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ON WEYL SUMS AND SKEW PRODUCTS OVER IRRATIONAL ROTATIONS

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Abstract. Let $\varphi:[0,1]\to\mathbb{R}$ have a Lipschitz-continuous derivative on $[0,1]$, $\int_0^1\varphi(t)\,\mathrm{d}t=0$, let $\alpha\in\mathbb{R}\backslash\mathbb{Q}$ and write $\varphi_n(x)=\varphi(x)+\varphi(\{x+\alpha\})+\cdots+\varphi(\{x+(n-1)\alpha\})$, $n\in\mathbb{N}$, $x\in[0,1[$. In this paper results on the boundedness and the limit points of the sequence $(\varphi_n(x))_{n\geq 1}$ are given. Further, ergodicity of the skew product $(x,y)\mapsto(x+\alpha,y+\varphi(x))$ on $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$ is proved for certain classes of φ and α.

Contents

Introduction

In [4] the following result was shown.

Theorem. *Let $\varphi:[0,1]\to\mathbb{R}$ be continuously differentiable on the closed interval $[0,1]$, let $\int_0^1\varphi(t)\,\mathrm{d}t=0$ and suppose that α is irrational. If $\varphi(0)\neq\varphi(1)$, then $T_\varphi(x,y)=(x+\alpha,y+\varphi(x))$ is ergodic on $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$ (with respect to $\lambda\times\lambda_\infty$, λ the normalized Haar measure on $\mathbb{R}/\mathbb{Z}$, λ_∞ the Lebesgue measure on $\mathbb{R}$).*

In this paper we shall study whether "$\varphi(0)\neq\varphi(1)$" is also a necessary condition for ergodicity of the skew product T_φ: see Corollary 1.3. Further, we shall obtain a result on the ergodicity of T_φ in the case $\varphi(0)=\varphi(1)$, φ K-times continuously differentiable on $[0,1]$, $K\geq 2$: see Theorem 1.4.

We shall use the following terminology. Let $\mathbb{R}/\mathbb{Z}$ be identified with the interval $[0,1[$ (with addition modulo 1) and let λ denote the normalized Haar measure on $\mathbb{R}/\mathbb{Z}\cong[0,1[$. If $n\in\mathbb{N}$ and $x\in[0,1[$, we write

$$\varphi_n(x):=\varphi(x)+\varphi(\{x+\alpha\})+\cdots+\varphi(\{x+(n-1)\alpha\}),$$

where $\{\cdot\}$ denotes the fractional part.

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It is well-known that the following two questions are closely related to ergodicity of skew products of the type T_φ (see, for example, [4, Lemmata 1, 2 and 3]).

0.1. Question. Under which conditions (for φ, x and α) do we have $\sup_n|\varphi_n(x)|<\infty$?

0.2. Question. What can be said about limit points of the sequence $(\varphi_n(x))_{n\geq 1}$?

These two questions are of interest in the theory of uniform distribution modulo 1 as well, see [9, 6, 7, 10] and, in particular, [2] for the first question and [8, 3] for the second one.

1. Results

Throughout this paper we shall assume that $\varphi:[0,1]\to\mathbb{R}$ is continuously differentiable on the closed interval $[0,1]$, $\int_0^1\varphi(t)\,\mathrm{d}t=0$, and φ' is Lipschitz-continuous on $[0,1]$ i.e.,

$$\sup\{|\varphi'(x)-\varphi'(y)|: 0\leq x, y\leq 1, |x-y|<\delta\}\leq C\cdot\delta,$$

C a positive constant, $\delta>0$ arbitrary. Let α be irrational with simple continued fraction expansion $\alpha=[a_0; a_1, a_2, \ldots]$, a_i the partial quotients. Let $(p_i/q_i)_{i\geq 0}$ be the sequence of the convergents to α.

1.1. Theorem. *Let* $\varphi(0)\neq\varphi(1)$. *Then*

(1) $\sup_n|\varphi_n(x)|=\infty\ \forall x\in[0,1[$;

(2) *for almost all* $x\in[0,1[$ *the sequence* $(\varphi_n(x))_{n\geq 1}$ *is dense in* $\mathbb{R}$;

(3) *statement* (2) *cannot be improved: there are* φ (*as above*), $\alpha\in\mathbb{R}\backslash\mathbb{Q}$ *and* $x\in[0,1[$ *such that* $(\varphi_n(x))_{n\geq 1}$ *is not dense in* $\mathbb{R}$.

1.2. Theorem. *Let* $\varphi(0)=\varphi(1)$. *Then we have the following statements.*

(1) For almost all $\alpha\in\mathbb{R}\backslash\mathbb{Q}$ *one has* $\sup_n|\varphi_n(x)|<\infty\ \forall x\in[0,1[$. *In particular, this is true for those* $\alpha\in\mathbb{R}\backslash\mathbb{Q}$ *with* $\sum_{i=0}^{\infty}a_{i+1}/q_i<\infty$ (*any irrational* α *with bounded partial quotients has this property*).

(2) *Let* $\sum_{k\in\mathbb{Z}}\hat{\varphi}(k)\,\mathrm{e}^{2\pi ik\cdot}$ *denote the Fourier series of* φ.

(2.1) *If* $\hat{\varphi}(k)\neq 0$ *for infinitely many* k, *then there are continuum-many* $\alpha\in\mathbb{R}\backslash\mathbb{Q}$ *such that* $\sup_n|\varphi_n(x)|=\infty\ \forall x\in[0,1[$.

(2.2) *If* $\hat{\varphi}(k)\neq 0$ *for finitely many* k *only, then for all* $\alpha\in\mathbb{R}\backslash\mathbb{Q}$ *one has* $\sup_n|\varphi_n(x)|<\infty\ \forall x\in[0,1[$.

1.3. Corollary. *Let* φ *be as above and let* $\alpha\in\mathbb{R}/\mathbb{Q}$ *be such that* $\sum_{i=0}^{\infty}a_{i+1}/q_i<\infty$. *From Theorem* 1.1(1) *and Theorem* 1.2(1) *it follows that* $\varphi(0)=\varphi(1)$ *if and only if* $\sup_n|\varphi_n(x)|<\infty$ *for some/all* $x\in[0,1[$. *Hence* T_φ *is ergodic on* $\mathbb{R}\backslash\mathbb{Z}\times\mathbb{R}$ *if and only if* $\varphi(0)\neq\varphi(1)$.

1.4. Theorem. *Suppose that $\varphi:[0,1]\to\mathbb{R}$ is K-times continuously differentiable on $[0,1]$, $K\geq 2$, $\varphi^{(j)}(0)=\varphi^{(j)}(1)$, $0\leq j\leq K-2$, $\varphi^{(K-1)}(0)\neq\varphi^{(K-1)}(1)$, and $\int_0^1\varphi(t)\,\mathrm{d}t=0$. Let $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ be such that $a_{i+1}\geq q_i^{2K}$ for $i=1,2,\ldots$ (there are continuum-many such α). Then $T_\varphi(x,y)=(x+\alpha, y+\varphi(x))$ is ergodic on $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$.*

1.5. Corollary. *If $\varphi:[0,1]\to\mathbb{R}$ is a polynomial of degree larger or equal to 1, $\int_0^1\varphi(t)\,\mathrm{d}t=0$, then T_φ is ergodic on $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$ for at least continuum-many $\alpha\in\mathbb{R}\setminus\mathbb{Q}$.*

Remark. Let φ and α be as in Theorem 1.4. Then the transformation $S_\varphi(x,y)=(x+\alpha, y+\varphi(x))$ is ergodic on $\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}$ (with respect to the invariant measure $\lambda\times\lambda$). This implies that S_φ is uniquely ergodic and hence minimal, even though the map $x\mapsto\varphi(x)$ mod 1 belongs to the trivial homotopy class of functions from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}/\mathbb{Z}$ (see [1]).

2. The proofs

Proof of Theorem 1.1. We shall give an indirect proof for (1). Suppose that $\sup_n|\varphi_n(x)|<\infty$ for some $x\in[0,1[$. The sequence $(T^kx)_{k\geq 0}$ is uniformly distributed modulo 1, hence (see [5, Corollary 1.1])

$$\|\varphi_n\|_2^2=\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}|\varphi_n(T^kx)|^2=\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}|\varphi_{n+k}(x)-\varphi_k(x)|^2.$$

Therefore

$$\sup_n\|\varphi_n\|_{L^2}<\infty\quad (L^2=L^2(\mathbb{R}/\mathbb{Z},\lambda)).$$

If $Tx:=x+\alpha$ mod 1 with $x\in\mathbb{R}/\mathbb{Z}\cong[0,1[$, then $\varphi=g-g\circ T$ in L^2 with some $g\in L^2$ (see [9, 6]). This implies

$$\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}|\hat{\varphi}(k)|^2/\|k\alpha\|^2<\infty, \tag{2.1}$$

where $\hat{\varphi}(k)$ stands for the kth Fourier coefficient of φ and $\|k\alpha\|$ denotes the distance to the nearest integer of $k\alpha$.

φ' is Lipschitz-continuous on $[0,1]$, so integration by parts gives ($k\neq 0$)

$$\hat{\varphi}(k)\geq C\cdot(|\varphi(1)-\varphi(0)|/|k|)\quad\text{with some constant } C>0.$$

It is $\|q_i\alpha\|<1/(a_{i+1}q_i)$ and the sum in (2.1) can be estimated from below by

$$\sum_{i=1}^{\infty}|\hat{\varphi}(q_i)|^2/\|q_i\alpha\|^2.$$

The latter sum diverges, which contradicts (2.1).

(2): We shall apply [4, Lemma 2]. Let $d\neq 0$ be an arbitrary real number. It will be shown that for almost all x in $[0,1[$ there is a sequence $(N_k)_{k\geq 1}$ such that $\lim_{k\to\infty}\varphi_{N_k}(x)=d$ (and even $\lim_{k\to\infty}\|N_k\alpha\|=0$). $(N_k)_{k\leq 1}$ will depend on x.

Write $m := [2|d| \cdot |\varphi(1)-\varphi(0)|^{-1}(1-\rho(\alpha))^{-1}]$ and $c := d/m$, $\rho(\alpha)$ as in [4]. Let K be a nonnegative integer and define $M(c, K) := \{x \in [0, 1[: \exists (q_{n_j})_{j\geq 1}$ subsequence of $(q_n)_{n\geq 1}$ such that $\lim_{j\to\infty} \varphi_{q_{n_j}}(x+K\alpha) = c\}$. It is $q_{n_j} = q_{n_j}(c, K, x)$. If $M(c) := \bigcap_{k=0}^{\infty} M(c, K)$ then [4, Lemma 2] implies $\lambda(M(c) = 1$.

For every x in $M(c)$ there are subsequences $(Q_k(i))_{k\geq 1}$, $i = 1, \ldots, m$, of the sequence $(q_n)_{n\geq 1}$ of denominators of the convergents to α such that

$$|\varphi_{Q_k(1)}(x) - c| < 1/(km),$$

$$|\varphi_{Q_k(i+1)}(x + (Q_k(1) + Q_k(2) + \cdots + Q_k(i))\alpha) - c| < 1/(km),$$

$i = 1, 2, \ldots, m-1$. Let $N_k := Q_k(1) + \cdots + Q_k(m)$. Then $\|N_k\alpha\|$ tends to zero and $|\varphi_{N_k}(x) - d| < 1/k$.

(3): If $\varphi(x) = x - \frac{1}{2}$ and $x = 0$, then Sós [10] has shown that $(\varphi_n(0))_{n\geq 1}$ is bounded from below for continuum-many α. □

2.1. Lemma. *Let* $s, q \in \mathbb{N}$, $s < q$, $(s, q) = 1$, *and let* s/q *have the simple continued fraction expansion* $[0; b_1, \ldots, b_n]$. *Then the discrepancy* D^*_{q-1} *of the* $q-1$ *points* $(k/q, \{ks/q\})_{k=0}^{q-2}$ *(and of the points* $(k/q, \{ks/q\})_{k=1}^{q-1}$ *as well) satisfies*

$$D^*_{q-1} \leq C \cdot \left(\sum_{i=1}^{n} b_i\right) \Big/ q$$

with some absolute constant C.

Proof. See [5, Chapter 2] for the definition of the discrepancy. It is elementary to show that

$$(q-1)D^*_{q-1} \leq 1 + \max_{1\leq m\leq q-1} m \cdot D^*_m((\{ks/q\})_{k=0}^{m-1}).$$

From [5, inequality (3.18)], it follows that

$$m \cdot D^*_m((\{ks/q\})_{k=0}^{m-1}) \leq 1 + 2\sum_{i=1}^{n} b_i. \qquad \square$$

Corollary. *The discrepancy* $D^*_{q_n-1}$ *of the* $q_n - 1$ *points* $(k/q_n, \{kq_{n-1}/q_n\})_{k=0}^{q_n-2}$ *satisfies*

$$D^*_{q_n-1} \leq C\left(\sum_{i=1}^{n} a_i\right) \Big/ q_n \tag{2.2}$$

with some absolute constant C.

Every $n \in \mathbb{N}$ has a unique representation of the form $n = \sum_{i=0}^{s} n_i q_i$ with digits $n_0 \in \{0, 1, \ldots, a_1 - 1\}$, $n_i \in \{0, 1, \ldots, a_{i+1}\}$, $i = 1, 2, \ldots$ We shall write $n(k) := \sum_{i=0}^{k-1} n_i q_i$, $k = 1, 2, \ldots$.

2.2. Lemma. *Let* $n \in \mathbb{N}$, $n = \sum_{i=0}^{s} n_i q_i$, $n_s \neq 0$, *and let* $x \in [0, 1[$. *Then*

$$\varphi_n(x) = (\varphi(1) - \varphi(0)) \sum_{k=0}^{s}{}' \sum_{l=0}^{n_k - 1} \sigma_{k,l} + \sum_{k=0}^{s}{}' \sum_{l=0}^{n_k-1} \tau_{k,l}$$
$$+ \mathrm{O}\left(\sum_{k=0}^{s}{}' n_k \cdot \max\left\{ 1/q_k, \sum_{i=0}^{k} a_i/(a_{k+1} q_k) \right\} \right), \tag{2.3}$$

where

$$\sum_{k=0}^{s}{}' \quad \textit{denotes} \quad \sum_{\substack{k=0 \\ k: n_k \neq 0}}^{s},$$

$$\alpha = \frac{p_k}{qk} + \frac{\theta_k}{(a_{k+1} q_k^2)}, \quad k = 0, 1, \ldots,$$

$$\theta_k = (-1)^k |\theta_k|, \qquad |\theta_k| < 1,$$

$$\sigma_{k,l} = \theta_k/(2a_{k+1}) + \{q_k x_{k,l}\} - \tfrac{1}{2},$$

$$x_{k,l} = \{x + (n(k) + l q_k)\alpha\},$$

$$\tau_{k,l} = \begin{cases} \varphi(\{1 - 1/q_k + \{q_k x_{k,l}\}/q_k + m_{q_k - 1}\theta_k/(a_{k+1} q_k^2)\}) - \varphi(1 - 1/q_k) & \\ & k \text{ even}, \\ \varphi(\{m_0 \theta_k/(a_{k+1} q_k^2) + \{q_k x_{k,l}\}/q_k\}) - \varphi(0) & k \text{ odd}, \end{cases}$$

$$m_{q_k - 1} \; \textit{such that} \; m_{q_k-1} \cdot p_k \equiv (q_k - 1) - r_{k,l} \bmod q_k,$$

$$m_0 \; \textit{such that} \; m_0 \cdot p_k \equiv -r_{k,l} \bmod q_k,$$

$$0 \leq m_0, \qquad m_{q_k - 1} \leq q_k - 1, \qquad r_{k,l} = [q_k x_{k,l}].$$

Proof. From identity (1) and Proposition 1 and 2 of [4] and from Lemma 2.1 it follows that ($q := q_k$, $a := a_{k+1}$, $\theta := \theta_k$)

$$\varphi_{q_k}(x) = (\varphi(1) - \varphi(0))(\theta/(2a) + \{qx\} - \tfrac{1}{2})$$
$$+ \varphi(\{1 - 1/q + \{qx\}/q + m_{q-1}\theta/(aq^2)\})$$
$$- \varphi(1 - 1/q) + (\theta/a)\mathrm{O}\left(\sum_{i=0}^{k} a_i/q \right) + \mathrm{O}(1/q)$$

for even k. For odd k, the second and the third term in this sum have to be replaced by $\varphi(\{m_0\theta/(aq^2) + \{qx\}/q\})$ and by $\varphi(0)$, respectively. Summation over k and l will give the result. □

2.3. Lemma. *For every* $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $\sum_{k=0}^{\infty} (\sum_{i=0}^{k} a_i)/q_k < \infty$.

Proof. We define $s(k) := \operatorname{card}\{i, 0 \leq i \leq k\colon a_i = 1\}$ and $t(k) := \operatorname{card}\{i, 0 \leq i \leq k-1\colon a_i = a_{i+1} = 1\}$. If $s(k) \leq \frac{3}{4}k$, then

$$q_k \geq \max\{a_i : 0 \leq i \leq k\} \cdot 2^{k/4-1}.$$

If $s(k) > \frac{3}{4}k$, then

$$q_k \geq \max\{a_i : 0 \leq i \leq k\} \cdot 2^{t(k)/2} > \max\{a_i : 0 \leq i \leq k\} \cdot 2^{3k/16}$$

From these estimates the result follows easily. □

Proof of Theorem 1.2. (1): Because of $\varphi(1) = \varphi(0)$ and of

$$|\tau_{k,l}| \leq \max\{|\varphi'(y)|: y \in [0, 1]\} \cdot 2/q_k$$

identity (2.3) implies

$$\varphi_n(x) = \mathrm{O}\left(\sum_{k=0}^{s}{}' \; a_{k+1} \max\left\{1/q_k, \sum_{i=0}^{k} a_i/(a_{k+1}q_k)\right\}\right).$$

From Lemma 2.3 and the fact that $\sum_{k=0}^{\infty} a_{k+1}/q_k$ is finite for almost all α the result follows.

(2.1): It is not difficult to show that there are continuum-many α such that

$$\sum_{k=1}^{\infty} |\hat{\varphi}(k)|^2/\|k\alpha\|^2$$

diverges. From the proof of Theorem 1.1(1) the result follows.

(2.2): By simple calculation we see that

$$|\varphi_n(x)| \leq C \sum_k |\hat{\varphi}(k)|/\|k\alpha\|$$

with some constant C. □

2.4. Lemma. *Let* φ *and* α *be as in Theorem* 1.4 *and let* $d \in \mathbb{R}$, $d > 0$, *be arbitrary. Then, for almost all* $x \in [0, 1[$, *there is a subsequence* $(q_{k_m})_{m \geq 1}$ *of the sequence* $(q_{2k})_{k \geq 1}$, $q_{k_m} = q_{k_m}(x)$, *such that*

$$\lim_{m \to \infty} \varphi_{t_m q_{k_m}}(x) = d \cdot \operatorname{sign}(\varphi^{(K-1)}(1) - \varphi^{(K-1)}(0))$$

and

$$\lim_{m \to \infty} \|t_m q_{k_m} \alpha\| = 0,$$

where

$$t_m := \begin{cases} [dq_{k_m}^{K-1}/C_K], & K \text{ even}, \\ [dq_{k_m}^{K-1}/(\tilde{C}_K\{q_{k_m}x\})], & K \text{ odd}, \end{cases}$$

$$C_K := |\varphi^{(K-1)}(1) - \varphi^{(K-1)}(0)| \cdot B_K/K!, \qquad \tilde{C}_K := KC_K B_{K-1}/B_K$$

and $(B_i)_{i \geq 1}$ *is the sequence of Bernoulli-numbers.*

Proof. It follows from Euler's summation formula that $(q := qk)$

$$\sum_{i=0}^{q-1} \varphi(i/q) = \frac{(\varphi^{(K-1)}(1) - \varphi^{(K-1)}(0))B_K}{K!\, q^{K-1}} + O(q^{-K})$$

if K is even, and

$$\sum_{i=0}^{q-1} \varphi(i/q) = O(q^{-K}), \tag{2.4}$$

as well as

$$\sum_{i=0}^{q-1} \varphi'(i/q) = \frac{(\varphi^{(K-1)}(1) - \varphi^{(K-1)}(0))B_{K-1}}{(K-1)!\, q^{K-2}} + O(q^{-K+1})$$

if K is odd.

The sequence $(\{q_{2k}x\})_{k\geq 1}$ is uniformly distributed modulo 1 for almost all $x \in [0, 1[$ (see [5, Theorem 4.1]). Hence, for almost all x, there exists a subsequence $(q_{k_m})_{m\geq 1}$—it depends on x—such that

$$\lim_{m\to\infty} \{q_{k_m}x\} = 0, \qquad \{q_{k_m}x\} \geq q_{k_m}^{-1/2}, \quad \forall m \in \mathbb{N}.$$

Let $q := q_{k_m}$, $a := a_{k_m+1}$, $t := t_m$ and $\theta := \theta_{k_m}$. Then $\theta > 0$, $0 < t \leq a^{1/2}$ and $\{qx\}/q + 1/(aq) + t/(aq) < 1/q$ for sufficiently large m. Hence (see [4, proof of Lemma 2])

$$\varphi_{tq}(x) = \sum_{b=0}^{t-1} \sum_{l=0}^{q-1} \varphi(l/q + m_l\theta/(aq^2) + \{qx\}/q + b\theta/(aq)),$$

where m_l is defined by the congruence $m_l p_{k_m} \equiv l - [q_{k_m}x] \bmod q_{k_m}$, $0 \leq m_1 < q_{k_m}$. We apply Taylor's formula and find that

$$\varphi_{tq}(x) = t \sum_{l=0}^{q-1} \varphi(l/q) + t\{qx\}/q \sum_{l=0}^{q-1} \varphi'(l/q) + O(t^2/a) + O(t\{qx\}^2 q^{-K+1}). \tag{2.5}$$

If K is even, then (2.4) and (2.5) imply

$$\varphi_{tq}(x) = tC_K/q^{K-1} + O(t^2/a) + O(t\{qx\}q^{-K+1}).$$

If K is odd, then (2.4) and (2.5) imply

$$\varphi_{tq}(x) = O(tq^{-K}) + t\{qx\}\tilde{C}_K q^{-K+1} + O(t^2/a) + O(t\{qx\}^2 q^{-K+1})$$

Thus, in both cases,

$$\lim_{m\to\infty} \varphi_{tq}(x) = d \cdot \operatorname{sign}(\varphi^{(K-1)}(1) - \varphi^{(K-1)}(0)).$$

It is trivial that $\lim_{m\to\infty} \|t_m q_{k_m}\alpha\| = 0$. $\square$

Proof of Theorem 1.4. Let $c \neq 0$ be an arbitrary real number with the same sign as $\varphi^{(K-1)}(1) - \varphi^{(K-1)}(0)$. Define $A(c) := \{x \in \mathbb{R}/\mathbb{Z} \cong [0, 1[: \exists (Q_m)_{m\geq 1}$, subsequence of

$\mathbb{N}$, such that $\lim_{m\to\infty} \varphi_{Q_m}(x) = c$ and $\lim_{m\to\infty} \|Q_m\alpha\| = 0\}$. The set $A(c)$ is invariant under the ergodic transformation $x \mapsto x + \alpha \bmod 1$ on $\mathbb{R}/\mathbb{Z}$, hence $\lambda(A(c)) \in \{0, 1\}$. From Lemma 2.4 it follows that the λ-measure of $A(c)$ is 1. We follow the proof of [4, Lemma 3] to show that c is a period of the skew product T_φ. The set P_φ of periods of T_φ is a group, hence $P_\varphi = \mathbb{R}$. This implies that T_φ is ergodic on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$. □

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