Abstract

In this paper, the problem of reconstructing a surface, given a set of scattered data points is addressed. First, a precise formulation of the reconstruction problem is proposed. The solution is mathematically defined as a particular mesh of the surface called the normalized mesh. This solution has the property to be included inside the Delaunay graph. A criterion to detect faces of the normalized mesh inside the Delaunay graph is proposed. This criterion is proved to provide the exact solution in 2D for points sampling a \( r \)-regular shapes with a sampling path \( \varepsilon < \sin(\pi/8)r \). In 3D, this result cannot be extended and the criterion cannot retrieve every face. A heuristic is proposed in order to complete the surface. © 1998 Published by Elsevier Science B.V.

Keywords: Shape reconstruction; Interpolation; Delaunay graph; Voronoi graph; Normalized mesh; \( r \)-regular shape; Sample points

1. Introduction

This paper addresses the problem of meshing a surface only known by an unorganized set of points. Such a problem may occur in many domains including pattern recognition, computer vision, and graphics. The initial data is a set of points with no structure, which samples a surface. These points can be measured directly on the boundary of an object using 3D sensors or laser range scanning systems. They can also result from the segmentation of a 3D volumetric image. Starting from this mere set of points only known by their coordinates, we are interested in building a mesh induced by the proximity relationships of the points on the surface.

To solve this problem, it is common to compute a neighbourhood graph of the data points and to restrict the search for a mesh inside this neighbourhood graph. Many neighbourhood graphs for which efficient algorithms from computational geometry exist have been tried [2]. In [10], the \( k \)-nearest neighbour graph is used to estimate a signed distance to the surface. After this step, a set of cubes
crossing the surface is computed and a marching-cube algorithm [11] provides the searched triangulation. Another appealing idea starts from the convex hull of the points and sculptures this convex hull until it passes through all the points [4,12]. The sculpture removes tetrahedra located on the current surface in such a way that at each step, the current surface remains a polyhedron. A drawback of such an approach is that no change of the topology is allowed and, consequently, it is impossible to get a surface formed of several connected components or having holes. The Voronoi graph is sometimes used to get additional information on the skeleton of the object [6,9,13].

More complex graphs have also been introduced like $\alpha$-hulls and $\alpha$-shapes [7,8]. $\alpha$-shapes are a generalization of the convex hull of a point set. An $\alpha$-shape is a polytope surrounding the set of points. The parameter $\alpha$ controls the maximum “curvature” of any cavity of the polytope. The choice of the parameter $\alpha$ might be tricky.

Although a large amount of work has been done on reconstruction, few theoretical results [14] exist. This is mainly due to the difficulty in formalizing the problem. In most papers, no mathematical definition of the searched mesh is given. In Section 2, we try to fill this deficiency by formulating the problem more precisely. In particular, the searched solution is defined as a particular mesh of the surface called the normalized mesh. In Section 3, a method is proposed for building this solution. The convergence of our approach is proved in the two-dimensional case for $r$-regular shapes. Unfortunately, the demonstration cannot be extended to the three-dimensional case. In practice, our algorithm is enable to retrieve every face of the surface. In Section 4, an heuristic is proposed to complete the surface.

## 2. Problem statement

In this section, definitions are given in three-dimensional space but they can easily be extended to a space of any dimension. $d$ represents the Euclidean distance and $\bar{X}$ designates the closure of the set $X$. Let $S$ be the unknown surface to mesh and $E$ a finite set of points located on $S$. To measure the quality of the sample, we introduce the notion of a sampling path.

**Definition 1** (Sampling path). $E \subset S$ is said to sample $S$ with the sampling path $\varepsilon$ if any sphere with radius $\varepsilon$ and center on $S$ contains at least one sample point in $E$ (Fig. 1).

![Fig. 1](image_url)

(a) (b) (c) (d) (e)

Fig. 1. (a) Curve. (b) Sample of the curve with sampling path $\varepsilon$. (c) Disks containing the curve. (d) Voronoi graph of $E$. (e) Normalized mesh.
The rest of this section is concerned with building up a *normalized mesh* from certain elementary spaces called *cells*. A cell is just a generalization of a simplex to an arbitrary number of vertices. The normalized mesh is a particular mesh of the surface $S$ defined as a union of cells with vertices at the data points $E$.

**Definition 2 (Cell).** A $k$-cell $[a_1, \ldots, a_k]$ is the set of points $\sum_{i=1}^{k} \lambda_i a_i$, where $a_1, \ldots, a_k$ are points, and the $\lambda_i$ are real numbers such that $\lambda_i \geq 0$ for all $i$ and $\sum_{i=1}^{k} \lambda_i = 1$. The points $a_1, \ldots, a_n$ are called the vertices of the cell.

**Definition 3 (Normalized mesh).** The normalized mesh of $S$ associated with $E$ is the set of cells $[p_1, \ldots, p_k]$ with $p_i \in E$ for which there exists a point $m \in S$ such that $d(m, p_1) = \cdots = d(m, p_k) = d(m, E)$.

By definition, the normalized mesh is included in the Delaunay diagram of $E$. It consists of the Delaunay elements (edges, faces and simplices) whose dual intersects the surface (Fig. 1). Generally, the normalized mesh consists of edges in 2D and triangular facets in 3D but it may also happen that the normalized mesh has a non-zero thickness if it includes a cell with no linearly independent vertices. One can remark that if $E$ samples $S$ with the sampling path $\epsilon$, then each face of the normalized mesh has diameter lower than $2\epsilon$. Normalized meshes are particularly attractive as they provide a piecewise linear interpolant of the surface that converges to the surface when the sampling path tends to 0.

**Proposition 1 (Convergence).** Let $S$ be a surface and $E_n$ be a sample of $S$ with sampling path $1/n$. Let $S_n$ be the normalized interpolant of $S$ associated with $E_n$. We have \( \lim_{n \to \infty} S_n = S \).

Using the previously defined notions, it is now possible to give a precise formulation of the initial reconstruction problem. Given a surface $S$ and a set of points $E$ located on $S$, the problem can be expressed as finding the normalized mesh of $S$ associated with $E$. Such a mesh provides a natural solution. It is a piecewise linear interpolant of the surface which converges to $S$ when the sampling path tends to 0.

In order to simplify the search for the normalized mesh, we will assume in the next section that $S$ is the boundary of an $r$-regular shape. $B(x, r)$ designates the ball with radius $r$ and center $x$. $B_0$ is the unit ball.

![Fig. 2. (a) r-regular shape. (b) In 2D, any circle passing through 3 distinct points has radius greater than $r$. (c) In 3D, this property does not hold anymore.](image-url)
Definition 4 \((r\text{-regular shape})\). A shape \(X\) is said to be \(r\)\text{-regular if it is morphologically open and closed with respect to a disk of radius } r > 0. \text{ In other words,}\]

\[
X = (X \ominus rB_0) \oplus rB_0 = (X \oplus rB_0) \ominus rB_0.
\]

The concept of \(r\)-regular shapes was first introduced in the field of mathematical morphology [15]. Since the parameter \(r\) can be arbitrarily small, \(r\)-regular shapes offer a relevant model to represent objects arising in image analysis [5]. \(r\)-regular shapes have many elegant geometric properties (Fig. 2).

1. The boundary of an \(r\)-regular shape has at each point a tangent and a radius of curvature greater or equal to \(r\).
2. The boundary of an \(r\)-regular shape divides any ball with radius \(2r\) and center on the boundary in exactly two connected components.
3. From the previous property, it follows that if \(\varepsilon < 2r\), the normalized mesh provides a tiling of the surface. In other words, the normalized mesh retains all the topological properties of the surface.
4. In the two-dimensional space, any circle passing through three distinct boundary points has radius greater than \(r\) (Fig. 2(b)). This property will be crucial in the next section to validate our algorithm. Unfortunately, this property has no 3D equivalent (Fig. 2(c)).

3. Characterizing boundary faces

In this section, our goal is to characterize faces of the normalized mesh. From the previous section, we know that this search can be restricted to the Delaunay diagram of the sample points. Therefore, the addressed problem is to determine which faces of the Delaunay diagram belong to the normalized mesh. Hereafter, a Delaunay sphere refers to a sphere circumscribed to a Delaunay simplex. The surface is assumed to be the boundary of an \(r\)-regular shape.

To detect boundary faces, one can take advantage of the following remark. In 2D, Delaunay discs tend to maximal discs of the object and of the complement of the object when the sampling density tends to 0 [5]. Consequently, Delaunay discs become tangent to the boundary (Fig. 3). In 3D, to evaluate the belonging of a Delaunay triangle to the surface, we are going to measure the angle formed by the two Delaunay spheres passing on both sides of this triangle. More precisely, if \(T\) is a Delaunay triangle, \(c_1\) and \(c_2\) the centers of the two Delaunay spheres passing through \(T\) and \(p\) a vertex of \(T\), we define (Fig. 3):

\[
\delta(T) = \pi - c_1p c_2.
\]

Fig. 3. Angle formed by Delaunay disks.
Broadly speaking, when the angle $\delta(T)$ is near 0, there is every chance that the triangle $T$ belongs to the boundary. On the contrary, if this angle is near $\pi$, the triangle has every chance to be inside or outside the object. Let us remark that an analogous criterion was proposed in [4]. The distance from a face $T$ to the Delaunay sphere passing through $T$ was used in order to measure the belonging of this face to the surface. One advantage of our criterion is that it is symmetric with respect to the boundary.

To reconstruct the surface, we are going to consider the set of triangles $T$ for which $\delta(T) \leq \delta_0$. Hereafter, this set is called $S_{\delta_0}$. In 2D, $S_{\delta_0}$ is a set of edges selected from the Delaunay diagram of $E$. The main result of this paper is that if $\varepsilon < \sin(\pi/8)r$, then the set of edges $S_{\pi/2}$ is the normalized mesh of $S$ associated with the points $E$. To demonstrate this result, one has to establish the following proposition.

**Proposition 2.** In 2D, let $X$ be an $r$-regular shape and $E$ be a sample of $\partial X$ with sampling path $\varepsilon$. For each edge $(pq)$ of the Delaunay diagram of $E$:

1. If $(pq)$ belongs to the normalized mesh and $\varepsilon < r/2$, then $\delta(pq) < \pi/2$.
2. If $(pq)$ does not belong to the normalized mesh and $\varepsilon < \sin(\pi/8)r$, then $\delta(pq) > \pi/2$.

**Proof.** Let $(pq)$ be an edge of the normalized mesh (Fig. 4(a)). Let $B(x_1, r_1)$ and $B(x_2, r_2)$ be the two Delaunay disks intersecting at points $p$ and $q$. If $m$ designates the middle point of $[pq]$ and $\delta$ the angle formed by $B(x_1, r_1)$ and $B(x_2, r_2)$, we have $\delta = \pi - x_1pm - mpx_2$. A disk intersecting the boundary of an $r$-regular shape in 3 distinct points has radius greater than $r$. Consequently, $r_1 \geq r$ and $r_2 \geq r$. Furthermore, edges of the normalized mesh have length smaller than $2\varepsilon$. It follows that $d(p, m) < \varepsilon$. Thus, for $i \in \{1, 2\}$,

$$\cos(x_1pm) = \frac{d(p, m)}{r_i} < \frac{\varepsilon}{r} < \frac{1}{2}.$$  

Consequently, $x_1pm > \pi/3$, $mpx_2 > \pi/3$ and $\delta < \pi/3 < \pi/2$.

The rest of the proof is devoted to establish the second point of the proposition. Hereafter, a Delaunay disk with center in $X_w$ is said to be inside. Let $p$ be a point of $E$. Let $[pp_1]$ and $[pp_2]$ be the two cells of the normalized mesh passing through $p$. Let $B_1$ and $B_2$ denote the two inside Delaunay disks passing through $[pp_1]$ and $[pp_2]$. For every inside Delaunay disk $B$ and $B'$ through $p$, the angle formed by $B$ and $B'$ is greater or equal to the angle $\delta$ formed by $B_1$ and $B_2$ (Fig. 4(b)). Let $m_i$ denote the

![Fig. 4. Illustration of the proof.](image-url)
middle point of \( \{pp_i\} \) and \( x_i \) the center of \( B_i \) (Fig. 4(c)). We have \( \delta = p_1pp_2 - m_1x_1p - px_2m_2 \). As above, \( d(p, x_1) \geq r \), \( d(p, x_2) \geq r \), \( d(p, m_1) < \varepsilon \) and \( d(p, m_2) < \varepsilon \). Thus

\[
\sin(m_1x_1p) = \frac{d(p, m_1)}{d(p, x_1)} < \frac{w^{-1}}{r}.
\]

In order to minorate the angle \( p_1pp_2 \), let us consider the two disks \( D^+ \) and \( D^- \) respectively included in \( X \) and its complement which are tangent to the boundary at \( p \) and having radius \( r \) (Fig. 4(d)). \( p_1 \) and \( p_2 \) are located outside disks \( D^+ \) and \( D^- \). Let \( \tau \) be the tangent at \( p \). Let \( p'_1 \) (respectively \( p'_2 \)) be the intersection of \( D^+ \) with the disk with center \( p \) and radius \( d(p, p_1) \) (respectively \( p_2' \)). We have

\[
\sin(\tau, pp'_1) = \frac{d(p, p'_1)}{2r} < \frac{w^{-1}}{r}.
\]

It follows that

\[
p_1pp_2 > p'_1pp'_2 = \pi - (\tau, pp'_1) - (\tau, pp'_2) > \pi - 2 \arcsin(\frac{w^{-1}}{r}).
\]

Combining previous equations, we have \( \delta > \pi - 4 \arcsin(\frac{w^{-1}}{r}) \). Finally, the assumption \( w^{-1}/r < \sin(\pi/8) \approx 0.38 \) makes it possible to conclude \( \delta > \pi/2 \).

**Proposition 3.** In 2D, let \( X \) be an \( r \)-regular shape and \( E \) be a sample of \( \partial X \) with sampling path \( \varepsilon \). If \( \varepsilon < \sin(\pi/8)r \), then \( S_{\pi/2} \) is the normalized mesh.

The beauty of this result lies in the fact that there is no need to know the two parameters \( r \) and \( \varepsilon \) in order to compute the normalized mesh. Moreover, the sampling path need not to be close to zero in
order to find the correct result. It is enough that the sampling path $\varepsilon$ and the constant $r$ characterizing the searched shape have approximately the same order. The computational time is also very attractive. The detection of boundary faces requires only the computation of a Voronoi graph. Given $n$ sample points in an $N$-dimensional space, the computational time is $O(n \log n + n^{\lceil N/2 \rceil})$. Fig. 5(a) illustrates the robustness of the proposed method on several examples. The method is able to detect curves even when the curves are not closed and even when the points are not regularly spaced on the curves. A few questions stay open. Are the assumptions on $r$ and $\varepsilon$ optimal? It would also be interesting to know which values of $\delta$ (but $\pi/2$) lead to the normalized mesh.

In 3D, contrary to the 2D case, it may happen that some Delaunay spheres intersect the boundary without being tangent to it as illustrated in Fig. 2(c). This new phenomenon makes it impossible to miniorate the radius of the Delaunay spheres. Consequently, Proposition 2 cannot be generalized. Experimentally, one can remark that the set $S_{\pi/2}$ does not capture all faces of the normalized mesh. Some triangles are missing and the surface might contain holes, depending on the distribution of sample points (Fig. 5). Nevertheless, as the proposed criterion seems not to produce erroneous triangles, one can start from this first approximation of the surface in order to find the solution.

4. Closing the surface

In this section, a heuristic is proposed in order to complete surfaces from the previous section. Hereafter, the term “surface” refers to the set $S_{\pi/2}$. The surface computed in the previous section is not closed. It has a border which consists of polygons. A first idea would be to triangulate each polygon of the border [3]. Nevertheless, as illustrated in Fig. 5, the holes on the surface are generally too complicated to be closed independently from the others. For this reason, a volume-based approach has been preferred. Instead of computing the surface directly, we first try to find a collection of objects whose boundary provides the searched surface. This approach assumes surfaces to be closed, and consequently, is less general than the previous one. On the other hand, it will always produce consistent surfaces.

The main idea is to merge Delaunay tetrahedra until space is partitioned into a satisfactory number of objects (Fig. 6(b)). Our algorithm aggregates Delaunay tetrahedra as follows.

1. **Initialization.** Delaunay tetrahedra and the complement of the convex hull of the sample points form the initial set of objects. Delaunay triangles are put in a list $L$ that is sorted according to the diameter of triangles.

2. **Merge.** While $L$ is not empty, the triangle $T$ having largest diameter is taken out from $L$. If $T$ separates two different objects $O_1$ and $O_2$, those two objects are merged providing that:
   - no triangle from $S_{\pi/2}$ disappears,
   - the merge does not isolate sample points inside $O_1 \cup O_2$. In other words, after the merge, every sample point must still belong to the boundary of at least one object (Fig. 6(a)).

   Rules chosen for merging tetrahedra differ from the rules used in [4] to sculpture the convex hull. In particular, more operations are permitted in order to allow topological change of the surface. The result is a set of closed surfaces of arbitrary topology. If $n$ designates the number of sample points and $k$ the number of Delaunay faces in the list at the beginning of the algorithm, the complexity is $O(k \log k + kn^2)$. Indeed, sort is done in $O(k \log k)$. To detect a possible merge between two objects,
it is necessary to consider the neighbours of each sample point, which can be \( n \) in the worst case. Consequently, the complexity of the algorithm is \( O(n^3) \) in 2D and \( O(n^4) \) in 3D. Fig. 7 presents some results, with a range of complexity from 90 points to 1100 points. The running time on an SGI Indy R4000 varies from 20 s to 240 s. When the sampling path is not sufficient, the algorithm is not able to find the expected surface as illustrated in Fig. 7 for the smallest set of points. Other heuristics based on curvature will be studied in future works.

One advantage of this approach is that the skeleton of objects can easily be deduced (Fig. 7). The skeleton is a very famous representation of objects from image analysis and computer vision. It is a thin figure which retains all the topological properties of the object. It is useful to describe and analyze objects. In 3D, it consists of surfaces and curves passing through the middle of the object. The skeleton can be computed by considering the dual of the Delaunay tetrahedra included in the object. A simplification method is presented in [1].

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**Fig. 6.** (a) Rules to merge objects. (b) Illustration of the merge process.

**Fig. 7.** Sample points, uncomplete surfaces \( S_{\pi/2} \). Surfaces resulting from the merge of Delaunay tetrahedra. Skeletons.
5. Conclusion

In this paper, a precise formulation of the reconstruction problem has been given. We have defined the solution as a particular subset of the Delaunay diagram. A first method has been proposed to construct this solution. The convergence of this method has been established in 2D. For 3D objects, the method does not detect every boundary face. A method has been proposed for completing surfaces and detecting missing faces.

A weak point of our study is that points must be located on or near the boundary of the objects. In the case of real data, this assumption cannot always be fulfilled. Sample points may form a dense cluster around the objects. A new formulation of the reconstruction problem must then be proposed. Current research is directed towards this direction. An idea would be to combine $\alpha$-shapes and the set of boundary faces $S_{\pi/2}$. Indeed, the two approaches seem to complement one another. While $\alpha$-shapes can capture noisy set of points, our approach enables to connect points located on curves and surfaces.

References


