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Stability properties for nonlinear diffusion in porous and other media

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1. Introduction

The paper is concerned with nonlinear diffusion in a spatial region where the diffusivity is a given function of the dependent variable, and where the dependent variable is specified on the boundary as a time-independent function of position. The steady (or equilibrium) state is specified by a boundary value problem, while the unsteady state is determined by an initial boundary value problem in which the dependent variable is specified at a given (initial) time. (The initial value corresponds to an initial perturbation from the steady state.)

The paper commences (Section 2) with the specification of the steady and unsteady problem as aforesaid, and with the initial boundary value problem for the perturbation (the difference between the dependent variables in the unsteady and steady states). This is followed (Section 3) by an analysis of the perturbation problem in the particular case of the *porous medium* model, which is characterized by a diffusivity which vanishes when the dependent variable vanishes. *The dependent variable in this context is usually interpreted as a mass concentration*. An integral, positive definite measure of the perturbation (Liapunov functional) at any time is introduced (cf. [1]) and we derive a differential inequality for this. We deduce an upper inequality estimate for the Liapunov functional descriptive of the global decay to zero, with time, of the perturbation. Another upper estimate for the decay of the aforesaid functional is obtained in the case where the boundary value and the initial perturbation are both bounded below by a positive constant; this is done by combining a previous result (obtained in [1]) together with the maximum principle. The first decay result is shown to be better than the second for sufficiently small times provided that the

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initial value of the unsteady (perturbed) state is sufficiently far away from equilibrium. The section concludes with an analysis of the corresponding 'backwards in time' initial boundary value problem for the perturbation: an inequality estimate for the Liapunov functional is obtained which shows that it grows with time, and one deduces that the solution can only exist for finite time.

The estimates referred to in the last paragraph imply properties that can be expressed in 'stability' terms. The first two estimates obtained imply global/unconditional stability, in a particular (global) positive definite measure, of the steady state, and these are valid however far the initial state is removed from equilibrium. Moreover, the phenomenon described by the 'backwards in time estimate' is an example of an 'explosive instability.'

Section 4 concerns diffusion in a medium whose (nonlinear) diffusivity in the steady state is bounded below by a positive constant, and, in some circumstances, also bounded above by a positive constant; this contrasts with the Section 3 where a particular diffusivity appropriate to a porous medium was assumed. Theorems are proved concerning the pointwise asymptotic exponential stability of the steady state. The paper concludes (Section 5) with some remarks concerning the porous medium in the one-dimensional case. On the one hand, it is shown explicitly that the assumptions underlying Section 4 are satisfied, while, on the other, an intuitive picture is given of the evolution of the unsteady to the steady state in circumstances where the initial state is far from equilibrium. In the latter case, we point out the relevance of Theorems 1 and 2 to this picture.

The results obtained in this paper represent a development of results obtained for nonlinear diffusion in a previous paper [1]. In the latter paper similar issues were addressed but the basic assumption therein was that the diffusivity was bounded below by a given positive constant. One of the (conceptually simple) Liapunov functionals introduced therein is also used here. The results obtained in this paper provide further evidence of its versatility—already in evidence in other contexts, both thermal and thermo-mechanical [2–4,6].

2. Steady, unsteady, and perturbation problems

Consider a spatial region Ω with smooth boundary $\partial\Omega$. Consider $T(\mathbf{x},t)$ satisfying (with $k(T)$ denoting the diffusivity at T)

$$\frac{\partial T}{\partial t} = \nabla^2\left[\int_0^T k(\tau)\,d\tau\right] \quad \text{in } \Omega \tag{1}$$

subject to

$$T(\mathbf{x},t) = \overline{T}(\mathbf{x}) \quad \text{on } \partial\Omega \tag{2}$$

and subject to

$$T(\mathbf{x},0) = f(\mathbf{x}) \quad \text{in } \Omega. \tag{3}$$

This is referred to as the unsteady state problem.

The corresponding steady state solution $U(\mathbf{x})$ satisfies

$$\nabla^2\left[\int\limits_0^{U(\mathbf{x})} k(\tau)\,d\tau\right] = 0 \quad \text{in } \Omega \tag{4}$$

subject to

$$U(\mathbf{x}) = \overline{T}(\mathbf{x}) \quad \text{on } \partial\Omega. \tag{5}$$

The perturbation defined by

$$u = T - U \tag{6}$$

satisfies, with

$$\Phi(u; U) = \int\limits_0^u d\bar{u} \int\limits_0^{\bar{u}} k(\tau + U)\,d\tau, \tag{7}$$

the initial boundary value problem

$$\frac{\partial u}{\partial t} = \nabla^2 \Phi_u \quad \text{in } \Omega \tag{8}$$

subject to

$$u(\mathbf{x}, t) = 0 \quad \text{on } \partial\Omega \tag{9}$$

and

$$u(\mathbf{x}, 0) = f(\mathbf{x}) - U(\mathbf{x}). \tag{10}$$

In (8) and subsequently the subscript u means partial differentiation with respect to u.

3. Results for the porous medium

The diffusivity $k(\tau)$—appropriate to the porous medium—is taken to be

$$k(\tau) = \tau^{n-1}, \tag{11}$$

n being a constant such that $n > 1$; τ is always positive, *supposing that the dependent variables T, U are positive throughout*, including, in particular,

$$\overline{T} > 0, \qquad f > 0. \tag{12}$$

In addition, we shall suppose throughout that *the solutions considered are classical*. However, the results of Theorem 1 continue to hold under less stringent assumptions, which allow for the possibility that T may be zero and that it may have discontinuous derivatives (see Remark 1).

The function Φ defined by (7) plays a central role in the ensuing analysis. Explicit calculations using (7), (11) gives

$$\Phi(u;U) = \{n(n+1)\}^{-1}\big[(u+U)^{n+1} - U^{n+1} - (n+1)U^n u\big], \tag{13}$$

$$\Phi_u(u;U) = n^{-1}\big[(u+U)^n - U^n\big]. \tag{14}$$

Some properties of Φ are now noted. It is clear that

$$\Phi(0,\cdot) = \Phi_u(0,\cdot) = 0. \tag{15}$$

Using the foregoing properties of Φ, Taylor's theorem (remainder form) gives

$$\Phi(u;U) = \big[\theta(u+U) + (1-\theta)U\big]^{n-1}u^2/2, \tag{16}$$

where θ is such that $0 < \theta < 1$. It follows from this, bearing in mind the assumptions $U + u \geqslant 0$, $U \geqslant 0$, that Φ is positive definite in u. The following important property is proved in Appendix A:

$$\Phi_u^2 \geqslant K_n \Phi^{2n/(n+1)}, \tag{17}$$

where

$$K_n = (n+1)^{2n(n+1)^{-1}} n^{-2}. \tag{17$'$}$$

We now define the Liapunov functional

$$E(t) = \int_V \Phi(u;U)\, dV. \tag{18}$$

In view of the positive-definiteness property of Φ this is a satisfactory global measure of the perturbation u at any time t. We wish to obtain an inequality estimate for $E(t)$ in terms of data, and to deduce therefrom that $E(t) \to 0$ as $t \to \infty$, giving global convergence, in the measure E, of the unsteady to the steady state.

Now

$$\frac{dE}{dt} = \int_V \Phi_u u_t\, dV = \int_V \Phi_u \nabla^2 \Phi_u\, dV = -\int_V (\nabla \Phi_u)^2\, dV \tag{19}$$

using (8), (9) together with the divergence theorem. Application of the Poincaré inequality to the term on the right-hand side of (19) gives

$$\frac{dE}{dt} \leqslant -\lambda_1 \int \Phi_u^2\, dV \tag{20}$$

where λ_1 is the lowest 'fixed-membrane' eigenvalue of V. Applying (17), we obtain

$$\frac{dE}{dt} = -\lambda_1 K_n \int \Phi^{2n(n+1)^{-1}}\, dV. \tag{21}$$

Denoting the volume of the region Ω by V, application of Hölder's inequality to (21) gives

$$\frac{dE}{dt} \leqslant -\lambda_1 K_n V^{-(n-1)(n+1)^{-1}} E^{2n(n+1)^{-1}}, \tag{22}$$

recalling (18). Integration of the differential inequality (22) gives

$$E(t) \leqslant \big[E^{-\sigma}(0) + \lambda_1 K_n \sigma V^{-\sigma} t\big]^{-\sigma^{-1}}, \tag{23}$$

where we have, for convenience, put

$$\sigma = \sigma(n) = (n-1)(n+1)^{-1}.$$

One deduces from (23) that

$$E(t) \to 0, \quad t \to \infty, \tag{24}$$

giving convergence, in the measure E, of the unsteady to the steady state. We embody the foregoing results in a theorem.

Theorem 1. *The unsteady stage converges to the steady state in the case of a porous medium for which* (1)–(5), (11) *hold, in accordance with* (23), (24) *bearing in mind definitions* (6)–(8).

Remark 1. It is easily verified that Theorem 1 continues to hold when the fundamental assumptions (in italics) in the first paragraph of Section 3 are relaxed somewhat, as follows: (i) $f = 0$ within a regular closed surface contained within the region, $f > 0$ is a smooth function defined outside it, and f is continuous across it; (ii) for a continuous range of time subsequent to $t = 0$, $T = 0$ within a regular (moving) closed surface, outside the surface T (> 0) is a classical solution subject to the relevant boundary conditions, and T is continuous across the surface; across the moving surface, the partial derivatives of T may be discontinuous but are such that

$$\partial T^n / \partial \nu_+ = 0,$$

$\partial/\partial\nu_+$ denoting the normal derivative to the surface from the side in which $T > 0$. The latter condition—which may be interpreted physically as a zero mass flux condition across the surface, or as a consequence of the Rankine–Hugoniot condition (e.g., [5])—arises in the application of the divergence theorem in connection with (19). Other relaxations of the fundamental assumptions are also possible.

Remark 2. In the terminology of stability (e.g., [6]), the steady (or equilibrium) state is globally (or unconditionally) stable in the measure E.

Suppose that the boundary and initial values are strictly positive, i.e., that there exists a positive number δ such that $\overline{T}(x) > \delta$, $f(x) > \delta$. Then, by the maximum principle, one has, for $0 \leqslant \alpha < 1$,

$$U + \alpha u = (1-\alpha)U + \alpha(u+U) \geqslant \delta.$$

It follows that the diffusivity $k(\cdot)$ is such that

$$k(U + \alpha u) \geqslant \delta^{n-1}.$$

It follows from previous work [1] that, in these circumstances, we have

Theorem 2. *If*

$$\overline{T}(x) > \delta, \qquad f(x) > \delta,$$

δ being a positive constant, the convergence of the unsteady to the steady state for a porous medium, as specified, is in accordance with

$$E(t) \leqslant E(0)\exp\left(-2\delta^{n-1}\lambda_1 t\right),$$

bearing in mind definitions (18), (6), (7).

Remark 3. The question arises as to when the upper bound given by (23) is better than the exponential one given in Theorem 2. It is easily verified that this is, in fact, so for sufficiently small t, provided that

$$E^{\sigma}(0)V^{-\sigma} > 2\delta^{n-1}(K_n\sigma)^{-1},$$

i.e., if, in a sense, the initial state is sufficiently far from equilibrium. If, however, $E(0)$ satisfies an inequality complementary to the latter, then the exponential bound is always better than that given by (23), i.e., if, in a sense, the initial state is sufficiently close to equilibrium.

Stated otherwise, the decay arising in (23) is slower than that arising in Theorem 2, except, possibly, in an initial time interval.

Further, by letting $n \to 1$, one finds from (24) that

$$E(t) \leqslant E(0)\exp(-2\lambda_1 t), \tag{25}$$

in agreement with a previous result (e.g., [1]).

If we now consider the *backwards in time* version of the initial boundary value problem in u, this means formally replacing (8) by

$$-\frac{\partial u}{\partial t} = \nabla^2 \Phi_u \tag{26}$$

the remaining specifications remaining formally unchanged, a virtual repetition of the previous argument gives

$$\left[E(t)\right]^{-\sigma} \leqslant E^{-\sigma}(0) - \lambda_1 K_n \sigma V^{-\sigma} t.$$

It is plain, in view of the nonnegativity of $E(t)$, that

$$t \leqslant \left[E(0)V^{-1}\right]^{-\sigma}\left[\lambda_1 K_n \sigma\right]^{-1}, \tag{27}$$

i.e., the time interval for which a solution exists for the backwards in time i.b.v.p. for u, is bounded (in terms of data). Naturally, for times for which there is existence, one has

$$E(t) \geqslant \left[E^{-\sigma}(0) - \lambda_1 K_n \sigma V^{-\sigma} t\right]^{-\sigma^{-1}}. \tag{28}$$

These results are embodied in the following theorem.

Theorem 3. *For the backwards in time initial boundary value problem in u, specified by* (26), (9), (10), *etc., the solution fails to exist for times t in excess of right-hand side of* (27), *but for times for which it does exist the estimate* (28) *holds.*

Remark 4. The phenomenon to which Theorem 3 refers is an example of an “explosive instability” in the terminology of Straughan [7].

4. Pointwise, local stability results for porous and other media

In the remaining part of the paper, the formulation of the steady and unsteady states is as already given in (1)–(10), smooth solutions being envisaged, but restrictions on $k(\cdot)$ at the steady state are introduced in connection with theorems proved, but these do not assume a particular functional form for $k(\cdot)$. It proves convenient, henceforward, however, to use a slightly different notation: instead of Φ (defined by (7)), we write

$$F(U+u) = \int_0^{U+u} k(\tau)\,d\tau, \qquad F'(U+u) = k(U+u).$$

Theorem 4. *Let $F \in C^2(\Re)$ and*

$$\begin{cases} \nabla^2 F(U) = 0, & \mathbf{x} \in \Omega, \\ U = \overline{T}(\mathbf{x}), & \mathbf{x} \in \partial\Omega, \end{cases} \tag{29}$$

$$\begin{cases} U_1 = \inf_\Omega U(\mathbf{x}) > -\infty, \\ U_2 = \sup_\Omega U(\mathbf{x}) < \infty, \end{cases} \tag{30}$$

Ω being bounded regular domain. If there exists a positive constant m such that

$$F'\big[U(\mathbf{x})\big] \geqslant 2m, \quad \forall \mathbf{x} \in \Omega, \tag{31}$$

then U is stable in the pointwise norm.

Proof. We begin by noticing that, in view of $F' \in C(\Re)$, it follows that on any closed subset $[a,b]$ with $a = \text{const} < U_1$, $b = \text{const} > U_2$, F' is uniformly continuous and hence $\forall \eta > 0$, $\exists \varepsilon(\eta) > 0$ such that

$$|U'' - U'| < \varepsilon$$

implies

$$F'(U') - \eta \leqslant F'(U'') \leqslant F'(U') + \eta, \quad U', U'' \in [a,b]. \tag{32}$$

Therefore by choosing

$$U' = U(\mathbf{x}), \qquad \eta = m, \tag{33}$$

and setting

$$U'' = U' + u, \tag{34}$$

from (31)–(34) it turns out that there exists a positive constant ε such that

$$|u| < \varepsilon \quad \Rightarrow \quad F'\big[U(\mathbf{x}) + u\big] \geqslant m > 0, \quad \forall \mathbf{x} \in \Omega. \tag{35}$$

Assume now—by way of contradiction—that U is unstable in the pointwise norm. Then there exists $u(\mathbf{x}, t)$ and $\bar{t}$ such that

$$\begin{cases} \sup_\Omega |u(\mathbf{x}, 0)| < \varepsilon/2, \\ \sup_\Omega |u(\mathbf{x}, \bar{t})| = \varepsilon, \\ \sup_\Omega |u(\mathbf{x}, t)| < \varepsilon, \quad t \in [0, \bar{t}[. \end{cases} \tag{36}$$

Therefore—in view of (36)—(35) holds $\forall t \in [0, \bar{t}[$ and hence (39) of [1][1] holds $\forall t \in [0, \bar{t}\,]$.

Starting from (39) of [1] and continuing as in [1] it turns out that

$$\sup_{\Omega}\big|u(\mathbf{x}, t)\big| \leqslant \sup_{\Omega}\big|u(\mathbf{x}, 0)\big| \leqslant \varepsilon/2, \quad \forall t \in [0, \bar{t}[,$$

hence in contradiction with (36) it follows that

$$\sup_{\Omega}\big|u(\mathbf{x}, \bar{t}\,)\big| = \lim_{t\to\bar{t}} \sup\big|u(\mathbf{x}, t)\big| \leqslant \varepsilon/2. \qquad \square$$

Theorem 5. *Let the assumption of Theorem* 4 *hold. Then* U *is pointwise attractive.*

Proof. One has to show that

$$\lim_{t\to\infty}\big|u(\mathbf{x}, t)\big| = 0 \quad \text{a.e. in } \Omega. \tag{37}$$

Theorem 4 ensures that

$$\sup_{\Omega}\big|u(\mathbf{x}, 0)\big| < \varepsilon \quad \Rightarrow \quad \sup_{\Omega}\big|u(\mathbf{x}, t)\big| < \varepsilon, \quad \forall \Omega \times \Re^{\div}, \tag{38}$$

hence (35) implies

$$F'\big[U(x) + u(\mathbf{x}, t)\big] \geqslant m, \quad \forall(\mathbf{x}, t) \in \Omega \times \Re^{\div}.$$

By taking into account (34) of [1] it turns out that U is asymptotically exponentially stable in the $L^2(\Omega)$-norm:

$$\int_{\Omega} u^2\, d\Omega \leqslant \frac{2}{m} E(0) e^{-2m\lambda_1 t}, \tag{39}$$

where λ_1 and $E(t)$ are as previously defined.

The proof of Theorem 5 is then completed by recalling that in (iii) of Remark 4 of [8] it has been shown that the asymptotic exponential stability in the $L^p(\Omega)$-norm $(p > 0)$ implies the pointwise attractivity. In fact, in the case at hand, denoting by $\Omega^*(t)$ the largest subset of Ω such that at time t

$$u^2(\mathbf{x}, t) \geqslant e^{-m\lambda_1 t} \quad \text{a.e. in } \Omega^*, \tag{40}$$

and setting

$$\Omega_1(t) = \Omega - \Omega^*(t), \tag{41}$$

from (40)–(41), it turns out that

$$\begin{cases} |u(\mathbf{x}, t)| \leqslant e^{-(m\lambda_1 t)/2} & \text{a.e. in } \Omega_1(t), \\ \text{meas}(\Omega - \Omega_1) \leqslant (2/m) E(0) e^{-m\lambda_1 t}, \end{cases} \tag{42}$$

hence, by letting $t \to \infty$, (36) immediately follows. $\square$

Although (40)–(41) essentially already show that, under (31), the pointwise decay is of exponential type, let us show that on requiring that $F'(U + v)$ be also bounded when $|u|$ is small, the following theorem of asymptotic decay holds.

[1] It should be noted, however, that T, u of this paper appear as u, v, respectively, in [1].

Theorem 6. *Let the assumption of Theorem* 4 *hold and let*

$$F'\big[U(\mathbf{x})\big] \leqslant m_1, \quad \forall \mathbf{x} \in \Omega, \tag{43}$$

m_1 *being a* (*positive*) *constant. Then* U *is asymptotically exponentially stable in the pointwise norm.*

Proof. From (32)–(34) it turns out that there exists a positive constant ε such that

$$|u| < \varepsilon, \quad \text{e.g., } M \geqslant F'(U) \geqslant m,\ \forall \mathbf{x} \in \Omega,$$

with $M = m_1 + m$. Let

$$\sup_{\Omega} \big|u(\mathbf{x}, 0)\big| < \varepsilon.$$

Then Theorem 4 implies

$$\big|u(\mathbf{x}, t)\big| < \varepsilon, \quad \forall (\mathbf{x}, t) \in \Omega \times \Re^+,$$

and hence, in view of (31) and (43), it follows that

$$M \geqslant F'\big[U(\mathbf{x}) + u(\mathbf{x}, t)\big] \geqslant m, \quad \forall (\mathbf{x}, t) \in \Omega \times \Re^+. \tag{44}$$

Therefore the assumptions of Theorem 3b of [1] are satisfied and it turns out that

$$\big\|u(\mathbf{x}, t)\big\|_\infty \leqslant N m^{-1} \exp\big\{-\lambda_1 m(t - t_0)/2\big\}$$

with N and t_0 positive constants. □

Remark 5. The existence of three positive constants ε, M^*, m^* such that

$$|u| < \varepsilon \quad \Rightarrow \quad M^* \geqslant F'\big[U(\mathbf{x}) + u\big] \geqslant m^* > 0, \quad \forall \mathbf{x} \in \Omega, \tag{45}$$

plays a fundamental role in Theorem 6. We notice that often (45), with an explicit value for ε, can be obtained immediately from the expression of $F'(U + u)$ and

$$m_1 \geqslant F'\big[U(x)\big] \geqslant 2m, \quad \forall \mathbf{x} \in \Omega, \tag{46}$$

or by requiring appropriate behaviour of F''. For instance, if

$$\big|F''(T)\big| < \alpha = \text{const}, \quad \forall T \in \Re,$$

then from

$$F'(U + u) = F'(U) + F''(U + \theta u)u, \quad 0 < \theta < 1,$$

it immediately follows that

$$F'(U) - \alpha|u| \leqslant F'(U + u) \leqslant F'(U) + \alpha|u|, \quad \forall \mathbf{x} \in \Omega.$$

Hence—in view of (46)—it turns out that

$$|u| < m/\alpha \quad \Rightarrow \quad m \leqslant F'(U + u) \leqslant m_1 + m, \quad \forall \mathbf{x} \in \Omega.$$

5. Porous medium: one-dimensional case

We conclude the paper with a few remarks. These make a connection between the contexts of Sections 3 and 4: considering the steady state for a porous medium (as envisaged in Section 3) in the one-dimensional case, it is shown explicitly that the underlying assumptions of Theorem 4 of Section 4 are satisfied.

We consider

$$\begin{cases} T_t = [F(T)]_{xx}, & 0 \leqslant x \leqslant 1,\ t \in \Re^+, \\ T(x,0) = f(x), & x \in [0,1], \\ T(1,t) = T_2, & \forall t \in \Re^+, \\ T(0,t) = T_1, & \forall t \in \Re^+, \end{cases}$$

T_2 and T_1 $(< T_2)$ being positive constants and $F = T^n$ $(n > 1)$. The steady solution satisfies

$$\begin{cases} [F(U)]_{xx} = 0, \\ U(0) = T_1, \\ U(1) = T_2, \end{cases}$$

and thus

$$U^n = Ax + B$$

with

$$\begin{cases} B = T_1^n, \\ A = T_2^n - T_1^n, \end{cases}$$

leading to

$$U = \left[\left(T_2^n - T_1^n\right)x + T_1^n\right]^{1/n}$$

and

$$T_1 = \inf_{[0,1]} U, \qquad T_2 = \sup_{[0,1]} U$$

with $F'(T) = n(U+u)^{n-1}$. It immediately follows that

$$|u| \leqslant \frac{T_1}{2} \quad \Rightarrow \quad M = \left(T_2 + \frac{T_1}{2}\right)^{n-1} \geqslant F'(U+u) > n\left(\frac{T_1}{2}\right)^{n-1} = 2m > 0.$$

Remark 6. One may similarly show that, in the one-dimensional case, the assumptions underlying Theorem 4 are satisfied in the case $0 < n < 1$.

We conclude this section with an intuitive picture of the likely evolution, to the steady state, of an initially perturbed state with compact support, again in the one-dimensional case of a porous medium. Specifically, we suppose that (with x_1, x_2 being constants)

$$f(x) = 0 \quad (x_1 < x < x_2), \qquad f(x) > 0 \quad \text{elsewhere in } [0,1].$$

As suggested by the evolution of the point source profile (e.g., [9]), the extremities of the support are expected to move towards one another with finite speed, and eventually to meet. Referring to the time at which they meet as the ‘critical time,’ one expects the evolution of the profile prior to the critical time to be ‘slow’ while its evolution subsequent to the critical time is expected to be ‘fast.’ One expects the evolution in the latter case to be governed by Theorem 2 (assuming that the (new) time origin is taken subsequent to the critical time), while that in the former case is expected to be governed by Theorem 1. Moreover, the evolution of the profile discussed above is consistent with what is envisaged in Remark 1.

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Appendix A

We here prove the inequality

$$\Phi_u^2 \geqslant K_n \Phi^{2n(n+1)^{-1}}, \tag{A.1}$$

where

$$K_n = (n+1)^{2n(n+1)^{-1}} n^{-2}, \tag{A.2}$$

assuming that

$$U + u \geqslant 0, \quad U \geqslant 0,\ n \geqslant 1. \tag{A.3}$$

Excluding the trivial case $U = 0$ for which the inequality plainly holds, we write

$$p = u/U + 1, \tag{A.4}$$

and (A.1) is equivalent to

$$(p^n - 1)^2 \geqslant n^{-2n(n+1)^{-1}} \big[p(p^n - 1) - n(p-1)\big]^{2n(n+1)^{-1}}. \tag{A.5}$$

Excluding the case $p = 1$ for which (A.5) is trivially true, the proof of (A.5) is equivalent to proving

$$\inf_{p>0} h(p) = n^{-2n(n+1)^{-1}}, \tag{A.6}$$

where

$$h(p) = (p^n - 1)^2 \big[p(p^n - 1) - n(p-1)\big]^{-2n(n+1)^{-1}}. \tag{A.7}$$

To prove the latter, note that

$$\begin{cases} \lim_{p\to 0} h(p) = n^{-2n(n+1)^{-1}}, \\ \lim_{p\to 1} h(p) = +\infty, \\ \lim_{p\to\infty} h(p) = 1; \end{cases} \tag{A.8}$$

the proof is completed if we can prove that $h(p)$ is decreasing for $p > 1$, and increasing for $1 > p > 0$.

This latter is equivalent to proving that $(p \neq 1)$

$$g(p) = (p^n - 1)\big[p(p^n - 1) - n(p-1)\big]^{-n(n+1)^{-1}} \tag{A.9}$$

is decreasing with respect to p $(p > 1)$, increasing with respect to p $(0 < p < 1)$. Now differentiation establishes that

$$P(p)g'(p) = -(n-1)p^n + np^{n-1} - 1, \tag{A.10}$$

where $P(p)$ is a positive quantity which does not need to concern us. Now write

$$r(p) = -n(n-1)p^n + np^{n-1} - 1, \tag{A.11}$$

whence

$$r'(p) = -(n-1)p^{n-2}(p-1). \tag{A.12}$$

Now $r(1) = 0$, $r'(p) < 0$ for $p > 1$, follow from (A.11), (A.12), whence $g'(p) < 0$ $(p > 1)$. For $p < 1$, $r'(0) > 0$ whence $g'(p) > 0$ $(0 < p < 1)$. This completes the proof. □

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