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Characterization of Generalized Haar Spaces

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We say that a subset G of $C_0(T, \mathbb{R}^k)$ is rotation-invariant if $\{Qg: g \in G\} = G$ for any $k \times k$ orthogonal matrix Q. Let G be a rotation-invariant finite-dimensional subspace of $C_0(T, \mathbb{R}^k)$ on a connected, locally compact, metric space T. We prove that G is a generalized Haar subspace if and only if $P_G(f)$ is strongly unique of order 2 whenever $P_G(f)$ is a singleton. © 1998 Academic Press

1. INTRODUCTION

Let T be a locally compact Hausdorff space and G a finite-dimensional subspace of $C_0(T, \mathbb{R}^k)$, the space of vector-valued functions f on T which vanish at infinity, i.e., the set $\{t \in T: \|f(t)\|_2 \geqslant \varepsilon\}$ is compact for every $\varepsilon > 0$. Here $\|y\|_2 := (\sum_{i=1}^k |y_i|^2)^{1/2}$ denotes the 2-norm on the k-dimensional Euclidean space $\mathbb{R}^k$ (of column vectors). For f in $C_0(T, \mathbb{R}^k)$, the norm of f is defined as

$$\|f\| := \sup_{t \in T} \|f(t)\|_2.$$

The metric projection P_G from $C_0(T, \mathbb{R}^k)$ to G is given by

$$P_G(f) = \{g \in G: \|f - g\| = \text{dist}(f, G)\}, \qquad \text{for} \quad f \in C_0(T, \mathbb{R}^k),$$

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where

$$\mathrm{dist}(f, G) = \inf \{ \|f - g\| : g \in G\}.$$

A subspace G of $C_0(T, \mathbb{R}^k)$ is said to be a Chebyshev subspace if $P_G(f)$ is a singleton for every $f \in C_0(T, \mathbb{R}^k)$. In the Banach space of real-valued continuous functions $C_0(T) \equiv C_0(T, \mathbb{R}^1)$, it is well-known that G is an n-dimensional Chebyshev subspace of $C_0(T)$ if and only if G satisfies the Haar condition (i.e., every nonzero g in G has at most $(n-1)$ zeros). The Haar condition not only provides an intrinsic characterization of Chebyshev subspaces of $C_0(T)$, but also ensures strong unicity and Lipschitz continuity of the metric projection P_G, as shown in the following theorem.

THEOREM 1. *Suppose that G is an n-dimensional subspace of $C_0(T)$. Then the following are equivalent*:

(i) *G satisfies the Haar condition*;

(ii) *G is a Chebyshev subspace of $C_0(T)$*;

(iii) *for every f in $C_0(T)$, $P_G(f)$ is strongly unique, i.e., there exists a constant $\gamma(f) > 0$ such that*

$$\|f - g\| \geqslant \mathrm{dist}(f, G) + \gamma(f) \cdot \|g - P_G(f)\|, \qquad \textit{for} \quad g \in G;$$

(iv) *for every f in $C_0(T)$, $P_G(f)$ is a singleton and P_G is Lipschitz continuous at f, i.e., there exists a constant $\lambda(f) > 0$ such that*

$$\|P_G(f) - P_G(h)\| \leqslant \lambda(f) \cdot \|f - h\|, \qquad \textit{for} \quad h \in C_0(T).$$

Furthermore, if $T = [a, b]$ is a closed subinterval of $\mathbb{R}$, then all the above are equivalent to the following statement:

(v) *$P_G(f)$ is strongly unique whenever $P_G(f)$ is a singleton.*

The equivalence of (i) and (ii) is due to Haar [6]. Newman and Shapiro [11] proved that (i) implies (iii). Lipschitz continuity of P_G was proved by Freud in [5] and the equivalence condition (v) was given by McLaughlin and Sommers [10]. See [8] for more details. The above theorem summarizes the implications of the Haar condition in $C_0(T)$. One natural question is what are the implications of the Haar condition for a finite-dimensional subspace of the Banach space, $C_0(T, \mathbb{C})$, of all complex-valued continuous functions on T that vanish at infinity. Newman and Shapiro [11] proved that if $G := \{\sum_{i=1}^n c_i g_i(x) : c_i \in \mathbb{C}\}$ is an n-dimensional subspace of $C_0(T, \mathbb{C})$ and satisfies the Haar condition, then G is a Chebyshev subspace of $C_0(T, \mathbb{C})$ and, for every $f(x) \in C_0(T, \mathbb{C})$, there exists a constant $\gamma(f) > 0$ such that

$$\|f-g\|^2 \geqslant \operatorname{dist}(f, G)^2 + \gamma(f) \cdot \|g - P_G(f)\|^2, \qquad \text{for} \quad g \in G. \tag{1}$$

The inequality (1) is also referred to as strong unicity of order 2 and is equivalent to the following original form given by Newman and Shapiro:

$$\|f-g\| \geqslant \operatorname{dist}(f, G) + \beta(f) \cdot \|g - P_G(f)\|^2,$$
$$\text{for} \quad g \in G \quad \text{with} \quad \|g - P_G(f)\| \leqslant 1,$$

where $\beta(f)$ is some positive constant. Moreover, the Haar condition is also necessary for a finite-dimensional Chebyshev subspace of $C_0(T, \mathbb{C})$. In fact, an analog of (i)–(iv) of Theorem 1 holds for finite-dimensional Chebyshev subspaces of $C_0(T, \mathbb{R}^k)$, due to the following intrinsic characterization, which we call the generalized Haar condition, of finite-dimensional Chebyshev subspaces of $C_0(T, \mathbb{R}^k)$ given by Zukhovitskii and Stechkin [13].

DEFINITION 2. Let G be an n-dimensional subspace of $C_0(T, \mathbb{R}^k)$ and let m be the maximum integer less than n/k (i.e., $mk < n \leqslant (m+1)k$). Then G is called a generalized Haar space if

(i) every nonzero g in G has at most m zeros;

(ii) for any m distinct points t_i in T and any m vectors $\{x_1, ..., x_m\}$ in $\mathbb{R}^k$, there is a vector-valued function p in G such that $p(t_i) = x_i$ for $1 \leqslant i \leqslant m$.

The following analog in $C_0(T, \mathbb{R}^k)$ for parts (i)–(iv) of Theorem 1 was given in [1]. The equivalence (i) $\Leftrightarrow$ (ii) in the following theorem belongs to Zukhovitskii and Stechkin [13].

THEOREM 3. *Let G be a finite-dimensional subspace of $C_0(T, \mathbb{R}^k)$. Then the following are equivalent*:

(i) *G is a generalized Haar subspace.*

(ii) *G is a Chebyshev subspace of $C_0(T, \mathbb{R}^k)$.*

(iii) *P_G is strongly unique of order 2 at each f in $C_0(T, \mathbb{R}^k)$.*

(iv) *for every f in $C_0(T, \mathbb{R}^k)$, $P_G(f)$ is a singleton and P_G satisfies a Hölder continuity condition of order $\frac{1}{2}$.*

Here the Hölder condition is the analog in $C_0(T, \mathbb{R}^k)$ for Lipschitz continuity in Theorem 1. The metric projection P_G is said to satisfy a Hölder continuity condition of order $\frac{1}{2}$ at f if $P_G(\phi)$ is a singleton for every

ϕ in $C_0(T, \mathbb{R}^k)$ and there exists a positive number $\lambda = \lambda(f)$ such that $\|P_G(f) - P_G(h)\| \leqslant \lambda \, \|f - h\|^{1/2} \, (1 + \|f + h\|)^{1/2}$ for all h in $C_0(T, \mathbb{R}^k)$.

The main goal of this paper is to present an analog of part (v) of Theorem 1 for finite-dimensional subspaces in $C_0(T, \mathbb{R}^k)$. However, we can only do so under the assumption that G is rotation invariant.

DEFINITION 4. A subspace G of $C_0(T, \mathbb{R}^k)$ is said to be rotation-invariant if $\{Qg : g \in G\} = G$ for any $k \times k$ orthogonal matrix Q.

Note that $C_0(T, \mathbb{C}) \equiv C_0(T, \mathbb{R}^2)$, since

$$f_1(x) + \mathbf{i} f_2(x) \equiv \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}.$$

Here $\mathbf{i} = \sqrt{-1}$. An n-dimensional subspace of $C_0(T, \mathbb{C})$ can be identified with a $(2n)$-dimensional subspace of $C_0(T, \mathbb{R}^2)$. In fact, one can prove that any rotation invariant finite-dimensional subspace in $C_0(T, \mathbb{R}^2)$ can be identified with a finite-dimensional subspace in $C_0(T, \mathbb{C})$ (cf. Lemma 9). In fact, we consider rotation-invariant subspaces of $C_0(T, \mathbb{R}^k)$ as the natural generalization of complex-valued function subspaces. Now we state the main theorem and present its proof in the next section.

THEOREM 5. *Let G be a rotation-invariant finite-dimensional subspace of $C_0(T, \mathbb{R}^k)$, where T is a connected and locally compact metric space. If $P_G(f)$ is strongly unique with order 2 whenever $P_G(f)$ is a singleton, then G is a generalized Haar subspace.*

Remark. Theorem 5 holds for any space T which is connected, locally compact, first countable, and Hausdorff because these are the only properties of T used in the proof.

In Lemma 9, we will show that G is rotation-invariant if and only if G is the tensor product of k-copies of a subspace G_1 of $C_0(T)$, i.e., $G = G_1 \times \cdots \times G_1$. Thus, G is a rotation-invariant Chebyshev subspace of $C_0(T, \mathbb{R}^k)$ if and only if G is the tensor product of k-copies of a Haar subspace G_1 of $C_0(T)$.

Note that for $k = 1$ the result of McLaughlin and Sommers [10] follows from Theorem 5 and, in fact, Theorem 5 gives the following stronger result than that of McLaughlin and Sommers, since the strong unicity of $P_G(f)$ implies the strong unicity of order 2.

COROLLARY 6. *Suppose that G is an n-dimensional subspace of $C_0(T)$ and T is a connected and locally compact metric space. Then G is a Haar subspace if $P_G(f)$ is strongly unique of order 2 whenever $P_G(f)$ is a singleton.*

2. PROOF OF THE MAIN THEOREM

The proof of Theorem 5 will follow after five lemmas are given. We use $\langle x, y\rangle := \sum_{i=1}^{n} x_i y_i$ to denote the dot product of vectors x and y in $\mathbb{R}^k$, $\operatorname{supp}(\sigma) := \{t \in T : \sigma(t) \neq 0\}$ for any mapping $\sigma: T \to \mathbb{R}^k$, $Z(g) := \{t \in T : g(t) = 0\}$ for any function g in $C_0(T, \mathbb{R}^k)$, and $Z(K) := \bigcap_{g \in K} Z(g)$ for any subset K of $C_0(T, \mathbb{R}^k)$. For any subset K of T, $G|_K$ denotes the restriction of G on K as a subspace of $C(K, \mathbb{R}^k)$. The boundary and closure of K are denoted by $\operatorname{bd}(K)$ and $\operatorname{cl}(K)$, respectively. For a finite subset T_0 of T, let $\operatorname{card}(T_0)$ be the cardinality of T_0 (i.e., $\operatorname{card}(T_0)$ is the number of points in T_0). A mapping σ from T to $\mathbb{R}^k$ is called an annihilator of G if

$$\sum_{t \in \operatorname{supp}(\sigma)} \langle \sigma(t), g(t)\rangle = 0 \qquad \text{for} \quad g \in G.$$

LEMMA 7. *Suppose that $G \neq \{0\}$ is a finite-dimensional subspace of $C_0(T, \mathbb{R}^k)$. Then there exists a mapping σ from T to $\mathbb{R}^k$ that has the following properties*:

(a) $\operatorname{supp}(\sigma)$ *is a finite subset of* T;

(b) $\langle \sigma(t), g(t)\rangle \equiv 0$ *whenever* $\langle \sigma(t), g(t)\rangle \geqslant 0$ *for* $t \in \operatorname{supp}(\sigma)$;

(c) $G_\sigma := \{g \in G : \operatorname{supp}(\sigma) \subset Z(g)\}$ *satisfies the generalized Haar condition on* $T \backslash Z(G_\sigma)$.

(d) $\dim G_\sigma \geqslant 1$.

Proof. We prove the lemma by induction. If $\dim(G) = 1$, then G does satisfy the generalized Haar condition on $T \backslash Z(G)$ and $\sigma(t) \equiv 0$ satisfies the conditions (a)–(d). Suppose that the lemma holds for subspaces of $C_0(T, \mathbb{R}^k)$ with dimension $< n$ and $\dim G = n$. Let $(m+1)$ be the smallest integer that is greater or equal to n/k. If G satisfies the generalized Haar condition on $T \backslash Z(G)$, let $\sigma(t) \equiv 0$. Then $G_\sigma = G$ and we are done. If G does not satisfy the generalized Haar condition on $T \backslash Z(G)$, then either there exist m points $t_1, ..., t_m$ in $T \backslash Z(G)$ such that $\dim G|_{\{t_1, ..., t_m\}} < km$ or there exists a nonzero function $g \in G$ such that $Z(g)$ contains $(m+1)$ points $t_1, ..., t_m, t_{m+1}$. In either case, there exists a finite subset T_0 of $T \backslash Z(G)$ and a nonzero function $g_0(t)$ such that $\dim G|_{T_0} < k \operatorname{card}(T_0)$ and $T_0 \subset Z(g_0)$. Since in the Banach space $C(T_0, \mathbb{R}^k)$ $\dim G|_{T_0} < k \operatorname{card}(T_0)$, there exists an annihilator τ of $G|_{T_0}$ with $\operatorname{supp}(\tau) \subset T_0$, so τ annihilates G also. Since $\operatorname{supp}(\tau) \subset T \backslash Z(G)$, $\dim G_\tau < \dim G$. Since $g_0 \in G_\tau$, $\dim G_\tau \geqslant 1$. Consider G_τ as a subspace defined on $T \backslash Z(G_\tau)$. By the induction assumption, there exists a mapping μ from $T \backslash Z(G_\tau)$ to $\mathbb{R}^k$ such that the conditions (a)–(d) hold for $\sigma \equiv \mu$ and $G \equiv G_\tau|_{T \backslash Z(G_\tau)}$.

Let $\sigma(t) := \tau(t)$ for $t \in \operatorname{supp}(\tau)$ and $\sigma(t) = \mu(t)$ for $t \notin \operatorname{supp}(\tau)$. Then it is easy to verify that the conditions (a)–(d) hold for σ. ∎

LEMMA 8. *Let K be a subset of G, $t_0 \in \operatorname{bd} Z(K)$, $t_0^i \in T \setminus Z(K)$ such that $t_0^i \to t_0$ as $i \to \infty$, and τ is a continuous mapping defined on $\{t_0, t_0^i : i = 1, 2, ...\}$. Then there exists a function $\bar{g} \in K$ and an index $\bar{\imath}$ such that $\langle \tau(t_0^i), g(t_0^i) \rangle = 0$ for $i \geqslant \bar{\imath}$ whenever*

$$\lim_{t \to \infty} \sup \frac{|\langle \tau(t_0^i), g(t_0^i) \rangle|}{|\langle \tau(t_0^i), \bar{g}(t_0^i) \rangle|^{3/2}} \leqslant 1, \tag{2}$$

where we define $0/0 := 0$.

Proof. Let $\langle \tau, K \rangle := \{\langle \tau(t), g(t) \rangle : g \in K\}$. Since $\dim \operatorname{span} \langle \tau, K \rangle|_{\{t_0^j t_0^{j+1}, ...\}}$ is a nonincreasing function of j and has finitely many values, there exists $\bar{\imath}$ such that $\dim \operatorname{span} \langle \tau, K \rangle|_{\{t_0^{\bar{\imath}}, t_0^{\bar{\imath}+1}, ...\}} = \dim \operatorname{span} \langle \tau, K \rangle|_{\{t_0^j, t_0^{j+1}, ...\}}$ for $j \geqslant \bar{\imath}$. That is, if $j \geqslant \bar{\imath}$ and $\langle \tau(t_0^i), g(t_0^i) \rangle = 0$ for $i \geqslant j$, then $\langle \tau(t), g(t) \rangle = 0$ on $T_0 := \{t_{\bar{\imath}}, t_{\bar{\imath}+1, ...}\}$.

Let g_1 be a nonzero function in K. (If $K = \{0\}$, then the lemma is trivially true.) If g_1 can not be used as $\bar{g}$, then there exists a function g_2 in K such that

$$\lim_{i \to \infty} \sup \frac{|\langle \tau(t_0^i), g_2(t_0^i) \rangle|}{|\langle \tau(t_0^i), g_1(t_0^i) \rangle|^{3/2}} \leqslant 1,$$

but $\langle \tau(t_0^i), g_2(t_0^i) \rangle \neq 0$ for infinitely many i's. By the choice of g_2, for all i sufficiently large, we have

$$|\langle \tau(t_0^i), g_2(t_0^i) \rangle| \leqslant 2\, |\langle \tau(t_0^i), g_1(t_0^i) \rangle|^{3/2}. \tag{3}$$

We claim that g_1 and g_2 are linearly independent. Let $c_1 g_1 + c_2 g_2 = 0$. Then for all i sufficiently large

$$\begin{aligned} 0 &= |c_1 \langle \tau(t_0^i), g_1(t_0^i) \rangle + c_2 \langle \tau(t_0^i), g_2(t_0^i) \rangle| \\ &\geqslant |c_1 \langle \tau(t_0^i), g_1(t_0^i) \rangle| - |c_2 \langle \tau(t_0^i), g_2(t_0^i) \rangle| \\ &\geqslant |c_1 \langle \tau(t_0^i), g_1(t_0^i) \rangle| - 2\, |c_2 \langle \tau(t_0^i), g_1(t_0^i) \rangle|^{3/2} \\ &= |\langle \tau(t_0^i), g_1(t_0^i) \rangle| (|c_1| - 2\, |c_2| \langle \tau(t_0^i), g_1(t_0^i) \rangle^{1/2}). \end{aligned} \tag{4}$$

Since $|c_2 \langle \tau(t_0^i), g_1(t_0^i) \rangle|^{1/2} \to 0$ as $i \to \infty$, it follows from (3) and (4) using those i for which $\langle \tau(t_0^i), g_2(t_0^i) \rangle \neq 0$ that $c_1 = 0$. Since g_2 is a nonzero function, $c_2 g_2 = 0$ implies $c_2 = 0$. Hence, g_1 and g_2 are linearly independent.

If g_2 can not be used as $\bar{g}$, then there exists a function g_3 in K such that $\langle \tau(t_0^i), g_3(t_0^i)\rangle \neq 0$ for infinitely many i (possibly different from where $\langle \tau(t_0^i), g_2(t_0^i)\rangle \neq 0$) and

$$\limsup_{i\to\infty} \frac{|\langle \tau(t_0^i), g_3(t_0^i)\rangle|}{|\langle \tau(t_0^i), g_2(t_0^i)\rangle|^{3/2}} \leqslant 1. \tag{5}$$

By (2) and (5), for all i sufficiently large, we have

$$|\langle \tau(t_0^i), g_3(t_0^i)\rangle| \leqslant 2\, |\langle \tau(t_0^i), g_2(t_0^i)\rangle|^{3/2} \leqslant 4\, |\langle \tau(t_0^i), g_1(t_0^i)\rangle|^{9/4}. \tag{6}$$

Now suppose that $c_1 g_1 + c_2 g_2 + c_3 g_3 = 0$. By (4) and (6), we get that, for all i sufficiently large,

$$\begin{aligned} 0 &= |c_1\langle \tau(t_0^i), g_1(t_0^i)\rangle + c_2\langle \tau(t_0^i), g_2(t_0^i)\rangle + c_3\langle \tau(t_0^i), g_3(t_0^i)\rangle| \\ &\geqslant |\langle \tau(t_0^i), g_1(t_0^i)\rangle|(|c_1| - 2\,|c_2|\,|\langle \tau(t_0^i), g_1(t_0^i)\rangle|^{1/2} \\ &\quad - 4\,|c_3|\,|\langle \tau(t_0^i), g_1(t_0^i)\rangle|^{5/4}). \end{aligned}$$

Using those i for which $\langle \tau(t_0^i), g_1(t_0^i)\rangle \neq 0$ as above we obtain $c_1 = 0$. Then from $c_2 g_2 + c_3 g_3 = 0$ we obtain as above

$$\begin{aligned} 0 &= |c_2\langle \tau(t_0^i), g_2(t_0^i)\rangle + c_3\langle \tau(t_0^i), g_3(t_0^i)\rangle| \\ &\geqslant |\langle \tau(t_0^i), g_2(t_0^i)\rangle|(|c_2| - 2\,|c_3| \cdot |\langle \tau(t_0^i), g_2(t_0^i)\rangle|^{1/2}) \end{aligned}$$

and using those i for which $\langle \tau(t_0^i), g_2(t_0^i)\rangle \neq 0$ we obtain $c_2 = 0$. Since $g_3 \neq 0$ and $c_3 g_3 = 0$, we have $c_3 = 0$. Therefore, g_1, g_2, g_3 are linearly independent.

If no function in K can be used as $\bar{g}$ then continuing in this manner we can construct infinitely many linearly independent functions $g_1, g_2, \ldots$ in K. Since G is finite-dimensional, this is impossible. ∎

LEMMA 9. *Suppose that G is a rotation-invariant finite dimensional subspace of $C_0(T, \mathbb{R}^k)$. Then $m := n/k$ is an integer and G is the tensor product of k-copies of an m-dimensional subspace G_1 of $C_0(T)$, i.e.,*

$$G = \left\{ \sum_{i=1}^{k} g_i e_i : g_i \in G_1 \text{ for } 1 \leqslant i \leqslant k \right\},$$

where e_i is the ith canonical basis vector for $\mathbb{R}^k$ (i.e., all components of e_i are zero except that the ith component is 1).

Proof. If g is in G, then $g = \sum_{i=1}^{k} g_i e_i$ where $g_i \in C_0(T, \mathbb{R}^1)$. Let $G_i := \{g_i : g = \sum_{j=1}^{k} g_j e_j \in G\}$ for $i = 1, \ldots, k$. Then it is obvious that

$G \subset G_1 \times \cdots \times G_k$ ($\equiv \{\sum_{i=1}^k g_i e_i : g_i \in G_i\}$). For a fixed i, let Q_i be the $k \times k$ orthogonal matrix whose jth column is e_j for $j \neq i$ and $-e_i$ for $j = i$. For any $g_i \in G_i$, there exists $g_j \in G_j$ for $j \neq i$ such that $g := \sum_{j=1}^k g_j e_j \in G$. Then $Q_i g \in G$ and $g_i e_i = \frac{1}{2}(g - Q_i g) \in G$. Thus, $G_1 \times ... \times G_k \subset G$, which implies $G = G_1 \times \cdots \times G_k$.

Now we show that $G_i \equiv G_1$ for $1 \leqslant i \leqslant k$. Let B_i be the orthogonal matrix that as e_i has its first column, e_1 as its ith column, e_j as its jth column for $j \neq 1$ or i. For any $g_1 \in G_1$ and $g_i \in G_i$, we have $g_i e_1 = B_i(g_i e_i) \in G$ and $g_1 e_i = B_i(g_1 e_1) \in G$. Hence, $g_i \in G_1$ and $g_1 \in G_i$. So $G_1 = G_2 = \cdots = G_k$ and $G = \{\sum_{i=1}^k g_i e_i : g_i \in G_1 \text{ for } 1 \leqslant i \leqslant k\}$. ∎

Lemma 10. *Suppose that G is a generalized Haar subspace of $C_0(T, \mathbb{R}^k)$ and* $\dim G = mk$. *Then, for any given m distinct points $t_1, ..., t_m$ in T and m vectors $x_1, ..., x_m$ in $\mathbb{R}^k$, there exists a function g in G such that $g(t_i) = x_i$ for $i = 1, ..., m$.*

Proof. We show that $\dim G|_{\{t_1, ..., t_m\}} = \dim G$. If not then there exists a $\bar{g} \neq 0$ in G such that $\bar{g}(t_i) = 0$, $i = 1, ..., m$. But this contradicts the fact that G is a generalized Haar set and therefore any function in G has at most $(m-1)$ zeroes. Since $\dim G = mk$ and $\dim C(\{t_1, ..., t_m\}, \mathbb{R}^k) = mk$ the result follows. ∎

Proof of Theorem 5. By Lemma 7, there exists a mapping σ from T into $\mathbb{R}^k$ such that the conditions (a)–(d) in Lemma 7 hold. It follows that if G is not a generalized Haar subspace, then $Z(G_\sigma) \neq \emptyset$. Since $Z(G_\sigma)$ is closed and T is connected, bd $Z(G_\sigma)$ contains at least one point, say t_0. Let $\{t_0^i\}_{i=1}^\infty$ be a sequence of distinct points in $T \backslash Z(G_\sigma)$ such that $\lim_{i \to \infty} t_0^i = t_0$.

Since G is rotation-invariant, it is easy to verify that G_σ is rotation-invariant and hence $\dim G_\sigma = km$ for some integer m. Choose m distinct points $\{t_1, ..., t_m\}$ in $T \backslash (\{t_0, t_0^i, i = 1, ...\} \cup Z(G_\sigma))$. Notice then that $\{t_1, ..., t_m\} \cap \text{supp}(\sigma) = \emptyset$. Then by Lemma 10, for any vectors $x_1, ..., x_m$ in $\mathbb{R}^k$, there exists a function g in G_σ such that $g(t_j) = x_j$, for $j = 1, ..., m$. Since, for fixed i, $\dim G_\sigma|_{\{t_1^i, t_1, ..., t_m\}} < k(m+1)$, there exists an annihilator τ_i of G_σ such that $\text{supp}(\tau_i) \subset \{t_0^i, t_1, ..., t_m\}$. By the interpolation property of G_σ on any m points of $T \backslash Z(G_\sigma)$, $\text{supp}(\tau_i)$ must have $(m+1)$ points. Thus,

$$\text{supp}(\tau_i) = \{t_0^i, t_1, ..., t_m\}.$$

Without loss of generality, we may assume that there exist unit vectors in $\mathbb{R}^k$, $\tau(t_0), \tau(t_1), ..., \tau(t_m)$, such that

$$\lim_{i \to \infty} \text{sgn}(\tau_i(t_0^i)) = \tau(t_0)$$

and

$$\lim_{i\to\infty} \operatorname{sgn}(\tau_i(t_j)) = \tau(t_j), \qquad \text{for} \quad j = 1, ..., m.$$

If t_0 is in supp(σ), we may assume that $\operatorname{sgn}(\sigma(t_0)) = \tau(t_0)$. Otherwise, we can replace τ_i by $Q\tau_i$, where Q is an orthogonal matrix such that $Q\tau(t_0) = \operatorname{sgn}(\sigma(t_0))$. (Here the rotation-invariance of G is used.)

Let $\tau(t_0^i) = \operatorname{sgn}(\tau_i(t_0^i))$. Then τ is a continuous function on the closed set

$$A := \{t_0, t_0^i : i = 1, 2, ...\}. \tag{7}$$

Let $K = \{g$ in $G_\sigma : g \not\equiv 0$ and, for $1 \leqslant j \leqslant m$ either $g(t_j) = 0$ or $\langle g(t_j), \tau(t_j)\rangle > 0\}$. Since G_σ is a generalized Haar set on $T \backslash Z(G_\sigma)$, it follows that $K \neq \varnothing$ and if g is in K then for at least one j, $g(t_j) \neq 0$. Let $\bar{g}$ in K be the function given by Lemma 8.

Now follows a lengthy construction of a function f in $C_0(T, \mathbb{R}^k)$. First let $\bar{t} \in (\operatorname{supp}(\sigma) \cup \{t_1, ..., t_m\}) \backslash \{t_0\}$. Then, for t in a sufficiently small neighborhood of each such $\bar{t}$, define

$$\bar{f}(t) = \begin{cases} \operatorname{sgn}(\sigma(\bar{t}))(1 - \|g(\bar{t})\|_2) & \text{if} \quad \bar{t} \in \operatorname{supp}(\sigma) \backslash \{t_0\} \\ \tau(\bar{t})(1 - \|\bar{g}(t)\|_2) & \text{if} \quad \bar{t} = t_j \in \{t_1, ..., t_m\},\ \bar{g}(t_j) = 0 \\ \tau(\bar{t}) & \text{if} \quad \bar{t} = t_j \in \{t_1, ..., t_m\},\ \bar{g}(t_j) \neq 0. \end{cases}$$

Then, for $\varepsilon > 0$ small enough and t near $\bar{t}$ in supp(σ),

$$\begin{aligned} \|\bar{f}(t) - \varepsilon\bar{g}(t)\|_2 &= \|(1 - \|\bar{g}(t)\|_2) \operatorname{sgn} \sigma(\bar{t}) - \varepsilon\bar{g}(t)\|_2 \\ &\leqslant 1 - \|\bar{g}(t)\|_2 + \varepsilon\, \|\bar{g}(t)\|_2 \leqslant 1, \end{aligned}$$

and, similarly, $\|\bar{f}(t) - \varepsilon\bar{g}(t)\| \leqslant 1$ for t near t_j in $\{t_1, ..., t_m\}$ if $\bar{g}(t_j) = 0$. If $\bar{g}(t_j) \neq 0$, then $\langle \bar{g}(t_j), \tau(t_j)\rangle = \langle \bar{g}(t_j), \bar{f}(t_j)\rangle > 0$ and, by the continuity of $\bar{g}$, $\langle \bar{g}(t), \tau(t_j)\rangle > \delta > 0$ for t near t_j. Thus, for t near t_j,

$$\begin{aligned} \|\bar{f}(t) - \varepsilon\bar{g}(t)\|_2^2 &= \|\tau(t_j)\|^2 - 2\varepsilon\langle \bar{g}(t_j), \tau(t_j)\rangle + \varepsilon^2\, \|\bar{g}(t)\|_2^2 \\ &\leqslant 1 - 2\,\delta\varepsilon + \varepsilon^2\, \|\bar{g}\| < 1, \end{aligned}$$

if $\varepsilon > 0$ is small enough. Therefore, for a sufficiently small neighborhood W_1 of $[(\operatorname{supp}(\sigma) \cup \{t_1, ..., t_m\}) \backslash \{t_0\}$, $\bar{f}$ is continuous and

$$\|\bar{f}(t) - \varepsilon\bar{g}(t)\|_2 \leqslant 1, \tag{8}$$

if $t \in W_1$ and $\varepsilon > 0$ is small enough.

Let $\bar{f}(t)$ and $\bar{h}(t)$ be defined on the closed set A (cf. (7)) by

$$\bar{h}(t) = \operatorname{sgn}(\tau_i(t)) \equiv \tau(t),$$

and

$$\bar{f}(t) = \bar{h}(t)(1 - |\langle \bar{h}(t), \bar{g}(t) \rangle|^{3/2}),$$

where $t = t_0, t_0^i$ for $i = 1, 2, ...$. Now we show that there exists an index i_0 such that $\langle \bar{h}(t_0^i), \bar{g}(t_0^i) \rangle \neq 0$ for $i \geqslant i_0$. First observe that since $\bar{g}$ is in G_σ, τ_i annihilates G_σ, and $\operatorname{supp}(\tau_i) = \{t_0^i, t_1, ..., t_m\}$, we get

$$0 = \langle \tau_i(t_0^i), \bar{g}(t_0^i) \rangle + \sum_{j=1}^{m} \langle \tau_i(t_j), \bar{g}(t_j) \rangle. \tag{9}$$

Since $\bar{g}$ is in K, by the definition, we have either $\bar{g}(t_j) = 0$ or

$$0 < \langle \bar{g}(t_j), \tau(t_j) \rangle = \lim_{t \to \infty} \frac{\langle \bar{g}(t_j), \tau_i(t_j) \rangle}{\|\tau_i(t_j)\|_2}. \tag{10}$$

However, (10) implies that $\langle \bar{g}(t_j), \tau_i(t_j) \rangle > 0$ for i large enough whenever $\bar{g}(t_j) \neq 0$. Since there is at least one j with $\bar{g}(t_j) \neq 0$, it follows from (9) that $\langle \tau_i(t_0^i), \bar{g}(t_0^i) \rangle < 0$ (i.e., $\langle \bar{h}(t_0^i), \bar{g}(t_0^i) \rangle < 0$) for i large enough. Thus, for $i \geqslant i_0$, $\|\bar{f}(t_0^i)\|_2 < 1$ and $\|\bar{f}(t_0)\|_2 = 1$. Since $\lim_{i \to \infty} t_0^i = t_0$ and T is locally compact Hausdorff, there exist open sets W and V with compact closures such that $t_0 \in V$, $[(\operatorname{supp}(\sigma) \cup \{t_1, ..., t_m\}) \setminus \{t_0\}] \subset W$, and $\operatorname{cl}(W) \cap \operatorname{cl}(V) = \varnothing$. Choose i_0 large enough such that $t_0^i \in V$ for $i \geqslant i_0$. Choose $W \subset W_1$ so that (8) holds for $t \in W$ and $\varepsilon > 0$ small enough. By relabeling of t_0^i, we may assume without loss of generality that $t_0^i \in V$ for all i and

$$\|\bar{f}(t_0^i)\|_2 < 1, \qquad \text{for} \quad i = 1, 2, ... \tag{11}$$

Now $\bar{h}$ can be extended from the closed set A (cf. (7)) to a continuous function $h(t)$ on the open set V with $A \subseteq V$ and $\|h(t)\|_2 \equiv 1$, $t \in V$, by Tietze's Extension Theorem for locally compact Hausdorff spaces [12, p. 385] and the proof of Corollary 5.3 [4, p. 151]. Let $\bar{f}(t) = h(t)(1 - |\langle h(t), \bar{g}(t) \rangle|^{3/2})$ for t in V. Since $B = \operatorname{cl}(V) \cup \operatorname{cl}(W)$ is compact, we can extend $\bar{f}$ from B to a function F on all of T with F in $C_c(T, \mathbb{R}^k)$ (the collection of functions in $C_0(T, \mathbb{R}^k)$ whose supports are compact) and $\|F(t)\|_2 \leqslant 1$. Let

$$D := \{t_0\} \cup \{t_1, ..., t_m\} \cup \operatorname{supp}(\sigma) \cup \{t_0^i : \langle h(_0^i), g(\bar{t}_0^i) \rangle \neq 0\}.$$

Then D is a G_δ set, there exists [4, p. 148] a function ϕ in $C_c(T, \mathbb{R})$ with $0 \leqslant \phi(t) \leqslant 1$ and $\phi^{-1}(1) = D$. Thus $f = \phi F$ is an extension of $\bar{f}$ from $W \cup V$ to T which satisfies the following conditions:

$$f(t) = \begin{cases} \operatorname{sgn}(\sigma(t)) & \text{for} \quad t \text{ in } \operatorname{supp}(\sigma), \\ \operatorname{sgn}(\tau(t)) & \text{for} \quad t \text{ in } \{t_1, ..., t_m\}, \end{cases} \tag{12}$$

$$\|f(t)\|_2 < 1 \quad \text{if } t \neq 0 \text{ and } t \in V, \qquad \text{and} \qquad \|f(t_0)\|_2 = 1, \tag{13}$$

$$\|f(t_0^i)\|_2 = 1 - |\langle h(t_0^i), \bar{g}(t_0^i)\rangle|^{3/2} \text{ if } \langle h(t_0^i), g(\bar{t}_0^i)\rangle \neq 0, \tag{14}$$

$$\|f(t)\|_2 \leqslant \|h(t)\|_2 \,(1 - |\langle h(t), \bar{g}(t)\rangle|^{3/2}) \qquad \text{for} \quad t \in V, \tag{15}$$

$$\|h(t)\|_2 = 1 \qquad \text{if} \quad t \text{ is in } V, \tag{16}$$

$$h(t_0^i) = \tau_i(t_0^i) = \tau(t_0^i), \qquad i \geqslant i_0, \tag{17}$$

and

$$\|f(t) - \varepsilon \bar{g}(t)\|_2 \leqslant 1 \qquad \text{if} \quad \varepsilon \leqslant \varepsilon_0 \text{ and } t \notin V, \tag{18}$$

where $\varepsilon_0 > 0$ is a small positive number. Note that (18) was verified for $\bar{f}$ and t in W, now $\bar{f}(t)$ is replaced by $\phi(t)\,\bar{f}(t)$ for $0 < \phi(t) \leqslant 1$ and the same calculation shows (18) still holds for $t \in W$. However, $\sup\{\|f(t)\|_2 : t \notin (V \cup W)\} < 1$ since V and W are open sets containing the only points where f has norm 1. Thus, (18) holds for f and $t \notin V$.

We claim that $P_G(f) = 0$. First it is shown that if g is in $P_G(f)$ and $g \not\equiv 0$, then g is in K and thus in G_σ. If g is in $P_G(f)$, it is easy to verify that since $\|f - g\| \leqslant 1$ it follows that $\langle g(t), \sigma(t)\rangle \geqslant 0$ for t in $\operatorname{supp}(\sigma)$. Thus, by Lemma 7(b), $\langle g(t), \sigma(t)\rangle = 0$ for t in $\operatorname{supp}(\sigma)$. Thus, for t in $\operatorname{supp}(\sigma)$, we have $(g(t), f(t)) = 0$, and

$$1 = \|f\| \geqslant \|f(t) - g(t)\|_2^2 = \|f(t)\|_2^2 + \|g(t)\|_2^2 = 1 + \|g(t)\|_2^2.$$

As a result, $g(t) = 0$ for t in $\operatorname{supp}(\sigma)$ and g is in G_σ. Similarly, one can show that $\langle g(t_j), \tau(t_j)\rangle \geqslant 0$ for $j = 1, ..., m$, and $g(t_j) = 0$ whenever $\langle g(t_j), \tau(t_j)\rangle = 0$. Hence if $g \neq 0$, then g is in K.

Now we show that for any nonzero g in $P_G(f)$,

$$\limsup_{i \to \infty} \frac{|\langle \tau(t_0^i), g(t_0^i)\rangle|}{|\langle \tau(t_0^i), \bar{g}(t_0^i)\rangle|^{3/2}} > 2. \tag{19}$$

If not, then

$$\limsup_{i \to \infty} \frac{|\langle \tau(t_0^i), \frac{1}{2} g(t_0^i)\rangle|}{|\langle \tau(t_0^i), \bar{g}(t_0^i)\rangle|^{3/2}} \leqslant 1.$$

Since $g \in K$, it is easy to verify that $\frac{1}{2}g \in K$. By Lemma 8, $\langle \tau(t_0^i), g(t_0^i) \rangle = 0$, for $i \geqslant \bar{\imath}$. (We may assume that $\bar{\imath} \geqslant i_0$.) Now τ_i annihilates G_σ, $g \in G_\sigma$, and $\text{supp}(\tau_i) = \{t_0^i, t_1, ..., t_m\}$. Thus,

$$0 = \sum_{t \in \text{supp}(\tau_i)} \langle g(t), \tau_i(t) \rangle = \langle \tau_i(t_0^i), g(t_0^i) \rangle + \sum_{j=1}^{m} \langle \tau_i(t_j), g(t_j) \rangle,$$

and by the definition of $\tau(t)$,

$$0 = \langle \tau(t_0^i), g(t_0^i) \rangle = \langle \text{sgn}(\tau_i(t_0^i), g(t_0^i)) \rangle = \frac{\langle \tau_i(t_0^i), g(t_0^i) \rangle}{\|\tau_i(t_0^i)\|}.$$

Hence, $\langle \tau_i(t_0^i), g(t_0^i) \rangle = 0$. Since τ_i is an annihilator of G_σ, $g \in K \subset G_\sigma$, and $\text{supp}(\tau_i) = \{t_0^i, t_1, ..., t_m\}$, we obtain

$$\sum_{j=1}^{m} \langle \tau_i(t_j), g(t_j) \rangle = \sum_{t \in \text{supp}(\tau_i)} \langle \tau_i(t), g(t) \rangle = 0. \tag{20}$$

Since g is in K, $g(t_j) = 0$ or $\langle g(t_j), \tau(t_j) \rangle > 0$ for $j = 1, ..., m$. If $\langle g(t_j), \tau(t_j) \rangle > 0$ then for i sufficiently large $\langle \tau_i(t_j), g(t_j) \rangle > 0$. Thus from (20) it follows that $g(t_j) = 0$, $j = 1, ..., m$. But then $g \equiv 0$ since G_σ is a generalized Haar set on $G \backslash Z(G_\sigma)$ and this contradicts the assumption that $g \not\equiv 0$, and thus (19) holds.

Now with nonzero g in $P_G(f)$ from (19) it follows that for infinitely many indices i,

$$|\langle \tau(t_0^i), g(t_0^i) \rangle| > 2\, |\langle \tau(t_0^i), \bar{g}(t_0^i) \rangle|^{3/2}. \tag{21}$$

Since τ_i is an annihilator of G_σ, the above inequality implies that, for infinitely many i's

$$\|\tau_i(t_0^i)\|_2 \langle \tau(t_0^i), g(t_0^i) \rangle = -\sum_{j=1}^{m} \langle \tau_i(t_j), g(t_j) \rangle < 0. \tag{22}$$

Thus,

$$\begin{aligned} \|f(t_0^i) - g(t_0^i)\|_2^2 &= \|\langle \tau(t_0^i), f(t_0^i) - g(t_0^i) \rangle\, \tau(t_0^i)\|_2^2 \\ &\quad + \|f(t_0^i) - g(t_0^i) - \langle \tau(t_0^i), f(t_0^i) - g(t_0^i) \rangle\, \tau(t_0^i)\|_2^2 \\ &= |\, \|f(t_0^i)\|_2 - \langle \tau(t_0^i), g(t_0^i) \rangle|^2 \\ &\quad + \|g(t_0^i) - \langle \tau(t_0^i), g(t_0^i) \rangle\, \tau(t_0^i)\|_2^2, \end{aligned} \tag{23}$$

where the first equality is an orthogonal decomposition of the error vector and then we use the definition of $f(t)$ to simplify the expression.

We continue the estimate of $\|f(t_0^i)-g(t_0^i)\|_2^2$ by using indices i for which (21) and (22) hold. Then

$$\|f(t_0^i)-g(t_0^i)\|_2^2 \geqslant (\|f(t_0^i)\|_2 + |\langle \tau(t_0^i), g(t_0^i)\rangle|)^2$$
$$\geqslant \|f(t_0^i)\|_2^2 + 2\,\|f(t_0^i)\|_2\,|\langle \tau(t_0^i), g(t_0^i)\rangle|. \tag{24}$$

Note that $\phi(t_0^i)=1$ and $f(t_0^i)=h(t_0^i)(1-|\langle h(t_0^i), \bar{g}(t_0^i)\rangle|^{3/2})$. Thus,

$$\|f(t_0^i)\|_2^2 = (1-|\langle h(t_0^i), \bar{g}(t_0^i)\rangle|^{3/2})^2$$
$$= (1-|\langle \tau(t_0^i), \bar{g}(t_0^i)\rangle|^{3/2})^2$$
$$\geqslant 1-2\,|\langle \tau(t_0^i), \bar{g}(t_0^i)\rangle|^{3/2}. \tag{25}$$

Since $\|f(t_0^i)\|_2 \to 1$ and $|\langle \tau(t_0^i), \bar{g}(t_0^i)\rangle|^{1/2} \to 0$ as $i\to\infty$, we have $2\,\|f(t_0^i)\|_2 \geqslant 1$ for i sufficiently large. Then, by (24), (25), and (21), we get that for infinitely many i's,

$$\|f(t_0^i)-g(t_0^i)\|_2^2 \geqslant 1-2\,|\langle \tau(t_0^i), \bar{g}(t_0^i)\rangle|^{3/2} + |\langle \tau(t_0^i), g(t_0^i)\rangle| > 1.$$

This is impossible, since $g\in P_G(f)$. The contradiction proves our claim that $P_G(f)=\{0\}$.

Next we show that $P_G(f)$ is not strongly unique of order 2 by estimating $\|f-\varepsilon\bar{g}\|$. By the definition of $f(t)$, for $\varepsilon>0$ small enough, $\|f(t)-\varepsilon\bar{g}(t)\|_2 \leqslant 1$ if $t\notin V$ (a neighborhood of t_0) (cf. (18)). If $P_G(f)$ is strongly unique of order 2, then there exists a positive constant γ such that

$$\|f-\varepsilon\bar{g}\|^2 \geqslant \operatorname{dist}(f, G)^2 + \gamma\,\operatorname{dist}(\varepsilon\bar{g}, P_G(f))^2,$$

i.e.,

$$\|f-\varepsilon\bar{g}\|^2 \geqslant 1+\gamma\varepsilon^2\,\|\bar{g}\|^2. \tag{26}$$

Let $t_\varepsilon\in V$ be such that

$$\|f(t_\varepsilon)-\varepsilon\bar{g}(t_\varepsilon)\|_2 = \|f-\varepsilon\bar{g}\| > 1.$$

Since $\|f(t)\|_2<1$ for $t\in V$ and $t\neq t_0$, it follows that $t_\varepsilon\to t_0$ as $\varepsilon\to 0^+$. Note that

$$\|f(t_\varepsilon)-\varepsilon\bar{g}(t_\varepsilon)\|_2^2$$
$$= \|f(t_\varepsilon)\|_2^2 - 2\varepsilon\langle f(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle + \varepsilon^2\|\bar{g}(t_\varepsilon)\|_2^2$$
$$\leqslant 1-|\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle|^{3/2} - 2\varepsilon\langle f(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle + \varepsilon^2\,\|\bar{g}(t_\varepsilon)\|_2^2.$$

By the above equality, (26), and $\bar{g}(t_\varepsilon) \to 0$ as $\varepsilon \to 0$, we obtain that, for $\varepsilon > 0$ small enough,

$$1 + \gamma\varepsilon^2 \|\bar{g}\|^2 \leqslant \|f - \varepsilon\bar{g}\|^2$$
$$\leqslant 1 - 2\varepsilon\langle f(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle - |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle|^{3/2} + \tfrac{1}{2}\gamma\varepsilon^2 \|\bar{g}\|^2,$$

which implies that

$$-2\varepsilon\langle f(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle - |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle|^{3/2} \geqslant \tfrac{1}{2}\gamma\varepsilon^2 \|\bar{g}\|^2.$$

As a consequence, $\langle f(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle < 0$ and

$$2\varepsilon\, |\langle f(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle| \geqslant |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle|^{3/2} + \tfrac{1}{2}\gamma\varepsilon^2 \|\bar{g}\|^2.$$

Since $f(t_\varepsilon) = \alpha h(t_\varepsilon)$ for some $0 \leqslant \alpha \leqslant 1$, the above inequality implies

$$2\varepsilon\, |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle| \geqslant |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle|^{3/2} + \tfrac{1}{2}\gamma\varepsilon^2 \|\bar{g}\|^2. \tag{27}$$

Since $|\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle|^{1/2} \to 0$, for $\varepsilon > 0$ small enough,

$$\frac{\gamma \|\bar{g}\|^2}{2} |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle|^{-1/2} > 1. \tag{28}$$

By (27) and (28),

$$2\varepsilon\, |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle| > \frac{2}{\gamma \|\bar{g}\|^2} |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle|^2 + \tfrac{1}{2}\gamma\varepsilon^2 \|\bar{g}\|^2. \tag{29}$$

Equivalently, we have

$$\left(\sqrt{\frac{2}{\gamma \|\bar{g}\|^2}} |\langle h(t_\varepsilon), \bar{g}(t_\varepsilon)\rangle| - \sqrt{\frac{\gamma \|\bar{g}\|^2}{2}}\, \varepsilon\right)^2 < 0,$$

which is impossible. Therefore, $P_G(f)$ is not strongly unique of order 2.

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