Convergence of Gauss–Christoffel Formula with Preassigned Node for Cauchy Principal-Value Integrals*

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The authors consider Gauss–Christoffel formulas with preassigned node 0 for evaluating Cauchy singular integrals with an even generalized smooth Jacobi weight. A convergence theorem is given, and some asymptotic estimates of the remainder are established. © 1987 Academic Press, Inc.

1. INTRODUCTION

Let \( \Phi(f; t) \) be a Cauchy principal-value (V.P.) integral of the function \( f \), namely,

\[
\Phi(f; t) = \int_{-1}^{1} \frac{f(x)}{x-t} w(x) \, dx
\]

\[
= \lim_{\varepsilon \to 0} \left\{ \int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1} \right\} \frac{f(x)}{x-t} w(x) \, dx, \quad |t| < 1, \quad (1.1)
\]

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where

(i) \( f \in TD := \{ f \in C^0(I) | \int_I u^{-1} \omega(f; u) du < \infty \} \). Here \( I = [-1, 1] \) and \( \omega(f; \cdot) \) is the modulus of continuity of \( f \) in \( I \);

(ii) \( w(x) := \psi(x)(1 - x^2)^\mu; \mu > -1, \psi(x) = \psi(-x), 0 \leq \psi \in TD, 1/\psi \) is (Lebesgue) integrable on \( I \).

It is well known that (i) and (ii) imply the existence and continuity of \( \Phi(f; \cdot) \) moreover [2]:

\[
\omega(\Phi f; \delta) = O(\omega(f; \delta)) \quad (\delta \to 0).
\]

In order to approximate \( \Phi f \), we construct a Gauss–Christoffel type formula \( \Phi_{2m} \) with a fixed node at 0 of a given multiplicity \( 2s \). This formula is a generalization of those contained in \([11, 6]\).

The principal aim of this work is to examine the convergence of the sequence \( \{\Phi_{2n}f\} \) under the assumption that the function \( f \in TD \) has derivatives of whatever order will be needed at 0. We prove there is a subsequence \( \{\Phi_{2m_n}f\} \) that converges uniformly to \( \Phi f \) on some closed subset of \((-1, 1) - \{0\}\). However, there exist also divergent subsequences \( \{\Phi_{2m}f\} \). Consequently, one concludes that when the function \( f \) is not sufficiently smooth, this type of quadrature formula is unsuitable for applications.

We also prove that if we omit in \( \Phi_{2m}f \) a term corresponding to a node nearest to the singularity, then we obtain a quadrature formula \( \Phi^*_{2m}f \) \((m \in \mathbb{N})\) that converges uniformly on a closed subset of \((-1, 1) - \{0\}\) to \( \Phi f \), under the previous assumptions on \( f \in TD \).

Moreover, we determine some asymptotic estimates of the remainders corresponding to the formulas \( \Phi_{2m} \) and \( \Phi^*_{2m} \).

2. A Gauss–Christoffel Type Formula for the Evaluation of Principal Value Integral

Let

\[
v(x) = x^{2s}w(x)
\]

where \( s \in \mathbb{N} \) and \( w \) is defined by (ii).

Moreover, let \( \{p_n\} \) be the sequence of the orthogonal polynomials in \( I \) associated with the weight function \( v \) defined by (2.1). Then, we denote the zeros of \( p_n(x) \) by \( x_{n,i} = x_{n,i}(v) \) \((i = 1, 2, \ldots, n)\), and the corresponding Christoffel numbers by \( \lambda_{n,i} = \lambda_{n,i}(v) \) \((i = 1, 2, \ldots, n)\). The function \( v \) is a "generalized smooth Jacobi" weight \((v \in GSJ)\), and the properties of the orthogonal polynomials \( p_n \) have been extensively studied by Stancu \([13]\), Rothmann \([12]\), Badkov \([1]\) and Nevai \([9]\).

Now, we consider the Gauss–Christoffel quadrature formula with respect
to the weight function $w$ with the fixed knot 0, given with its order of multiplicity $2s$ [13]

$$G_{2m}(g) = \sum_{i=1}^{2m} A_{2m,i} g(x_{2m,i}) + \sum_{j=0}^{s} B_{2m,2j} g^{(2j)}(0) \quad (2.2)$$

where $x_{2m,i} = x_{2m,i}(v) \ (i = 1, 2, ..., 2m)$, and

$$A_{2m,i} = \frac{\gamma_{2m,i}(v)}{\chi_{2m,i}(v)}, \quad (i = 1, 2, ..., 2m), \quad (2.3)$$

$$B_{2m,2j} = \int_{-1}^{1} \beta_{2m,2j}(x) w(x) \, dx, \quad (j = 0, 1, ..., s - 1),$$

where the function $g$ is defined on $I$. As the weight function $v$ is even, we have

$$\gamma_{2m,i} = -\gamma_{2m,2m-i+1}, \quad (i = 1, 2, ..., m),$$

$$A_{2m,i} = A_{2m,2m-i+1}, \quad (i = 1, 2, ..., m).$$

Let $R_{2m}(g)$ be the remainder that is defined by

$$\int_{-1}^{1} g(x) w(x) \, dx = G_{2m}(g) + R_{2m}(g) \quad (2.5)$$

It is well known that the quadrature formula (2.2) has degree of exactness $4m + 2s - 1$.

Now, we may construct a Gauss–Christoffel quadrature formula for the approximate evaluation of $\Phi f$ by the formula (2.2). In order to approximate the integral $\Phi f$, we write

$$\Phi(f; t) = f(t) \int_{-1}^{1} \frac{w(x)}{x-t} \, dx + \int_{-1}^{1} \frac{f(x) - f(t)}{x-t} w(x) \, dx \quad (7.6)$$

Hence, by (2.5) we have

$$\Phi(f; t) = f(t) \int_{-1}^{1} \frac{w(x)}{x-t} \, dx + G_{2m} \left( \frac{f-f(t)}{e_1-t} \right) + R_{2m} \left( \frac{f-f(t)}{e_1-t} \right), \quad (2.7)$$

where $e_k(x) = x^k \ (k \in N)$, and we have assumed that $t \neq 0$. Then (2.7) can be rewritten in the following form

$$\Phi(f; t) = \Phi_{2m}(f; t) + E_{2m}(f; t), \quad (2.8)$$
where we assume

$$E_{2m}(f; t) = R_{2m} \left( \frac{f-f(t)}{e_1-t} \right),$$  \hfill (2.9)$$

$$\Phi_{2m}(f; t) = f(t) \int_{-1}^{1} \frac{w(x)}{x-t} dx + \sum_{i=1}^{2m} A_{2m,i} \frac{f(x_{2m,i}) - f(t)}{x_{2m,i}-t} + \frac{B_{2m,0}(t)}{t} f(t) - \sum_{j=0}^{2s-2} \frac{B_{2m,j}(t)}{t} f^{(j)}(0),$$  \hfill (2.10)$$

$$B_{2m,2s-2}(t) = B_{2m,2s-2},$$  \hfill (2.11)$$

$$B_{2m,j}(t) = B_{2m,j} + (j+1) \frac{B_{2m,j+1}(t)}{t} \quad (j = 2s-3, \ldots, 1, 0),$$

where $B_{2m,2s-1} = 0$. Moreover, by

$$\int_{-1}^{1} \frac{w(x)}{x-t} dx - \sum_{i=1}^{2m} \frac{A_{2m,i}}{x_{2m,i}-t} + \frac{B_{2m,0}(t)}{t} = R_{2m} \left( \frac{1}{e_1-t} \right) = A_{2m}(t),$$  \hfill (2.10)$$

(2.10) becomes

$$\Phi_{2m}(f; t) = A_{2m}(t) f(t) + \sum_{i=1}^{2m} \frac{A_{2m,i}}{x_{2m,i}-t} f(x_{2m,i}) - \sum_{j=0}^{2s-2} \frac{B_{2m,j}(t)}{t} f^{(j)}(0).$$  \hfill (2.12)$$

We have established (2.12) under the assumption that $t \neq 0$; however, by an easy limit calculus, we may have the following quadrature formula

$$\int_{-1}^{1} \frac{f(x)}{x} \, w(x) \, dx = \Phi_{2m}(f; 0) = \sum_{i=1}^{m} \frac{A_{2m,i}}{x_{2m,i}} \left[ f(x_{2m,i}) - f(-x_{2m,i}) \right] + \sum_{j=0}^{s-1} \frac{B_{2m,2j}}{2j+1} f^{(2j+1)}(0).$$  \hfill (2.13)$$

We note that formulas (2.12) and (2.13) have degree of exactness $4m + 2s$, but (2.12) depends on $m + 2s$ coefficients, whereas only $m + s$ coefficients are in (2.13); furthermore, the calculation of these is easier.

Finally, we observe that in the special case in which $w \equiv 1$ and $s = 1$, the formula (2.13) becomes the formula given by Hunter in [6], and for $w \equiv 1$ and $s = 0$, we obtain the Piessen's formula [11].

Now, we want examine the convergence of the formulas that we have introduced. It is clear that the convergence of the formula (2.13) follows
easily; in fact, the function \( f \) has a derivative at point 0 and so the function \( (f(x))/x \) is Riemann-integrable. Therefore, we shall attend to the convergence of the quadrature formula (2.10).

3. NOTATIONS AND BASIC LEMMAS

The symbol "const" stands for some positive constant taking a different value each time it is used. If \( A \) and \( B \) are two expressions depending on some variables, then we write

\[ A \sim B \quad \text{if} \quad |AB^{-1}| \leq \text{const} \quad \text{and} \quad |A^{-1}B| \leq \text{const} \]

uniformly for the variables under consideration. Throughout this paper, \( C_c(I) \) denotes the class of the continuous functions on \( I \) with \( 2s-2 \) derivatives at point 0, and \( A \) denotes a closed set such that \( A \subset (-1, 1) - \{0\} \).

As mentioned above, \( v \) is a generalized smooth Jacobi weight function; the properties of the corresponding orthogonal polynomials and of the Christoffel numbers are well known [1, 9].

From among these, we recall the important relations

\[ H_{2m,i} - H_{2m,i+1} \sim (2m)^{-1}, \quad (i = 1, 2,..., 2m - 1), \quad (3.1) \]

where \( x_{2m,i} = \cos \theta_{2m,i} \) \((i = 1, 2,..., 2m)\) are the zeros of \( p_{2m} \), so ordered

\[-1 < x_{2m,1} < x_{2m,2} < \cdots < x_{2m,2m} < 1, \]

(see [9, Theorem 9.22, p. 166]), and

\[ \tilde{\lambda}_{2m,i}(v) \sim (2m)^{-1}(1-x_{2m,i}^{2}) + \frac{1}{2}(|x_{2m,i}| + (2m)^{-1})^{2}, \quad (i = 1, 2,..., m). \quad (3.2) \]

(See [9, Theorem 6.3.28, p. 120].)

Then, we set: \( N^* = \{ m \in N: x_{2m,i} \neq t, \ i = 1, 2,..., 2m \} \), and \( x_{2m,c} \) denotes the closest knot to \( t \); more precisely

\[ |t - x_{2m,c}| = \min \{ t - x_{2m,d}, x_{2m,d+1} - t \}, \]

where: \( x_{2m,d} \leq t \leq x_{2m,d+1}, \ (0 \leq d \leq 2m) \). It is obvious that

\[ N^* = \{ m \in N^*: t - x_{2m,d} \sim x_{2m,d+1} - t \} \subseteq N^*, \]

and both sets are infinite [3].

At this point we prove the following lemmas, we need later for the main theorem.
**Lemma 3.1.** The inequalities

\[
\frac{\lambda_{2m,i}}{|x_{2m,i} - t|} < \frac{A_{2m,i}}{|x_{2m,i} - t|} < \frac{C}{(1 - t^2)^{1/2}} \left\{ (2m)^{-1} |x_{2m,i} - t|^{-1} + (1 - x_{2m,i}^{-2})^{-1/2} \right\}
\]

\[ (i = 1, 2, ..., 2m), \quad (3.3) \]

hold for some constant \( C > 0 \) independent of \( m \).

**Proof.** By (2.3) and (2.4), we have

\[
A_{2m,i} = \lambda_{2m,i} x_{2m,i}^{2i} \approx (2m)^{-1} (1 - x_{2m,i}^2)^{1/2} \left( 1 + \frac{1}{2m |x_{2m,i}|} \right)^{2i},
\]

\[ (i = 1, 2, ..., 2m). \]

Moreover, as the weight function \( v \in GSJ \) is even, and by (3.1), we obtain

\[
\frac{1}{2m |x_{2m,i}|} \leq \frac{1}{2m |x_{2m,m}|} \sim 1, \quad (i = 1, 2, ..., 2m).
\]

Thus

\[
A_{2m,i} \leq \text{const} (2m)^{-1} (1 - x_{2m,i}^2)^{-1/2}, \quad (i = 1, ..., 2m),
\]

from which the second inequality in (3.3) easily follows, and also first inequality, from

\[
A_{2m,i} \leq \lambda_{2m,i}, \quad (i = 1, 2, ..., 2m). \]

**Lemma 3.11.** Let \( x_{2m,d} \) be the closest knot to \( t \). If we set

\[
\sigma_m^*(t) = \sum_{i=1}^{2m} \frac{A_{2m,i}}{|x_{2m,i} - t|},
\]

\[ (3.4) \]

then

\[
\sigma_m^*(t) \sim \log m, \quad (m \in N), \quad (3.5)
\]

holds uniformly on \( A \).

**Proof.** Without any loss of generality, we suppose that

\[
x_{2m,d} = x_{2m,d} \leq t < x_{2m,d+1} \quad \text{thus} \quad t - x_{2m,d-1} \sim x_{2m,d+1} - t \sim (2m)^{-1}
\]

Then, by (3.3), we have

\[
\sigma_m^*(t) > \sum_{i=1}^{2m} \frac{\lambda_{2m,i}}{|x_{2m,i} - t|} = \sum_{i=1}^{d-1} \frac{\lambda_{2m,i}}{t - x_{2m,i}} + \sum_{i=d+1}^{2m} \frac{\lambda_{2m,i}}{x_{2m,i} - t},
\]

\[ (3.6) \]
Further, the function \( u(x) = |x - t|^{-1} \) is such that
\[
u^{(i)}(x) > 0, \quad x \in [-1, x_{2m,d + 1}], \quad i = 0, 1, \ldots
\]
and
\[
(-1)^i \nu^{(i)}(x) > 0, \quad x \in [x_{2m,d + 1}, 1], \quad i = 0, 1, \ldots
\]
Then, by the generalized Markov-Stieltjes inequalities (see Lemma 1.5.3 in [5, p. 301] and Lemmas 3.2, 3.3 in [8, p. 222]), we obtain
\[
\sum_{i = 1}^{d} \sum_{i \neq i'} \frac{\lambda_{2m,i}^{2}}{|x_{2m,i} - t|} \geq \int_{S_{2m,d}} v(x) \frac{\nu(x)}{|t - x|} \, dx,
\]
\[
\sum_{i = d + 1}^{2m} \lambda_{2m,i}^{2} \geq \int_{S_{2m,d+1}} v(x) \frac{\nu(x)}{|x - t|} \, dx.
\]
From these inequalities, we easily get
\[
\sigma^{*}_{m}(t) > 2v(t) \log m + v(t) \log(1 - t^2) + V(t),
\]
where
\[
V(t) = \int_{-1}^{1} \left| \frac{v(x) - v(t)}{x - t} \right| \, dx.
\]
From \( v \in TD \), we have that the function \( V \) is continuous on \( A \), and thus
\[
\sigma^{*}_{m}(t) > a_{1} \log m + b_{1}, \quad (\forall t \in A),
\]
where \( a_{1}, b_{1} \) are two constants, independent of \( m \).
Further, by (3.3) we obtain
\[
\sigma^{*}_{m}(t) \leq \text{const} \frac{1}{1 - t^2} \left[ \sum_{i = 1}^{2m} \frac{1}{2m|x_{2m,i} - t|} + \sum_{i = 1}^{2m} \frac{1}{2m \sqrt{1 - x_{2m,i}^2}} \right].
\]
By (3.1), we represent the two summations in the last inequality as Riemann Darboux ones. Thus
\[
\sigma^{*}_{m}(t) \leq \text{const} \frac{1}{1 - t^2} \left[ \int_{-1}^{x_{2m,d - 1}} \frac{dx}{|t - x|} + \int_{x_{2m,d + 1}}^{1} \frac{dx}{x - t} + \pi \right].
\]
Again, by (3.1) and \( t \in A \), we have
\[
\sigma^{*}_{m}(t) \leq a_{2} \log m + b_{2},
\]
where \( a_{2}, b_{2} \) are two constants, independent of \( m \).
Hence the lemma is proved. \( \square \)
**Lemma 3.11.** For the coefficients $B_{2m,2r}^{\pm}$ in (2.2) the inequalities

$$|B_{2m,2r}^{\pm}| \leq \frac{K}{(2r)!} \frac{m^{2r+1}}{m^{2r+1}+1}, \quad (r = 0, 1, \ldots, s - 1), \quad (3.7)$$

hold, where $K$ is a constant, independent of $m$ and $r$.

**Proof.** If one set $g(x) = x^{2r}$ ($r = 0, 1, \ldots, s - 1$) in (2.5), we obtain

$$B_{2m,2r}^{\pm} = \frac{1}{(2r)!} \int_{-1}^{1} x^{2r} w(x) \, dx - \sum_{i=1}^{2m} \frac{\lambda_{2m,i}}{x^{2(s-r)}}.$$ 

Now, the function $\gamma(x) = x^{2(r-s)} (r < s)$ is such that

$$\gamma^{(i)}(x) > 0, \quad x < 0, \quad i = 0, 1, \ldots,$$

$$(-1)^i \gamma^{(i)}(x) > 0, \quad x > 0, \quad i = 0, 1, \ldots,$$

and

$$v(x) \gamma(x) = x^{2r} w(x).$$

Then, by the generalized Markov–Stieltjes inequalities (see [8, p. 222]), we have

$$\sum_{i=1}^{m+1} \frac{\lambda_{2m,i}}{x^{2(s-r)}} \leq \int_{-1}^{1} x^{2r} w(x) \, dx \leq \sum_{i=1}^{m+1} \frac{\lambda_{2m,i}}{x^{2(s-r)}}.$$ 

From these inequalities, we deduce

$$\alpha_m - \beta_m \leq B_{2m,2r}^{\pm} \leq \alpha_m,$$

where

$$\alpha_m = \frac{1}{(2r)!} \int_{-1}^{1} x^{2r} w(x) \, dx,$$

and

$$\beta_m = \frac{2}{(2r)!} \frac{\lambda_{2m,m+1}}{x^{2(s-r)}}.$$ 

Thus

$$|B_{2m,2r}^{\pm}| \leq \max \{\alpha_m, \beta_m\}.$$
Further, we have

\[ (2r)! \, \alpha_m = w(\xi) \int_{x_{2m,m}}^{x_{2m,m+1}} x^{2r} \, dx = w(\xi) \frac{x_{2m,m+1}^{2r+1} - x_{2m,m}^{2r+1}}{2r + 1} \]

\[ - 2\omega(\xi) \, x_{2m,m+1}^{2r+1} \sim m^{-2r-1}, \]

where \( \xi \in (x_{2m,m}, x_{2m,m+1}) \). Moreover, by (3.2), we see

\[ (2r)! \beta_m \sim m^{-1} (1 - x_{2m,m+1}^2)^{n+1/2} \left( x_{2m,m+1} + \frac{1}{2m} \right)^{2s} x_{2m,m+1}^{2(r-s)} \]

\[ = m^{-1} (1 - x_{2m,m+1}^2)^{n+1/2} \left( x_{2m,m+1} + \frac{1}{2m} \right)^{2r} \]

\[ \times \left( x_{2m,m+1} + \frac{1}{2m} \right)^{2s} x_{2m,m+1}^{2(r-s)} \]

\[ = m^{-1} (1 - x_{2m,m+1}^2)^{n+1/2} \left( x_{2m,m+1} + \frac{1}{2m} \right)^{2r} \left( 1 + \frac{1}{2mx_{2m,m+1}} \right)^{2s} x_{2m,m+1} \]

\[ \approx m^{-1} \]

Hence, the lemma is proved. 

Before proceeding any further, we observe that the result of Lemma 3.11 is sufficient to prove the convergence of the formula (2.2) when \( g \in C^0(I) \) and we suppose the existence of \( 2s - 1 \) derivatives of the function \( g \) at point 0.

Now, we note that the functions \( B_{2m,j}(t) \) depend on \( B_{2m,j} \) by (2.11). From these relations, the equalities

\[ B_{2m,2j}(t) = \frac{1}{(2j)!} \sum_{i=1}^{j} (2j + 2i - 2)! \frac{B_{2m,2j}^{2i+2} t^{2i-2}}{t^{2i-2}}, \quad (j = 0, 1, ..., s - 1), \]

\[ B_{2m,2j+1}(t) = \frac{1}{t(2j+1)!} \sum_{i=1}^{j} (2j + 2i)! \frac{B_{2m,2j+2i}^{2i+2} t^{2i-2}}{t^{2i-2}}, \quad (j = 0, 1, ..., s - 2), \]

casily follow.

Thus, by (3.7) we deduce

\[ \left| \frac{B_{2m,j}(t)}{t^{j+1}} \right| \leq \frac{K}{j! \, m^{j+1}}, \quad (j = 0, 1, ..., 2s - 2), \]

where \( K > 0 \) is a constant, independent on \( m, j, \) and \( t \in A \).
4. On the Convergence of Rule (2.10)

We set
\[ P_{2s-2}(x) = \sum_{k=0}^{2s-2} \frac{f^{(k)}(0)}{k!} x^k, \]
\[ g = f - P_{2s-2}, \]
\[ r_m = g - q_m, \]
where \( q_m \) is the best approximation polynomial with respect to the function \( g \). We remark that \( \omega(g; \delta) \leq \text{const} \omega(f; \delta) \) when \( f \in C^0(I) \).

Under these assumptions, we may prove the following.

**Lemma 4.1.** Given any function \( f \in C_c(I) \), there is a constant \( L \) independent on \( f \) and \( m \in \mathbb{N} \) such that
\[ |E_{2m}(f; t)| \leq L \delta_m + |a_{2m,c}(t)|, \quad t \in A, \tag{4.1} \]
where
\[ a_{2m,c}(t) = A_{2m,c} \frac{r_m(x_{2m,c}) - r_m(t)}{x_{2m,c} - t}, \quad (m \in \mathbb{N}^*), \]
\[ \delta_m = \|r_m\|(\log m + m^{-1}) + \left[ \int_0^1 \frac{\omega(f; u)}{u} \, du + \frac{\|q_m\|}{m} \right] \]
\[ + m \sum_{j=1}^{2s} \frac{|q_m^{(j)}(0)|}{j! m^j}, \quad (m \in \mathbb{N}), \]
and \( A \) is a closed set such that \( A \subset (-1, 1) \).

**Proof.** Having rule \( \phi_{2m} \) degree of exactness \( 4m + 2s \), we may claim that
\[ |E_{2m}(f; t)| \leq I_1 + I_2 + I_3, \]
where
\[ I_1 = |\phi(r_m - r_m(t); t)|, \tag{4.2} \]
\[ I_2 = \left| \sum_{i=1}^{2m} A_{2m,i} \frac{r_m(x_{2m,i}) - r_m(t)}{x_{2m,i} - t} \right|, \tag{4.3} \]
\[ I_3 = \left| \frac{B_{2m,0}(t)}{t} (r_m(t) - r_m(0)) + \sum_{j=1}^{2s} \frac{B_{2m,j}(t)}{t} q_m^{(j)}(0) \right|. \tag{4.4} \]
Now, we obtain

\[ I_1 \leq 2 \left\| r_m \right\| W(t) + w(t) \times \left\{ 4 \left\| r_m \right\| \log m + \int_{\|x\|_A \leq 1} \frac{r_m(x) - r_m(t)}{x - t} \, dx \right\}, \]

where

\[ W(t) = \int_{-1}^{1} \frac{w(x) - w(t)}{x - t} \, dx \]

By

\[ \left| \int_{\|x\|_A \leq 1} \frac{r_m(x) - r_m(t)}{x - t} \, dx \right| \leq \text{const} \int_{0}^{1/m} \omega(f; u) u^{-1} \, du + \frac{\|q_m\|_A}{m}, \]

as \( w \in TD \) implies \( W \in C^0(A) \), we obtain

\[ I_1 \leq \text{const} \left\{ \|r_m\| (\log m + 1) + \int_{0}^{1/m} \omega(f; u) u^{-1} \, du + \frac{\|q_m\|_A}{m} \right\}. \]

Furthermore, because of Lemma 3.11,

\[ I_2 \leq \text{const} \|r_m\| \log m + |a_{2m,n}(t)|. \]

Finally, by (3.8), we have

\[ I_3 \leq Km^{-1} \left\{ 2 \|r_m\| + \sum_{j=1}^{2} \frac{\|q_{m,j}(0)\|}{j! \, m^j} \right\}. \]

The combination of these inequalities proves the lemma.

At this point, we may prove the following

**Theorem 4.1.** For any function \( f \in C_s(I) \cap TD \), there exists a subsequence \( \{ \Phi_{2m_k} \}_{k \in N} \) uniformly convergent to \( \Phi f \) on \( A \).

**Proof.** First we remark that from \( f \in C_s(I) \cap TD \) follows

\[ \|r_m\| \leq \text{const} \omega(f; m^{-1}), \]

\[ \lim_{m \in N} \int_{0}^{1/m} \omega(f; u) u^{-1} \, du = 0, \]

and

\[ \lim_{m \in N} \omega(f; m^{-1}) \log m = 0. \]
Furthermore, it is well known [7] that for any function \( F \in C^r(I) \) \((r \geq 0)\), if we denote by \( q_m \) the best approximation polynomial with respect to the function \( F \), for any \( k > r \) there exists a constant \( M_k \) such that

\[
\| q_m^{(k)} \|_A \leq M_k m^{k-r} \omega(F^{(r)}; m^{-1}),
\]

where \( A \) is a closed set such that \( A \subset (-1, 1) \).

Thus, from \( g \in C^0(I) \), we obtain

\[
m^{-1} \left\{ \| q_m' \|_A + \sum_{j=1}^{2k-2} \frac{|q_m^{(j)}(0)|}{j! m^j} \right\} \leq \text{const} \omega(f; m^{-1}).
\]

Finally, we have

\[
\lim_{m \to \infty} \delta_m = 0.
\]

At this point, we introduce the set \( N' = \{ n \in N^*: |x_{2n, \epsilon} - t| \sim n^{-1} \log^{-1} n \} \).

By \( N^* \subset N' \subset N^* \), we obtain that \( N' \) is an infinite set and \( N' = \{ m_k \}_{k \in N} \).

Then, for any sufficiently large \( k \in N \), there exists a constant \( H > 0 \), independent of \( f \) and \( k \) such that

\[
\| a_{2m,i} \| \leq H \omega(f; m_k^{-1}) \log m_k = o(1), \quad (k \to \infty),
\]

and the theorem is proved. \( \blacksquare \)

Now, let \( LD(\lambda), \ (\lambda > 0) \), be the class of functions \( f \in C^0(I) \), such that \( \omega(f, \delta) \log^\lambda \delta^{-1} = o(1), \ (\delta \to 0^+) \). Obviously, we have

\[
LD(\lambda) \supset TD \quad \text{if} \quad \lambda \in (0, 1],
\]

and

\[
LD(\lambda) \subset TD \quad \text{if} \quad \lambda > 1.
\]

Note that by Lemma 4.1 and Theorem 4.1 the corollaries immediately follows:

**Corollary 4.1.** For any function \( f \in LD(\lambda), \ (\lambda > 1) \), there exists a subsequence \( \{ E_{2m_i, f} \}_{i \in N} \) such that

\[
\| E_{2m_i, f} \| = o(\log^{-1} m_i), \quad (i \to \infty).
\]

**Corollary 4.11.** For any function \( f \in \text{Lip}_s \alpha \), there exists a subsequence \( \{ E_{2m_j, f} \}_{j \in N} \) such that

\[
\| E_{2m_j, f} \| = O(m_j^{-2} \log m_j), \quad (j \to \infty).
\]
COROLLARY 4.III. For any function \( f \in \text{Lip}_M \), we have
\[
\| E_{2n} f \| = O(n^{-1} \log n), \quad (n \in N^*).
\]

COROLLARY 4.IV. For any function \( f \in C^{(k)}(I) \), we have
\[
\| E_{2n} f \| = O(n^{-k} \log n \omega(f; n^{-1})), \quad (n \in N^*).
\]

Furthermore, we observe that obvious changes in the proof of Theorem 4.1 are sufficient to prove the following:

THEOREM 4.11. If the integral \( \Phi_f \) exists, then, for any function \( f \in C_1(I) \cap LD(1) \), there exists a subsequence \( \{ \Phi_{m_k} f \}_{k \in N} \) convergent to \( \Phi_f \) in \((-1, 1) - \{0\} \).

At this point, we remark that by Lemma 3.11 the norm of \( \Phi_m \) is not bounded; then the continuity of the function \( f \) is not sufficient for the convergence of the sequence \( \{ \Phi_{2m} \} \), which besides is defined just for \( m \in N^* \subset N \). Yet, in some cases we have \( N^* = N \). This is true when \( s = 0 \), \( \psi(x) \equiv 1 \), \( \mu = \pm \frac{1}{2} \), and \( t = \cos(\pi p/q) \), where \( p/q \) is a rational number in \((0, 1)\), [8].

Further, it is not difficult to find a subsequence \( \{ \Phi_{2m} \}_{v \in N} \) not convergent when the function \( f \in \text{Lip}_M \), \( (x < 1) \). In fact, following the example \( s = 0 \), \( \psi(x) \equiv 1 \), \( \mu = \pm \frac{1}{2} \), if we suppose \( t = \cos(\theta \pi) \), where \( \theta \) is an irrational number in \((0, 1)\), we have \( |a_{2m,v}| \sim (2m)^{1-x} \) for a particular sequence \( \{m_v\} \subset N^* \) (see [10, p. 231]).

From the proof of Theorem 4.1, we have that the term that causes difficulties to the convergence of \( \Phi_{2m} f \) is \( a_{2m,v}(t) \), corresponding to the closest knot to the singularity. Now, let us omit this term and consider the quadrature formula
\[
\Phi_{2m}^*(f; t) = f(t) \int_{-1}^{1} \frac{w(x)}{x - t} \, dx + \sum_{i=1}^{2m} \frac{A_{2m,i} f(x_{2m,i}) - f(t)}{x_{2m,i} - t} \left( \frac{B_{2m,0}(t)}{t} f(t) - \sum_{j=0}^{2x-2} \frac{B_{2m,j}(t)}{t} f^{(j)}(0) \right),
\]
that can be rewritten in the following form
\[
\Phi_{2m}^*(f; t) = A_{2m}^*(f; t) + \sum_{i=1}^{2m} \frac{A_{2m,i} f(x_{2m,i}) - \sum_{j=0}^{2x-2} \frac{B_{2m,j}(t)}{t} f^{(j)}(0)}{x_{2m,i} - t}.
\]
where
\[
A_{2m}^*(t) = \int_{-1}^{1} \frac{w(x)}{x - t} \, dx - \sum_{i=1}^{2m} \frac{A_{2m,i}}{x_{2m,i} - t} + \frac{B_0(t)}{t}.
\]
Then, the corresponding remainder is defined by
\[ E_{2m}^*(f; t) = \Phi(f; t) - \Phi_{2m}^*(f; t). \]

One can easily prove that \( \Phi_{2m}^* \) has degree of exactness 0, however we may consider \( \Phi_{2m}^* \) for any \( m \in \mathbb{N} \), and we may prove the following:

**Theorem 4.I.** For any function \( f \in C_{s}(I) \cap TD \), the sequence \( \{ \Phi_{2m}^* f \}_{m \in \mathbb{N}} \) converges uniformly to \( \Phi f \) on \( A \).

**Proof:** Proceeding as in [4], we obtain
\[ E_{2m}^*(f; t) = E_{2m}(r_m; t) + A_{2m,c} \frac{q_m(x_{2m,c}) - q_m(t)}{x_{2m,c} - t}, \]
and recalling (3.4), (4.2), (4.4), we have
\[ |E_{2m}^*(f; t)| \leq I_1 + 2 \| r_m \| \sigma_m(t) + I_3 + A_{2m,c} \| q_m \|_A. \]

At this point, if we proceed as for the proof of Theorem 4.1, we may obtain
\[ \| E_{2m}^* f \|_A \leq C(\delta_m + \omega(f; m^{-1})), \tag{4.8} \]
where \( C \) is a constant, independent on \( f \) and \( m \).

This completes the proof of the theorem.

Furthermore, by (4.8) we obtain
\[ \| E_{2m}^* f \|_A = O(m^{-\frac{1}{2}}) \quad \text{if} \quad f \in LD(\hat{\alpha}), \quad (\hat{\alpha} > 1), \tag{4.9} \]
\[ \| E_{3m}^* f \|_A = O(m^{-\frac{1}{2}} \log m) \quad \text{if} \quad f \in Lip_M \alpha, \quad (0 < \alpha \leq 1). \tag{4.10} \]

On the other side, we have the following:

**Theorem 4.IV.** If the integral \( \Phi f \) exists, then for any function \( f \in C_{s}(I) \subset LD(1) \), the sequence \( \{ \Phi_{2m}^* f \} \) converges to \( \Phi f \) in \( (-1, 1) - \{0\} \):

Further, we point out that Theorem 4.III, Theorem 4.IV and the relations (4.9), (4.10) also hold for the quadrature formula \( \Phi_{2m}^* \) that we may obtain from \( \Phi_{2m} \) omitting the two terms that correspond to both knots \( x_{2m,d} \) and \( x_{2m,d+1} \).

Finally, note that in the special case in which \( s = 0 \), we obtain results established in [4].

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REFERENCES