Double exponential transformation in the Sinc-collocation method for a boundary value problem with fourth-order ordinary differential equation

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Received 9 March 2004

Abstract

In this paper we consider a Sinc-collocation method for the two-point boundary value problem of fourth-order ordinary differential equation incorporated with the double exponential transformation (abbreviated as the DE transformation). By this method a convergence rate $O\left(\exp\left(-cN/\log N\right)\right)$ where $N$ is a parameter representing the number of terms in the Sinc approximation is attained. We compared the result with ones based on the single exponential transformation which made us confirm the high efficiency of the present method.

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Keywords: Double exponential transformation; Sinc-collocation; Boundary value problem

1. Introduction

The double exponential transformation, abbreviated as the DE transformation, was first proposed by Takahasi and Mori \cite{10} in 1974 for one dimensional numerical integration and it has come to be widely

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used in applications in the last three decades. In 1997, Sugihara [7] established the “meta-optimality” of the DE formula in a mathematically rigorous manner, and since then it has turned out that the DE transformation is also useful for other various kinds of numerical methods. Indeed, it has been demonstrated in [2,4,8] that the use of the Sinc method incorporated with the DE transformation gives highly efficient numerical methods for approximation of functions, indefinite numerical integration and the solution of differential equations.

The Sinc method for the boundary value problem of fourth-order ordinary differential equation has been considered in [3] and [5] based on the single exponential transformation (abbreviated as the SE transformation) in the sense that after the transformation the solution decays single exponentially, and it has been shown that its convergence rate is \( O(\exp(-c' \sqrt{N})) \) with some \( c' > 0 \) under certain mild conditions where \( N \) is a parameter representing the number of terms in the Sinc approximation. On the other hand Sugihara was successful in solving the boundary value problem of second-order differential equation using the Sinc-collocation method based on the DE transformation [8].

In the present paper, we propose a Sinc-collocation method solving the boundary value problem of fourth-order differential equation incorporated with the DE transformation and show that the convergence rate is \( O(\exp(-cN/\log N)) \) with some \( c > 0 \). Specifically we consider the Sinc-collocation method for the two point boundary value problem of linear fourth-order ordinary differential equation

\[
Ly = y''''(x) + \mu_3(x)y'''(x) + \mu_2(x)y''(x) + \mu_1(x)y'(x) + \mu_0(x)y(x) = \sigma(x),
\]

\( y(a) = y(b) = 0, \quad y'(a) = y'(b) = 0, \quad x \in (a, b), \)

(1.1)

based on the DE transformation where \( a \) and \( b \) are real and finite. In order to carry the arguments smoothly, we first give some preliminaries.

2. The Sinc function preliminaries and the DE transformation

The sinc function is defined on the whole real line by

\[
sinc(t) = \frac{\sin \pi t}{\pi t}, \quad -\infty < t < \infty. \tag{2.1}
\]

In the Sinc method, we use a set of functions

\[
S(k, h)(t) = \frac{\sin[\pi(t - kh)/h]}{\pi(t - kh)/h}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{2.2}
\]

where \( h \) is the step size which characterizes the set [6]. We refer to \( S(k, h)(t) \) as the \( k \)th Sinc function with step size \( h \) evaluated at \( t \), and \( t_k = kh \), \( k = 0, \pm 1, \pm 2, \ldots \) are called the Sinc grid points. Let \( g \) be an analytic function defined on the real line. Then the series

\[
\hat{g}(t) = \sum_{j=-\infty}^{\infty} g(jh)S(j, h)(t) \tag{2.3}
\]

is called the Whittaker cardinal expansion of \( g \) whenever this series converges, and it gives an interpolation since \( \hat{g}(kh) = g(kh) \) holds at the Sinc grid points. Also it is known that under some mild analytic conditions it usually gives a good approximation to \( g \). Most properties of the cardinal expansion may be found in
At this stage, in view of such favorable properties of the cardinal expansion, an expansion in terms of $S(j, h)$

$$\sum_{j=-\infty}^{\infty} \tilde{w}_j S(j, h)(t)$$

(2.4)

may be used to obtain a good approximation to the solution $g(t)$ of a problem defined on $(-\infty, \infty)$.

In actual applications, however, we want to use the expansion for problems defined on an arbitrary interval $(a, b)$. Approximations on an interval $(a, b)$ can be developed by using some relevant transformation. Let $\mathcal{D}_d$ be the strip region of width $2d$ ($d > 0$) in the $t$-plane, i.e.,

$$\mathcal{D}_d \equiv \{ t \in \mathbb{C} : |\text{Im } t| < d \}. \quad (2.5)$$

Let $t = \phi(z)$ denote a conformal mapping which maps $(a, b)$ onto $(-\infty, \infty)$ in such a way that $\phi(a) = -\infty$ and $\phi(b) = +\infty$. Conversely, we assume that the inverse $z = \psi(t) = \phi^{-1}(t)$ maps $\mathcal{D}_d$ onto a simply connected domain $\mathcal{D}$ with a boundary $\partial \mathcal{D}$ on which the points $a$ and $b$ lie. Corresponding to the Sinc grid points in the $t$-plane we define

$$z_k = \psi(kh) = \phi^{-1}(kh), \quad k = 0, \pm 1, \pm 2, \ldots, \quad (2.6)$$

in the $z$-plane which we also call the Sinc grid points. In the $z$-plane we also have the cardinal expansion

$$\hat{f}(z) = \sum_{j=-\infty}^{\infty} f(z_j) S(j, h) \circ \phi(z),$$

(2.7)

which defines an interpolation because $\hat{f}(z_k) = f(z_k)$ holds. Under some mild analytic conditions it also gives a good approximation to $f(z)$.

We introduce a function space $H^1(\mathcal{D})$ as defined

$$H^1(\mathcal{D}) = \{ f \mid f \text{ is analytic in } \mathcal{D}, N_1(f \phi', \mathcal{D}) < \infty \},$$

where

$$N_1(f, \mathcal{D}) = \int_{\partial \mathcal{D}} |f(z)||\phi'|dz.$$ 

Let $K_2(\mathcal{D})$ be the family of all functions in $H^1(\mathcal{D})$ such that

$$|f(z)| \leq \exp(-\alpha \exp |\phi(z)|), \quad (\alpha > 0) \text{ as } z \to a, b,$$

(2.8)

with $\phi^{-1} : \mathcal{D}_d \to \mathcal{D}$ which we say that $f(z)$ decays double exponentially. The reason why we call it the double exponential decay is that (2.8) is equivalent to

$$|f(\phi^{-1}(t))| \leq \exp(-\alpha \exp |t|) \quad \text{as } t \to \pm \infty.$$ 

Now we propose here specifically a transformation $\phi$, whose inverse $\phi^{-1} = \psi$ maps $\mathcal{D}_d$ onto $\mathcal{D}$, defined by

$$t = \phi(z) = \log \left(\frac{1}{\pi} \log \left(\frac{z-a}{b-z}\right) + \frac{1}{\pi^2} \left(\log \frac{z-a}{b-z}\right)^2 + 1\right). \quad (2.9)$$
The function \( \psi(t) = \phi^{-1}(t) \) in (2.10) defines a DE transformation because \( \psi'(t) = \phi^{-1'}(t) \) behaves as

\[
|\psi(t)| = |\phi^{-1'}(t)| = \frac{(b - a)}{2} \frac{\frac{\pi}{2} \cosh t}{\cosh^2(\frac{\pi}{2} \sinh t)} = O\left(\exp\left(-\frac{\pi(1 - \varepsilon)}{2} \exp|t|\right)\right) \text{ as } t \to \pm \infty,
\]

for a sufficiently small \( \varepsilon \). Incidentally, Morlet [3] and Smith et al. [5] used

\[
\psi_1(t) = \phi_1^{-1}(t) = \frac{(b - a)}{2} \frac{\frac{\pi}{2} \cosh t}{\cosh^2(\frac{\pi}{2} \sinh t)} + \frac{(b + a)}{2}
\]

whose derivative decays single exponentially, i.e.,

\[
|\psi_1'(t)| = |\phi_1^{-1'}(t)| = \frac{b - a}{2} \frac{\frac{1}{2}}{\cosh^2\frac{t}{2}} = O(\exp(-|t|)) \text{ as } t \to \pm \infty,
\]

so that we call it the SE transformation.

Corresponding to the uniform grid points \( t_k = kh, k = 0, \pm 1, \pm 2, \ldots \) in \( D_d \) we specifically have the Sinc grid points \( z_k = \psi(kh) = \phi^{-1}(kh), k = 0, \pm 1, \pm 2, \ldots \) for the DE transformation with \( \phi^{-1}(t) \) defined as (2.10). Hereafter we write \( x_k \) instead of \( z_k \) because they are real in the present case. Incidentally, an explicit image in the \( z \)-plane of the lines \( |\text{Im } t| = d, d = 0.1, 0.2, \ldots \) parallel to the real axis in the \( t \)-plane in case of the DE transformation (2.10) with \( a = -1 \) and \( b = 1 \) is shown in Fig. 1 [4].
3. Sinc-collocation method based on the DE transformation

In this section we will propose a Sinc-collocation method based on the DE transformation for fourth-order ordinary differential equation. In view of favorable properties of the cardinal expansion (2.7), we approximate the solution \( y(x) \) of (1.1) by an expansion

\[
y_N(x) = \frac{1}{\phi'(x)} \sum_{j=-N}^{N} w_j S(j, h) \circ \phi(x),
\]

(3.1)

with \( \phi \) defined as (2.9) which maps \((a, b)\) onto \((-\infty, \infty)\). We multiplied the term \(1/\phi'(x)\) in (3.1) in order that \(y_N(x)\) meets the boundary condition \(y_N'(a) = y_N'(b) = 0\) [3]. The unknown coefficients \(w_j, j = -N, -N+1, \ldots, N\) in (3.1) are determined from the following requirement using the collocation method in which the collocation points are the Sinc grid points \(x_k, k = -N, -N + 1, \ldots, N:\)

\[
Ly_N(x_k) = \sigma(x_k), \quad x_k = \phi^{-1}(kh), \quad k = -N, -N + 1, \ldots, N.
\]

(3.2)

Thus, the actual number of the Sinc basis functions in the approximation (3.1), as well as that of the collocation points, is \(2N + 1\). Substituting \(y_N\) of the expansion (3.1) into \(y_N\) in the left-hand side of (3.2) and multiplying \(h^4/[\phi']^3\) on both sides, we have [3]

\[
h^4 \sum_{j=-N}^{N} \left\{ \omega_{4,j}(x_k) + \frac{\mu_2}{[\phi']^2} \omega_{3,j}(x_k) + \frac{\mu_2}{[\phi']^3} \omega_{2,j}(x_k) + \frac{\mu_1}{[\phi']^4} \omega_{1,j}(x_k) + \frac{\mu_0}{[\phi']^5} \omega_{0,j}(x_k) \right\} w_j
\]

\[= h^4 \frac{1}{[\phi']^3} \sigma(x_k), \quad k = -N, -N + 1, \ldots, N,\]

(3.3)

where

\[
\omega_{m,j}(x) = \frac{1}{[\phi'(x)]^{m-1}} \frac{d^m}{dx^m} \left( \frac{S(j, h) \circ \phi(x)}{\phi'(x)} \right).
\]

(3.4)

It should be noted that

\[
\omega_{0,j}(x) = S(j, h) \circ \phi(x)
\]

(3.5)

holds, and hence

\[
\eta_N(x) = \frac{1}{\phi'(x)} \sum_{j=-N}^{N} y(x_j) \phi'(x_j) \omega_{0,j}(x) = \frac{1}{\phi'(x)} \sum_{j=-N}^{N} y(x_j) \phi'(x_j) S(j, h) \circ \phi(x)
\]

(3.6)

defines a modified finite cardinal expansion corresponding to the solution \(y(x)\) satisfying

\[
\eta_N(x_k) = y(x_k), \quad x_k = \phi^{-1}(kh), \quad k = -N, -N + 1, \ldots, N.
\]

(3.7)
After a little hard algebraic manipulation, (3.3) becomes a system of linear equations for $w_j$, $j = -N, -N + 1, \ldots, N$ as follows:

$$
\sum_{j=-N}^{N} \left( \delta^{(4)}_{jk} + h g_3(x_k) \delta^{(3)}_{jk} + h^2 g_2(x_k) \delta^{(2)}_{jk} + h^3 g_1(x_k) \delta^{(1)}_{jk} + h^4 g_0(x_k) \delta^{(0)}_{jk} \right) w_j
= h^4 \left( \frac{\sigma}{[\phi']^3} \right)(x_k), \quad k = -N, -N + 1, \ldots, N,
$$

(3.8)

where

$$
g_3 = -2 \left( \frac{1}{\phi'} \right)' + \frac{\mu_3}{\phi'},
$$

$$
g_2 = 2 \left( \frac{1}{\phi'} \right)'' - \left( \left( \frac{1}{\phi'} \right)' \right)^2 + \frac{\mu_2}{[\phi']^2},
$$

$$
g_1 = 3 \left( \frac{1}{\phi'} \right)^2 \left( \frac{1}{\phi'} \right)'' - 4 \left( \frac{1}{\phi'} \right)' \left( \frac{1}{\phi'} \right)'' + 2 \left( \left( \frac{1}{\phi'} \right)' \right)^3
+ \frac{\mu_3}{\phi'} \left( 2 \left( \frac{1}{\phi'} \right)'' - \left( \left( \frac{1}{\phi'} \right)' \right)^2 \right) + \frac{\mu_2}{[\phi']^2} \left( \frac{1}{\phi'} \right)'
+ \frac{\mu_1}{[\phi']^3},
$$

$$
g_0 = \frac{1}{[\phi']^3} \left( \frac{1}{\phi'} \right)'''
+ \frac{\mu_3}{[\phi']^3} \left( \frac{1}{\phi'} \right)'''
+ \frac{\mu_2}{[\phi']^3} \left( \frac{1}{\phi'} \right)''
+ \frac{\mu_1}{[\phi']^3} \left( \frac{1}{\phi'} \right)'
+ \frac{\mu_0}{[\phi']^4}
$$

(3.9)

and

$$
\delta^{(m)}_{jk} = h^m \frac{d^m}{dt^m} S(j, h(t)|_{t=kh}), \quad m = 0, 1, \ldots, 4.
$$

(3.10)

Specifically,

$$
\delta^{(0)}_{jk} = \begin{cases} 1; & j = k, \\ 0; & j \neq k, \end{cases}
\quad \delta^{(1)}_{jk} = \begin{cases} 0; & j = k, \\ (-1)^{k-j}/(k-j); & j \neq k, \end{cases}
$$

$$
\delta^{(2)}_{jk} = \begin{cases} -\pi^2/3; & j = k, \\ -2(-1)^{k-j}/(k-j)^2; & j \neq k, \end{cases}
\quad \delta^{(3)}_{jk} = \begin{cases} 0; & j = k, \\ [6 - \pi^2(k-j)^2]/(k-j)^3; & j \neq k, \end{cases}
$$

$$
\delta^{(4)}_{jk} = \begin{cases} \pi^4/5; & j = k, \\ -4[6 - \pi^2(k-j)^2]/(k-j)^4; & j \neq k. \end{cases}
$$

(3.11)
Note that
\[ \delta^{(4)}_{jk} = \delta^{(4)}_{kj}, \quad \delta^{(3)}_{jk} = -\delta^{(3)}_{kj}, \quad \delta^{(2)}_{jk} = \delta^{(2)}_{kj}, \quad \delta^{(1)}_{jk} = -\delta^{(1)}_{kj}, \quad \delta^{(0)}_{jk} = \delta^{(0)}_{kj}. \] (3.12)

Also, explicit forms of the derivatives of \(1/\phi'\) at \(x = \psi(t)\) are as follows:

\[
\left( \frac{1}{\phi'} \right)(x) = \left( \frac{1}{\phi'} \right)(\psi(t)) = \frac{b - a}{2} \frac{\pi \cosh t}{\cosh^2(\frac{\pi}{2} \sinh t)},
\] (3.13)

\[
\left( \frac{1}{\phi'} \right)' = \frac{d}{dx} \left( \frac{1}{\phi'} \right) = \phi'(x) \frac{d}{dx} \left( \frac{1}{\phi'} \right) = \tanh t - \pi \cosh t \tanh \left( \frac{\pi}{2} \sinh t \right),
\] (3.14)

\[
\left( \frac{1}{\phi'} \right)'' = \frac{1}{\cosh^2 t} - \pi \sinh t \tanh \left( \frac{\pi}{2} \sinh t \right) - \frac{\pi^2}{2} \frac{\cosh^2 t}{\cosh^2(\frac{\pi}{2} \sinh t)},
\] (3.15)

\[
\left( \frac{1}{\phi'} \right)^2 \left( \frac{1}{\phi'} \right)''' = -\frac{3 \tanh t}{\cosh^2 t} - \pi^2 \sinh t \cosh t,
\] (3.16)

\[
\left( \frac{1}{\phi'} \right)^3 \left( \frac{1}{\phi'} \right)'''' = \frac{12 \tanh^2 t}{\cosh^2 t} - \frac{3}{\cosh^4 t} - \frac{\pi^2}{2} \left( \frac{6\pi}{\cosh t} + 2\pi^3 \cosh^3 t \right) \tanh t \tanh \left( \frac{\pi}{2} \sinh t \right).
\] (3.17)

Then by solving the system of linear equations (3.8) for \(w_j\)'s, we obtain an approximate solution \(y_N(x)\) of (1.1) from (3.1).

4. Convergence analysis

Now we proceed to a convergence analysis. We follow the same line as in [3], but we need to change some of the assumptions on the coefficients of the given differential equation. Using the notations

\[ D(g) = \text{diag}(g(z_N), \ldots, g(z_N)) \] (4.1)

and

\[ I^{(m)} = [\delta^{(m)}_{kj}], \quad m = 0, 1, 2, 3, 4, \] (4.2)
the system of linear equations (3.8) for unknown coefficients \( w_j, j = -N, -N + 1, \ldots, N \) can be written in a matrix form
\[
Aw = p,
\]
where the matrix \( A \), the vector \( p \) and the vector \( w \) are given by
\[
A = I^{(4)} + hD(g_3)I^{(3)} + h^2D(g_2)I^{(2)} + h^3D(g_1)I^{(1)} + h^4D(g_0),
\]
\[
w = (w_{-N}, \ldots, w_N)^T, \quad p = h^4D \left( \frac{1}{[\phi']^3} \right) (\sigma(x_{-N}), \ldots, \sigma(x_N))^T.
\]

In order to establish a bound of \( |y(x) - y_N(x)| \) in the maximum norm, we first need to get a bound of \( \|A\tilde{y} - p\| \) where \( \tilde{y} \) is a vector defined by
\[
\tilde{y} = (\tilde{y}(x_{-N}), \tilde{y}(x_{-N+1}), \ldots, \tilde{y}(x_N))^T,
\]
with
\[
\tilde{y}(x) = y(x)\phi'(x),
\]
y\( x \) being the exact solution of (1.1), and next we need to get a bound of \( A^{-1} \). To this aim, we show the following lemma.

**Lemma 4.1.** (i) Suppose that \( \mu_j, j = 0, 1, 2, 3 \) and \( \sigma \) are analytic in \( \mathcal{D} \), that the differential equation (1.1) has a unique solution \( y \) which belongs to \( K_{\phi^d}(\mathcal{D}) \), and that \( \frac{\mu_3}{\phi'}, \frac{\mu_2}{[\phi']^2}, \frac{\mu_1}{[\phi']^3} \) are bounded from above as follows with positive constants \( B \) and \( \beta \):
\[
\left| \frac{\mu_3}{\phi'} \right|, \left| \frac{\mu_2}{[\phi']^2} \right|, \left| \frac{\mu_1}{[\phi']^3} \right| \leq B \exp(\beta|\phi(x)|), \quad \text{for } x \in \phi^{-1}((-\infty, \infty)).
\]

Furthermore, suppose that the mesh size \( h \) and \( N \) satisfy
\[
h = \frac{\log(\pi dN/\varepsilon')}{N}, \quad \varepsilon' = \varepsilon - \frac{\pi}{2}
\]
and that \( \tilde{y} = y\phi' \) belongs to \( K_{\phi^d}(\mathcal{D}) \). Then there exists a constant \( C_1 \), independent of \( N \), such that
\[
\|A\tilde{y} - p\| \leq C_1 N^{\beta + 4} \exp \left( -\frac{\pi dN}{\log(\pi dN/\varepsilon')} \right).
\]

(ii) Let the assumptions in (i) be satisfied. If the eigenvalues of the matrix
\[
E = \{h[D(g_3)I^{(3)} - I^{(3)}D(\tilde{g}_3)] + h^2[D(g_2)I^{(2)} + I^{(2)}D(\tilde{g}_2)]
\]
\[
+ h^3[D(g_1)I^{(1)} - I^{(1)}D(\tilde{g}_1)] + h^4D(2 \Re g_0)\}
\]

are analytic in \( \mathcal{D} \), that the differential equation (1.1) has a unique solution \( y \) which belongs to \( K_{\phi^d}(\mathcal{D}) \), and that \( \frac{\mu_3}{\phi'}, \frac{\mu_2}{[\phi']^2}, \frac{\mu_1}{[\phi']^3} \) are bounded from above as follows with positive constants \( B \) and \( \beta \):
\[
\left| \frac{\mu_3}{\phi'} \right|, \left| \frac{\mu_2}{[\phi']^2} \right|, \left| \frac{\mu_1}{[\phi']^3} \right| \leq B \exp(\beta|\phi(x)|), \quad \text{for } x \in \phi^{-1}((-\infty, \infty)).
\]

are non-negative for \( x \in \phi^{-1}((\infty, \infty)) \), then

\[
\|A^{-1}\| \leq \frac{16N^4}{\pi^4} (1 + C_2 N^{-1})
\]  

(4.11)

holds for a constant \( C_2 \) and for sufficiently large \( N \).

**Proof.** (i) See Appendix A.

(ii) Let \( \delta_i \), \( i = 1, 2, \ldots, 2N + 1 \) be the singular values of the matrix \( A \) with \( \delta_i \leq \delta_{i+1} \), and \( \lambda_i(\cdot) \) be the eigenvalues of a matrix as ordered \( \lambda_i(\cdot) \leq \lambda_{i+1}(\cdot) \). It is well-known that the eigenvalues of the matrix \( I^{(4)} \) are bounded from below by \( 16 \sin^4 (\pi/(4N + 4)) \). From [1] and the assumptions, we have

\[
\delta_1 \geq \min \left| \lambda_i \left( \frac{A + A^*}{2} \right) \right| = \min \left| \lambda_i \left( I^{(4)} + \frac{1}{2} E \right) \right| \\
\geq \min \lambda_i(I^{(4)}) \geq 16 \sin^4 (\pi/(4N + 4)).
\]

Therefore, we have the bound for \( \|A^{-1}\| \) and Lemma 4.1 is proved. \( \square \)

Now we are ready to prove a bound of the function \( y - y_N \) in the maximum norm. The result is summarized as follows:

**Theorem 4.2.** Let \( y \) be the exact solution of (1.1), and let \( y_N \) be its Sinc approximation (3.1). Let \( w = (w_{-N}, \ldots, w_N)^T \) be the exact solution of the system of Eq. (4.3). If the assumptions in Lemma 4.1 are satisfied, then there exist constant \( C \), which is independent of \( N \), such that

\[
\sup_{x \in \phi^{-1}((\infty, \infty))} |y(x) - y_N(x)| \leq C' N^\beta + 8 \exp \left( -\frac{\pi dN}{\log(\pi dN/\alpha')} \right),
\]  

(4.12)

with the mesh size \( h \) and \( N \) satisfying (4.8).

**Proof.** First we note (3.6), i.e.,

\[
\eta_N(x) = \frac{1}{\phi'(x)} \sum_{j=-N}^{N} y(x_j) \phi'(x_j) S(j, h) \circ \phi(x).
\]

On the other hand, from the triangle inequality, we have

\[
|y(x) - y_N(x)| \leq |y(x) - \eta_N(x)| + |\eta_N(x) - y_N(x)|.
\]  

(4.13)

By the assumption \( \tilde{y} \in K_{\varphi}(\mathcal{D}) \), and therefore by [9], there exists a constant \( C_3 \) independent of \( N \) such that

\[
\sup_{x \in \phi^{-1}((\infty, \infty))} |y(x) - \eta_N(x)| \leq C_3 \exp \left( -\frac{\pi dN}{\log(\pi dN/\alpha')} \right).
\]  

(4.14)
The term on the extreme right-hand side of (4.13) satisfies

\[ |\eta_N(x) - y_N(x)| = \left| \frac{1}{\phi'(x)} \sum_{j=-N}^{N} [\tilde{y}(x_j) - w_j] S(j, h) \circ \phi(x) \right| \]

\[ \leq \sum_{j=-N}^{N} |\tilde{y}(x_j) - w_j| \left| \frac{S(j, h) \circ \phi(x)}{\phi'(x)} \right| \]

\[ \leq \left( \sum_{j=-N}^{N} |\tilde{y}(x_j) - w_j|^2 \right)^{1/2} \left( \sum_{j=-N}^{N} \left| \frac{S(j, h) \circ \phi(x)}{\phi'(x)} \right|^2 \right)^{1/2} \]

\[ \leq C_3' \left( \sum_{j=-N}^{N} |\tilde{y}(x_j) - w_j|^2 \right)^{1/2} = C_3' \|\tilde{y} - w\|, \quad (4.15) \]

where we used the fact that if \( x \in \phi^{-1}((-\infty, \infty)) \) then \( \sum_{j=-\infty}^{\infty} |S(j, h) \circ \phi(x)/\phi'(x)|^2 \) is bounded to obtain the last line. Finally, from Lemma 4.1(ii), we have

\[ \|\tilde{y} - w\| = \|A^{-1}(A\tilde{y} - p)\| \]

\[ \leq \|A^{-1}\| \|A\tilde{y} - p\| \]

\[ \leq C_3'' N^{\beta+8} \exp \left( -\frac{\pi dN}{\log(\pi dN/x')} \right), \quad (4.16) \]

with \( C_3'' \) a constant independent of \( N \). Combining (4.14) and (4.16), we thus conclude Theorem 4.2. \( \square \)

**Remark 4.3.** (a) The assumption for (4.10) of Lemma 4.1 is not a necessary condition in order that Eq. (1.1) can be solved by the present method. It is only a technical assumption to prove Theorem 4.2. There are a lot of examples in which the approximate solution converges as fast as \( \exp(-cN/\log N) \) even if the assumption for (4.10) is not satisfied. To establish the convergence theorem without the assumption for (4.10) is an important future work.

(b) Although the condition in (i) of Lemma 4.1 is required for all \( x \in \phi^{-1}((-\infty, \infty)) \), it suffices in numerical experiments to verify that the assumption on (4.10) be satisfied for Sinc points \( x_k = \phi^{-1}(kh), k = -N, \ldots, N \) with the selected mesh size satisfying (4.8), because the matrix \( A \) is evaluated only at these Sinc points.

5. Numerical examples

In this section we present numerical results obtained by applying the DE transformation to illustrate the analytical result discussed in the previous sections. We summarize here the procedure for numerical solution of Eq. (1.1). Since we look for a solution \( y \) of (1.1) belonging to \( K_{\mathcal{X}}(\mathcal{X}) \), namely
\( \tilde{y} = y \psi' \in \mathbf{K}_x(D) \), we first fix \( N \) and choose the step size as \( h = \log(\pi d N / x') / N \) with \( d = \pi / 2 - \varepsilon \) and \( x' = x - \pi / 2 \) where \( \varepsilon \) is an arbitrarily small positive number. Then by solving the system of linear equations (3.8) for \( w_j \)'s, we obtain an approximate solution (3.1) of (1.1). We note here that when the problem (1.1) has no singularity in \( D \) except \( a, b \) we should use \( d = \pi / 2 - \varepsilon \) with a small positive number \( \varepsilon \) for Theorem 4.2 to hold. However, in actual computation, we use \( d = \pi / 2 \) instead of \( d = \pi / 2 - \varepsilon \).

The examples reported in this section are selected from [3] and [5] except Example 5.1. We also apply the SE transformation (2.12) to all the examples and compare the efficiency between the DE and the SE transformations. In every example, we use \( N = 2, 4, 8, 16, \ldots \) as in [3] and [5]. The error is reported on the set of uniform gridpoints \( x_i \in (a, b), x_i = a + (b - a)i/1000, i = 1, 2, \ldots, 999 \), i.e.,

\[
\max_{1 \leq i \leq 999} |y(x_i) - y_N(x_i)|. \tag{5.1}
\]

The error by the DE transformation together with the SE transformation are shown in Figs. 2–5. In each figure the abscissa corresponds to \( N \), \( 2N + 1 \) being the actual number of basis functions, and the ordinate “max error” corresponds to the maximum of the absolute value of the error in logarithmic scale. The curve marked as DE is for the error by the double exponential Sinc-approximation and the curve marked as SE is for the error by the single exponential one. The solutions of all the problems are computed with double precision accuracy using Compaq Visual Fortran Compiler on a Pentium IV personal computer, and also with quadruple precision accuracy using Fujitsu Fortran compiler when we want to obtain a result with accuracy higher than double precision.

We first consider a homogeneous boundary value problem of fourth-order linear differential equation:

**Example 5.1.**

\[
y^{(iv)}(x) + \frac{1}{x^{10}} y(x) = \sigma(x),
\]
\[
y(0) = y(1) = 0, \quad y'(0) = y'(1) = 0,
\]
\[
\sigma(x) = x^{-3/2} \left[ x^{-6} (\log x)^3 - \frac{15}{16} (\log x)^3 - 3 (\log x)^2 + 21 \log x + 24 \right].
\]

The exact solution \( y(x) = x^{5/2} (\log x)^3 \) has an algebraic and logarithmic singularity at the same time at \( x = 0 \). In this example, since \( a = 0 \) and \( b = 1 \), we employ

\[
x = \phi^{-1}(t) = \frac{1}{2} \tanh \left( \frac{\pi}{2} \sinh t \right) + \frac{1}{2} \tag{5.2}
\]

We choose the optimal mesh size \( h = [\log(2\pi N / 3)]/N \) since \( \bar{y} = y \psi' \in \mathbf{K}_{3\pi/4}(D) \) with \( x' = 3\pi/4 \). With some calculations we can verify that this problem meets all the assumptions in Theorem 4.2. In fact, we computed all the eigenvalues of matrix \( E \) with the selected mesh size satisfying (4.8) and for Sinc points \( x_k = \phi^{-1}(kh), k = -N, N + 1, \ldots, N \) and confirmed their positivity, so that we can say that the assumption for (4.10) is satisfied. The result of numerical computation by the DE transformation is shown in Fig. 2 together with the one by the SE transformation. In all the figures in this section max error at the ordinate is the error (5.1) in logarithmic scale. The curve indicated as DE corresponds to the result by the DE transformation, while the one indicated as SE corresponds to the SE transformation. The curve indicated as DE clearly shows the expected convergence rate.
In the numerical computation of the next three examples the convergence rate of approximate solutions is observed as $O(\exp(-cN)/\log N)$ as we expected though there are a few negative eigenvalues of matrix $E$, namely the problem does not meet the assumption for (4.10) in Lemma 4.1. Also, the error behavior observed using the Sinc-collocation method based on the SE transformation is similar to the errors reported in [3] and [5].
Example 5.2 (Morlet [3]).

\[ y^{(4)}(x) = \sigma(x), \]
\[ y(0) = y(1) = 0, \quad y'(0) = y'(1) = 0, \]
\[ \sigma(x) = \frac{9}{16} x^{-5/2} (1 - x)^{-5/2}. \]

Note that the exact solution of the present problem is \( y(x) = x^{3/2} (1 - x)^{3/2} \), and it presents an algebraic singularity both at \( x = 0 \) and \( x = 1 \). In this example, since \( a = 0 \) and \( b = 1 \), we again used (5.2) for the function \( \tilde{y} = y \phi' \) belongs to \( K_{\pi/4}(\mathcal{D}) \) with \( \alpha' = \pi/4 \). The error by the DE transformation together with the one by the SE transformation is shown in Fig. 3. We there observe that the error by the DE transformation converges to zero as \( \exp(-cN/\log N) \) as is expected from Theorem 4.2.

Next example considered here is a fourth-order inhomogeneous boundary value problem.

Example 5.3 (Morlet [3]).

\[ y^{(4)}(x) + y(x) = \sigma(x), \]
\[ y(-1) = y(1) = 0, \quad y'(-1) = y'(1) = 0, \]
\[ \sigma(x) = \left( \left( \frac{\pi}{2} \right)^4 + 1 \right) \cos \left( \frac{\pi}{2} x \right). \]

The exact solution of the problem is \( y(x) = \cos(\pi x/2) \). It can be transformed to a problem with a homogeneous boundary condition via \( u(x) = y(x) + \pi(x^2 - 1)/4 \), which gives

\[ u^{(4)}(x) + u(x) = \left( \left( \frac{\pi}{2} \right)^4 + 1 \right) \cos \left( \frac{\pi}{2} x \right) + \frac{\pi}{4} (x^2 - 1), \]
\[ u(-1) = u(1) = 0, \quad u'(-1) = u'(1) = 0. \]

In this example, since \( a = -1 \) and \( b = 1 \), we employ

\[ x = \phi^{-1}(t) = \tanh \left( \frac{\pi}{2} \sinh t \right). \quad (5.3) \]

It is clear that \( \tilde{u} = u \phi' \) belongs to \( K_{\pi/2}(\mathcal{D}) \) and hence we have the optimal mesh size \( h = [\log(2\pi N)]/N \) since \( \alpha' = \pi/2 \). The result is shown in Fig. 4. It is observed that the expected convergence rate \( \exp(-cN/\log N) \) is achieved by the DE transformation.

Example 5.4 (Smith et al. [5]).

\[ y^{(4)}(x) + \frac{1}{x} y^{(3)}(x) + \frac{1}{x^2} y^{(2)}(x) + \frac{1}{x^3} y'(x) + \frac{1}{x^4} y(x) = \sigma(x), \]
\[ y(0) = y(1) = 0, \quad y'(0) = y'(1) = 0, \]
\[ \sigma(x) = \frac{[x(1 - x)]^{3/2}}{16} \{ 131 (1 - x)^4 - 990 x (1 - x)^3 \}
\[ + 1860 x^2 (1 - x)^2 - 330 x^3 (1 - x) - 15 x^4 \}. \]
The exact solution of this problem is $y(x) = [x(1 - x)]^{5/2}$ which is singular both at $x = 0$ and $x = 1$. $x = 0$ is a regular singular point. We used again (5.2) for this problem. We take the optimal mesh size $h = \log(2\pi N/3)/N$ because $\tilde{y} = y\phi'$ belongs to $K_{3\pi/4}(\phi')$. The result is shown in Fig. 5. This problem also illustrates the high efficiency of the Sinc-collocation method incorporated with the DE transformation.

In every example it is observed that if we fix $N$ and use

$$h = \frac{\log(\pi d N/\alpha)}{N},$$

(5.4)

we obtain a more accurate result than the one obtained by using (4.8), i.e.,

$$h = \frac{\log(\pi d N/\alpha')}{N}, \quad \alpha' = \alpha - \pi/2$$

with the same $N$. In Fig. 6 we show an error curve for Example 5.1 using (5.4) with $\alpha = 5\pi/4$ which is marked as $\alpha = 5\pi/4$ together with the error curve DE in Fig. 2 which is marked as $\alpha' = 3\pi/4$. The error is computed also by (5.1) on the set of uniform grid points. It is evident that this improvement corresponds to that in the error behavior given by the right-hand side of (4.12) with $\alpha = 5\pi/4$ instead of $\alpha' = 3\pi/4$. The situation is the same in other examples. Although it is necessary to select $h$ according to (4.8) for Theorem 4.2 to hold, we can employ a smaller $h$ given by (5.4) to get a more accurate result in actual numerical computation. To justify the selection of $h$ in a more rigorous way is left to a future work.

The solution of the examples shown above has, except of Example 5.3, a singularity at either end of the interval $(a, b)$. For such problems the Sinc-collocation method based on the DE transformation works very well. For problems in which the solution is regular on $[a, b]$ including both end points, on the other hand, while the present method gives also good result, other methods, say a high order Chebyshev approximation, may work more efficiently.
Appendix A.

A.1. Proof of Lemma 4.1(i)

For the proof of Lemma 4.1(i), we need here an error expression of the cardinal expansion. First define

\[ K_m(x, z) = \frac{1}{2\pi i[\phi'(x)]^{m-1}} \frac{\partial_{\phi'}^{m}}{\partial_{x^{m}}} \left( \frac{\sin[\pi\phi(x)/h]}{\phi'(x)[\phi(z) - \phi(x)]} \right). \]  

(A.1)

Since we assumed that \( \tilde{y} = y\phi' \in K_d(\mathcal{D}) \), then \( \tilde{y}\phi' \in H^1(\mathcal{D}) \) and we have the cardinal series expansion for \( y(x)\phi'(x) \) with an error term

\[ y(x) = \sum_{j=-\infty}^{\infty} \tilde{y}(x_j) \oint_{\partial D} \frac{K_m(x, z)\tilde{y}(z)\phi'(z)}{\phi'(\phi(z) - \phi(x))\sin[\pi\phi(z)/h]} \, dz. \]  

(A.2)

Thus we obtain in general

\[ \frac{d^m}{dx^m} y(x) - \sum_{j=-\infty}^{\infty} \tilde{y}(x_j) \phi_{m,j}(x) = \frac{1}{2\pi i} \oint_{\partial D} \frac{K_m(x, z)\tilde{y}(z)\phi'(z)}{\phi'(\phi(z) - \phi(x))\sin[\pi\phi(z)/h]} \, dz, \quad m = 0, 1, 2, 3, 4, \]  

(A.3)

where \( \omega_{m,j}(x) \) is defined by (3.4).

Let \( r_k \) denote the \( k \)th component of the residual vector \( r = A\tilde{y} - p \). Then, by replacing \( w_j \) with \( \tilde{y}(x_j) \) in (3.3) we have

\[ r_k \equiv r_k = \{ A\tilde{y} - p \}_k \]

\[ = h^4 \sum_{j=-N}^{N} \left\{ \omega_{4,j}(x_k) + \frac{\mu_3}{[\phi']^3} \omega_{3,j}(x_k) + \frac{\mu_2}{[\phi']^2} \omega_{2,j}(x_k) \right. \]

\[ + \frac{\mu_1}{[\phi']^3} \omega_{1,j}(x_k) + \frac{\mu_0}{[\phi']^4} \omega_{0,j}(x_k) \left. \right\} \tilde{y}(x_j) - h^4 \left( \frac{\sigma}{[\phi']^3} \right)(x_k). \]  

(A.4)

Since \( Ly - \sigma = 0 \), we subtract it from (A.4). Then from (3.6) the sum

\[ h^4 \sum_{j=-N}^{N} \frac{\mu_0(x_j)\tilde{y}(x_j)}{[\phi']^4} \omega_{0,j}(x_k) \]
in (A.4) cancels with \( h^4/[\phi'(x_k)]^3 \mu_0(x_k) y(x_k) \) in \( L_y \), and we have

\[
\begin{align*}
\lambda_k &= \{ A\tilde{y} - p \}_k = \left\{ A\tilde{y} - p - h^4 \frac{(L_y - \sigma)(x)}{[\phi'(x)]^3} \right\}_k \\
&= h^4 \sum_{j=-N}^{N} \left\{ \omega_{4,j}(x_k) + \frac{\mu_3}{[\phi']^3} \omega_{3,j}(x_k) + \frac{\mu_2}{[\phi']^2} \omega_{2,j}(x_k) + \frac{\mu_1}{[\phi']^3} \omega_{1,j}(x_k) \right\} \tilde{y}(x_j) \\
&- \frac{h^4}{[\phi'(x_k)]^3} [y'''(x_k) + \mu_3(x_k) y''(x_k) + \mu_2(x_k) y''(x_k) + \mu_1(x_k) y'(x_k)] = \lambda_k^{(1)} + \lambda_k^{(2)}.
\end{align*}
\]

Here we wrote the summation as \( \sum_{j=-N}^{N} = \sum_{j=-\infty}^{\infty} - \sum_{|j|>N} \), i.e.,

\[
\lambda_k^{(1)} = -\frac{h^4}{[\phi'(x_k)]^3} [y'''(x_k) + \mu_3(x_k) y''(x_k) + \mu_2(x_k) y''(x_k) + \mu_1(x_k) y'(x_k)] \\
+ \frac{\mu_1}{[\phi'(x_k)]^3} \omega_{1,j}(x_k) \tilde{y}(x_j) \\
= -h^4 \int_{\mathbb{R}} \left[ K_4(x_k, z) + \frac{\mu_3(x_k)}{[\phi'(x_k)]^2} K_3(x_k, z) + \frac{\mu_2(x_k)}{[\phi'(x_k)]^3} K_2(x_k, z) + \frac{\mu_1(x_k)}{[\phi'(x_k)]^3} K_1(x_k, z) \right] \\
\times \frac{\phi'(z) y(z)}{\sin[\pi\phi(z)/h]} \, dz, \\
\lambda_k^{(2)} = -h^4 \sum_{|j|>N} \left( \omega_{4,j}(x_k) + \frac{\mu_3(x_k)}{[\phi'(x_k)]^3} \omega_{3,j}(x_k) + \frac{\mu_2(x_k)}{[\phi'(x_k)]^2} \omega_{2,j}(x_k) + \frac{\mu_1(x_k)}{[\phi'(x_k)]^3} \omega_{1,j}(x_k) \right) \tilde{y}(x_j).
\]

The explicit forms of \( K_m(x_k, z), m = 0, 1, 2, 3, 4 \) are as follows:

\[
K_0(x_k, z) = 0, \quad K_1(x_k, z) = \frac{(-1)^k}{2ih[\phi(z) - kh]},
\]

\[
K_2(x_k, z) = \frac{(-1)^k}{2ih[\phi(z) - kh]^2} \left[ 2 + (\phi(z) - kh) \left( \frac{1}{\phi'} \right)'(x_k) \right],
\]

\[
K_3(x_k, z) = \frac{(-1)^k}{2ih[\phi(z) - kh]^3} \left[ 6 + (\phi(z) - kh)^2 \left( \frac{\pi}{h} \right)^2 + 2 \frac{1}{\phi'} \left( \frac{1}{\phi'} \right)''(x_k) \right] \\
- (\phi(z) - kh)^2 \left( \frac{1}{\phi'} \right)^2(x_k)
\]
\[ K_4(x_k, z) = \frac{(-1)^k}{2ih[\phi(z) - kh]^4} \left[ 24 - 12(\phi(z) - kh) \left( \frac{1}{\phi'} \right)'(x_k) 
+ (\phi(z) - kh)^2 \left( 4 \frac{1}{\phi'} \left( \frac{1}{\phi'} \right)'' - 2 \left( \frac{1}{\phi'} \right)' + 4 \left( \frac{\pi}{h} \right)^2 \right)(x_k) 
+ (\phi(z) - kh)^3 \left( -4 \frac{1}{\phi'} \left( \frac{1}{\phi'} \right)' + 2 \left( \frac{1}{\phi'} \right)'' \right)(x_k) 
+ (\phi(z) - kh)^3 \left( 3 \left( \frac{1}{\phi'} \right)'' + 2 \left( \frac{\pi}{h} \right)^2 \left( \frac{1}{\phi'} \right)' \right)(x_k) \right]. \tag{A.5} \]

It should be remarked that, while \( K_m(x_k, z) \), \( m = 0, 1, 2, 3 \) are exactly the same as given in [3], \( K_4(x_k, z) \) is different for the present transformation.

Note that \(|\text{Im } \phi(z)| = d\) and \(|\phi(z) - kh| \geq d\) on \( \partial \mathcal{D} \) since \(|\text{Im } t| = d\) and \(|t - kh| \geq d\) on \( \partial \mathcal{D}_d \), and that, from the assumptions on the coefficients of the differential equation and on the mapping function \( \phi \),

\[
|K_1(x_k, z)| \leq \frac{C'_1h^{-1}}{[(u(z) - kh)^2 + d^2]^{1/2}}, \\
|K_2(x_k, z)| \leq \frac{C'_2h^{-1} \exp(|kh|)}{[(u(z) - kh)^2 + d^2]^{1/2}}, \\
|K_3(x_k, z)| \leq \frac{C'_3h^{-3} \exp(2|kh|)}{[(u(z) - kh)^2 + d^2]^{1/2}}, \\
|K_4(x_k, z)| \leq \frac{C'_4h^{-3} \exp(3|kh|)}{[(u(z) - kh)^2 + d^2]^{1/2}},
\]

where \( u(z) = \text{Re } \phi(z) \). Then we have

\[
h^4 \left| K_4(x_k, z) + \frac{\mu_3(x_k)}{\phi'(x_k)} K_3(x_k, z) + \frac{\mu_2(x_k)}{[\phi'(x_k)]^2} K_2(x_k, z) + \frac{\mu_1(x_k)}{[\phi'(x_k)]^3} K_1(x_k, z) \right| \\
\leq \frac{C_5h \exp((\beta + 3)|kh|)}{[(u(z) - kh)^2 + d^2]^{1/2}}, \tag{A.6} \]

with a constant \( C_5 \) which depends on the bounds for the coefficients of the differential equation and for the derivatives of \( 1/\phi' \) defined in (2.9), on mesh size \( h \), and on \( d \leq \pi/2 \). Therefore, we have

\[
\| A\hat{y} - p \| = \left( \sum_{k=-N}^{N} |r_k|^2 \right)^{1/2} \leq \left( \sum_{k=-N}^{N} |r_k^{(1)}|^2 \right)^{1/2} + \left( \sum_{k=-N}^{N} |r_k^{(2)}|^2 \right)^{1/2}.
\]
The first norm in the extreme right-hand side term satisfies
\[
\sum_{k=-N}^{N} |r_k^{(1)}|^2 \leq \sum_{k=-N}^{N} \left| \int_{\partial \Omega} \frac{C_5 h \exp((\beta + 3)|kh|)}{[u(z) - kh]^2 + d^2]^1/2 |\sin[\pi \phi(z)/h]| |\phi'(z)\tilde{y}(z)| |dz| \right|^2 \\
\leq C_5' h^{-1} \frac{\exp(2(\beta + 3)Nh)}{[\sinh[\pi d/h]^2]} \left( \int_{\partial \Omega} |\phi'(z)\tilde{y}(z)| dz \right)^2 \\
\leq C_5'' h^{-1} \frac{\exp(2(\beta + 3)Nh)}{[\sinh[\pi d/h]^2]}
\]
by (A.6), from the bound \(\sinh[\pi d/h] \leq \sinh[\pi \phi(z)/h]\) on \(\partial \Omega\) and from the existence of the integral of \(|\tilde{y}|\). For the second norm we use the assumptions on the mapping \(\phi\) and on the expressions for \(\delta_{jk}^{(m)}, j = -N, \ldots, N, k = -N, \ldots, N, m = 1, 2, 3, 4\), and have
\[
\sum_{k=-N}^{N} |r_k^{(2)}|^2 = \sum_{k=-N}^{N} \left| \sum_{|j|>N} \{\delta_{jk}^{(4)} + h g_3(x_k)\delta_{jk}^{(3)} + h^2 g_2(x_k)\delta_{jk}^{(2)} + h^3 g_1(x_k)\delta_{jk}^{(1)}\} \tilde{y}(x_j) \right|^2 \\
\leq C_6 \sum_{k=-N}^{N} \left( \exp(2(\beta + 3)|kh|) \sum_{|j|>N} \gamma_{jk}^2 \sum_{|j|>N} |\tilde{y}(x_j)|^2 \right) \\
\leq C_6' h^{-2} \exp(2(\beta + 3)Nh) \exp(-2x \exp(Nh)),
\]
where \(\gamma_{jk}\) is defined
\[
\gamma_{jk} = \max\{|\delta_{jk}^{(4)}|, |\delta_{jk}^{(3)}|, |\delta_{jk}^{(2)}|, |\delta_{jk}^{(1)}|\}.
\]

For fixed \(k\), it follows that \(\sum_{j=-\infty}^{\infty} \gamma_{jk}^2 = C_7 < \infty\). We also used the fact that \(|\tilde{y}(x_j)|\) is bounded by the double exponential decaying factor. Finally gathering the bounds for \(r_k^{(1)}\) and \(r_k^{(2)}\), and replacing \(h\) with its value \(h = \log(\pi dN/z')/N\), we conclude that
\[
\|A\tilde{y} - p\| = \left( \sum_{k=-N}^{N} |r_k|^2 \right)^{1/2} \leq C_1 N^{\beta+4} \exp \left( -\frac{\pi dN}{\log(\pi dN/z')} \right). \quad \Box
\]

References


