Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 258 (2010) 534–558
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Quasi-potentials of the entropy functionals for scalar conservation laws

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Received 13 April 2009; accepted 6 July 2009

Available online 22 July 2009

Communicated by C. Villani

Abstract

We investigate the quasi-potential problem for the entropy cost functionals of non-entropic solutions to scalar conservation laws with smooth fluxes. We prove that the quasi-potentials coincide with the integral of a suitable Einstein entropy.

Keywords: Quasi-potential; Conservation laws; Entropy

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2009.07.003

1. Introduction

For a real function f, consider the scalar conservation law in the unknown $u \equiv u(t,x)$

$$u_t + f(u)_x = 0 \tag{1.1}$$

where $t \in [0,T]$ for some $T > 0$, $x \in \mathbb{T}$ (the one-dimensional torus), and subscripts denote partial derivatives. Eq. (1.1) does not admit in general classical solutions for the associated Cauchy problem, even if the initial datum is smooth. On the other hand, if f is non-linear, there exist in general infinitely many weak solutions. An admissibility condition, the so-called entropic condition, is then required to recover uniqueness for the Cauchy problem in the weak sense [6]. The unique solution satisfying such a condition is called the *Kruzkhov solution*.

A classical result [6, Chapter 6.3] states that the Kruzkhov solution can be obtained as limit for $\varepsilon \downarrow 0$ of the solution u^ε to the Cauchy problem associated with the equation

$$u_t + f(u)_x = \frac{\varepsilon}{2}\big(D(u)u_x\big)_x \tag{1.2}$$

provided that the initial data also converge. Here the *diffusion coefficient* D is a uniformly positive smooth function, and we remark that convergence takes place in the strong $L_p([0,T]\times\mathbb{T})$ topology. The Kruzkhov solution to (1.1) has also been proved to be the hydrodynamical limit of the empirical density of stochastic particles systems under hyperbolic scaling, when the number of particles diverges to infinity [11, Chapter 8]. These results legitimize the Kruzkhov solution as the physically relevant solution to (1.1), and the entropic condition as the appropriate selection rule between the infinitely many weak solutions to (1.1).

Provided the flux f and the diffusion coefficient D are chosen appropriately (depending on the particles system considered), one may say that (1.2) is a continuous version for the evolution of the empirical density of particles system, in which the small stochastic effects are neglected (or averaged) and ε is the inverse number of particles. The convergence of both (1.2) and the empirical measure of the density of particles to the same solution of (1.1) confirms somehow that this approximation is reliable.

In [10,15], the long standing problem of providing a large deviations principle for the empirical measure of the density of stochastic particles systems under hyperbolic scaling is addressed. In particular, the *totally asymmetric simple exclusion process* is investigated (which in particular corresponds to $f(u) = u(1-u)$ in the hydrodynamical limit equation (1.1)), and the large deviations result partially established. Roughly speaking, when the number of particles N diverges to infinity, the asymptotic probability of finding the density of particles in a small neighborhood of a path $u : [0,T]\times\mathbb{T} \to \mathbb{R}$ is $e^{-NH^{JV}(u)}$, where H^{JV} is a suitable large deviations rate functional (see Section 2).

A continuous mesoscopic mean field counterpart of this large deviations result is provided in [2,14]. In [14] a large deviations principle for a stochastic perturbation to (1.2) (driven by a *fluctuation coefficient* σ) is investigated in the limit of jointly vanishing stochastic noise and (deterministic) diffusion. In [2] a purely variational problem is addressed, namely the investigation of the Γ-limit of a family of functionals H_ε associated with (1.2) (see Section 2). The candidate large deviations functional H introduced in [14] and the candidate Γ-limit introduced in [2] coincide, and in the case $f(u) = u(1-u)$ they are expected to coincide with the functional H^{JV} introduced in [10,15] (the equality can be proved on functions of bounded variations, but it is missing in the general case). The functional H thus provides a generalization of the functional H^{JV}, for arbitrary fluxes f (in particular, not necessarily convex or concave), diffusion coefficients D and fluctuation coefficients σ. The functionals H_ε, H and H^{JV} are nonnegative; H_ε vanishes only on solutions to (1.2), so that H_ε can be interpreted as the cost of violating the flow (1.2). On the other hand, H and H^{JV} are $+\infty$ off the set of weak solutions to (1.1), they vanish only on Kruzkhov solution to (1.1), and thus they can be interpreted as the cost of violating the entropic condition for the flux (1.1). Section 2 of the paper is devoted to the precise definition of the functionals H_ε, H and H^{JV}.

Redirecting the reader to Section 3 for a more detailed discussion, here we briefly recall a general definition of the quasi-potential associated with a family of functionals. Suppose we are given a topological space U, and for each $T > 0$ a set $\mathcal{X}_T \subset C([0,T];U)$ and a functional $I_T : \mathcal{X}_T \to [0,+\infty]$. For the sake of simplicity, let us also fix a point $m \in U$. The quasi-potential $V : U \to [0,+\infty]$ associated with $\{I_T\}$ is then defined as $V(u) := \inf_{T>0} \inf_w I_T(w)$, where the infimum is carried over all the $w \in \mathcal{X}_T$ such that $w(0) \equiv m$ and $w(T) = u$. A natural choice for the reference point m should be an attractive point for the minima of the functionals I_T (see e.g. Theorems 4.4 and 5.5 for the case of H_ε, H and H^{JV}). Indeed, in such a case, the investigation of the quasi-potential is a classical subject both in dynamical optimal control theory and in large deviations theory, as it quantifies "the cheapest cost" to move from the stable point m to a general one u. Moreover, from the optimal control theory point of view, the quasi-potential describes the long time limit of the functionals I_T, see e.g. [5]. Furthermore, there is a broad family of stochastic processes for which the quasi-potential is expected to be the large deviations rate functional of their invariant measures, provided I_T is the large deviations rate functional of the laws of such processes up to time $T > 0$ (see e.g. [9, Theorem 4.4.1] for the classical finite-dimensional case, and [4] for a more general discussion and applications to particles systems). Moreover, see [9, Chapter 4], the quasi-potential of the large deviations rate functionals provides a valuable tool to investigate long time behavior of the processes (e.g. average time to be waited for the process to leave an attractive point, and the path to follow when the process performs such a deviation). Finally, in the context of non-equilibrium statistical mechanics in which the functionals H_ε, H and H^{JV} have been introduced, the quasi-potential has been proposed as a dynamical definition of the free energy functional for systems out of equilibrium [3].

Since H_ε is a functional associated with a control problem (see [2]) and it can be also retrieved as large deviations rate functional of some particles systems (e.g. weakly asymmetric particle systems, see [11]) and stochastic PDEs (see [13]), the quasi-potential problem is relevant for such a functional. Similarly, H and H^{JV} are the (candidate) large deviations rate functionals for both particles systems processes and stochastic PDEs, see [10,15,14].

Given a bounded measurable map $u_i : \mathbb{T} \to \mathbb{R}$, it is well known that the (entropic) solutions to the Cauchy problems for (1.1) and (1.2) with initial datum u_i will converge to the constant $m = \int_{\mathbb{T}} dx\, u_i(x)$, namely constant profiles are attractive points for the zeros of the functionals H_ε, H and H^{JV}. Given $m \in \mathbb{R}$ and two positive smooth maps on $\mathbb{T}$, interpreted as the diffusion coefficient D and fluctuation coefficient σ, the *Einstein entropy* is defined as the unique nonnegative function h_m on $\mathbb{R}$ such that $h_m(m) = 0$ and $\sigma h_m'' = D$. In this paper, we establish an explicit formula for the quasi-potential problem associated with the functionals H_ε, H and H^{JV} (which of course will depend on a time parameter T) with reference point the constant maps on the torus, proving that these three quasi-potential functionals coincide and are equal to the integral of the Einstein entropy. More precisely, given $u_f \in L_\infty(\mathbb{T})$, the quasi-potential $V(m, u_f)$ of H_ε, H and H^{JV} with reference constant $m \in \mathbb{R}$ is equal to $\int_{\mathbb{T}} dx\, h_m(u_f(x))$ if $\int_{\mathbb{T}} dx\, u_f(x) = m$ and it is $+\infty$ otherwise (see Theorem 3.1).

As remarked above, both the large deviations results in [10,15,14] and the Γ-limit results in [2] are incomplete, due to little knowledge of structure theorems and regularity results for weak solutions to conservation laws (1.1). These results on the quasi-potential give therefore an additional heuristic argument supporting the actual identification of H as the Γ-limit of H_ε. Similarly, since it is easily seen that the large deviations rate functional (in the hydrodynamical limit) of the invariant measures of the totally asymmetric simple exclusion process is also given by the integral of the Einstein entropy, these results also support the conjecture that H and H^{JV} may coincide at least in the case $f(u) = u(1-u)$ and that they are in fact the large deviations rate functional of the totally asymmetric simple exclusion process. Finally, we remark that the integral of the Einstein entropy is expected to rule the long time behavior of the well-behaving physical systems, and the result provided in this paper thus also supports the universality of the Jensen and Varadhan functional H^{JV} (or in general of H) as a relevant universal *entropy functional* for asymmetric conservative, closed physical systems.

In Section 2 we recall some preliminary notions. In Section 3 the main definitions and results are stated, while Sections 4 and 5 are devoted to the proofs. The techniques used to prove the results vary from calculus of variations (Remark 3.2 and Corollary 5.4), large deviations theory (Lemmas 4.2, 4.3 and 5.1) and conservation laws (Lemma 5.8).

2. Preliminaries

Our analysis will be restricted to equibounded "densities" u, and for the sake of simplicity we let u take values in $[-1, 1]$. Let U denote the compact Polish space of measurable functions $u : \mathbb{T} \to [-1, 1]$, equipped with the $H^{-1}(\mathbb{T})$ metric

$$d_U := \sup\bigl\{\langle u, \varphi\rangle,\ \varphi \in C_c^\infty(\mathbb{T}),\ \langle \varphi_x, \varphi_x\rangle + \langle \varphi, \varphi\rangle = 1\bigr\}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L_2(\mathbb{T})$. Given $T > 0$, let $\mathcal{X}_T$ be the Polish space $C([0, T]; U)$ endowed with the metric

$$d_{\mathcal{X}_T}(u, v) := \sup_{t\in[0,T]} d_U\bigl(u(t), v(t)\bigr) + \|u - v\|_{L_1([0,T]\times\mathbb{T})}.$$

Hereafter we assume f a Lipschitz function on $[-1, 1]$. Moreover we let $D, \sigma \in C([-1, 1])$ with D uniformly positive, and σ strictly positive in $(-1, 1)$.

2.1. The functional H_ε

For $\varepsilon > 0, T > 0$, we define $H_{\varepsilon;T} : \mathcal{X}_T \to [0, +\infty]$ as (hereafter we may drop the explicit dependence on integration variables inside integrals when no misunderstanding is possible)

$$H_{\varepsilon;T}(u) := \begin{cases} \sup_{\varphi\in C_c^\infty((0,T)\times\mathbb{T})} \varepsilon^{-1}[-\int_0^T dt\langle u, \varphi_t\rangle + \langle f(u), \varphi_x\rangle - \frac{\varepsilon}{2}\langle D(u)u_x, \varphi_x\rangle \\ \quad -\frac{1}{2}\int_0^T dt\langle \sigma(u)\varphi_x, \varphi_x\rangle] \quad \text{if } u_x \in L^2([0, T] \times \mathbb{T}), \\ +\infty \quad \text{otherwise.} \end{cases} \tag{2.1}$$

Note that $H_{\varepsilon;T}(u) = 0$ iff $u \in \mathcal{X}_T$ is a weak solution to (1.2). $H_{\varepsilon;T}$ is a lower-semicontinuous and coercive functional on $\mathcal{X}_T$ (see [2, Proposition 3.3, Theorem 2.5]). Moreover if $H_{\varepsilon;T}(u) < +\infty$ then $u \in C([0, T]; L^1(\mathbb{T}))$ (see [2, Lemma 3.2]).

2.2. Entropy-measure solutions

We say that $u \in \mathcal{X}_T$ is a weak solution to (1.1) iff for each $\varphi \in C_c^\infty((0,T)\times\mathbb{R})$

$$\int_0^T dt\,\langle u, \varphi_t\rangle + \big\langle f(u), \varphi_x\big\rangle = 0.$$

A function $\eta \in C^2([-1,1])$ is called an *entropy*, and its *conjugated entropy flux* $q \in C([-1,1])$ is defined up to an additive constant by $q(w) := \int^w dv\,\eta'(v) f'(v)$. For $u \in \mathcal{X}_T$ a weak solution to (1.1), for (η, q) an entropy – entropy flux pair, the *η-entropy production* is the distribution $\wp_{\eta,u}$ acting on $C_c^\infty((0,T)\times\mathbb{R})$ as

$$\wp_{\eta,u}(\varphi) := -\int_0^T dt\,\big\langle \eta(u), \varphi_t\big\rangle + \big\langle q(u), \varphi_x\big\rangle.$$

Let $C_c^{2,\infty}([-1,1]\times(0,T)\times\mathbb{T})$ be the set of compactly supported maps $\vartheta : [-1,1]\times(0,T)\times\mathbb{R} \ni (v,t,x) \mapsto \vartheta(v,t,x) \in \mathbb{R}$, that are C^2 in the v variable, with derivatives continuous up to the boundary of $[-1,1]\times(0,T)\times\mathbb{T}$, and C^∞ in the (t,x) variables. For $\vartheta \in C_c^{2,\infty}([-1,1]\times(0,T)\times\mathbb{T})$ let ϑ' and ϑ'' denote its partial derivatives with respect to the v variable. We say that a function $\vartheta \in C_c^{2,\infty}([-1,1]\times(0,T)\times\mathbb{T})$ is an *entropy sampler*, and its *conjugated entropy flux sampler* $Q : [-1,1]\times(0,T)\times\mathbb{T}$ is defined up to an additive function of (t,x) by $Q(w,t,x) := \int^w \vartheta'(v,t,x) f'(v)\,dv$. Finally, given a weak solution u to (1.1), the *ϑ-sampled entropy production* $P_{\vartheta,u}$ is the real number

$$P_{\vartheta,u} := -\int_{(0,T)\times\mathbb{T}} dt\,dx\,\big[(\partial_t\vartheta)\big(u(t,x),t,x\big) + (\partial_x Q)\big(u(t,x),t,x\big)\big]. \tag{2.2}$$

If $\vartheta(v,t,x) = \eta(v)\varphi(t,x)$ for some entropy η and some $\varphi \in C_c^\infty((0,T)\times\mathbb{T})$, then $P_{\vartheta,u} = \wp_{\eta,u}(\varphi)$.

The next proposition introduces a suitable class of solutions to (1.1) which will be needed in the sequel. We denote by M_T the set of finite measures on $(0,T)\times\mathbb{T}$ that we consider equipped with the weak* topology. In the following, for $\varrho \in M_T$ we denote by $\varrho^\pm$ the positive and negative part of ϱ.

Proposition 2.1. *(See [2, Proposition 2.3], [7].) Let $u \in \mathcal{X}_T$ be a weak solution to* (1.1). *The following statements are equivalent*:

- (i) *for each entropy η, the η-entropy production $\wp_{\eta,u}$ can be extended to a Radon measure on $(0,T)\times\mathbb{T}$, namely $\|\wp_{\eta,u}\|_{\mathrm{TV}} := \sup\{\wp_{\eta,u}(\varphi), \varphi \in C_c^\infty((0,T)\times\mathbb{T}), |\varphi| \leqslant 1\} < +\infty$;*
- (ii) *there exists a bounded measurable map $\varrho_u : [-1,1] \ni v \to \varrho_u(v; dt, dx) \in M_T$ such that for any entropy sampler ϑ*

$$P_{\vartheta,u} = \int_{[-1,1]\times(0,T)\times\mathbb{T}} dv\,\varrho_u(v; dt, dx)\,\vartheta''(v,t,x). \tag{2.3}$$

We say that a weak solution $u \in \mathcal{X}_T$ is an *entropy-measure solution* to (1.1) iff it satisfies the equivalent conditions of Proposition 2.1. The set of entropy-measure solutions to (1.1) is denoted by $\mathcal{E}_T \subset \mathcal{X}_T$. In general $\mathcal{E}_T \not\subseteq BV([0,T] \times \mathbb{T})$, the main regularity result for $\mathcal{E}_T$ being $\mathcal{E}_T \subset C([0,T]; L^1(\mathbb{T}))$, provided $f \in C^2([-1,1])$ is such that there is no interval in which f is affine (see [2, Lemma 5.1]). A *Kruzkhov solution* to (1.1) is a weak solution $u \in C([0,T]; L_1(\mathbb{T}))$ such that $\wp_{\eta,u} \leqslant 0$ in distributional sense, for each convex entropy η. Since a weak solution u such that $\wp_{\eta,u} \leqslant 0$ can be shown to be an entropy-measure solution, the entropic condition is equivalent to $\varrho_u(v; dt, dx) \leqslant 0$ for a.e. $v \in [-1,1]$.

2.3. *Γ-entropy cost of non-entropic solution*

For $T > 0$, we introduce the functional $H_T : \mathcal{X}_T \to [0, +\infty]$ as

$$H_T(u) := \begin{cases} \int dv\, \frac{D(v)}{\sigma(v)} \varrho_u^+(v; dt, dx) & \text{if } u \in \mathcal{E}_T, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4}$$

In [2, Proposition 2.6] it is proved that H_T is coercive and lower semicontinuous, and that it vanishes only on Kruzkhov solutions to (1.1).

As noted in [2, Remark 2.7], if $u \in \mathcal{X}_T \cap BV([0,T] \times \mathbb{T})$ is a weak solution to (1.1), then $u \in \mathcal{E}_T$. Let J_u be the jump set of $u \in \mathcal{E}_T \cap BV([0,T] \times \mathbb{T})$, $\mathcal{H}^1 \llcorner J_u$ the one-dimensional Hausdorff measure restricted to J_u and, at a point $(s,y) \in J_u$, let $n = (n^t, n^x) \equiv n(s,y)$ be the normal to J_u and $u^- \equiv u^-(s,y)$ (respectively $u^+ \equiv u^+(s,y)$) be the left (respectively right) trace of u (these are well defined $\mathcal{H}^1 \llcorner J_u$ a.e., since n^x can be chosen uniformly positive, see [2, Remark 2.7]). Then

$$H_T(u) = \int_{J_u} d\mathcal{H}^1 |n^x| \int dv\, \frac{D(v)}{\sigma(v)} \frac{\rho^+(v, u^+, u^-)}{|u^+ - u^-|} \tag{2.5}$$

where

$$\begin{aligned} \rho(v, u^+, u^-) := \big[& f(u^-)(u^+ - v) + f(u^+)(v - u^-) \\ & - f(v)(u^+ - u^-)\big] \mathbb{1}_{[u^- \wedge u^+,\, u^- \vee u^+]}(v) \end{aligned} \tag{2.6}$$

and ρ^+ denotes the positive part of ρ.

In [2] a suitable set $\mathcal{S}_T \subset \mathcal{E}_T$ of *entropy-splittable* solutions to (1.1) is also introduced, and the next result is proved.

Theorem 2.2. *(See [2, Theorem 2.5].) For each $T > 0$, the following statements hold.*

(i) *The sequence of functionals $\{H_{\varepsilon;T}\}$ satisfies the Γ-liminf inequality $\Gamma\text{-}\underline{\lim}_\varepsilon H_{\varepsilon;T} \geqslant H_T$ on $\mathcal{X}_T$.*
(ii) *Assume that there is no interval where f is affine. Then the sequence of functionals $\{H_{\varepsilon;T}\}$ is equicoercive on $\mathcal{X}_T$.*

(iii) *Assume furthermore that* $f \in C^2([-1,1])$*, and* $D, \sigma \in C^\alpha([-1,1])$ *for some* $\alpha > 1/2$*. Define*

$$\overline{H}_T(u) := \inf\Big\{\underline{\lim}_n H_T(u_n),\ \{u_n\} \subset \mathcal{S}_T\colon\ u_n \to u \text{ in } \mathcal{X}_T\Big\}.$$

Then the sequence of functionals $\{H_{\varepsilon;T}\}$ *satisfies the* Γ*-limsup inequality* $\Gamma\text{-}\overline{\lim}_\varepsilon H_{\varepsilon;T} \leqslant \overline{H}_T$ *on* $\mathcal{X}_T$*.*

Note that Γ-limsup inequality is not complete, as it is not known that $\overline{H}_T = H_T$.

2.4. The Jensen and Varadhan functional

Suppose that σ is such that there exists $h \in C^2([-1,1])$ such that $\sigma h'' = D$, and let g be such that $g' = h' f'$. For $T > 0$, we also introduce the Jensen and Varadhan functional $H_T^{JV} : \mathcal{X}_T \to [0,+\infty]$ as

$$H_T^{JV}(u) := \begin{cases} \sup_{\varphi \in C^\infty([0,T]\times\mathbb{T};[0,1])}\{\int_{\mathbb{T}} dx\,[h(u(T,x))\varphi(T,x) - h(u(0,x))\varphi(0,x)] \\ \quad - \int_0^T dt\,\langle h(u),\varphi_t\rangle + \langle g(u),\varphi_x\rangle\} \quad \text{if } u \text{ is a weak solution to (1.1)}, \\ +\infty \quad \text{otherwise}. \end{cases} \tag{2.7}$$

Note that the definition of H_T^{JV} does not depend on the choice of h, provided it satisfies $\sigma h'' = D$. This functional has been introduced in [10] (in the case $D \equiv 1$ and $f(u) = \sigma(u) = u(1-u)$). In [2] it is proved that $H_T^{JV} \leqslant H_T$, that $H_T^{JV}(u) = H_T(u)$ if f is convex or concave and u has bounded variation, and that $H_T^{JV} < H_T$ if f is neither convex or concave.

3. Quasi-potentials

We want to study the *quasi-potentials* $V_\varepsilon, V, V^{JV} : [-1,1] \times U \to [0,+\infty]$ associated respectively with $H_{\varepsilon;T}$, H_T and H_T^{JV}, and defined as

$$V_\varepsilon(m,u_f) := \inf\big\{H_{\varepsilon;T}(u),\ T > 0,\ u \in \mathcal{X}_T\colon\ u(0) \equiv m,\ u(T) = u_f\big\}, \tag{3.1}$$

$$V(m,u_f) := \inf\big\{H_T(u),\ T > 0,\ u \in \mathcal{X}_T\colon\ u(0) \equiv m,\ u(T) = u_f\big\}, \tag{3.2}$$

$$V^{JV}(m,u_f) := \inf\big\{H_T^{JV}(u),\ T > 0,\ u \in \mathcal{X}_T\colon\ u(0) \equiv m,\ u(T) = u_f\big\}. \tag{3.3}$$

Note that, if $u_f \equiv m$, then $V_\varepsilon(m,u_f) = V(m,u_f) = V^{JV}(m,u_f) = 0$. On the other hand, whenever $m = +1$ or $m = -1$, if $u_f \not\equiv m$, then $\int dx\, u_f(x) \neq m$ and thus $V_\varepsilon(m,u_f) = V(m,u_f) = V^{JV}(m,u_f) = +\infty$, since $H_{\varepsilon;T}(u) = H_T(u) = H_T^{JV}(u) = +\infty$ whenever $u \in \mathcal{X}_T$ is such that $\int_{\mathbb{T}} dx\, u(s,x) \neq \int_{\mathbb{T}} dx\, u(t,x)$ for some $s,t \in [0,T]$. Therefore, in the following we focus on the case $m \in (-1,1)$.

Our main result is the following. For $m \in (-1,1)$ define the *Einstein entropy* $h_m \in C([-1,1];[0,+\infty]) \cap C^2((-1,1))$ as the unique function such that $\sigma(v)h_m''(v) = D(v)$ for $v \in (-1,1)$, $h_m(m) = 0$, $h_m'(m) = 0$, and let

$$W_m(u_f) := \int_{\mathbb{T}} dx\, h_m\big(u_f(x)\big) \in [0,+\infty].$$

Note that, if $\int_{\mathbb{T}} dx\, u_f(x) = m$, $W_m(u_f)$ can also be written by the more explicit but less evocative formula

$$\int_{\mathbb{T}} dx \int_m^{u_f(x)} dw\, \big[u_f(x) - w\big] \frac{D(w)}{\sigma(w)}.$$

Theorem 3.1.

(i) *Assume*

$$\lim_{\alpha \downarrow 0} \alpha^2 \left[\frac{1}{\sigma(-1+\alpha)} + \frac{1}{\sigma(1-\alpha)} \right] = 0. \tag{3.4}$$

Then

$$V_\varepsilon(m, u_f) = \begin{cases} W_m(u_f) & \text{if } \int_{\mathbb{T}} dx\, u_f = m, \\ +\infty & \text{otherwise} \end{cases}$$

for any $\varepsilon > 0$, *for any* $m \in (-1, 1)$ *and* $u_f \in U$.

(ii) *Assume* $f \in C^2([-1, 1])$ *is such that there is no interval in which* f *is affine. Assume also that for some* $\delta_0 > 0$ *the set* $\{v \in [-1, 1]\colon\ f''(v) = 0\} \cap [m - \delta_0, m + \delta_0]$ *is finite. Then*

$$V(m, u_f) = \begin{cases} W_m(u_f) & \text{if } \int_{\mathbb{T}} dx\, u_f = m, \\ +\infty & \text{otherwise} \end{cases}$$

for any $m \in (-1, 1)$ *and* $u_f \in U$.

(iii) *Assume the same hypotheses of* (ii) *and furthermore that there exists* $h \in C^2([-1, 1])$ *such that* $\sigma h'' = D$. *Then*

$$V^{JV}(m, u_f) = \begin{cases} W_m(u_f) & \text{if } \int_{\mathbb{T}} dx\, u_f = m, \\ +\infty & \text{otherwise} \end{cases}$$

for any $m \in (-1, 1)$ *and* $u_f \in U$.

Note that (3.4) is verified if σ does not vanish, or vanishes slower than quadratically in -1 and $+1$.

Observe that $H_{\varepsilon;T}$ has a quadratic structure (see (4.1)), so that the proof of Theorem 3.1(i) is an infinite-dimensional version of Freidlin–Wentzell theorem [9, Theorem 4.3.1]. However this is not the case for H_T. In particular, since $H_T(u) = +\infty$ if u is not an (entropy-measure) solution to (1.1), the main technical difficulty in the proof of Theorem 3.1(ii) is to show that one can find a solution u to (1.1) such that u connects *in finite time* a profile $v \in U$ close in $L_\infty(\mathbb{T})$ to the constant profile m, to m itself. We remark that the quasi-potential problem for H_T is at this time being addressed in [1] in the case of Dirichlet boundary conditions. While this setting is quite similar to ours, the difficulties are completely different. In the boundary driven case, the entropic evolution connects a non-constant profile to a constant in finite time, so for T large it is not difficult to solve the minimization problem (3.2) far from the boundaries. On the other hand, new challenging difficulties appear in (3.2) when dealing with weak solutions to (1.1) featuring discontinuities at the boundary (boundary layers). Of course, this problem does not appear at all on a torus.

Remark 3.2. Let $T_1, T_2 > 0$, and let $u_1 \in \mathcal{X}_{T_1}$, $u_2 \in \mathcal{X}_{T_2}$. Define the measurable function $u : [0, T_1 + T_2] \times \mathbb{T} \to [-1, 1]$ by $u(t, x) = u_1(t, x)$ if $t \in [0, T_1]$, and $u(t, x) = u_2(t - T_1, x)$ if $t \in (T_1, T_1 + T_2]$. Then $u \in \mathcal{X}_{T_1+T_2}$ iff $u_1(T_1) = u_2(0)$ and in such a case

$$H_{\varepsilon;T_1+T_2}(u) = H_{\varepsilon;T_1}(u_1) + H_{\varepsilon;T_2}(u_2).$$

Furthermore if the hypotheses of Theorem 3.1(ii) hold, then

$$H_{T_1+T_2}(u) = H_{T_1}(u_1) + H_{T_2}(u_2).$$

Proof. A change of variables in the definition (2.1) shows that $H_{\varepsilon;T_1}(u_1) + H_{\varepsilon;T_2}(u_2)$ can be still written in the form (2.1) with $T = T_1 + T_2$, where now the supremum is carried over all the test functions $\varphi \in C_c^\infty((0, T_1) \cup (T_1, T_1 + T_2) \times \mathbb{T})$. However, if $u_1(T_1) = u_2(0)$, the supremum carried over such test functions coincides with the supremum carried over $C_c^\infty((0, T_1 + T_2) \times \mathbb{T})$. Namely, $H_{\varepsilon;T_1}(u_1) + H_{\varepsilon;T_2}(u_2)$ equals the definition of $H_{\varepsilon;T_1+T_2}(u)$.

By (2.4) it follows that $H_{T_1+T_2}(u) = +\infty$ whenever $H_{T_1}(u_1) = +\infty$ or $H_{T_2}(u_2) = +\infty$. Assume instead $H_{T_1}(u_1), H_{T_2}(u_2) < \infty$. Under the assumptions of Theorem 3.1(ii), the boundedness of H_T implies strong continuity in $L_1(\mathbb{T})$ as remarked below Proposition 2.1. Therefore if $u_1(T_1) = u_2(0)$ then $u \in C([0, T_1 + T_2]; L_1(\mathbb{T}))$ and $u \in \mathcal{E}_{T_1+T_2}$. By (2.2), (2.3) and the $L_1(\mathbb{T})$ continuity of u_1, u_2 and u, it follows that $\varrho_u(v; \{T_1\} \times \mathbb{T}) = \varrho_{u_1}(v; \{T_1\} \times \mathbb{T}) = \varrho_{u_2}(v; \{0\} \times \mathbb{T}) = 0$ for a.e. $v \in [-1, 1]$. Thus $\varrho_u(v; dt, dx) = \varrho_{u_1}(v; dt, dx)$ in $[0, T_1] \times \mathbb{T}$ and $\varrho_u(v; dt, dx) = \varrho_{u_2}(v; d(t - T_1), dx)$ in $[T_1, T_1 + T_2] \times \mathbb{T}$, and the equality follows from (2.4). □

Since $H_{\varepsilon;T}(m) = H_T(m) = 0$, by Remark 3.2, the infima in (3.1), (3.2) are attained in the limit $T \to +\infty$.

4. Proof of Theorem 3.1 for V_ε

Given a bounded measurable function $a \geqslant 0$ on $[0, T] \times \mathbb{T}$ let $\mathcal{D}^1_{a;T}$ be the Hilbert space obtained by identifying and completing the functions $\varphi \in C^\infty([0, T] \times \mathbb{T})$ with respect to the seminorm $[\int_0^T dt \langle \varphi_x, a\varphi_x\rangle]^{1/2}$. Let $\mathcal{D}^{-1}_{a;T}$ be its dual space. The corresponding norms are denoted respectively by $\|\cdot\|_{\mathcal{D}^1_{a;T}}$ and $\|\cdot\|_{\mathcal{D}^{-1}_{a;T}}$.

Remark 4.1. Let $a \geqslant 0$ be a bounded measurable function on $[0, T] \times \mathbb{T}$. Let $F, G \in L_2([0, T] \times \mathbb{T})$ be such that $F_x, (aG)_x \in \mathcal{D}^{-1}_{a;T}$. Assume that $\int_{\mathbb{T}} dx\, G(t, x) = 0$ for a.e. $t \in [0, T]$. Then

$$\big(F_x, (aG)_x\big)_{\mathcal{D}^{-1}_{a;T}} = \int_0^T dt \langle F, G\rangle$$

where $(\cdot,\cdot)_{\mathcal{D}^{-1}_{a;T}}$ denotes the scalar product in $\mathcal{D}^{-1}_{a;T}$.

By a standard application of the Riesz representation theorem (see [2, Lemma 3.1]), we have that if $H_{\varepsilon;T}(u) < +\infty$ then

$$H_{\varepsilon;T}(u) = \frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)_x - \frac{\varepsilon}{2}\big(D(u)u_x\big)_x \right\|^2_{\mathcal{D}^{-1}_{\sigma(u);T}}. \tag{4.1}$$

If $\int_{\mathbb{T}} dx\, u_f(x) \neq m$, then Theorem 3.1(i) follows from the conservation of the total mass of functions $u \in \mathcal{X}_T$ with $H_{\varepsilon;T}(u) < +\infty$. On the other hand, if $\int_{\mathbb{T}} dx\, u_f(x) = m$, the proof of the theorem is a consequence of the following lemmas. In fact from Lemma 4.2 we get $V_\varepsilon(m, u_f) \geqslant W_m(u_f)$, and from Lemmas 4.2 and 4.3 we have $V_\varepsilon(m, u_f) \leqslant W_m(u_f) + \gamma$ for each $\gamma > 0$.

Lemma 4.2. *Assume* (3.4)*, let $T > 0$ and $u \in \mathcal{X}_T$ be such that $H_{\varepsilon;T}(u) < +\infty$, $u(0,x) \equiv m$, $u(T,x) = u_f(x)$. Then $\int_{\mathbb{T}} dx\, h_m(u_f(x)) < +\infty$, $u_t + f(u)_x, (D(u)u_x)_x \in \mathcal{D}^{-1}_{\sigma(u);T}$ and*

$$H_{\varepsilon;T}(u) = \int_{\mathbb{T}} dx\, h_m\big(u_f(x)\big) + \frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)_x + \frac{\varepsilon}{2}\big(D(u)u_x\big)_x \right\|^2_{\mathcal{D}^{-1}_{\sigma(u);T}}.$$

Lemma 4.3. *For each $\gamma > 0$, there exist $T > 0$ and $u \in \mathcal{X}_T$ such that $H_{\varepsilon;T}(u) < +\infty$, $u(0) \equiv m$, $u(T) = u_f$ and*

$$\frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)_x + \frac{\varepsilon}{2}\big(D(u)u_x\big)_x \right\|^2_{\mathcal{D}^{-1}_{\sigma(u);T}} \leqslant \gamma.$$

Proof of Lemma 4.2. We first assume that there exists $\delta > 0$ such that for a.e. $(t,x) \in [0,T] \times \mathbb{T}$, $-1 + \delta \leqslant u(t,x) \leqslant 1 - \delta$, so that $\sigma(u)$ is uniformly positive. It follows that $(D(u)u_x)_x, f(u)_x \in \mathcal{D}^{-1}_{\sigma(u);T}$ so that, since $H_{\varepsilon;T}(u) < +\infty$, by (4.1) we also have $u_t \in \mathcal{D}^{-1}_{\sigma(u);T}$. In particular there exists $\theta \in L_2([0,T] \times \mathbb{T})$ such that $u_t = \theta_x$ weakly. Therefore

$$\begin{aligned}
H_{\varepsilon;T}(u) &= \frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)_x - \frac{\varepsilon}{2}\big(D(u)u_x\big)_x \right\|^2_{\mathcal{D}^{-1}_{\sigma(u);T}} \\
&= \frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)_x + \frac{\varepsilon}{2}\big(D(u)u_x\big)_x \right\|^2_{\mathcal{D}^{-1}_{\sigma(u);T}} - \big(\theta_x + f(u)_x, \big(D(u)u_x\big)_x\big)_{\mathcal{D}^{-1}_{\sigma(u);T}} \\
&= \frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)_x + \frac{\varepsilon}{2}\big(D(u)u_x\big)_x \right\|^2_{\mathcal{D}^{-1}_{\sigma(u);T}} \\
&\quad - \int_0^T dt \left\langle \theta, \frac{D(u)}{\sigma(u)} u_x \right\rangle + \left\langle f(u), \frac{D(u)}{\sigma(u)} u_x \right\rangle
\end{aligned}$$

where in the last line we used Remark 4.1, as for each $t \in [0,T]$

$$\int_{\mathbb{T}} dx\, \frac{D(u(t,x))}{\sigma(u(t,x))} u_x(t,x) = \int_{\mathbb{T}} dx\, h'_m\big(u(t,x)\big)_x = 0.$$

Similarly we have $\langle f(u(t)), \frac{D(u(t))}{\sigma(u(t))} u_x(t)\rangle = 0$ and integrating by parts:

$$-\int_0^T dt \left\langle \theta, \frac{D(u)}{\sigma(u)} u_x \right\rangle = \int_0^T dt \left\langle \theta_x, h'_m(u) \right\rangle = \int_0^T dt \left\langle u_t, h'_m(u) \right\rangle$$

$$= \int_{\mathbb{T}} dx\, h_m\big(u(T,x)\big) - h_m\big(u(0,x)\big).$$

Lemma 4.2 is therefore established for each $u \in \mathcal{X}_T$ bounded away from -1 and $+1$. For a general $u \in \mathcal{X}_T$ such that $u(0,\cdot) \equiv m \in (-1,1)$, and $\delta > 0$, let us define

$$u^\delta(t,x) = (1-\delta)u(t,x) + \delta m.$$

Provided (3.4) holds, the sequence $\{u^\delta\} \subset \mathcal{X}_T$ converges to u as $\delta \to 0$, and is such that: for $\delta > 0$, u^δ is bounded away from -1 and $+1$; $u^\delta(0,\cdot) \equiv m$, $\int_{\mathbb{T}} dx\, h(u^\delta(T,x)) \to \int_{\mathbb{T}} dx\, h(u(T,x))$; $H_{\varepsilon;T}(u^\delta) \to H_{\varepsilon;T}(u)$;

$$\left\| u^\delta_t + f\big(u^\delta\big)_x + \frac{\varepsilon}{2}\big(D\big(u^\delta\big)u^\delta_x\big)_x \right\|^2_{\mathcal{D}^{-1}_{\sigma(u^\delta);T}} \to \left\| u_t + f(u)_x + \frac{\varepsilon}{2}\big(D(u)u_x\big)_x \right\|^2_{\mathcal{D}^{-1}_{\sigma(u);T}}.$$

Therefore, since Lemma 4.2 holds for u^δ for each $\delta > 0$, it also holds for u. □

The following result is well known [8].

Theorem 4.4. *Let $u_f \in U$ and let $v : [0,\infty) \times \mathbb{T} \to \mathbb{R}$ be the solution to* (1.2) *with initial datum u_f. Then $\lim_{t\to\infty} \|v(t) - m\|_{L_\infty([0,T]\times\mathbb{T})} = 0$ where $m = \int_{\mathbb{T}} dx\, u_f(x)$.*

Proof of Lemma 4.3. Let $v : [0,\infty) \times \mathbb{T} \to \mathbb{R}$ be the solution to (1.2) with initial datum $v(0,x) = u_f(-x)$, and for $T_1, T_2 > 0$ let $u \in \mathcal{X}_{T_1+T_2}$ be defined as

$$u(t,x) = \begin{cases} (1 - \frac{t}{T_1})m + \frac{t}{T_1} v(T_2, -x) & \text{for } t \in [0, T_1], \\ v(T_1 + T_2 - t, -x) & \text{for } t \in [T_1, T_1 + T_2]. \end{cases}$$

Since u satisfies $u_t + f(u)_x + \frac{\varepsilon}{2}(D(u)u_x)_x = 0$ for $t \in [T_1, T_1 + T_2]$, we have by Remark 3.2

$$\begin{aligned} &\frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)_x + \frac{\varepsilon}{2}\big(D(u)u_x\big)_x \right\|^2_{\mathcal{D}^{-1}_{\sigma(u);T_1+T_2}} \\ &\quad = \frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)_x + \frac{\varepsilon}{2}\big(D(u)u_x\big)_x \right\|^2_{\mathcal{D}^{-1}_{\sigma(u);T_1}} \\ &\quad \leqslant \frac{3\varepsilon^{-1}}{2} \left[\|u_t\|^2_{\mathcal{D}^{-1}_{\sigma(u);T_1}} + \|f(u)_x\|^2_{\mathcal{D}^{-1}_{\sigma(u);T_1}} + \left\| \frac{\varepsilon}{2}\big(D(u)u_x\big)_x \right\|^2_{\mathcal{D}^{-1}_{\sigma(u);T_1}} \right]. \end{aligned} \tag{4.2}$$

Let now $\delta > 0$ (to be chosen below) be small enough to have $-1 < m - \delta < m + \delta < 1$, and define

$$C_{\sigma,\delta} := \max_{v\in[m-\delta,m+\delta]} \frac{1}{\sigma(v)} < +\infty,$$

$$c_f(t) := \int_{\mathbb{T}} dx\, \sigma\big(u(t,x)\big) \int_{\mathbb{T}} dx\, \frac{f(u(t,x))}{\sigma(u(t,x))},$$

$$C_{f,\delta} := \max_{v\in[m-\delta,m+\delta]} f(v) - \min_{v\in[m-\delta,m+\delta]} f(v),$$

$$C_D := \max_{v\in[-1,1]} \frac{D(v)^2}{2}.$$

Let also $\theta \in L_2([0,T_1]\times\mathbb{T})$ be defined by

$$\theta_x(t,x) = \frac{v(T_2,-x) - m}{T_1},$$

$$\int_{\mathbb{T}} \frac{\theta(t,x)}{\sigma(u(t,x))} = 0.$$

By Theorem 4.4, there exists $\tau_\delta > 0$ such that $\|v(t) - m\|_{L_\infty(\mathbb{T})} \leqslant \delta$ for each $t \geqslant \tau_\delta$. By Remark 4.1 and (4.2), since $u_t = \theta_x$ weakly, we have for each $T_2 \geqslant \tau_\delta$

$$\begin{aligned}
&\frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)_x + \frac{\varepsilon}{2}\big(D(u)u_x\big)_x \right\|^2_{\mathcal{D}^{-1}_{\sigma(u);T_1+T_2}} \\
&\quad\leqslant \frac{3\varepsilon^{-1}}{2} \int_0^{T_1} dt \left\langle \theta, \frac{\theta}{\sigma(u)} \right\rangle + \left\langle f(u) - c_f, \frac{f(u) - c_f}{\sigma(u)} \right\rangle + \left\langle D(u)u_x, \frac{D(u)u_x}{\sigma(u)} \right\rangle \\
&\quad\leqslant \frac{3\varepsilon^{-1}}{2} C_{\sigma,\delta} \left[\int_0^{T_1} dt \langle\theta,\theta\rangle + T_1 C_{f,\delta}^2 + T_1 C_D \big\langle v(T_2)_x, v(T_2)_x \big\rangle \right].
\end{aligned} \tag{4.3}$$

By standard parabolic estimates we have

$$\int_0^{+\infty} dt \langle v_x, v_x\rangle < +\infty.$$

In particular there exists $T_{2,\delta} > \tau_\delta$ such that $\langle v(T_{2,\delta})_x, v(T_{2,\delta})_x\rangle < \delta$. Note that, as $\delta \to 0$, $C_{\sigma,\delta}$ stays bounded, while $C_{f,\delta}$, $\langle\theta,\theta\rangle$ and $\langle v(T_{2,\delta})_x, v(T_{2,\delta})_x\rangle$ vanish. Therefore the right-hand side of (4.3) can be made arbitrarily small provided δ is small enough. □

5. Proof of Theorem 3.1 for V and V^{JV}

Define the parity operator $P : U \to U$ by $Pu(x) = u(-x)$ and for $T > 0$ the time–space parity operator $P^T : \mathcal{X}_T \to \mathcal{X}_T$ by $P^T u(t,x) = u(T-t,-x)$. Define the time reversed quasi-potential $V : U \times [-1,1] \to [0,\infty]$ as

$$V(u_i, m) := \inf\bigl\{H_T(u),\ T > 0,\ u \in \mathcal{X}_T\colon\ u(0) = u_i,\ u(T) \equiv m\bigr\}.$$

Lemma 5.1. *Assume $f \in C^2([-1,1])$ is such that there is no interval in which f is affine. Let $T > 0$, $u_f \in U$ and $m = \int_{\mathbb{T}} dx\, u_f(x)$. Then*

$$V(m, u_f) = V(Pu_f, m) + W_m(u_f).$$

Proof. By the assumptions on f, as remarked below Proposition 2.1, $\mathcal{E}_T \subset C([0,T]; L^1(\mathbb{T}))$. In particular Eqs. (2.2)–(2.3) extend to any $\vartheta \in C^{2,\infty}([-1,1] \times [0,T] \times \mathbb{T})$ (now $\vartheta(0)$ and $\vartheta(T)$ need not to vanish) as

$$\begin{aligned}
&\int_{\mathbb{T}} dx\, \vartheta\bigl(u(T,x), T, x\bigr) - \vartheta\bigl(u(0,x), 0, x\bigr) \\
&\qquad - \int_{[0,T]\times\mathbb{T}} dt\, dx\, \bigl[(\partial_t \vartheta)\bigl(u(t,x), t, x\bigr) + (\partial_x Q)\bigl(u(t,x), t, x\bigr)\bigr] \\
&\quad = \int_{[-1,1]\times[0,T]\times\mathbb{T}} dv\, \varrho_u(v; dt, dx)\, \vartheta''(v, t, x).
\end{aligned} \tag{5.1}$$

Note that for $u \in \mathcal{E}_T$ and $v \in [-1,1]$

$$\varrho_{P^T u}(v; dt, dx) = -\varrho_u\bigl(v; d(T-t), d(-x)\bigr).$$

Therefore assuming also $u(0) \equiv m$, $u(T) = u_f$, we have for each $\eta \in C^2([-1,1])$ with $\eta(m) = 0$

$$\begin{aligned}
&\int dv\, \eta''(v) \varrho_u^+(v; dt, dx) - \int dv\, \eta''(v) \varrho_{P^T u}^+(v; dt, dx) \\
&\quad = \int dv\, \eta''(v) \varrho_u^+(v; dt, dx) - \int dv\, \eta''(v) \varrho_u^-\bigl(v; d(T-t), -dx\bigr) \\
&\quad = \int dv\, \eta''(v) \varrho_u^+(v; dt, dx) - \int dv\, \eta''(v) \varrho_u^-(v; dt, dx) \\
&\quad = \int dv\, \eta''(v) \varrho_u(v; dt, dx) = \int dx\, \eta\bigl(u(T,x)\bigr) - \eta\bigl(u(0,x)\bigr) \\
&\quad = \int dx\, \eta\bigl(u_f(x)\bigr)
\end{aligned} \tag{5.2}$$

where we used (5.1) with $\vartheta(v,t,x)=\eta(v)$. If σ is bounded away from 0, then (5.2) evaluated for $\eta = h_m$ immediately yields

$$H_T(u) = H_T\big(P^T u\big) + W_m(u_f). \tag{5.3}$$

If σ vanishes at -1 or $+1$, then (5.3) is obtained by monotone convergence, when considering in (5.2) a sequence $\{\eta^n\} \subset C^2([-1,1])$ such that: $\eta^n(m)=0$; $0 \leqslant (\eta^n)'' \leqslant h_m''$; and for all $v \in [-1,1]$, $\eta^n(v) \uparrow h_m(v)$ and $(\eta^n)''(v) \uparrow h_m''(v)$.

Optimizing in (5.3) over T and u we get $V(m,u_f) \geqslant V(Pu_f, m) + W_m(u_f)$. Replacing u_f by Pu_f and thus Pu_f by $P(Pu_f) = u_f$, we get the reverse inequality. □

Definition 5.2. We say that $u_i \in U$ is *piecewise constant* iff there is a finite partition of $\mathbb{T}$ in intervals such that u_i is constant on each interval. For $T>0$, we say that $u \in \mathcal{X}_T$ is *piecewise constant* iff $u \in C([0,T]; L_1(\mathbb{T}))$ and there exists a finite partition of $[0,T]\times\mathbb{T}$ in connected sets with Lipschitz boundary such that u is constant on each set of these.

The following lemma is the main technical difficulty of this paper, and its proof is postponed at the end of this section.

Lemma 5.3. *Assume the same hypotheses of Theorem* 3.1(ii). *For each $\gamma>0$, there exist T^γ, $\delta^\gamma>0$ such that the following holds. For each* piecewise constant *$u_i \in U$ satisfying $\int_{\mathbb{T}} dx\, u_i(x) = m$ and $\|u_i - m\|_{L_\infty(\mathbb{T})} \leqslant \delta^\gamma$, there exists $u^\gamma \in \mathcal{X}_{T^\gamma}$ such that $u^\gamma(0)=u_i$, $u^\gamma(T^\gamma)\equiv m$ and $H_{T^\gamma}(u^\gamma) \leqslant \gamma$.*

The next corollary relaxes the condition in Lemma 5.3 requiring u_i to be piecewise constant.

Corollary 5.4. *Assume the same hypotheses of Theorem* 3.1(ii). *For each $\gamma>0$, there exist T^γ, $\delta^\gamma>0$ such that the following holds. For each $u_i \in U$ satisfying $\int_{\mathbb{T}} dx\, u_i(x) = m$ and $\|u_i - m\|_{L_\infty(\mathbb{T})} \leqslant \delta^\gamma$, there exists $u^\gamma \in \mathcal{X}_{T^\gamma}$ such that $u^\gamma(0)=u_i$, $u^\gamma(T^\gamma)\equiv m$ and $H_{T^\gamma}(u^\gamma) \leqslant \gamma$.*

Proof. For a fixed $\gamma>0$, let T^γ and $\delta^\gamma>0$ be as in Lemma 5.3. For $u_i \in U$ such that $\|u_i - m\|_{L_\infty(\mathbb{T})} \leqslant \delta^\gamma$, where $m = \int_{\mathbb{T}} dx\, u_i(x)$, let $\{u_i^n\} \subset U$ be a sequence of piecewise constant functions converging to u_i in U and satisfying $\int_{\mathbb{T}} dx\, u_i^n(x) = m$ and $\|u_i - m\|_{L_\infty(\mathbb{T})} \leqslant \delta^\gamma$. For each n, by Lemma 5.3 there exist $u^{n,\gamma}$ such that $u^{n,\gamma}(0) = u_i^n$, $u^{n,\gamma}(T^\gamma) \equiv m$ and $H_{T^\gamma}(u^{n,\gamma}) \leqslant \gamma$. Therefore, since H_{T^γ} has compact sublevel sets (see [2, Proposition 2.6]), there is a (not relabeled) subsequence $\{u^{n,\gamma}\}$ converging to a u^γ in $\mathcal{X}_{T^\gamma}$, and $H_{T^\gamma}(u^\gamma) \leqslant \gamma$. By the definition of convergence in $\mathcal{X}_{T^\gamma}$, $u^{n,\gamma}(0)$ and $u^{n,\gamma}(T^\gamma)$ converge in U to $u^\gamma(0)$ and $u^\gamma(T^\gamma)$ respectively, and thus $u^\gamma(0) = u_i$ and $u^\gamma(T^\gamma) \equiv m$. □

We recall a result in [6, Chapter 11.5], [12].

Theorem 5.5. *Assume $f \in C^2([-1,1])$, and that there is no interval in which f is affine. Let $u_i \in U$ and let $\bar{u} : [0,\infty)\times\mathbb{T} \to \mathbb{R}$ be the Kruzkhov solution to* (1.1) *with initial datum $u_i \in U$. Then*

$$\lim_{t\to\infty} \big\|\bar{u}(t) - m\big\|_{L_\infty([0,T]\times\mathbb{T})} = 0$$

where $m = \int_{\mathbb{T}} dx\, u_i(x)$.

Proof of Theorem 3.1(ii). Fix $u_i \in U$ and $\gamma > 0$. Let T^γ and δ^γ be as in Corollary 5.4, let $\bar{u} : [0, +\infty) \to U$ be the Kruzhkov solution to (1.1) with initial datum u_i, and let $m = \int_{\mathbb{T}} dx\, u_i(x)$. By Theorem 5.5, there exists τ^γ such that $|\bar{u}(\tau^\gamma) - m|_{L_\infty(\mathbb{T})} \leqslant \delta^\gamma$. By Corollary 5.4 there exists $\tilde{u} \in \mathcal{X}_{T^\gamma}$ such that $\tilde{u}(0) = \bar{u}(\tau^\gamma)$, $\tilde{u}(T^\gamma) \equiv m$ and $H_{T^\gamma}(\tilde{u}) \leqslant \gamma$. Define $u \in \mathcal{X}_{\tau^\gamma + T^\gamma}$ by

$$u(t,x) := \begin{cases} \bar{u}(t,x) & \text{if } t \leqslant \tau^\gamma, \\ \tilde{u}(t - \tau^\gamma, x) & \text{if } \tau^\gamma \leqslant t \leqslant \tau^\gamma + T^\gamma. \end{cases}$$

Then, by Remark 3.2, $H_{\tau^\gamma + T^\gamma}(u) = H_{\tau^\gamma}(\bar{u}) + H_{T^\gamma}(\tilde{u}) \leqslant \gamma$. Therefore $V(u_i, m) = 0$ and the proof is thus complete since Lemma 5.1 holds. □

The remaining of this section is devoted to the proof of Lemma 5.3.

Remark 5.6. Let $m \in (-1, 1)$, assume the same hypotheses of Theorem 3.1(ii), and let δ_0 be defined accordingly. Then, taking perhaps a smaller δ_0, one can assume $[m - \delta_0, m + \delta_0] \subset (-1, 1)$ and that one (and only one) of the following holds:

(A) in the interval $[m - \delta_0, m + \delta_0]$, f is either strictly convex or strictly concave.
(B) f is either strictly convex in $[m - \delta_0, m]$ and strictly concave in $[m, m + \delta_0]$, or strictly concave in $[m - \delta_0, m]$ and strictly convex in $[m, m + \delta_0]$.

With no loss of generality, we will assume f convex in $[m - \delta_0, m + \delta_0]$ if case (A) holds, and f concave in $[m - \delta_0, m]$ and convex in $[m, m + \delta_0]$ if (B) holds.

Remark 5.7. Let $T > 0$ and assume $u \in \mathcal{E}_T$ to be piecewise constant according to Definition 5.2. Then the jump set of u consists of a finite number of segments in $[0, T] \times \mathbb{T}$. In particular there exist a finite sequence $0 = T^0 < T^1 < \cdots < T^n = T$, and, for $k = 1, \dots, n$, finite sequences $\{w_j^k\}_{j=1,\dots,N_k} \subset [-1, 1]$ such that for $t \in (T^{k-1}, T^k)$, $u(t)$ is piecewise constant with jump set consisting of a finite set of points $\{x_j^k(t)\}_{k=1,\dots,N_k} \in \mathbb{T}$, and the traces of $u(t)$ at $x_j^k(t)$ are w_j^k (from the right) and w_{j-1}^k (from the left, where we understand $w_0^k \equiv w_{N_k}^k$).

In particular, by (2.5) we have that

$$H_T(u) = \sum_{k=1}^{n} \left(T^k - T^{k-1}\right) \sum_{j=1}^{N_k} \int dv\, \frac{D(v)}{\sigma(v)} \frac{\rho^+(v, w_j^k, w_j^{k-1})}{|w_j^k - w_j^{k-1}|}. \tag{5.4}$$

If $u \in \mathcal{E}_T$ is piecewise constant, and u^-, u^+ are the left and right traces of u at a given point in the jump set of u, we say that the shock between u^- and u^+ is *entropic* iff $\rho(v, u^-, u^+) \leqslant 0$ for almost every v, while it is *anti-entropic* iff $\rho(v, u^-, u^+) \geqslant 0$ for almost every v. If f is convex or concave, each shock is either entropic or anti-entropic, but in the general case the sign of $\rho(v, u^-, u^+)$ may depend on v.

Lemma 5.8. *Let $m \in (-1, 1)$, and $\delta_0 \equiv \delta_0(m) > 0$ be as in Remark 5.6. Let $u_i \in U$ be piecewise constant and such that $\int_{\mathbb{T}} dx\, u_i(x) = m$ and $\|u_i - m\|_{L_\infty(\mathbb{T})} \leqslant \delta_0$. Then for each $\overline{T}, \gamma > 0$ there exists $w \in \mathcal{X}_{\overline{T}}$ piecewise constant such that $\|w - m\|_{L_\infty([0,T] \times \mathbb{T})} \leqslant \|u_i - m\|_{L_\infty(\mathbb{T})}$, $w(0) = u_i$, and $H_{\overline{T}}(w) \leqslant \gamma$.*

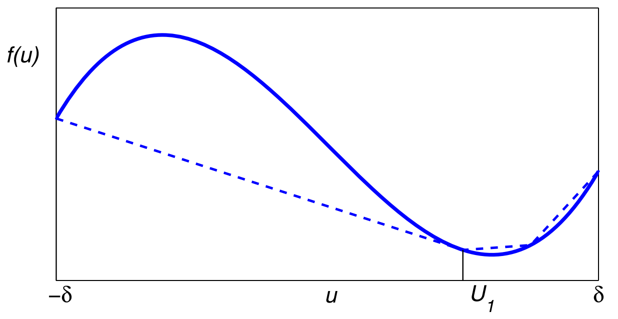

Fig. 1. In the figure, we have $f(u) = u^3 - u$, $m = 0$ and $M = 2$. Consider a discontinuity between the values $u_1^- = -\delta$ and $u_1^+ = \delta$. Then U_1 is chosen as the abscissa of the point at which a line passing in $(-\delta, f(-\delta))$ is tangent to the graph of f. The values $U_1 + \frac{k}{M}(u_1^+ - U_1)$ for $k = 1, 2$ are the abscissas of the intersections of the dashed lines with the graph of f.

The proof of Lemma 5.8 will be divided in three steps. The main idea is to construct a piecewise smooth weak solution w, by splitting each shock appearing in the initial datum in an entropic part and an anti-entropic part, the anti-entropic part being split itself in M small anti-entropic shocks, with M a large integer, see Fig. 1. For such a solution to exist, the points at which the shocks are split have to be carefully chosen. We are then able to define w up to the first time at which two (or more) shocks collide. Defining then w recursively, we prove that there can be only a finite number of times at which the shocks collide, and thus w is well defined globally in time. Finally, we show that $H_{\overline{T}}(w)$ can be made arbitrarily small by choosing M large.

Proof of Lemma 5.8. As noted in Remark 5.6, we assume f to be strictly convex in $[m, m+\delta_0]$. Hereafter we let $\delta := \|u_i - m\|_{L_\infty(\mathbb{T})} \leqslant \delta_0$.

Step 1 *(Evolution of shocks).* Let $x_1, \ldots, x_N \in \mathbb{T}$ be the points at which the discontinuities of u_i are located. With a little abuse of notation, we also denote by $u_i : \mathbb{R} \to [-1, 1]$ and $x_j \in [0, 1] \subset \mathbb{R}$ the lift of u_i and x_j to $\mathbb{R}$, and we assume $x_j < x_{j+1}$ for $j = 1, \ldots, N-1$. For $j = 1, \ldots, N$ and $n \in \mathbb{Z}$, let $x_{j+nN} = x_j + n \in \mathbb{R}$, and for $j \in \mathbb{Z}$ let u_j^- and u_j^+ be respectively the left and right traces of u_i at x_j. Define

$$U_j := \begin{cases} \max\{w \in [u_j^-, u_j^+]\colon\ \rho(v, w, u_j^-) \leqslant 0,\ \forall v \in [-1, 1]\} & \text{if } u_j^- < u_j^+, \\ \min\{w \in [u_j^+, u_j^-]\colon\ \rho(v, w, u_j^-) \leqslant 0,\ \forall v \in [-1, 1]\} & \text{if } u_j^+ < u_j^-. \end{cases}$$

Since f is convex in $[m, m+\delta_0]$, if $u_j^- < u_j^+$ and $U_j \leqslant v \leqslant v' \leqslant u_j^+$, or if $u_j^+ < u_j^-$ and $u_j^+ \leqslant v' \leqslant v \leqslant U_j$ then

$$\frac{f(u_j^-) - f(U_j)}{u_j^- - U_j} \leqslant \frac{f(U_j) - f(v)}{U_j - v} \leqslant \frac{f(v) - f(v')}{v - v'}$$

where we understand $\frac{f(w)-f(w)}{w-w} = f'(w)$ for $w \in [-1, 1]$.

Therefore, fixed an integer $M \geqslant 2$, it is possible to define the map $w_j^{u_i} : [0, +\infty) \times \mathbb{R} \to [m-\delta, m+\delta]$ as (see Fig. 2)

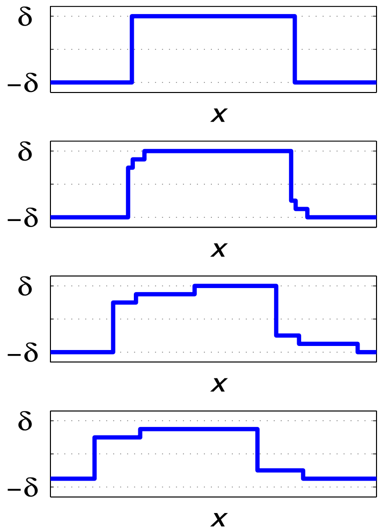

Fig. 2. In the figure, we have $f(u) = u^3 - u$, $m = 0$, and the initial datum u_i having two jumps between the values $-\delta$ and δ. The figure shows w at different times $t \in [0, T(u_i)]$.

$$w_j^{u_i}(t,x) := \begin{cases} u_j^- & \text{if } x - x_j \leqslant \frac{f(u_j^-)-f(U_j)}{u_j^- - U_j} t, \\ U_j & \text{if } x - x_j \in [\frac{f(u_j^-)-f(U_j)}{u_j^- - U_j} t, M \frac{f(U_j) - f(U_j + \frac{u_j^+ - U_j}{M})}{U_j - u_j^+} t], \\ U_j + \frac{k}{M}(u_j^+ - U_j) & \text{if } x - x_j \in [M \frac{f(U_j + \frac{k-1}{M}(u_j^+ - U_j)) - f(U_j + \frac{k}{M}(u_j^+ - U_j))}{U_j - u_j^+} t, \\ & \quad M \frac{f(U_j + \frac{k}{M}(u_j^+ - U_j)) - f(U_j + \frac{k+1}{M}(u_j^+ - U_j))}{U_j - u_j^+} t] \\ & \quad \text{for } k \in \{1, \ldots, M-1\}, \\ u_j^+ & \text{if } x - x_j \geqslant M \frac{f(U_j + \frac{M-1}{M}(u_j^+ - U_j)) - f(u_j^+)}{U_j - u_j^+} t. \end{cases} \tag{5.5}$$

Note that this definition makes sense in the case $U_j = u_j^-$ or $U_j = u_j^+$. We also let $X_j^-(t) := x_j + \frac{f(u_j^-)-f(U_j)}{u_j^- - U_j} t$, $X_j^+(t) := x_j + \frac{f(U_j + \frac{M-1}{M}(u_j^+ - U_j)) - f(u_j^+)}{U_j - u_j^+} t$ and

$$T(u_i) := \inf\Big\{t \geqslant 0\colon \min_j \big[X_j^-(t) - X_{j-1}^+(t)\big] = 0\Big\}.$$

We next define $w^{u_i} : [0, T(u_i)] \times \mathbb{R} \to [m - \delta, m + \delta]$ as

$$w^{u_i}(t,x) = w_j^{u_i}(t,x) \quad \text{if } x \in \big[X_j^+(t), X_{j+1}^-(t)\big].$$

w^{u_i} is a weak solution to (1.1) in $[0, T(u_i)] \times \mathbb{R}$, since it is piecewise constant and satisfies the Rankine–Hugoniot condition along the shocks. Since it is also 1-periodic on $\mathbb{R}$, it defines a map $w^{u_i} : [0, T(u_i)] \times \mathbb{T} \to [m - \delta, m + \delta]$ such that $w^{u_i}(0) = u_i$ and $w^{u_i} \in \mathcal{E}_{T(u_i)}$.

Step 2 *(There is a finite number of shocks merging).* We next define recursively, for $k \geqslant 1$, $T^k \in [0, \overline{T}]$ and $w^k : [T^{k-1}, T^k] \times \mathbb{T} \to [m-\delta, m+\delta]$ (where $T^0 = 0$) by

$$\begin{aligned} T^1 &:= T(u_i) \wedge \overline{T}, \\ w^1 &:= w^{u_i}, \\ T^k &:= \left[T^{k-1} + T\left(w^{k-1}\left(T^{k-1}\right)\right)\right] \wedge \overline{T} \quad \text{for } k \geqslant 2, \\ w^k(t,x) &:= w^{w^{k-1}(T^{k-1})}\left(t - T^{k-1}, x\right) \quad \text{for } k \geqslant 2. \end{aligned}$$

We want to show that there exists a $K \in \mathbb{N}$ such that $T^K = \overline{T}$.

By definition, for each $k \geqslant 1$ and $t \in (T^{k-1}, T^k)$, the discontinuities of $w^k(t)$ are either entropic or non-entropic. On the other hand, at the times T^k at which two or more shocks collide, one and only one of the following may happen.

- At a point $y \in \mathbb{T}$, two or more entropic shocks of w^k merge at time T^k. Then w^{k+1} has one entropic shock in $[T^k, T^{k+1}]$ starting at y.
- At a point $y \in \mathbb{T}$ one or more entropic shocks of w^k merge with one or more anti-entropic shocks. Then either w^{k+1} has one entropic shock starting at y, or w^{k+1} has a anti-entropic shocks starting at y, for some integer a, $0 \leqslant a \leqslant M$.

Note that, at a time T^k, one or more of the merging here described may happen at different points $y \in \mathbb{T}$, but at no point there can be a shock merging involving only anti-entropic shocks. The last statement can be proved by exhaustion, and – roughly speaking – it is a consequence of the well-known instability of anti-entropic shocks. Let us detail the case of convex f (corresponding to (A) in Remark 5.6). Anti-entropic shocks are then increasing, and according to the Rankine–Hugoniot condition, the higher the shock the higher its speed. Therefore, two or more anti-entropic shocks that are close enough, will separate as time increases rather than colliding. The case (B) of Remark 5.6 is treated similarly.

Summarizing the previous remark, at a given shocks merging: either the number of entropic shocks stays constant and the number of anti-entropic shocks decreases by at least one; or the number of entropic shocks decreases by at least one, and the number of anti-entropic shocks may either decrease, or increase (by at most M). It follows that there can be at most a finite number of shocks merging, and in particular a finite number of times at which shocks merge. Recalling that N was the total number of discontinuity points of u_i, and that by definition w^1 has at most N entropic shocks and NM anti-entropic shocks, it follows that for each k, w^k has at most $(2N-1)M$ anti-entropic shocks, the remaining shocks being entropic. Therefore the sequence $\{T^k\}$ has no accumulation points before $\overline{T}$, and it will hit $\overline{T}$ for k large enough.

Step 3 *(Computing H_T).* We can thus define $w : [0, \overline{T}] \times \mathbb{T} \to [m-\delta, m+\delta]$ by requiring $w(t,x) = w^k(t,x)$ for $t \in [T^{k-1}, T^k]$. w is piecewise constant and it satisfies the Rankine–Hugoniot condition along the shocks, therefore, since $w^{k-1}(T^k) = w^k(T^k)$, $w \in \mathcal{E}_{\overline{T}}$. As noted above, in each time interval $[T^{k-1}, T^k]$, w has at most $(2N-1)M$ shocks. Moreover, by the definition (5.5), the left and the right traces of w at an anti-entropic shock differ at most by $2\delta_0/M$. Therefore we can bound the sum in (5.4) as

$$
\begin{aligned}
H_{\overline{T}}(w) \leqslant \overline{T}(2N-1)\, M & \sup_{\substack{u^-,u^+\in[m-\delta_0,m+\delta_0]\\ |u^+-u^-|\leqslant \frac{2\delta_0}{M}}} \int dv\, \frac{D(v)}{\sigma(v)} \frac{\rho^+(v,u^+,u^-)}{|u^+-u^-|} \\
&\leqslant \overline{T}(2N-1)\, M \bigg[\max_{v\in[m-\delta_0,m+\delta_0]} \frac{D(v)}{\sigma(v)}\bigg] \\
&\quad \times \sup_{|u^+-u^-|\leqslant \frac{2\delta_0}{M}} \Bigg[\frac{|u^+-u^-|}{2}\big[f\big(u^+\big)+f\big(u^-\big)\big] - \int_{u^-}^{u^+} dv\, f(v)\Bigg] \\
&= \overline{T}(2N-1)\, M \bigg[\max_{v\in[m-\delta_0,m+\delta_0]} \frac{D(v)}{\sigma(v)}\bigg] \\
&\quad \times \sup_{|u^+-u^-|\leqslant \frac{2\delta_0}{M}} \Bigg[\frac{|u^+-u^-|}{2}\big[f\big(u^+\big)-f\big(u^-\big)-f'\big(u^-\big)\big(u^+-u^-\big)\big] \\
&\quad - \int_{u^-}^{u^+} dv\, \big[f(v)-f\big(u^-\big)-f'\big(u^-\big)\big(v-u^-\big)\big]\Bigg] \\
&\leqslant C\overline{T}\,(2N-1)M^{-2}
\end{aligned}
$$

where in the last inequality we used the standard Taylor remainder estimate and C is a constant depending only on f, D, σ. Namely, $H_{\overline{T}}(w)$ is arbitrarily small provided M is chosen sufficiently large. □

In the following, whenever $m+\delta, m+\delta' \in [-1,1]$, we introduce the short hand notation $R(\delta,\delta')$ for the Rankine–Hugoniot velocity of a shock between the values $m+\delta$ and $m+\delta'$, namely

$$
R(\delta,\delta') := \frac{f(m+\delta)-f(m+\delta')}{\delta-\delta'}
$$

and we understand $R(\delta,\delta) = f'(m+\delta)$. We also introduce

$$
\begin{aligned}
C(\delta,\delta') &:= \int dv\, \frac{\rho(v,m+\delta,m+\delta')}{|\delta-\delta'|} \\
&= \frac{|\delta-\delta'|}{2}\big[f(m+\delta)+f(m+\delta')\big] - \int_{m+\delta'}^{m+\delta} dv\, f(v).
\end{aligned}
$$

The following lemma introduces an explicit solution to (1.1), with initial datum having only two discontinuities and final datum being constant.

Lemma 5.9. *Assume the same hypotheses of Theorem* 3.1(ii) *and let* $\gamma > 0$. *Let* $m \in (-1,1)$ *and let* $\delta_0 = \delta_0(m)$ *be defined as in Remark* 5.6. *Then for each* $\delta_1 \in (0,\delta_0)$ *there exists* $\overline{\delta}_2 \equiv \overline{\delta}_2(\delta_1) \in (0,\delta_0)$ *such that for each* $\delta_2 \in (0,\overline{\delta}_2)$ *the following holds. For a fixed arbitrary* $x_0 \in \mathbb{T}$

let $u_d \equiv u_d^{\delta_1,\delta_2} \in U$ *be defined as*

$$u_d(x) := \begin{cases} m+\delta_1 & \text{if } |x-x_0| \leqslant \frac{\delta_2}{2(\delta_1+\delta_2)}, \\ m-\delta_2 & \text{otherwise,} \end{cases}$$

and let

$$\tau \equiv \tau^{\delta_1,\delta_2} := \frac{1}{|R(\delta_1,0)-R(0,-\delta_2)|}.$$

Then $\tau < \infty$, *and there exists* $u \in \mathcal{X}_\tau$ *such that* $u(0)=u_d$, $u(\tau)\equiv m$ *and*

$$H_\tau(u) \leqslant \left[\max_{v\in[m-\delta_0,m+\delta_0]} \frac{D(v)}{\sigma(v)}\right] \frac{C(\delta_1,0)^+ + C(0,-\delta_2)^+}{|R(\delta_1,0)-R(0,-\delta_2)|}. \tag{5.6}$$

Proof. Fix $\delta_1 \in (0,\delta_0)$. By the definition of δ_0, $R(\delta_1,0) \neq f'(m)$ and assuming f strictly convex in $[m,m+\delta_0]$ (see Remark 5.6), we have $R(\delta_1,0) > f'(m)$. Recalling the definition of ρ in (2.6), still by the convexity of f in $[m,m+\delta_0]$, we have $\rho(v,m,m+\delta_1)<0$ for $v\in(m,m+\delta_1)$ and $C(\delta_1,0)>0$. In particular there exists $\overline{\delta}_2$ small enough such that for each $\delta_2 \in (0,\overline{\delta}_2)$ and each $v \in (m-\delta_2,m+\delta_1)$

$$R(\delta_1,0)-R(0,-\delta_2) > 0, \tag{5.7}$$

$$\rho(v,m-\delta_2,m+\delta_1) < 0. \tag{5.8}$$

Let us now fix $\delta_2 \in (0,\overline{\delta}_2)$. By (5.7) τ^{δ_1,δ_2} is finite. With no loss of generality we may assume $x_0 = \frac{\delta_2}{2(\delta_1+\delta_2)}$, as the general case is obtained by a space translation of the solution u given below by the quantity $x_0 - \frac{\delta_2}{2(\delta_1+\delta_2)}$. Define

$$u(t,x) := \begin{cases} m & \text{if } |x-[R(\delta_1,0)+R(0,-\delta_2)]\frac{t}{2}| \leqslant [R(\delta_1,0)-R(0,-\delta_2)]\frac{t}{2}, \\ m+\delta_1 & \text{if } |x-\frac{\delta_2}{2(\delta_1+\delta_2)}-[R(\delta_1,0)+R(\delta_1,-\delta_2)]\frac{t}{2}| \\ & \quad \leqslant \frac{\delta_2}{2(\delta_1+\delta_2)}-[R(\delta_1,0)-R(\delta_1,-\delta_2)]\frac{t}{2}, \\ m-\delta_2 & \text{otherwise.} \end{cases}$$

It follows that $u(0)=u_d$ and $u(\tau)\equiv m$. Moreover u is a piecewise constant weak solution to (1.1). For a fixed $t\in(0,T)$, $u(t)$ has three discontinuity points, where its value jumps from m to $m+\delta_1$, from $m+\delta_1$ to $m-\delta_2$ and from $m-\delta_2$ to m. In particular $H_\tau(u)$ can be calculated by (5.4). The shock between the values $m+\delta_1$ and $m-\delta_2$ is entropic by (5.8), and thus it gives no contributions to the sum (5.4). By the convexity assumption on f in $[m,m+\delta_0]$, the shock between m and $m+\delta_1$ is anti-entropic, namely $\rho(v,m+\delta_1,m)\geqslant 0$. Moreover the shock between $m-\delta_2$ and m is either entropic (if case (A) in Remark 5.6 holds) or anti-entropic (if case (B) in Remark 5.6 holds). Therefore (5.4) yields

$$
\begin{aligned}
H_\tau(u) &= \tau\left[\int dv\,\frac{D(v)}{\sigma(v)}\,\frac{\rho^+(v,m,m-\delta_2)}{\delta_2} + \int dv\,\frac{D(v)}{\sigma(v)}\,\frac{\rho^+(v,m+\delta_1,m)}{\delta_1}\right] \\
&= \frac{[\int dv\,\frac{D(v)}{\sigma(v)}\,\frac{\rho(v,m,m-\delta_2)}{\delta_2}]^+ + \int dv\,\frac{D(v)}{\sigma(v)}\,\frac{\rho(v,m+\delta_1,m)}{\delta_1}}{R(\delta_1,0)-R(0,-\delta_2)} \\
&\leqslant \left[\max_{v\in[m-\delta_0,m+\delta_0]}\frac{D(v)}{\sigma(v)}\right]\frac{[\int dv\,\frac{\rho(v,m,m-\delta_2)}{\delta_2}]^+ + \int dv\,\frac{\rho(v,m+\delta_1,m)}{\delta_1}}{R(\delta_1,0)-R(0,-\delta_2)}
\end{aligned}
$$

namely (5.6). □

Remark 5.10. Let $s_1,\dots,s_n : [0,T]\to\mathbb{T}$ be a finite collection of Lipschitz maps, and let $F:[0,T]\times\mathbb{T}\to\mathbb{R}$ be a bounded function, such that F is Lipschitz in each connected component of $[0,T]\times\mathbb{T}\setminus\bigcup_{i=1}^n \mathrm{Graph}(s_i)$ (F may feature discontinuities on the graphs of the curves s_i, $i=1,\dots,n$). Assume either $\inf_{t,x}F(t,x)\geqslant \operatorname{ess\,sup}_{i,t}\dot s_i(t)$ or $\sup_{t,x}F(t,x)\leqslant \operatorname{ess\,inf}_{i,t}\dot s_i(t)$. Then, for each fixed $s_0\in\mathbb{T}$, there exists a unique Lipschitz map $s:[0,T]\to\mathbb{T}$ such that $s(0)=s_0$ and $\dot s(t)=F(t,s)$ for a.e. $t\in[0,T]$.

Proof of Lemma 5.3. Fix $\gamma>0$. Recall the definition of $\delta_0\equiv\delta_0(m)$ in Remark 5.6; as noted in Remark 5.6 we may assume f to be strictly convex in $[m,m+\delta_0]$. We thus have $R(\delta_1,0)>f'(m)$, $\rho(v,m+\delta_1,m)\geqslant 0$ for each $\delta_1\in(0,\delta_0)$. Then by explicit computation

$$
\lim_{\delta_1\downarrow 0}\lim_{\delta_2\downarrow 0}\lim_{\delta\downarrow 0}\sup_{\delta',\delta''\in[-\delta,\delta]}\frac{|C(\delta_1,\delta')|+|C(\delta'',-\delta_2)|+|C(\delta_1,-\delta_2)|}{R(\delta_1,0)-f'(m)}=0.
$$

In particular, defining $\bar\delta_2(\cdot)$ as in Lemma 5.9, there exist $\delta_1\equiv\delta_1^\gamma\in(0,\delta_0)$, $\delta_2\equiv\delta_2^\gamma\in(0,\bar\delta_2(\delta_1))$ and $\delta\equiv\delta^\gamma\in(0,\delta_1\wedge\delta_2)$ such that

$$
\left[\max_{v\in[m-\delta_0,m+\delta_0]}\frac{D(v)}{\sigma(v)}\right]\sup_{\delta'\in[-\delta,\delta]}\frac{C(\delta_1,0)+|C(\delta',-\delta_2)|+C(\delta_1,-\delta_2)}{R(\delta_1,0)-f'(m)}\leqslant\frac{\gamma}{8}, \tag{5.9}
$$

$$
\begin{aligned}
\frac{R(\delta_1,0)-f'(m)}{4} &\leqslant \frac{R(\delta_1,0)-R(0,-\delta_2)}{2} \\
&\leqslant \inf_{\delta',\delta''\in[-\delta,\delta]} R(\delta_1,\delta')-R(\delta'',-\delta_2),
\end{aligned}\tag{5.10}
$$

$$
\inf_{\delta',\delta''\in[-\delta,\delta]}\left|R(\delta',-\delta_2)-R(\delta',\delta'')\right|>0, \tag{5.11}
$$

$$
\inf_{\delta',\delta''\in[-\delta,\delta]}\left|R(\delta_1,\delta'')-R(\delta',\delta'')\right|>0, \tag{5.12}
$$

$$
\rho(v,m-\delta,m+\delta_1)>0\quad\text{for } v\in(m-\delta,m+\delta_1), \tag{5.13}
$$

$$
\left|\rho(v,m+\delta,m-\delta_2)\right|>0\quad\text{for } v\in(m-\delta_2,m+\delta). \tag{5.14}
$$

Let now $u_i\in U$ be an arbitrary piecewise constant profile such that $\|u_i-m\|_{L_\infty(\mathbb{T})}\leqslant\delta$. Fix

$$
\bar T := \frac{4}{R(\delta_1,0)-f'(m)}.
$$

By Lemma 5.8 there exists a piecewise constant map $w \equiv w^{\overline{T},\gamma/4} : [0,\overline{T}] \times \mathbb{T} \to [m-\delta, m+\delta]$ such that $w(0) = u_i$ and $H_{\overline{T}}(w) \leqslant \gamma/4$.

Let the Lipschitz map $s_1, s_2 : [0,+\infty) \to \mathbb{T}$ be defined as the solutions to the Cauchy problems

$$\begin{cases} \dot{s}_1(t) = \dfrac{f(m+\delta_1) - f(w(t,s_1(t)))}{m+\delta_1 - w(t,s_1(t))} \equiv R\big(\delta_1, w\big(t,s_1(t)\big) - m\big), \\ s_1(0) = 0, \end{cases}$$

$$\begin{cases} \dot{s}_2(t) = \dfrac{f(m-\delta_2) - f(w(t,s_2(t)))}{m-\delta_2 - w(t,s_2(t))} \equiv R\big(w\big(t,s_2(t)\big) - m, -\delta_2\big), \\ s_2(0) = 0. \end{cases}$$

Despite the right-hand sides are discontinuous, these equations are well posed since w is piecewise constant and conditions (5.11)–(5.12) hold, so that Remark 5.10 applies.

With a little abuse of notation, we also denote by s_1 and s_2 the lift of s_1 and s_2 on $\mathbb{R}$. Note that, by (5.10), $s_1(t) - s_2(t)$ is increasing in t and letting $T > 0$ be the first time t at which $s_1(t) - s_2(t) = 1$, we have still by (5.10)

$$T \leqslant \overline{T}. \tag{5.15}$$

We also set $x_0 := s_1(T) \equiv s_2(T) \in \mathbb{T}$, and let $u_d \equiv u_d^{\delta_1,\delta_2}$, $\tau \equiv \tau^{\delta_1,\delta_2}$ be defined as in Lemma 5.9 (with δ_1, δ_2 and x_0 defined as above in this proof), and let $v \in \mathcal{X}_\tau$ be the solution to (1.1) whose existence is proved in Lemma 5.9. We finally let (see Figs. 3 and 4)

$$u^\gamma(t,x) := \begin{cases} m+\delta_1 & \text{if } t \in [0,T] \text{ and } x \in A_1(t), \\ m-\delta_2 & \text{if } t \in [0,T] \text{ and } x \in A_2(t), \\ w(t,x) & \text{if } t \in [0,T] \text{ and } x \notin A_1(t) \cup A_2(t), \\ v(t-T,x) & \text{if } t \in [T, T+\tau] \end{cases}$$

where for $t \geqslant 0$

$$\begin{aligned} A_1(t) &:= \Big\{x \in \mathbb{T}\colon \Big|x - \frac{1}{2}\big[s_1(t) + R(\delta_1,-\delta_2)t\big]\Big| \\ &\qquad \leqslant \frac{1}{2}\big[s_1(t) - R(\delta_1,-\delta_2)t\big]\Big\}, \\ A_2(t) &:= \Big\{x \in \mathbb{T}\colon \Big|x - \frac{1}{2}\big[R(\delta_1,-\delta_2)t + s_2(t)\big]\Big| \\ &\qquad \leqslant \frac{1}{2}\big[R(\delta_1,-\delta_2)t - s_2(t)\big]\Big\}. \end{aligned}$$

Note that $u^\gamma_{|[0,T]} \in \mathcal{X}_T$ is piecewise constant, and it is the gluing of solutions to (1.1) satisfying the Rankine–Hugoniot condition at the borders of $\{(t,x) \in [0,T] \times \mathbb{T}\colon x \in A_i(t)\}$ (for $i = 1,2$). We thus have $u^\gamma_{|[0,T]} \in \mathcal{E}_T$ and $u^\gamma \in \mathcal{E}_{T+\tau}$.

In order to calculate $H_T(u^\gamma_{|[0,T]})$, we will use Remark 5.7. Note that for each $t \in [0,T]$ the set of discontinuity points of $u^\gamma(t)$ consists of the discontinuity points of $w(t)$, and the discontinuities at $s_1(t)$, at $s_2(t)$ and at $R(\delta_1,-\delta_2)t$. Because of assumptions (5.11)–(5.12), there is at most

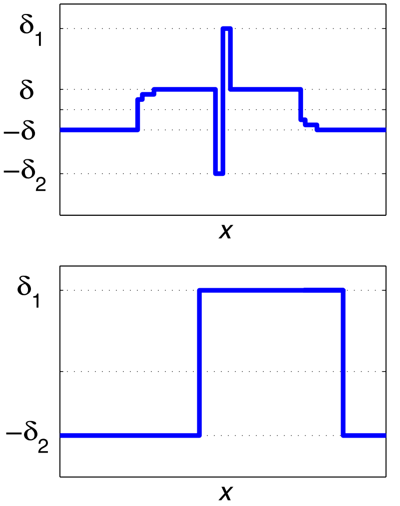

Fig. 3. In the figure, we have $f(u) = u^3 - u$, $m = 0$, and the initial datum u_i having two jumps between the values $-\delta$ and δ, so that w is the same as in Fig. 2. Here the figure shows u^γ at a small time $0 < t < T$ and at time T.

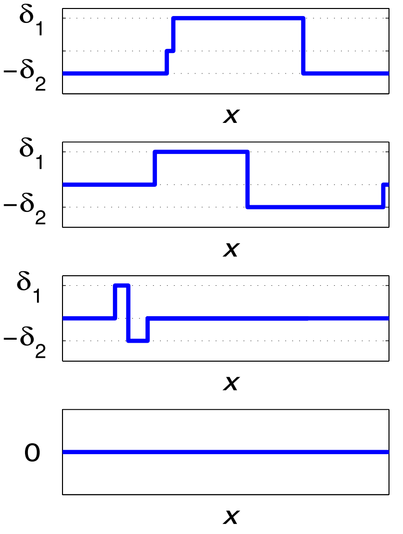

Fig. 4. In the figure, we have $f(u) = u^3 - u$, $m = 0$, and the initial datum u_i having two jumps between the values $-\delta$ and δ. The figure shows u^γ at different times $t \in (T, T + \tau]$.

a finite number of times $t \in [0, T]$ at which $s_1(t)$ and $s_2(t)$ may overlap with a discontinuity of $w(t, \cdot)$. Note that assumption (5.13) implies $\rho(v, w, m + \delta_1) \leqslant 0$, for each $v \in [-1, 1]$ and $w \in [m - \delta, m + \delta]$, so that the shock of u^γ at s_1 is entropic and it does not appear in the sum (5.4). Conversely $\rho(v, m - \delta_2, m + \delta_1) \geqslant 0$, so that the shock along the curve $t \mapsto R(\delta_1, -\delta_2)t$ appears in the sum (5.4). Finally, by (5.14), $\rho(v, w(t, s_2(t)), m - \delta_2)$ is either negative or positive for each $t \in [0, T]$ and $v \in [-1, 1]$, depending on whether case (A) or (B) of Remark 5.6 holds for f. By Remark 3.2 and recalling that v satisfies (5.6)

$$
\begin{aligned}
H_{T+\tau}\big(u^\gamma\big) &= H_\tau(v) + H_T\big(u^\gamma_{|[0,T]}\big) \leqslant H_\tau(v) + H_{\overline{T}}(w) \\
&\quad + \int_0^T dt \Bigg[\int dv\, \frac{D(v)}{\sigma(v)} \frac{\rho^+(v, m-\delta_2, w(t, s_2(t)))}{\delta_2} \\
&\quad + \int dv\, \frac{D(v)}{\sigma(v)} \frac{\rho^+(v, m+\delta_1, m-\delta_2)}{\delta_1+\delta_2}\Bigg] \\
&\leqslant \frac{\gamma}{4} + \Bigg[\max_{v\in[m-\delta_0, m+\delta_0]} \frac{D(v)}{\sigma(v)}\Bigg] \frac{C(\delta_1,0)^+ + C(0,-\delta_2)^+}{|R(\delta_1,0) - R(0,-\delta_2)|} \\
&\quad + \int_0^T dt \Bigg[\int dv\, \frac{D(v)}{\sigma(v)} \frac{\rho(v, m-\delta_2, w(t, s_2(t)))}{\delta_2}\Bigg]^+ \\
&\quad + \int dv\, \frac{D(v)}{\sigma(v)} \frac{\rho(v, m+\delta_1, m-\delta_2)}{\delta_1+\delta_2} \\
&\leqslant \frac{\gamma}{4} + \Bigg[\max_{v\in[m-\delta_0, m+\delta_0]} \frac{D(v)}{\sigma(v)}\Bigg] \\
&\quad \times \Bigg[\frac{C(\delta_1,0)^+ + C(0,-\delta_2)^+}{|R(\delta_1,0) - R(0,-\delta_2)|} + T\, C(\delta_1,-\delta_2) + T \sup_{\delta'\in[-\delta,\delta]} C(\delta',-\delta_2)^+\Bigg].
\end{aligned}
$$

By (5.15) and (5.10) we thus obtain

$$
\begin{aligned}
H_{T+\tau}\big(u^\gamma\big) &\leqslant \frac{\gamma}{4} + \Bigg[\max_{v\in[m-\delta_0, m+\delta_0]} \frac{D(v)}{\sigma(v)}\Bigg] \\
&\quad \times \frac{2C(\delta_1,0)^+ + 2C(0,-\delta_2)^+ + 4C(\delta_1,-\delta_2) + 4\sup_{\delta'\in[-\delta,\delta]} C(\delta',-\delta_2)^+}{R(\delta_1,0) - f'(m)} \\
&\leqslant \frac{\gamma}{4} + 6\Bigg[\max_{v\in[m-\delta_0, m+\delta_0]} \frac{D(v)}{\sigma(v)}\Bigg] \\
&\quad \times \frac{C(\delta_1,0) + C(\delta_1,-\delta_2) + \sup_{\delta'\in[-\delta,\delta]} |C(\delta',-\delta_2)|}{R(\delta_1,0) - f'(m)}.
\end{aligned}
$$

Therefore $H_{T+\tau}(u^\gamma) \leqslant \gamma$ by (5.9). □

Proof of Theorem 3.1(iii). We assume $\int_{\mathbb{T}} dx\, u_f(x) = m$, the proof being trivial otherwise. Since $H_T^{JV} \leqslant H_T$ we have $V^{JV} \leqslant W_m(u_f)$ by Theorem 3.1(ii). The converse inequality is obtained by taking $\varphi \equiv 1$ in the very definition of H^{JV}. □

Acknowledgments

We are indebted to Lorenzo Bertini and Matteo Novaga for enlighting discussions.

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