ON WEYL SUMS AND SKEW PRODUCTS OVER IRRATIONAL ROTATIONS

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Abstract. Let \( \varphi : [0,1] \to \mathbb{R} \) have a Lipschitz-continuous derivative on \([0,1]\), \( \int_0^1 \varphi(t) \, dt = 0 \), let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) and write \( \varphi_n(x) = \varphi(x) + \varphi(\{x + \alpha\}) + \cdots + \varphi(\{x + (n-1)\alpha\}) \), \( n \in \mathbb{N} \), \( x \in [0,1] \). In this paper results on the boundedness and the limit points of the sequence \((\varphi_n(x))_{n=1}^{\infty}\) are given. Further, ergodicity of the skew product \((x,y) \mapsto (x+\alpha, y+\varphi(x))\) on \(\mathbb{R}/\mathbb{Z} \times \mathbb{R}\) is proved for certain classes of \(\varphi\) and \(\alpha\).

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Introduction

In [4] the following result was shown.

Theorem. Let \( \varphi : [0,1] \to \mathbb{R} \) be continuously differentiable on the closed interval \([0,1]\), let \( \int_0^1 \varphi(t) \, dt = 0 \) and suppose that \( \alpha \) is irrational. If \( \varphi(0) \neq \varphi(1) \), then \( T_\alpha(x,y) = (x+\alpha, y+\varphi(x)) \) is ergodic on \(\mathbb{R}/\mathbb{Z} \times \mathbb{R}\) (with respect to \( \lambda \times \lambda \times \lambda \times \lambda \), \( \lambda \) the normalized Haar measure on \(\mathbb{R}/\mathbb{Z} \), \( \lambda \) the Lebesgue measure on \(\mathbb{R}\)).

In this paper we shall study whether "\( \varphi(0) \neq \varphi(1) \)" is also a necessary condition for ergodicity of the skew product \( T_\alpha \): see Corollary 1.3. Further, we shall obtain a result on the ergodicity of \( T_\alpha \) in the case \( \varphi(0) = \varphi(1) \), \( \varphi \) \( K \)-times continuously differentiable on \([0,1] \), \( K \geq 2 \): see Theorem 1.4.

We shall use the following terminology. Let \( \mathbb{R}/\mathbb{Z} \) be identified with the interval \([0,1]\) (with addition modulo 1) and let \( \lambda \) denote the normalized Haar measure on \( \mathbb{R}/\mathbb{Z} \equiv [0,1] \). If \( n \in \mathbb{N} \) and \( x \in [0,1] \), we write

\[ \varphi_n(x) = \varphi(x) + \varphi(\{x + \alpha\}) + \cdots + \varphi(\{x + (n-1)\alpha\}) , \]

where \( \{ \cdot \} \) denotes the fractional part.

It is well-known that the following two questions are closely related to ergodicity of skew products of the type $T_\varphi$ (see, for example, [4, Lemmata 1, 2 and 3]).

0.1. **Question.** Under which conditions (for $\varphi$, $x$ and $\alpha$) do we have $\sup_n |\varphi_n(x)| < \infty$?

0.2. **Question.** What can be said about limit points of the sequence $(\varphi_n(x))_{n>1}$?

These two questions are of interest in the theory of uniform distribution modulo 1 as well, see [9, 6, 7, 10] and, in particular, [2] for the first question and [8, 3] for the second one.

1. **Results**

Throughout this paper we shall assume that $\varphi : [0, 1] \to \mathbb{R}$ is continuously differentiable on the closed interval $[0, 1]$, $\int_0^1 \varphi(t) \, dt = 0$, and $\varphi'$ is Lipschitz-continuous on $[0, 1]$ i.e.,

$$\sup\{|\varphi'(x) - \varphi'(y)| : 0 \leq x, y \leq 1, |x - y| < \delta\} \leq C \cdot \delta,$$

where $C$ is a positive constant, $\delta > 0$ arbitrary. Let $\alpha$ be irrational with simple continued fraction expansion $\alpha = [a_0; a_1, a_2, \ldots]$, $a_i$ the partial quotients. Let $(p_i/q_i)_{i>0}$ be the sequence of the convergents to $\alpha$.

1.1. **Theorem.** Let $\varphi(0) \neq \varphi(1)$. Then

1. $\sup_n |\varphi_n(x)| = \infty \forall x \in [0, 1]$;
2. for almost all $x \in [0, 1]$ the sequence $(\varphi_n(x))_{n>1}$ is dense in $\mathbb{R}$;
3. statement (2) cannot be improved: there are $\varphi$ (as above), $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $x \in [0, 1]$ such that $(\varphi_n(x))_{n>1}$ is not dense in $\mathbb{R}$.

1.2. **Theorem.** Let $\varphi(0) = \varphi(1)$. Then we have the following statements.

1. For almost all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ one has $\sup_n |\varphi_n(x)| < \infty \forall x \in [0, 1]$. In particular, this is true for those $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with $\sum_{i=0}^{\infty} a_{i+1}/q_i < \infty$ (any irrational $\alpha$ with bounded partial quotients has this property).
2. Let $\sum_{k \in \mathbb{Z}} \hat{\varphi}(k) e^{2\pi i k \cdot x}$ denote the Fourier series of $\varphi$.
   
   (2.1) If $\hat{\varphi}(k) \neq 0$ for infinitely many $k$, then there are continuum-many $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $\sup_n |\varphi_n(x)| = \infty \forall x \in [0, 1]$.
   
   (2.2) If $\hat{\varphi}(k) \neq 0$ for finitely many $k$ only, then for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ one has $\sup_n |\varphi_n(x)| < \infty \forall x \in [0, 1]$.

1.3. **Corollary.** Let $\varphi$ be as above and let $\alpha \in \mathbb{R} / \mathbb{Q}$ be such that $\sum_{i=0}^{\infty} a_{i+1}/q_i < \infty$. From Theorem 1.1(1) and Theorem 1.2(1) it follows that $\varphi(0) = \varphi(1)$ if and only if $\sup_n |\varphi_n(x)| < \infty$ for some/all $x \in [0, 1]$. Hence $T_\varphi$ is ergodic on $\mathbb{R} \setminus \mathbb{Z} \times \mathbb{R}$ if and only if $\varphi(0) \neq \varphi(1)$. 


1.4. Theorem. Suppose that \( \varphi : [0, 1] \to \mathbb{R} \) is \( K \)-times continuously differentiable on \([0, 1]\), \( K \geq 2 \), \( \varphi^{(j)}(0) = \varphi^{(j)}(1) \), \( 0 \leq j \leq K - 2 \), \( \varphi^{(K-1)}(0) \neq \varphi^{(K-1)}(1) \), and \( \int_0^1 \varphi(t) \, dt = 0 \). Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) be such that \( a_{i+1} \geq q_i^2 \) for \( i = 1, 2, \ldots \) (there are continuum-many such \( \alpha \)). Then \( T_\alpha(x, y) = (x + \alpha, y + \varphi(x)) \) is ergodic on \( \mathbb{R}/\mathbb{Z} \times \mathbb{R} \).

1.5. Corollary. If \( \varphi : [0, 1] \to \mathbb{R} \) is a polynomial of degree larger or equal to 1, \( \int_0^1 \varphi(t) \, dt = 0 \), then \( T_\alpha \) is ergodic on \( \mathbb{R}/\mathbb{Z} \times \mathbb{R} \) for at least continuum-many \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \).

Remark. Let \( \varphi \) and \( \alpha \) be as in Theorem 1.4. Then the transformation \( S_\alpha(x, y) = (x + \alpha, y + \varphi(x)) \) is ergodic on \( \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \) (with respect to the invariant measure \( \lambda \times \lambda \)). This implies that \( S_\alpha \) is uniquely ergodic and hence minimal, even though the map \( x \mapsto \varphi(x) \mod 1 \) belongs to the trivial homotopy class of functions from \( \mathbb{R}/\mathbb{Z} \) to \( \mathbb{R}/\mathbb{Z} \) (see [1]).

2. The proofs

Proof of Theorem 1.1. We shall give an indirect proof for (1). Suppose that \( \sup_n |\varphi_n(x)| < \infty \) for some \( x \in [0, 1] \). The sequence \( (T^k \psi)_k \) is uniformly distributed modulo 1, hence (see [5, Corollary 1.11])

\[
\|\varphi_n\|_2^2 = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} |\varphi_n(I^k x)|^2 = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} |\varphi_{n+k}(x) - \varphi_k(x)|^2.
\]

Therefore

\[
\sup_n \|\varphi_n\|_L^2 < \infty \quad (L^2 = L^2(\mathbb{R}/\mathbb{Z}, \lambda)).
\]

If \( Tx := x + \alpha \mod 1 \) with \( x \in \mathbb{R}/\mathbb{Z} \equiv [0, 1], \) then \( \varphi = g - g \circ T \) in \( L^2 \) with some \( g \in L^2 \) (see [9, 6]). This implies

\[
\sum_{k \in \mathbb{Z}} |\hat{\psi}(k)|^2 / \|k\alpha\|^2 < \infty,
\]

where \( \hat{\psi}(k) \) stands for the \( k \)th Fourier coefficient of \( \varphi \) and \( \|k\alpha\| \) denotes the distance to the nearest integer of \( k\alpha \).

\( \varphi' \) is Lipschitz-continuous on \([0, 1]\), so integration by parts gives (\( k \neq 0 \))

\[
\hat{\varphi}(k) \geq C \cdot (|\varphi(1) - \varphi(0)|/|k|) \quad \text{with some constant} \ C > 0.
\]

It is \( \|q, k\| \leq 1/(a_{i+1}, a_i) \) and the sum in (2.1) can be estimated from below by

\[
\sum_{i=1}^\infty |\hat{\varphi}(q_i)|^2 / \|q, k\|^2.
\]

The latter sum diverges, which contradicts (2.1).

(2): We shall apply [4, Lemma 2]. Let \( d \neq 0 \) be an arbitrary real number. It will be shown that for almost all \( x \) in \([0, 1]\) there is a sequence \( (N_k)_k \) such that \( \lim_{k \to \infty} \varphi_{N_k}(x) = d \) (and even \( \lim_{k \to \infty} \|N_k\| = 0 \)). \( (N_k)_k \) will depend on \( x \).
Write $m := \lfloor 2d \cdot |\varphi(1) - \varphi(0)| \rfloor (1 - \rho(\alpha))^{-1}$ and $c := d/m$, $\rho(\alpha)$ as in [4]. Let $K$ be a nonnegative integer and define $M(c, K) := \{x \in [0, 1[: \exists (q_n)_{n \geq 1} \text{ subsequence of } (q_n)_{n \geq 1} \text{ such that } \lim_{n \to \infty} \varphi_{q_n}(x + K\alpha) = c\}$. It is $q_n = q_n(c, K, x)$. If $M(c) := \bigcap_{k=0}^{\infty} M(c, K)$ then [4, Lemma 2] implies $\lambda(M(c)) = 1$.

For every $x$ in $M(c)$ there are subsequences $(Q_k(i))_{k=1}^{m}$, $i = 1, \ldots, m$, of the sequence $(q_n)_{n \geq 1}$ of denominators of the convergents to $\alpha$ such that

$$|\varphi_{Q_k(i)}(x) - c| < 1/(km),$$

$$|\varphi_{Q_k(i+1)}(x + (Q_k(1) + Q_k(2) + \cdots + Q_k(i))\alpha) - c| < 1/(km),$$

$i = 1, \ldots, m - 1$. Let $N_k := Q_k(1) + \cdots + Q_k(m)$. Then $\|N_k\alpha\|$ tends to zero and $|\varphi_{N_k}(x) - d| < 1/k$.

(3): If $\varphi(x) = x - 1$ and $x = 0$, then Sós [10] has shown that $(\varphi_{-n}(0))_{n \geq 1}$ is bounded from below for continuum-many $\alpha$. \(\square\)

2.1. Lemma. Let $s, q \in \mathbb{N}$, $s < q$, $(s, q) = 1$, and let $s/q$ have the simple continued fraction expansion $[0; b_1, \ldots, b_n]$. Then the discrepancy $D^*_q, 1$ of the $q-1$ points $(k/q, \{ks/q\})_{k=1}^{q-2}$ (and of the points $(k/q, \{ks/q\})_{k=1}^{q-1}$ as well) satisfies

$$D^*_q, 1 \leq C \left( \sum_{i=1}^{\infty} b_i \right) / q$$

with some absolute constant $C$.

Proof. See [5, Chapter 2] for the definition of the discrepancy. It is elementary to show that

$$(q - 1)D^*_q, 1 \leq 1 + \max_{1 \leq m \leq q - 1} m \cdot D^*_m((\{ks/q\})_{k=0}^{m-1}).$$

From [5, inequality (3.18)], it follows that

$$m \cdot D^*_m((\{ks/q\})_{k=0}^{m-1}) \leq 1 + 2 \sum_{i=1}^{n} b_i.$$ \(\square\)

Corollary. The discrepancy $D^*_{q_n}$ of the $q_n - 1$ points $(k/q_n, \{kq_n^{-1}/q_n\})_{k=0}^{q_n-2}$ satisfies

$$D^*_{q_n} \leq C \left( \sum_{i=1}^{n} a_i \right) / q_n$$

with some absolute constant $C$.

Every $n \in \mathbb{N}$ has a unique representation of the form $n = \sum_{i=0}^{m} q_i$, with digits $n_0 \in \{0, 1, \ldots, a_1 - 1\}$, $n_i \in \{0, 1, \ldots, a_i - 1\}$, $i = 1, 2, \ldots$. We shall write $n(k) := \sum_{i=0}^{k-1} n_iq_i$, $k = 1, 2, \ldots$. 

2.2. **Lemma.** Let \( n \in \mathbb{N} \), \( n = \sum_{i=0}^{\infty} n_i q_i \), \( n_i \neq 0 \), and let \( x \in [0, 1[ \). Then

\[
\varphi_n(x) = (\varphi(1) - \varphi(0)) \sum_{k=0}^{n-1} \sum_{l=0}^{n_i-1} \sigma_{k,l} + \sum_{k=0}^{n-1} \sum_{l=0}^{n_i-1} \tau_{k,l}
\]

\[+ O \left( \sum_{k=0}^{n-1} n_k \cdot \max \left\{ 1/q_k, \sum_{l=0}^{k} a_i/(a_{k+1} q_k) \right\} \right),
\]

(2.3)

where

\[
\sum_{k=0}^{n-1} \text{ denotes } \sum_{k=0}^{n} \text{, for } k: n_k \neq 0.
\]

\[
\alpha = \frac{p_k}{q_k} + \frac{\theta_k}{(a_{k+1} q_k^2)}, \quad k = 0, 1, \ldots,
\]

\[
\theta_k = (-1)^k \theta_k, \quad |\theta_k| < 1,
\]

\[
\sigma_{k,l} = \theta_k/(2a_{k+1}) + \{q_k x_{k,l}\}^{-1},
\]

\[
x_{k,l} = \{x + (n(k) + \lfloor q_l \rfloor)\alpha\},
\]

\[
\tau_{k,l} = \begin{cases} 
\varphi(\{1 - 1/q_k + \{q_k x_{k,l}\}/q_k + m_{q_k-1} \theta_k/(a_{k+1} q_k^2)\}) - \varphi(1 - 1/q_k) & \text{if } k \text{ is even}, \\
\varphi(\{m_0 \theta_k/(a_{k+1} q_k^2) + \{q_k x_{k,l}\}/q_k\}) - \varphi(0) & \text{if } k \text{ is odd}, 
\end{cases}
\]

\[
m_{q_k-1} \text{ such that } m_{q_k-1} \cdot p_k = (q_k - 1) - r_{k,l} \mod q_k,
\]

\[
m_0 \text{ such that } m_0 \cdot p_k = -r_{k,l} \mod q_k,
\]

\[
0 \leq m_0, \quad m_{q_k-1} \leq q_k - 1, \quad r_{k,l} = [q_k x_{k,l}].
\]

**Proof.** From identity (1) and Proposition 1 and 2 of [4] and from Lemma 2.1 it follows that (\( q := q_k, a := a_{k+1}, \theta := \theta_k \))

\[
\varphi_{q_k}(x) = (\varphi(1) - \varphi(0))(\theta/(2a) + \{qx\} - \frac{1}{2})
\]

\[+ \varphi(\{1 - 1/q + \{qx\}/q + m_{q_k-1} \theta/(aq^2)\})
\]

\[- \varphi(1 - 1/q) + (\theta/a)O \left( \sum_{l=0}^{k} a_i/q \right) + O(1/q)
\]

for even \( k \). For odd \( k \), the second and the third term in this sum have to be replaced by \( \varphi(\{m_0 \theta/(aq^2) + \{qx\}/q\}) \) and by \( \varphi(0) \), respectively. Summation over \( k \) and \( l \) will give the result. \( \square \)
2.3. Lemma. For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, \( \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} a_i \right) / q_k < \infty \).

Proof. We define \( s(k) := \text{card}\{ i, 0 \leq i \leq k : a_i = 1 \} \) and \( t(k) := \text{card}\{ i, 0 \leq i \leq k - 1 : a_i = a_{i+1} = 1 \} \). If \( s(k) \leq \frac{3}{4} k \), then

\[
q_k \geq \max\{ a_i : 0 \leq i \leq k \} \cdot 2^{k/4 - 1}.
\]

If \( s(k) > \frac{3}{4} k \), then

\[
q_k \geq \max\{ a_i : 0 < i < k \} \cdot 2^{t(k)/2} \geq \max\{ a_i : 0 < i < k \} \cdot 2^{3k/16}.
\]

From these estimates the result follows easily.  

Proof of Theorem 1.2. (1): Because of $\varphi(1) = \varphi(0)$ and of

\[
|\tau_{k,\alpha}| \leq \max\{ |\varphi'(y)| : y \in [0, 1] \} \cdot 2/q_k
\]

identity (2.3) implies

\[
\varphi_n(x) = O\left( \sum_{k=0}^{\infty} a_{k+1} \max\left\{ 1/q_k, \sum_{i=0}^{k} a_i/(a_{k+1}q_k) \right\} \right).
\]

From Lemma 2.3 and the fact that $\sum_{k=0}^{\infty} a_{k+1}/q_k$ is finite for almost all $x$ the result follows.

(2.1): It is not difficult to show that there are continuum-many $\alpha$ such that

\[
\sum_{k=1}^{\infty} |\hat{\varphi}(k)|^2/\|k\alpha\|^2
\]

diverges. From the proof of Theorem 1.1(1) the result follows.

(2.2): By simple calculation we see that

\[
|\varphi_n(x)| \leq C \sum_{k} |\hat{\varphi}(k)|/\|k\alpha\|
\]

with some constant $C$.

2.4. Lemma. Let $\varphi$ and $\alpha$ be as in Theorem 1.4 and let $d \in \mathbb{R}$, $d > 0$, be arbitrary. Then, for almost all $x \in [0, 1]$, there is a subsequence $(q_{k_m})_{m \geq 1}$ of the sequence $(q_{2k})_{k \geq 1}$, $q_{k_m} = q_{k_m}(x)$, such that

\[
\lim_{m \to \infty} \varphi_{t_m,q_{k_m}}(x) = d \cdot \text{sign}(\varphi^{(K-1)}(1) - \varphi^{(K-1)}(0))
\]

and

\[
\lim_{m \to \infty} \| t_m q_{k_m} \alpha \| = 0,
\]

where

\[
t_m := \begin{cases} \left[ dq_{k_m}^{K-1}/C_K \right], & K \text{ even}, \\ \left[ dq_{k_m}^{K-1}/(\tilde{C}_K q_{k_m} x) \right], & K \text{ odd}, \end{cases}
\]

\[
C_K := |\varphi^{(K-1)}(1) - \varphi^{(K-1)}(0)| \cdot B_K / K!, \quad \tilde{C}_K := KC_K B_{K-1}/B_K
\]

and $\{B_k\}_{k \geq 1}$ is the sequence of Bernoulli-numbers.
Proof. It follows from Euler's summation formula that $(q := q^k)$

$$
\sum_{i=0}^{q-1} \varphi(i/q) = \frac{(\varphi^{(K-1)}(1) - \varphi^{(K-1)}(0)) B_K}{K! q^{K-1}} + O(q^{-K})
$$

if $K$ is even, and

$$
\sum_{i=0}^{q-1} \varphi(i/q) = O(q^{-K}),
$$

as well as

$$
\sum_{i=0}^{q-1} \varphi'(i/q) = \frac{(\varphi^{(K-1)}(1) - \varphi^{(K-1)}(0)) B_{K-1}}{(K-1)! q^{K-2}} + O(q^{-K+1})
$$

if $K$ is odd.

The sequence $(\{q_{2k}x\})_{k \geq 1}$ is uniformly distributed modulo 1 for almost all $x \in [0, 1[$ (see [5, Theorem 4.1]). Hence, for almost all $x$, there exists a subsequence $(q_{k_m})_{m \geq 1}$—it depends on $x$—such that

$$
\lim_{m \to \infty} \{q_{k_m}x\} = 0, \quad \{q_{k_m}x\} \geq q_{k_m}^{-1/2}, \quad \forall m \in \mathbb{N}.
$$

Let $q := q_{k_m}$, $\alpha := \alpha_{k_m+1}$, $t := t_m$ and $\theta := \theta_{k_m}$. Then $\theta > 0$, $0 < t \leq a^{1/2}$ and $(q_x)/q + 1/(aq) + t/(aq) < 1/q$ for sufficiently large $m$. Hence (see [4, proof of Lemma 2])

$$
\varphi_{tq}(x) = \sum_{b=0}^{l-1} \sum_{l=0}^{q-1} \varphi(l/q + m_l \theta/(aq^2) + \{qx\}/q + b \theta/(aq)),
$$

where $m_l$ is defined by the congruence $m_l q_{k_m} \equiv l - \lfloor q_{k_m}x \rfloor \mod q_{k_m}$, $0 \leq m_l < q_{k_m}$. We apply Taylor's formula and find that

$$
\varphi_{tq}(x) = t \sum_{l=0}^{q-1} \varphi(l/q) + t \{qx\}/q \sum_{l=0}^{q-1} \varphi'(l/q) + O(t^2/a) + O(t \{qx\}^2 q^{-K+1}).
$$

(2.5)

If $K$ is even, then (2.4) and (2.5) imply

$$
\varphi_{tq}(x) = t C_K / q^{K-1} + O(t^2/a) + O(t \{qx\} q^{-K+1}).
$$

If $K$ is odd, then (2.4) and (2.5) imply

$$
\varphi_{tq}(x) = O(t q^{-K}) + t \{qx\} \tilde{C}_K q^{-K+1} + O(t^2/a) + O(t \{qx\}^2 q^{-K+1})
$$

Thus, in both cases,

$$
\lim_{m \to \infty} \varphi_{tq}(x) = d \cdot \text{sign}(\varphi^{(K-1)}(1) - \varphi^{(K-1)}(0)).
$$

It is trivial that $\lim_{m \to \infty} \|t_m q_{k_m} \alpha\| = 0$. □

Proof of Theorem 1.4. Let $c \neq 0$ be an arbitrary real number with the same sign as $\varphi^{(K-1)}(1) - \varphi^{(K-1)}(0)$. Define $A(c) := \{x \in \mathbb{R}/\mathbb{Z} : [0, 1[ : \exists (Q_m)_{m \geq 1}, \text{subsequence of}$
N, such that \( \lim_{m \to \infty} \varphi_{Q_m}(x) = c \) and \( \lim_{m \to \infty} \|Q_m\alpha\| = 0 \). The set \( A(c) \) is invariant under the ergodic transformation \( x \mapsto x + \alpha \mod 1 \) on \( \mathbb{R}/\mathbb{Z} \), hence \( \lambda(A(c)) \in \{0, 1\} \). From Lemma 2.4 it follows that the \( \lambda \)-measure of \( A(c) \) is 1. We follow the proof of [4, Lemma 3] to show that \( c \) is a period of the skew product \( T_c \). The set \( P_c \) of periods of \( T_c \) is a group, hence \( P_c = \mathbb{R} \). This implies that \( T_c \) is ergodic on \( \mathbb{R}/\mathbb{Z} \times \mathbb{R} \). \( \square \)

References