x_{i}^{2} < 1 \}, \ N \geq 2 \). We have now

\[
  f(x, u, q, t) = 2N \exp \left\{ (g + t + 1)u + \sum_{i=1}^{N} x_{i}^{2} \right\} > 0
\]

and

\[
  u_{0}(x) = 1 - \sum_{i=1}^{N} x_{i}^{2}.
\]

Hopf’s maximum principle implies any solution \( u(x, t) \) of (3.1)-(3.3) must be positive in \( D \times (0, T) \). It is easy to check that (2.1) and (2.2) hold. In order to determine the constant \( \beta \), we take \( \alpha = 2 \). Then

\[
\frac{\bar{f}}{uf^{\alpha-1}} = \frac{1}{2}
\]

and for \( u_{0} \neq 0 \),

\[
\begin{aligned}
\frac{\Delta u_{0} + f(x, u_{0}, q_{0}, 0)}{u_{0}^{\alpha}} &= -2N + 2N \exp \left\{ (q_{0} + 1)u_{0} + \sum_{i=1}^{N} x_{i}^{2} \right\} \\
&= \frac{-2N + 2N \exp \left\{ -4 \left( \sum_{i=1}^{N} x_{i}^{2} \right)^{2} + 4 \sum_{i=1}^{N} x_{i}^{2} + 1 \right\}}{u_{0}^{2}} \\
&\geq \frac{-2N + 2N e}{u_{0}^{2}} = \frac{2N(e-1)}{u_{0}^{2}} \geq 2N(e-1)
\end{aligned}
\]

By (3.4) and (3.5), we take

\[
\beta = \min \left\{ \frac{1}{2}, 2N(e-1) \right\} = \frac{1}{2}.
\]

It follows from Theorem 1 that \( u(x, t) \) must blow-up in finite time \( T^{*} \) and

\[
T^{*} \leq \frac{M_{0}^{1-\alpha}}{\beta(\alpha - 1)} = 2.
\]

REFERENCES