A Multidimensional Generalization of the Barnes Integral and the Continuous Hahn Polynomial

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Submitted by Daniel Waterman

Received October 20, 1998

Employing a multidimensional generalization of the beta integral of Barnes, we derive an expression of the continuous Hahn polynomials. Motivation of the present work is to study such integrals from the viewpoint of a finite-difference version of the de Rham theory.

Key Words: continuous Hahn polynomials; multidimensional generalizations of Barnes' beta integral

1. INTRODUCTION

Recently, multidimensional generalizations of Barnes' integral have been treated in the framework of a finite-difference version of the de Rham theory developed by Tarasov and Varchenko [22]. These integrals appear to express the form factors arising in the integrable quantum field theory [13, 21] and the correlation functions of the integrable lattice models [11].

On the other hand, the original Barnes integral is used to give the orthogonality relation for the continuous Hahn polynomials. Henceforce, recalling that the continuous Hahn polynomials are the generalization of the Jacobi polynomials [4] and that the Jacobi polynomial has an integral representation of the Selberg type [1, 18], it is natural to expect that the continuous Hahn polynomial could be expressed in terms of a multidimensional generalization of the Barnes integral; the present work is devoted to showing that such an expression can be found.

We believe that the present work will be helpful in studies of the de Rham theory and in the calculation of such integrals in physics as those mentioned above.
2. CONTINUOUS HAHN POLYNOMIALS

Barnes [6] evaluated the integral
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma(\alpha + iz)\Gamma(\beta + iz)\Gamma(\gamma - iz)\Gamma(\delta - iz) \, dz
= \frac{\Gamma(\alpha + \gamma)\Gamma(\alpha + \delta)\Gamma(\beta + \gamma)\Gamma(\beta + \delta)}{\Gamma(\alpha + \beta + \gamma + \delta)},
\]
for the case Re(\(\alpha, \beta, \gamma, \delta\)) > 0, where \(i = \sqrt{-1}\) is the imaginary unit. The corresponding orthogonal polynomials are the continuous Hahn polynomials defined by
\[
Q_n(z; \alpha, \beta, \gamma, \delta) = i^n(\alpha + \gamma)_n(\alpha + \delta)_n
\times \binom{3F_2}{-n, n + \alpha + \beta + \gamma + \delta - 1, \alpha + iz}{\alpha + \gamma, \alpha + \delta; 1},
\]
where \((a)_n\) stands for the shifted factorial \(\prod_{0 \leq j \leq n-1}(a + j)\) and \(3F_2\) is the generalized hypergeometric series
\[
3F_2 \left[ \begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n(a_3)_n}{(b_1)_n(b_2)_n n!} z^n.
\]
Their orthogonality relation is
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} Q_n(z)Q_m(z)\Gamma(\alpha + iz)\Gamma(\beta + iz)\Gamma(\gamma - iz)\Gamma(\delta - iz) \, dz
= 0, \quad m \neq n
\]
\[
= \frac{\Gamma(n + \alpha + \gamma)\Gamma(n + \alpha + \delta)\Gamma(n + \beta + \gamma)\Gamma(n + \beta + \delta)n!}{\Gamma(2n + \alpha + \beta + \gamma + \delta - 1)\Gamma(n + \alpha + \beta + \gamma + \delta - 1)},
\]
for Re(\(\alpha, \beta, \gamma, \delta\)) > 0. The weight function is positive when \(\gamma = \overline{\alpha}, \delta = \overline{\beta}\).

The continuous Hahn polynomials \(Q_n(z)\) satisfy the finite-difference equation
\[
\left[ D_z - n(n + \alpha + \beta + \gamma + \delta - 1) \right] Q_n(z) = 0,
\]
\[
D_z = (\alpha + iz)(\beta + iz)(e^{\delta/iz} - 1) + (\gamma - iz)(\delta - iz)(e^{-\delta/iz} - 1),
\]
where \(e^{\delta/iz}f(z) = f(z + i^{-1})\).
It is known that the continuous Hahn polynomials reduce to the Jacobi polynomials through some limiting procedure. In this sense, the continuous Hahn polynomials extend the Jacobi polynomials (see [2, 5, 14, 15]).

3. MAIN RESULT

In [7], van Diejen introduces a multidimensional generalization of the Barnes integral (3.1) as a weight function in the orthogonality relation for the multivariable continuous Hahn polynomials.

\[ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi(t) \, dt_1 \cdots dt_n, \quad (3.1) \]

with

\[ \Phi(t) = \prod_{1 \leq k \leq n} \Gamma(\alpha + it_k)\Gamma(\beta + it_k)\Gamma(\gamma - it_k)\Gamma(\delta - it_k) \]
\[ \times \prod_{1 \leq k \neq l \leq n} \frac{\Gamma(\lambda + i(t_k - t_l))}{\Gamma(i(t_k - t_l))}, \]

where

\[ \text{Re}(\alpha, \beta, \gamma, \delta) > 0, \quad \lambda > 0. \]

It is known [8, 9] that (3.1) is equal to

\[ \frac{n! (2\pi)^n}{\Gamma(j \lambda)\Gamma(\alpha + \gamma + (j - 1) \lambda)\Gamma(\alpha + \delta + (j - 1) \lambda)} \]
\[ \times \prod_{1 \leq j \leq n} \frac{\Gamma(\beta + \gamma + (j - 1) \lambda)\Gamma(\beta + \delta + (j - 1) \lambda)}{\Gamma(\lambda)\Gamma(\alpha + \beta + \gamma + \delta + (n + j - 2) \lambda)} \]
\[ \times \frac{\Gamma(\alpha + \gamma + (j - 1) \lambda)\Gamma(\beta + \delta + (j - 1) \lambda)}{1}. \quad (3.2) \]

Our interest is in the study of functions in \( z \) expressed by

\[ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(z, t) \Phi(t) \, dt_1 \cdots dt_n, \]

where \( f(z, t) \) is invariant with respect to permutations of \( t_1, \ldots, t_n \).

In the present paper, in particular, we consider the case \( f(z, t) = \prod_{1 \leq k \leq n} (it_k - iz) \), in which case we obtain the following.
THEOREM.

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1 \leq k \leq n} (it_k - iz) \Phi(t) \, dt_1 \cdots dt_n / \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi(t) \, dt_1 \cdots dt_n \\
= \prod_{1 \leq j \leq n} \frac{(\alpha + \gamma + (j - 1) \lambda)(\alpha + \delta + (j - 1) \lambda)}{(\alpha + \beta + \gamma + \delta + (n + j - 2) \lambda)}
\times _3F_2 \left[ -n, \frac{\alpha + \beta + \gamma + \delta}{\lambda} + n - 1, \frac{\alpha + iz}{\lambda} ; 1 \right]
\times \left( \frac{\lambda}{i} \right)^n \frac{1}{\left( \frac{\alpha + \beta + \gamma + \delta}{\lambda} + n - 1 \right)_n} Q_n(z/\lambda; \alpha/\lambda, \beta/\lambda, \gamma/\lambda, \delta/\lambda).
\]

Here \( Q_n(z) \) is the continuous Hahn polynomial defined by (2.2).

4. PROOF OF THEOREM

In this section, we introduce \( x_k = it_k \) for \( 1 \leq k \leq n \), and write

\[
\langle \varphi(x) \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x) \Phi(x) \, dx_1 \cdots dx_n
\]

for the sake of brevity.

**Lemma 1.** We have

\[
\sum_{j=1}^{n} \frac{x_j^l}{\prod_{1 \leq k \leq n} (x_j - x_k)} = \begin{cases} 0, & 0 \leq l \leq n - 2, \\
1, & l = n - 1. \end{cases} \tag{4.1}
\]

**Proof.** The left-hand side of (4.1) can be expressed by means of the integral

\[
\sum_{j=1}^{n} \text{Res}_{y=x_j} \frac{y^l}{\prod_{1 \leq k \leq n} (y - x_k)} \, dy = \frac{1}{2\pi i} \int_C \frac{y^l}{\prod_{1 \leq k \leq n} (y - x_k)} \, dy
\]

where the contour \( C \) circles the origin in the counterclockwise direction so that all poles \( x_1, \ldots, x_n \) are inside the contour. Moreover, by using power
series expansions, we have

\[
\frac{1}{2\pi i} \oint_C \frac{y^{l-n}}{\prod_{1 \leq k \leq n} \left(1 - \frac{x_k}{y}\right)} \, dy
\]

\[= \sum_{m_1, \ldots, m_n \geq 0} \frac{1}{2\pi i} \oint_C y^{l-n} \left(\frac{x_1}{y}\right)^{m_1} \cdots \left(\frac{x_n}{y}\right)^{m_n} \, dy\]

\[= \left\{ \begin{array}{ll}
0, & 0 \leq l \leq n - 2, \\
\sum_{m_1, \ldots, m_n \geq 0} x_1^{m_1} \cdots x_n^{m_n}, & l \geq n - 1.
\end{array} \right. \tag{4.2}\]

Hence we reach the desired relation.

**Remark.** Equality (4.2) is Theorem 1.5 of [10] and is known to be crucial in the calculation of multivariable hypergeometric series well poised in \(SU(n)\).

**Lemma 2.**

\[
\sum_{j=1}^{n} \prod_{1 \leq k \leq n, \, k \neq j} y_k \left(1 + \frac{\lambda}{y_{kj}}\right) = \sum_{0 \leq l \leq n-1} e_{n-l-1}(y_1, \ldots, y_n) \lambda^l, \tag{4.3}
\]

where \(e_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} y_{i_1} \cdots y_{i_k}\) is the elementary symmetric polynomial of degree \(k\), and \(y_{kj}\) designates \(y_k - y_j\).

**Proof.** Note that

\[
\sum_{j=1}^{n} \prod_{1 \leq k \leq n, \, k \neq j} y_k \left(1 + \frac{\lambda}{y_{kj}}\right)
\]

\[= \sum_{j=1}^{n} y_1 \cdots \widehat{y_j} \cdots y_n \left(1 + \lambda \sum_{1 \leq k_1 \leq n, \, k_1 \neq j} \frac{1}{y_{k_1 j}} \right) + \lambda^2 \sum_{1 \leq k_1 < k_2 \leq n, \, k_1, k_2 \neq j} \frac{1}{y_{k_1 j} y_{k_2 j}}
\]

\[+ \cdots + \lambda^{n-1} \frac{1}{y_{1j} \cdots y_{jj} \cdots y_{nj}}\]

\[= e_{n-1}(y_1, \ldots, y_n) + \sum_{1 \leq l \leq n-1} \lambda^l \]

\[\times \left\{ \sum_{j=1}^{n} \sum_{1 \leq k_1 < k_2 < \cdots < k_l \leq n, \, k_1, k_2, \ldots, k_l \neq j} y_{k_1 j} y_{k_2 j} \cdots y_{k_l j} \right\}.
\]
Here, the coefficient of $\lambda^l$ for each $l$ satisfying $1 \leq l \leq n - 1$ is calculated as

$$
\sum_{j=1}^{n} \sum_{1 \leq k_1 < k_2 < \cdots < k_l < k_i \leq n} \frac{y_1 \cdots y_j \cdots y_n}{y_{k_1} y_{k_2} \cdots y_{k_i + 1}} y_{k_{i+1}} y_{k_{i+2}} \cdots y_{k_n} = 
$$

$$
= \sum_{1 \leq k_1 < k_2 < \cdots < k_{i+1} \leq n} \left( \frac{y_1 \cdots y_{k_1} \cdots y_n}{y_{k_1} y_{k_2} \cdots y_{k_{i+1} + 1}} + \frac{y_1 \cdots y_{k_2} \cdots y_n}{y_{k_2} y_{k_3} \cdots y_{k_{i+1} + 1}} + \cdots + \frac{y_1 \cdots y_{k_{i+1}} \cdots y_n}{y_{k_{i+1}} y_{k_{i+2}} \cdots y_{k_{n}} \cdots y_{k_{i+1} + 1}} \right)
$$

$$
= \sum_{1 \leq k_1 < k_2 < \cdots < k_{i+1} \leq n} \frac{y_1 \cdots y_n}{y_{k_1} y_{k_2} \cdots y_{k_{i+1}}} \left( \frac{y_{k_2} \cdots y_{k_{i+1}}}{y_{k_2} y_{k_3} \cdots y_{k_{i+1} + 1}} + \frac{y_{k_1} \cdots y_{k_{i+1}}}{y_{k_1} y_{k_2} \cdots y_{k_{i+1} + 1}} + \cdots + \frac{y_{k_1} \cdots y_{k_{i+1}}}{y_{k_{i+1}} y_{k_{i+2}} \cdots y_{k_{n}} \cdots y_{k_{i+1} + 1}} \right)
$$

$$
= \sum_{1 \leq k_1 < k_2 < \cdots < k_{i+1} \leq n} \frac{y_1 \cdots y_n}{y_{k_1} y_{k_2} \cdots y_{k_{i+1}}} = e_{n-l-1}(y_1, \ldots, y_n). \quad (4.4)
$$

The third equality in (4.4) is obtained by

$$
\sum_{1 \leq s \leq l + 1} \frac{y_{k_1} \cdots y_{k_s} \cdots y_{k_{i+1}}}{y_{k_1} k_1 y_{k_2} y_{k_3} \cdots y_{k_{i+1} + 1}} = \sum_{1 \leq s \leq l + 1} \frac{y_{k_s}^{l-1}}{y_{k_1} y_{k_2} \cdots y_{k_{i+1} + 1}} = 1,
$$

which is from Lemma 1. This completes the proof of Lemma 2.

**LEMMA 3.**

$$
\sum_{j=1}^{n} \prod_{k \neq j}^{n} (iz - x_k) \left( 1 + \frac{\lambda}{x_{jk}} \right) = \frac{1}{\lambda} \left( \prod_{j=1}^{n} (iz - x_j + \lambda) - \prod_{j=1}^{n} (iz - x_j) \right),
$$

where $x_{jk}$ denotes $x_j - x_k$.

**Proof.** Substitute $iz - x_k$ for $y_k$ in the equality in Lemma 2, and note that $\prod_{j=1}^{n} (y_j + \lambda) = \sum_{j=0}^{n} e_{n-j}(y) \lambda^j$. The desired equality follows. \[\square\]
Lemma 3 shows that
\[
n \left( (iz - x_2) \cdots (iz - x_n) \left( 1 + \frac{\lambda}{x_{12}} \right) \cdots \left( 1 + \frac{\lambda}{x_{1n}} \right) \right)
= \frac{1}{\lambda} \left( e^{\lambda \partial_z / i \partial_z} - 1 \right) \left( \prod_{j=1}^{n} (iz - x_j) \right),
\] (4.5)
where the left-hand side follows from the symmetry of the function \( \Phi(x) \). On the other hand, we have

**Lemma 4.**
\[
\left( (\alpha + x_1)(\beta + x_1)(iz - x_2) \cdots (iz - x_n) \left( 1 + \frac{\lambda}{x_{12}} \right) \cdots \left( 1 + \frac{\lambda}{x_{1n}} \right) \right)
= \left( (\gamma - x_1)(\delta - x_1)(iz - x_2) \cdots (iz - x_n) \left( 1 - \frac{\lambda}{x_{12}} \right) \cdots \left( 1 - \frac{\lambda}{x_{1n}} \right) \right),
\]
where \( x_{jk} \) is \( x_j - x_k \).

**Proof.** Since
\[
e^{\partial_x X_1} \Phi(x) = \frac{(\alpha + x_1)(\beta + x_1)}{(\gamma - x_1 - 1)(\delta - x_1 - 1)}
\times \prod_{k=2}^{n} \frac{\lambda + x_1 - x_k}{(x_1 - x_k)(\lambda + x_k - x_1 - 1)} \Phi(x),
\]
we obtain
\[
e^{\partial_x X_1} \left\{ \left( (\gamma - x_1)(\delta - x_1) \frac{\lambda + x_{21}}{x_{21}} \cdots \frac{\lambda + x_{n1}}{x_{n1}} \Phi(x) \right) \right\}
= \left( (\alpha + x_1)(\beta + x_1) \frac{\lambda + x_{12}}{x_{12}} \cdots \frac{\lambda + x_{1n}}{x_{1n}} \Phi(x) \right),
\]
which implies the desired relation. \( \square \)

Substituting the equalities
\[
(\alpha + x_1)(\beta + x_1) = (iz - x_1)^2 - \{(iz + \alpha) + (iz + \beta)\}
\times (iz - x_1) + (iz + \alpha)(iz + \beta),
\]
\[
(\gamma - x_1)(\delta - x_1) = (iz - x_1)^2 - \{(iz - \gamma) + (iz - \delta)\}
\times (iz - x_1) + (iz - \gamma)(iz - \delta)
\]
into the formula in Lemma 4 and using Lemma 1, we have
\[
(\alpha + \beta + \gamma + \delta + (n - 1)\lambda)(iz - x_1) \cdots (iz - x_n)
\]
\[
= (iz + \alpha)(iz + \beta)(iz - \gamma)(iz - \delta)
\]
\[
\times \left( (iz - x_2) \cdots (iz - x_n) \left( 1 + \frac{\lambda}{x_{12}} \right) \cdots \left( 1 + \frac{\lambda}{x_{1n}} \right) \right).
\]
(4.6)

Combining (4.5) and (4.6) leads to the equation
\[
\left[ D_z - n\lambda(\alpha + \beta + \gamma + \delta + (n - 1)\lambda) \right] \left( \prod_{1 \leq k \leq n} (iz - x_k) \right) = 0, \quad (4.7)
\]
where
\[
\widetilde{D}_z = (\alpha + iz)(\beta + iz)(e^{\lambda \partial / i \partial z} - 1) + (\gamma - iz)(\delta - iz)(e^{-\lambda \partial / i \partial z} - 1).
\]
The finite-difference equation (4.7) implies, by noting (2.3), that
\[
\left( \sum_{1 \leq k \leq n} (iz - x_k) \right)
\]
is a constant multiple of
\[
\binom{-n, \alpha + \beta + \gamma + \delta + n - 1, \alpha + iz}{\frac{\lambda}{\alpha + \gamma}, \frac{\alpha + \delta}{\lambda}} ; 1.
\]
(3F2)

If we put
\[
\left( \prod_{1 \leq k \leq n} (x_k - iz) \right) = C \cdot \binom{-n, \alpha + \beta + \gamma + \delta + n - 1, \alpha + iz}{\frac{\lambda}{\alpha + \gamma}, \frac{\alpha + \delta}{\lambda}} ; 1,
\]
(4.8)

and compare the coefficient of \(z^n\) in each side, the constant \(C\) is determined to be
\[
\prod_{1 \leq j \leq n} \frac{(\alpha + \gamma + (j - 1)\lambda)(\alpha + \delta + (j - 1)\lambda)}{(\alpha + \beta + \gamma + \delta + (n + j - 2)\lambda)} \times \langle 1 \rangle.
\]
(4.9)

This completes the proof of our theorem.
5. FINAL COMMENT

It is known [4, 15] that there are further sets of orthogonal polynomials which extend the continuous Hahn polynomials and the Jacobi polynomials. For instance, the Wilson polynomials reduce to the continuous dual Hahn polynomials, the continuous Hahn polynomials, and the Jacobi polynomials. Furthermore, the Askey–Wilson polynomials include all of them. The problem discussed in the present paper regarding these polynomials will be discussed in a separate work.

On the other hand, our work could be generalized to the cases of multivariable polynomials: The multivariable Wilson polynomials and the multivariable continuous (dual) Hahn polynomials studied by van Diejen [7], along with their $q$-cases. We refer the reader to [20] and [19] for the examples of the integral representation of Jack symmetric polynomials and Macdonald polynomials, respectively. The definition and some properties of the Macdonald polynomials and their generalizations (the Koornwinder–Macdonald polynomials) are found in [16, 17, 12], respectively.

REFERENCES