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Posets of Shuffles

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DEDICATED TO THE MEMORY OF HERBERT J. RYSER

We study properties of a lattice order defined on shuffles of subwords of a pair of fixed words.

1. Introduction

Let $\mathbf{x} = x_1 x_2 \cdots x_m$ and $\mathbf{y} = y_1 y_2 \cdots y_n$ be words of length m and n, respectively, with letters chosen from some sufficiently large alphabet $\mathscr{A}$. We assume that the $m+n$ letters occurring in $\mathbf{x}$ and $\mathbf{y}$ are all distinct.

For any pair of words $\mathbf{u}$ and $\mathbf{v}$ in $\mathscr{A}^*$, write $\mathbf{u} \leqslant \mathbf{v}$ if $\mathbf{u}$ is a subword of $\mathbf{v}$. Let $\bar{\mathbf{u}}$ denote the set of letters which occur in $\mathbf{u}$, and let $\mathbf{v}|\mathbf{u}$ denote the subword (possibly empty) of $\mathbf{v}$ obtained by restricting $\mathbf{v}$ to the letters in $\bar{\mathbf{u}}$.

Given $\mathbf{x}$ and $\mathbf{y}$ as defined above, let $\mathscr{W}_{\mathbf{x},\mathbf{y}}$ denote the set of all words $\mathbf{w}$ in $\mathscr{A}^*$ such that $\bar{\mathbf{w}} \subseteq \bar{\mathbf{x}} \cup \bar{\mathbf{y}}$, and such that $\mathbf{w}|\mathbf{x} \leqslant \mathbf{x}$ and $\mathbf{w}|\mathbf{y} \leqslant \mathbf{y}$. In other words, $\mathscr{W}_{\mathbf{x},\mathbf{y}}$ consists of all "shuffles" of subwords of $\mathbf{x}$ and subwords of $\mathbf{y}$. We introduce a partial order on $\mathscr{W}_{\mathbf{x},\mathbf{y}}$ (denoted by $\preccurlyeq$) by letting

$$\mathbf{w} \preccurlyeq \mathbf{w}' \text{ iff } \mathbf{w}|\mathbf{x} \geqslant \mathbf{w}'|\mathbf{x},\ \mathbf{w}|\mathbf{y} \leqslant \mathbf{w}'|\mathbf{y}, \text{ and } \mathbf{w}|\mathbf{w}' = \mathbf{w}'|\mathbf{w}.$$

According to this definition, $\mathbf{w} \preccurlyeq \mathbf{w}'$ if and only if $\mathbf{w}'$ can be obtained from $\mathbf{w}$ by deleting letters of $\mathbf{x}$ and adding letters of $\mathbf{y}$. For example, if $\mathbf{x} = ABCDEFG$ and $\mathbf{y} = xyzw$, then $BxDFwG \preccurlyeq xDywG$ in $\mathscr{W}_{\mathbf{x},\mathbf{y}}$.

Clearly the order structure of $\mathscr{W}_{\mathbf{x},\mathbf{y}}$ depends only on the number of letters in $\mathbf{x}$ and $\mathbf{y}$. Accordingly, we will write $\mathscr{W}_{\mathbf{x},\mathbf{y}} = \mathscr{W}_{m,n}$ when $\mathbf{x}$ and $\mathbf{y}$ have m and n letters, respectively. Note that $\mathscr{W}_{m,n}$ is dually isomorphic to $\mathscr{W}_{n,m}$ and $\mathscr{W}_{m,0}$ is isomorphic to the lattice of subsets of an m-element set. Figure 1 illustrates the Hasse diagram of $\mathscr{W}_{3,1}$.

In the next section, we will show that the ordering just defined is a lattice ordering, for all m and n. Perhaps not surprisingly, these lattices enjoy many "nice" combinatorial properties; for example, they are rank-sym-

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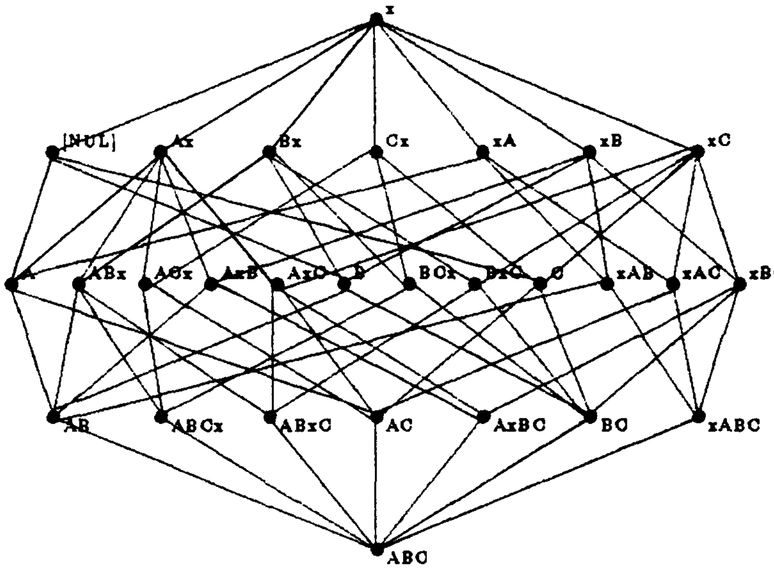

FIG. 1. The lattice $\mathscr{W}_{3,1}$.

metric (but not self-dual!), shellable, strongly Sperner, and have symmetric chain decompositions. We will prove these facts and derive formulas and generating functions for many familiar combinatorial invariants such as the number of elements, the rank generating function, the number of maximal chains, the Möbius function, the zeta polynomial, and the characteristic polynomial. Surprisingly, all of these invariants can be expressed in a simple way using evaluations of a certain family of polynomials $\Phi_{m,n}(x)$. The polynomials $\Phi_{m,n}(x)$ are in turn related to classical Jacobi polynomials, and as a consequence, many results from the theory of orthogonal polynomials can be applied to give useful information about the posets $\mathscr{W}_{m,n}$.

We first became interested in these posets as a (very) idealized model of some situations which arise in mathematical biology [10, 5]. The author thanks M. Waterman and L. Gordon for stimulating discussions during the development of this paper, and for generously sharing their computing resources during the author's visit at U.S.C.

2. $\mathscr{W}_{m,n}$ IS A LATTICE

It is trivial that $\mathscr{W}_{m,n}$ is a ranked poset, with rank function given by

$$\operatorname{rank}(\mathbf{w}) = \operatorname{length}(\mathbf{w} \mid \mathbf{y}) + (m - \operatorname{length}(\mathbf{w} \mid \mathbf{x})).$$

It is also clear that $\mathbf{x}$ is the unique minimal element, and $\mathbf{y}$ is the unique maximal element of $\mathscr{W}_{m,n}$. We assume that the letters of $\mathbf{x}$ and $\mathbf{y}$ are linearly ordered, and write $x < x'$, for example, if x precedes x' in $\mathbf{x}$.

THEOREM 2.1. *For any m and n, $\mathscr{W}_{m,n}$ is a lattice.*

Proof. Let $\mathbf{u}$ and $\mathbf{v}$ be arbitrary elements of $\mathscr{W}_{m,n} = \mathscr{W}_{\mathbf{x},\mathbf{y}}$. We will show how to construct $\mathbf{u} \vee \mathbf{v}$. First delete from both $\mathbf{u}$ and $\mathbf{v}$ all those letters x in $\mathbf{x}$ such that either

(1) $x \in \bar{\mathbf{u}} \cup \bar{\mathbf{v}} - \bar{\mathbf{u}} \cap \bar{\mathbf{v}}$, or

(2) $x \in \bar{\mathbf{u}} \cap \bar{\mathbf{v}}$ and there exist letters $y \leqslant y'$ in $\mathbf{y}$ such that y' precedes x in one of the words, and y follows x in the other.

If $\mathbf{u}_0$ and $\mathbf{v}_0$ denote the resulting words, we may write

$$\mathbf{u}_0 = \alpha_1 \xi_1 \alpha_2 \xi_2 \cdots \alpha_k \xi_k \alpha_{k+1}$$

$$\mathbf{v}_0 = \beta_1 \xi_1 \beta_2 \xi_2 \cdots \beta_k \xi_k \beta_{k+1},$$

where $\xi_1, \xi_2, ..., \xi_k$ are letters in $\mathbf{x}$, $\alpha_1, \alpha_2, ..., \alpha_{k+1}$ and $\beta_1, \beta_2, ..., \beta_{k+1}$ are subwords (possibly empty) of $\mathbf{y}$, and α_i and β_j have no common letters if $i \neq j$. For each i, let $\alpha_i \vee \beta_i$ denote the unique shortest subword of y which contains both α_i and β_i, and define

$$\mathbf{w} = (\alpha_1 \vee \beta_1)\, \xi_1 (\alpha_2 \vee \beta_2)\, \xi_2 \cdots \xi_{k-1} (\alpha_k \vee \beta_k)\, \xi_k (\alpha_{k+1} \vee \beta_{k+1}).$$

We claim that $\mathbf{w} \in \mathscr{W}_{\mathbf{x},\mathbf{y}}$ and $\mathbf{w} = \mathbf{u} \vee \mathbf{v}$. Clearly $\mathbf{w} \mid \mathbf{x} \leqslant \mathbf{x}$, and the deletions in step (2) above imply $\mathbf{w} \mid \mathbf{y} \leqslant \mathbf{y}$. Hence $\mathbf{w} \in \mathscr{W}_{\mathbf{x},\mathbf{y}}$. It is immediate that $\mathbf{u} \preccurlyeq \mathbf{u}_0 \preccurlyeq \mathbf{w}$ and $\mathbf{v} \preccurlyeq \mathbf{v}_0 \preccurlyeq \mathbf{w}$. If $\mathbf{w}' \in \mathscr{W}_{\mathbf{x},\mathbf{y}}$ is any other upper bound for $\mathbf{u}$ and $\mathbf{v}$ in $\mathscr{W}_{\mathbf{x},\mathbf{y}}$, then $\mathbf{w}'$ must contain all of the letters in the subwords $\alpha_i \vee \beta_i$ defined above, and it follows that $(\alpha_1 \vee \beta_1) \cdots (\alpha_{k+1} \vee \beta_{k+1}) = \mathbf{w} \mid \mathbf{y} \leqslant \mathbf{w}' \mid \mathbf{y}$. Furthermore, $\mathbf{w}'$ cannot contain any of the letters x of $\mathbf{u} \mid \mathbf{x}$ and $\mathbf{v} \mid \mathbf{x}$ deleted in steps (1) and (2) above. Hence $\mathbf{w}' \mid \mathbf{x} \leqslant \mathbf{w} \mid \mathbf{x} = \xi_1 \xi_2 \cdots \xi_k$. Finally, $\mathbf{w}' \geqslant \mathbf{u}$ and $\mathbf{w}' \geqslant \mathbf{v}$ implies $\mathbf{w} \mid \mathbf{w}' = \mathbf{w}' \mid \mathbf{w}$, i.e., the letters in $\mathbf{w} \mid \mathbf{y}$ and $\mathbf{w}' \mid \mathbf{x}$ occur in the same order in both words. Hence $\mathbf{w} \preccurlyeq \mathbf{w}'$, and $\mathbf{w} = \mathbf{u} \vee \mathbf{v}$, as claimed. The construction of $\mathbf{u} \wedge \mathbf{v}$ may be carried out by interchanging the roles of $\mathbf{x}$ and $\mathbf{y}$. ∎

We illustrate the construction of $\mathbf{u} \vee \mathbf{v}$ and $\mathbf{u} \wedge \mathbf{v}$ with an example.

EXAMPLE 2.2. Let $\mathbf{x} = ABCDEFG$ and $\mathbf{y} = wxyz$. If

$$\mathbf{u} = xBCyzEF \qquad \mathbf{v} = ABwCyEzG$$

then

$$\mathbf{u} \vee \mathbf{v} = wxCyz \qquad \mathbf{u} \wedge \mathbf{v} = ABCyEFG$$

3. Combinatorial Invariants

We begin by introducing notation for some of the standard combinatorial invariants associated with $\mathscr{W}_{m,n}$. A more detailed introduction to these invariants for general posets can be found, for example, in [9].

Let $\Omega_{m,n}$ denote the number of elements in $\mathscr{W}_{m,n}$, and let $\Omega_{m,n}(q)$ denote the rank generating function of $\mathscr{W}_{m,n}$,

$$\Omega_{m,n}(q) = \sum_{\mathbf{w} \in \mathscr{W}_{m,n}} q^{\operatorname{rank}(\mathbf{w})}.$$

Thus in this notation, $\Omega_{m,n}(1) = \Omega_{m,n}$.

Let $C_{m,n}$ denote the number of maximal chains in $\mathscr{W}_{m,n}$, and let $\mu_{m,n}$ denote the value of the Möbius function $\mu(\mathbf{0}, \mathbf{1})$ in $\mathscr{W}_{m,n}$.

Let $Z_{m,n}(s)$ denote the zeta polynomial of $\mathscr{W}_{m,n}$. Recall that $Z_{m,n}(s)$ counts the number of multichains $\mathbf{0} \leqslant \mathbf{z}_1 \leqslant \mathbf{z}_2 \leqslant \cdots \leqslant \mathbf{z}_s = \mathbf{1}$ in $\mathscr{W}_{m,n}$. It is well known (see, for example, [9]) that

$$Z_{m,n}(-1) = \mu_{m,n} \tag{1}$$

and

$$Z_{m,n}(s) \sim C_{m,n} \frac{s^{m+n}}{(m+n)!} \tag{2}$$

as $s \to \infty$.

Finally, let $\chi_{m,n}(\lambda)$ denote the characteristic polynomial of $\mathscr{W}_{m,n}$,

$$\chi_{m,n}(\lambda) = \sum_{\mathbf{w} \in \mathscr{W}_{m,n}} \mu(\mathbf{0}, \mathbf{w})\, \lambda^{m+n-\operatorname{rank}(\mathbf{w})}.$$

Thus

$$\chi_{m,n}(0) = \mu_{m,n}. \tag{3}$$

Our goal is to obtain explicit formulas for each of these invariants. It will be convenient to introduce the following family of polynomials:

Definition 3.3. Let m and n be nonnegative integers. Define

$$\Phi_{m,n}(x) = \sum_{j \geqslant 0} \binom{m}{j} \binom{n}{j} x^j.$$

These polynomials are closely related to the Jacobi polynomials, which are defined (for example, in [6]) by the formula

$$P_n^{(\alpha,\beta)}(x) = 2^{-n} \sum_{j=0}^{n} \binom{n+\alpha}{j} \binom{n+\beta}{n-j} (x-1)^{n-j} (x+1)^j,$$

where n is a nonnegative integer and α, β are arbitrary complex numbers. A simple substitution yields the formula

$$\Phi_{m,n}(x) = (x-1)^n P_n^{(m-n,0)}\left(\frac{x+1}{x-1}\right) \tag{4}$$

when $m \geqslant n$.

THEOREM 3.4. *Let m and n be nonnegative integers. Then*

$$\Omega_{m,n} = 2^{m+n}\Phi_{m,n}(\tfrac{1}{4}) \tag{5}$$

$$\Omega_{m,n}(q) = (1+q)^{m+n}\,\Phi_{m,n}\left(\frac{q}{(1+q)^2}\right) \tag{6}$$

$$C_{m,n} = (m+n)!\,\Phi_{m,n}(\tfrac{1}{2}) \tag{7}$$

$$\mu_{m,n} = (-1)^{m+n}\binom{m+n}{n} = (-1)^{m+n}\,\Phi_{m,n}(1) \tag{8}$$

$$Z_{m,n}(s) = s^{m+n}\Phi_{m,n}\left(\frac{s-1}{2s}\right) \tag{9}$$

$$\chi_{m,n}(\lambda) = (\lambda-1)^{m+n}\,\Phi_{m,n}\left(\frac{1}{1-\lambda}\right). \tag{10}$$

The proof of Theorem 3.4 will begiven in the next two sections. It follows from (4) that each of the formulas in Theorem 3.4 has a corresponding restatement in terms of Jacobi polynomials. For example,

$$\Omega_{m,n} = 2^{m-n}(-3)^n P_n^{(m-n,0)}(-\tfrac{5}{3})$$

and

$$C_{m,n} = (m+n)!\,(-2)^{-n} P_n^{(m-n,0)}(-3).$$

4. Classification of Shuffles

Most of the results in the previous section can be obtained by algebraic manipulation of generating functions, although the derivations are not all straightforward. We will instead present a series of combinatorial arguments, based on a certain canonical decomposition of $\mathscr{W}_{m,n}$ into Boolean sublattices. From this decomposition it is easy to derive formulas (5) and (6), and many other facts also follow easily. The essential idea is to classify shuffles $\mathbf{w} \in \mathscr{W}_{m,n}$ according to certain adjacent pairs which we call the *interface* of $\mathbf{w}$.

DEFINITION 4.5. Given $\mathbf{w} \in \mathscr{W}_{x,y}$, the *interface* of $\mathbf{w}$ (denoted $\mathscr{I}(\mathbf{w})$), is the set of all letters x, y, where $x \in \mathbf{x}$ and $y \in \mathbf{y}$, and x immediately follows y in $\mathbf{w}$. The *residue* of $\mathbf{w}$ (denoted $\mathscr{R}(w)$) is the set of letters in $\mathbf{w}$ which remain after the interface has been deleted, i.e., $\mathscr{R}(\mathbf{w}) = \bar{\mathbf{w}} - \mathscr{I}(\mathbf{w})$.

For example, if $\mathbf{x} = ABCDEFGH$, $\mathbf{y} = uvwxyz$, and $\mathbf{w} = BuvCEFwGxyH$, then

$$\mathscr{I}(\mathbf{w}) = \{v, C, w, G, y, H\}$$

$$\mathscr{R}(\mathbf{w}) = \{B, u, E, F, x\}.$$

Further, define

$$\mathscr{I}_{\mathbf{x}}(\mathbf{w}) = \mathscr{I}(\mathbf{w}) \cap \bar{\mathbf{x}} \qquad \mathscr{I}_{\mathbf{y}}(\mathbf{w}) = \mathscr{I}(\mathbf{w}) \cap \bar{\mathbf{y}}$$

$$\mathscr{R}_{\mathbf{x}}(\mathbf{w}) = \mathscr{R}(\mathbf{w}) \cap \bar{\mathbf{x}} \qquad \mathscr{R}_{\mathbf{y}}(\mathbf{w}) = \mathscr{R}(\mathbf{w}) \cap \bar{\mathbf{y}}.$$

Note that $\mathscr{R}_{\mathbf{x}}(\mathbf{w}) \subseteq \bar{\mathbf{x}} - \mathscr{I}_{\mathbf{x}}(\mathbf{w})$ and $\mathscr{R}_{\mathbf{y}}(\mathbf{w}) \subseteq \mathbf{y} - \mathscr{I}_{\mathbf{y}}(\mathbf{w})$. The following lemma is essentially trivial, but crucial for our arguments. We omit the straightforward proof.

LEMMA 4.6. *A shuffle* $\mathbf{w} \in \mathscr{W}_{x,y}$ *is determined uniquely by its interface* $\mathscr{I}(\mathbf{w})$ *and residue* $\mathscr{R}(\mathbf{x})$. *In other words, if* $\mathbf{w}, \mathbf{w}' \in \mathscr{W}_{\mathbf{x},\mathbf{y}}$, *and* $\mathscr{I}(\mathbf{w}) = \mathscr{I}(\mathbf{w}')$ *and* $\mathscr{R}(\mathbf{w}) = \mathscr{R}(\mathbf{w}')$, *then* $\mathbf{w} = \mathbf{w}'$.

As an immediate consequence we obtain the following result, which is a restatement of formula (5).

COROLLARY 4.7. $\Omega_{m,n} = \sum_{k \geq 0} \binom{m}{k}\binom{n}{k} 2^{m+n-2k}$.

Proof. For each $k \geq 0$, there are $\binom{m}{k}\binom{n}{k}$ ways to choose an interface of size $2k$, and 2^{m+n-2k} ways to choose the residue from the remaining letters. By Lemma 4.6, these choices uniquely determine a word $w \in \mathscr{W}_{x,y}$, and Corollary 4.7 follows. ∎

These arguments actually yield much more. If S is a subset of $\bar{\mathbf{x}} \cup \bar{\mathbf{y}}$ such that $|S \cap \bar{\mathbf{x}}| = |S \cap \bar{\mathbf{y}}|$, define

$$\mathscr{W}_{m,n}[S] = \{\mathbf{w} \in \mathscr{W}_{m,n} \mid \mathscr{I}(\mathbf{w}) = S\}.$$

Then the elements of $\mathscr{W}_{m,n}[S]$ are in one-to-one correspondence with subsets $T \subseteq \bar{\mathbf{x}} \cup \bar{\mathbf{y}} - S$. More precisely, if we write

$$\hat{\mathscr{R}}_x(\mathbf{w}) = (\bar{\mathbf{x}} - \mathscr{I}_{\mathbf{x}}(\mathbf{w})) - \mathscr{R}_{\mathbf{x}}(\mathbf{w}) \subseteq \bar{\mathbf{x}} - \mathscr{I}_{\mathbf{x}}(\mathbf{w})$$

then $\mathbf{w}$ corresponds to the pair $(\hat{\mathscr{R}}_{\mathbf{x}}(\mathbf{w}), \mathscr{R}_{\mathbf{y}}(\mathbf{w}))$ in such a way that if $\mathbf{w}$, $\mathbf{w}' \in \mathscr{W}_{m,n}[S]$, then $\mathbf{w} \preccurlyeq \mathbf{w}'$ if and only if $\hat{\mathscr{R}}_{\mathbf{x}}(\mathbf{w}) \subseteq \hat{\mathscr{R}}_{\mathbf{x}}(\mathbf{w}')$ and $\mathscr{R}_{\mathbf{y}}(\mathbf{w}) \subseteq \mathscr{R}_{\mathbf{y}}(\mathbf{w}')$. Hence $\mathscr{W}_{m,n}[S]$ is order isomorphic to the Boolean lattice

$$\mathbf{2}^{\bar{\mathbf{x}} - \mathscr{I}_{\mathbf{x}}(\mathbf{w})} \times \mathbf{2}^{\bar{\mathbf{y}} - \mathscr{I}_{\mathbf{y}}(\mathbf{w})} \approx \mathbf{2}^{\bar{\mathbf{x}} \cup \bar{\mathbf{y}} - \mathscr{I}(\mathbf{w})}.$$

If $|S| = 2k$ and $\mathbf{w} \in \mathscr{W}_{m,n}[S]$, then

$$\begin{aligned}\operatorname{rank}(\mathbf{w}) &= (m - |\mathscr{I}_{\mathbf{x}}(\mathbf{w})| - |\mathscr{R}_{\mathbf{x}}(\mathbf{w})|) + (|\mathscr{I}_{\mathbf{y}}(\mathbf{w})| + |\mathscr{R}_{\mathbf{y}}(\mathbf{w})|)\\ &= k + |\hat{\mathscr{R}}_{\mathbf{x}}(\mathbf{w})| + |\mathscr{R}_{\mathbf{y}}(\mathbf{w})|.\end{aligned}$$

It follows that if $\mathbf{w} \in \mathscr{W}_{m,n}[S]$, then $k \leqslant \operatorname{rank}(\mathbf{w}) \leqslant m + n - k$. Thus we can write $\mathscr{W}_{m,n}$ as a disjoint union

$$\mathscr{W}_{m,n} = \bigcup_{S} \mathscr{W}_{m,n}[S],$$

where each $\mathscr{W}_{m,n}[S]$ is a Boolean sublattice symmetric in rank about $(m+n)/2$. A number of results follow from these observations.

COROLLARY 4.8. $\Omega_{m,n}(q) = \sum_{k \geqslant 0} \binom{m}{k}\binom{n}{k} q^k (1+q)^{m+n-2k}$.

Proof. There are $\binom{m}{k}\binom{n}{k}$ ways to choose an interface S, and for each choice the words in $\mathscr{W}_{m,n}[S]$ contribute $q^k(1+q)^{m+n-2k}$ to $\Omega_{m,n}(q)$. ∎

COROLLARY 4.9. *For any m, $n \geqslant 0$, $\mathscr{W}_{m,n}$ is rank-symmetric, i.e., the number of elements of rank j in $\mathscr{W}_{m,n}$ is equal to the number of elements of rank $m+n-j$, for $0 \leqslant j \leqslant m+n$.*

Surprisingly, in view of Corollary 4.9, the lattices $\mathscr{W}_{m,n}$ are not self-dual. For example, in Fig. 1 one can see that $\mathscr{W}_{3,1}$ contains two elements of rank three, each of which covers four elements of rank two. But there are no elements of rank one covered by four elements of rank two. Hence $\mathscr{W}_{3,1}$ has no dual automorphisms.

It is possible to define an explicit bijection between elements of ranks j and $m+n-j$ in $\mathscr{W}_{m,n}$. If $\mathbf{w} \in \mathscr{W}_{m,n}$, let $\mathbf{w}^*$ denote the unique word in $\mathscr{W}_{m,n}$ such that $\mathscr{I}(\mathbf{w}^*) = \mathscr{I}(\mathbf{w})$ and $\mathscr{R}(\mathbf{w}^*) = \bar{\mathbf{x}} \cup \bar{\mathbf{y}} - \mathscr{I}(\mathbf{w}) - \mathscr{R}(\mathbf{w})$. Then $\mathbf{w} \mapsto \mathbf{w}^*$ is an involution on $\mathscr{W}_{m,n}$ which maps elements of rank j bijectively onto elements of rank $m+n-j$.

It is well known [3] that every finite Boolean algebra of rank N can be partitioned into saturated chains which are symmetric in rank about $N/2$. Such a partion is called a *symmetric chain decomposition.* It is also well known (see, for example, [4]) that any ranked poset with a symmetric chain decomposition satisfies the *strong Sperner property*: the maximum number of elements in a subset containing no chains of length $k+1$ is equal to the number of elements in the k largest ranks.

COROLLARY 4.10. *For any m, $n \geqslant 0$, the lattice $\mathscr{W}_{m,n}$ has a symmetric chain decomposition. Hence $\mathscr{W}_{m,n}$ satisfies the strong Sperner property.*

Proof. Each sublattice $\mathscr{W}_{m,n}[S]$ has a decomposition into symmetric chains, and each of these chains is symmetric in $\mathscr{W}_{m,n}$. ∎

It is possible to give a simple direct description of the symmetric chain decomposition of $\mathscr{W}_{m,n}$. Given $\mathbf{w} \in \mathscr{W}_{m,n}$, construct an $m+n$-tuple $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_{\mathbf{w}}$ with entries in $\{0, 1, *\}$, as follows: for each i, $1 \leqslant i \leqslant m$,

$$\varepsilon_i = \begin{cases} * & \text{if } \; x_i \in \mathscr{I}(\mathbf{w}) \\ 0 & \text{if } \; x_i \in \mathscr{R}(\mathbf{w}) \\ 1 & \text{otherwise} \end{cases}$$

and for each j, $1 \leqslant j \leqslant n$,

$$\varepsilon_{m+j} = \begin{cases} * & \text{if } \; y_j \in \mathscr{I}(\mathbf{w}) \\ 1 & \text{if } \; y_j \in \mathscr{R}(\mathbf{w}) \\ 0 & \text{otherwise.} \end{cases}$$

For example, in the example following Definition 4.5,

$$\begin{array}{cccccccccccccccc} & A & B & C & D & E & F & G & H & u & v & w & x & y & z \\ \boldsymbol{\epsilon} = & 1 & 0 & * & 1 & 0 & 0 & * & * & 1 & * & * & 1 & * & 0 \end{array}$$

Next pair up the 0's and 1's in $\boldsymbol{\epsilon}$, using the familiar "bracketing" rule (originally in [3], but rediscovered and studied by many authors). According to this rule, one iteratively brackets 0–1 pairs which are adjacent, or separated by $*$'s, or by other bracketed 0–1 pairs. An example is illustrated in the diagram above. Given a word $\mathbf{w}$, define the *bracket basis* $\mathscr{B}$ $(\mathbf{w})$ of $\mathbf{w}$ to be the set of letters in $\bar{\mathbf{x}} \cup \bar{\mathbf{y}}$ which correspond to bracketed 1's in $\boldsymbol{\epsilon}$. For example, if $\mathbf{w}$ is defined as in the example illustrated above, then $\mathscr{B}(\mathbf{w}) = \{D, u, x\}$. It is easy to see that $\boldsymbol{\epsilon}$ is uniquely determined by $\mathscr{I}(\mathbf{w})$ and $\mathscr{B}(\mathbf{w})$, and it is also clear that if $\mathbf{w}, \mathbf{w}' \in \mathscr{W}_{m,n}$ are such that $\mathscr{I}(\mathbf{w}) = \mathscr{I}(\mathbf{w}')$ and $\mathscr{B}(\mathbf{w}) = \mathscr{B}(\mathbf{w}')$, then either $\mathbf{w} \leqslant \mathbf{w}'$ or $\mathbf{w}' \leqslant \mathbf{w}$. For S, $B \subseteq \bar{\mathbf{x}} \cup \bar{\mathbf{y}}$, let

$$\mathscr{C}(S, B) = \{\mathbf{w} \in \mathscr{W}_{m,n} \mid \mathscr{I}(\mathbf{w}) = S, \; \mathscr{B}(\mathbf{w}) = B\}.$$

Then $\mathscr{C}(S, B)$ is a symmetric chain (possibly empty), and these chains decompose $\mathscr{W}_{m,n}$.

5. Recursions

In this section we will complete the proof of Theorem 3.4, by showing that each of the functions $C_{m,n}$, $\mu_{m,n}$, $Z_{m,n}(x)$, and $\chi_{m,n}(\lambda)$ satisfies an appropriate recursion. These recursions, as well as related formulae for

$\Phi_{m,n}(x)$, $\Omega_{m,n}$, and $\Omega_{m,n}(q)$ are of some independent combinatorial interest.

LEMMA 5.11. *For any $m, n \geqslant 1$,*

$$\Phi_{m,n}(x) = \Phi_{m-1,n}(x) + \Phi_{m,n-1}(x) - (1-x)\,\Phi_{m-1,n-1}(x). \tag{11}$$

Proof. This is essentially equivalent to the binomial identity

$$\binom{m}{k}\binom{n}{k} = \binom{m}{k}\binom{n-1}{k} + \binom{m-1}{k}\binom{n}{k} - \binom{m-1}{k}\binom{n-1}{k} + \binom{m-1}{k-1}\binom{n-1}{k-1} \tag{12}$$

which is easy to verify directly. Alternatively, one can interpret the left-hand side as counting pairs of sets (S, T), where $S \subseteq \{1, 2, ..., m\}$ and $T \subseteq \{1, 2, ..., n\}$, with $|S| = |T| = k$. On the right side, $\binom{m-1}{k-1}\binom{n-1}{k-1}$ counts pairs such that $m \in S$ and $n \in T$, while $\binom{m}{k}\binom{n-1}{k} + \binom{m-1}{k}\binom{n}{k} - \binom{m-1}{k}\binom{n-1}{k}$ counts the pairs such that $m \notin S$ or $n \notin T$. ∎

COROLLARY 5.12. *For any $m, n \geqslant 1$,*

$$\Omega_{m,n} = 2\Omega_{m-1,n} + 2\Omega_{m,n-1} - 3\Omega_{m-1,n-1} \tag{13}$$

and

$$\Omega_{m,n}(q) = (1+q)\,\Omega_{m-1,n}(q) + (1+q)\,\Omega_{m,n-1}(q) - (1+q+q^2)\,\Omega_{m-1,n-1}(q). \tag{14}$$

Proof. These follow from (5) and (6), after substituting $x = \frac{1}{4}$ and $x = q/(1+q)^2$, respectively, into formula (11). ∎

It is interesting to give a direct combinatorial proof of (13) and (14). For this we need the following terminology:

DEFINITION 5.13. Let $\mathbf{w} \in \mathscr{W}_{x,y}$. A letter $z \in \bar{\mathbf{x}} \cup \bar{\mathbf{y}}$ is said to be *bound* in $\mathbf{w}$ if $z \in \mathscr{I}(\mathbf{w})$, and *free* otherwise.

Alternate Proof of Corollary 5.12. In formula (14), $(1+q)\,\Omega_{m-1,n}(q)$ enumerates (by rank) words $\mathbf{w} \in \mathscr{W}_{m,n}$ such that x_m is free in $\mathbf{w}$. Every $\mathbf{w}' \in \mathscr{W}_{m-1,n}$ yields two such words, obtained by either adding or not adding x_m. Similarly, $(1+q)\,\Omega_{m,n-1}(q)$ enumerates words in which y_n is free, and $(1+q)^2\,\Omega_{m-1,n-1}(q)$ enumerates words in which both x_m and y_n are free. Hence the enumerator for words in which at least one of x_m or y_n is free is

$$(1+q)\,\Omega_{m-1,n}(q) + (1+q)\,\Omega_{m,n-1}(q) - (1+q)^2\,\Omega_{m-1,n-1}(q)$$

Finally,

$$q\Omega_{m-1,n-1}(q)$$

enumerates words in which both x_m and y_n are bound. Adding the last two expressions gives (14), and setting $q=1$ gives (13). ▮

Next we derive a recursion for the number $Z_{m,n}(s)$ of multichains of length s in $\mathscr{W}_{m,n}$, from which formulas (7), (8), and (9) will follow.

LEMMA 5.14. *For any m, $n \geqslant 1$,*

$$Z_{m,n}(s) = sZ_{m-1,n}(s) + sZ_{m,n-1}(s) - \binom{s+1}{2} Z_{m-1,n-1}(s). \qquad (15)$$

Proof. We wish to count multichains $\mathbf{x} = \mathbf{z}_0 \leqslant \mathbf{z}_1 \leqslant \cdots \leqslant \mathbf{z}_s = \mathbf{y}$ of length s in $\mathscr{W}_{m,n}$. Let us call such a chain $\mathbf{x}$-*terminal* if for each $i = 0, 1, ..., s$, the letter x_m is either not present in $\mathbf{z}_i$, or is the last letter of $\mathbf{z}_i$. Similarly, let us call a chain $\mathbf{y}$-*terminal* if y_n is either not present in $\mathbf{z}_i$, or is the last letter of $\mathbf{z}_i$, for each i. Note that a chain can be both $\mathbf{x}$-terminal and $\mathbf{y}$-terminal. It is easy to see that every multichain must be either $\mathbf{x}$-terminal or $\mathbf{y}$-terminal. We claim that the number of $\mathbf{x}$-terminal multichains of length s in $\mathscr{W}_{m,n}$ is equal to $sZ_{m-1,n}(s)$. Given a multichain $\mathbf{z}_0 \leqslant \mathbf{z}_1 \leqslant \cdots \leqslant \mathbf{z}_s$ in $\mathscr{W}_{m-1,n}$ and an integer i, with $0 \leqslant i \leqslant s-1$, one can construct an $\mathbf{x}$-terminal multichain in $\mathscr{W}_{m,n}$ by adding x_m at the end of each word $\mathbf{z}_j$, $0 \leqslant j \leqslant i$. The correspondence is one-to-one, and this proves the claim. By a similar argument, $sZ_{m,n-1}(s)$ counts the number of $\mathbf{y}$-terminal multichains in $\mathscr{W}_{m,n}$, and $\binom{s+1}{2}$ $Z_{m-1,n-1}(s)$ counts the number of multichains which are both $\mathbf{x}$-terminal and $\mathbf{y}$-terminal. The lemma follows immediately. ▮

COROLLARY 5.15.

$$Z_{m,n}(s) = s^{m+n}\Phi_{m,n}((s-1)/2s). \qquad (16)$$

Proof. Replacing x by $(s-1)/2s$ in (11) and multiplying by s^{m+n} shows that the right side of (16) satisfies recursion (15). Furthermore, (16) holds trivially when $m=0$ or $n=0$, and hence for all m, n, by induciton. ▮

COROLLARY 5.16. $C_{m,n} = (m+n)!\, \Phi_{m,n}(\frac{1}{2})$.

Proof. Using (2) and (16) we have

$$\begin{aligned} C_{m,n} &= \lim_{s \to \infty} \frac{(m+n)!}{s^{m+n}} Z_{m,n}(s) \\ &= \lim_{s \to \infty} (m+n)!\, \Phi_{m,n}\left(\frac{s-1}{2s}\right) \\ &= (m+n)!\, \Phi_{m,n}(\tfrac{1}{2}). \quad ▮ \end{aligned}$$

Using Corollary 5.16 and (11), it is easy to show that

$$C_{m,n} = (m+n)\, C_{m-1,n} + (m+n)\, C_{m,n-1} - \binom{m+n}{2} C_{m-1,n-1}. \quad (17)$$

There exists a combinatorial proof of formula (17) along the lines of Lemma 5.14, but we leave its discovery as an exercise for the reader.

COROLLARY 5.17. $\mu_{m,n} = (-1)^{m+n} \binom{m+n}{m}$.

Proof. Set $s = -1$ in (16). ∎

LEMMA 5.18.

$$\chi_{m,n}(\lambda) = (\lambda - 1)^{m+n}\, \Phi_{m,n}(1/(1-\lambda)). \quad (18)$$

Proof. It is an easy consequence of Weisner's formula (see [7]) that if L is any lattice with a rank function, then

$$\chi_L(\lambda) = \lambda \chi_{[\mathbf{0},\mathbf{b}]}(\lambda) - \sum_{\substack{\mathbf{z} > \mathbf{0} \\ \mathbf{z} \wedge \mathbf{b} = \mathbf{0}}} \chi_{[\mathbf{z},\mathbf{1}]}(\lambda), \quad (19)$$

where $\mathbf{b} > \mathbf{0}$ is an element of L. If $\mathbf{x} = x_1 x_2 \cdots x_m$ and $\mathbf{y} = y_1 y_2 \cdots y_n$, and we take $\mathbf{b} = y_1 y_2 \cdots y_{n-1}$, it is not difficult to see that $\mathbf{z} \wedge \mathbf{b} = \mathbf{0}$ (with $\mathbf{z} > \mathbf{0}$) if and only if $\bar{\mathbf{z}} \cap \bar{\mathbf{y}} = \{y_n\}$ and $\bar{\mathbf{z}} \cap \bar{\mathbf{x}} = \bar{\mathbf{x}}$. In other words, $\mathbf{z}$ is obtained by inserting y_n somewhere in $\mathbf{x}$. If y_n is followed in $\mathbf{z}$ by i letters of $\mathbf{x}$, the interval $[\mathbf{z}, \mathbf{1}]$ is isomorphic to $\mathcal{W}_{m-1,n-1} \times \mathcal{W}_{i,0}$. Hence

$$\chi_{[\mathbf{z},\mathbf{1}]}(\lambda) = (-1)^i \chi_{m-i,n-1}(\lambda)$$

and we have

$$\chi_{m,n}(\lambda) = \lambda \chi_{m,n-1}(\lambda) + \sum_{i=0}^{m} (-1)^i \chi_{m-i,n-1}(\lambda).$$

It is not difficult to show that the expression on the right-hand side of (18) satisfies this recursion, and the lemma follows by induction. ∎

6. GENERATING FUNCTIONS

In this section we derive several generating functions for the combinatorial invariants discussed in Sections 3–5. All of these results follow readily once we compute generating functions for the family of polynomials $\Phi_{m,n}(x)$.

THEOREM 6.19.

$$\sum_{m,n\geqslant 0} \Phi_{m,n} u^m v^n = \frac{1}{1-u-v+(1-x)\,uv}. \tag{20}$$

Proof. This is an immediate consequence of Lemma 11.

THEOREM 6.20. *For any integer $\delta \geqslant 0$,*

$$\sum_{n\geqslant 0} \Phi_{n+\delta,n}(x)\,z^n = 2^\delta R^{-1}(1-z+R)^{-\delta} \tag{21}$$

where

$$R = \sqrt{1-2z(x+1)+z^2(x-1)^2}. \tag{22}$$

Proof. Here we invoke a classical result from the theory of Jacobi polynomials (see [6]), which states that for all α, β,

$$\sum_{n\geqslant 0} P_n^{(\alpha,\beta)}(x)\,z^n = 2^{\alpha+\beta}R^{-1}(1-z+R)^{-\alpha}\,(1+z+R)^{-\beta},$$

where

$$R = \sqrt{1-2xz+z^2}.$$

Since by formula (4) we have

$$\Phi_{n+\delta,n}(x) = (x-1)^n\,P_n^{(\delta,0)}\left(\frac{x+1}{x-1}\right)$$

it follows that

$$\begin{aligned}\sum_{n\geqslant 0} \Phi_{n+\delta,n}(x)\,z^n &= \sum_{n\geqslant 0} P_n^{(\delta,0)}\left(\frac{x+1}{x-1}\right)((x-1)\,z)^n \\ &= 2^\delta R^{-1}(1-z+R)^{-\delta},\end{aligned}$$

where

$$R = \sqrt{1-2((x+1)/(x-1))(x-1)\,z+((x-1)\,z)^2}$$

which simplifies to (22). ∎

When $\delta = 0$ the generating function in Theorem 6.20 has an especially simple form, which is related to the familiar generating function for Legendre polynomials [6].

COROLLARY 6.21.

$$\sum_{n\geqslant 0} \Phi_{n,n}(x)\, z^n = \frac{1}{\sqrt{1-2z(x+1)+z^2(x-1)^2}}. \tag{23}$$

Next we specialize (20) to each of the invariants which appear in the statement of Theorem 3.4.

COROLLARY 6.22.

$$\sum_{m,n\geqslant 0} \Omega_{m,n} u^m v^n = \frac{1}{1-2u-2v+3uv} \tag{24}$$

$$\sum_{m,n\geqslant 0} \Omega_{m,n}(q)\, u_m v^n = \frac{1}{1-(1+q)\,u-(1+q)\,v+(1+q+q^2)\,uv} \tag{25}$$

$$\sum_{m,n\geqslant 0} \frac{C_{m,n}}{(m+n)!} u^m v^n = \frac{1}{1-u-v+(1/2)\,uv} \tag{26}$$

$$\sum_{m,n\geqslant 0} Z_{m,n}(s)\, u^m v^n = \frac{1}{1-su-sv+\binom{s+1}{2}\,uv} \tag{27}$$

$$\sum_{m,n\geqslant 0} \mu_{m,n} u^m v^n = \frac{1}{1+u+v} \tag{28}$$

$$\sum_{m,n\geqslant 0} \chi_{m,n}(\lambda)\, u^m v^n = \frac{1}{1-(\lambda-1)\,u-(\lambda-1)\,v+\lambda(\lambda-1)\,uv}. \tag{29}$$

Theorem 6.20 and Corollary 6.21 can be applied in a similar fashion. For simplicity, we will state the formulas only when $\delta=0$.

COROLLARY 6.23.

$$\sum_{n\geqslant 0} \Omega_{n,n} z^n = \frac{1}{\sqrt{1-10z+9z^2}} = (1-z)^{-1/2}\,(1-9z)^{-1/2} \tag{30}$$

$$\sum_{n\geqslant 0} \Omega_{n,n}(q)\, z^n = \frac{1}{\sqrt{1-2(1+3q+3q^2)\,z+(1+q+q^2)^2\, z^2}} = (1-(1-\sqrt{q}+q)^2\, z)^{-1/2}\,(1-(1+\sqrt{q}+q)^2\, z)^{-1/2} \tag{31}$$

$$\sum_{n\geqslant 0} \frac{C_{n,n}}{2n!} z^n = \frac{1}{\sqrt{1-3z+\frac{1}{4}z^2}} \tag{32}$$

$$\sum_{n \geqslant 0} Z_{n,n}(s)\, z^n = \frac{1}{\sqrt{1 - s(3s-1)\, z + \binom{s+1}{2}^2 z^2}} \tag{33}$$

$$\sum_{n \geqslant 0} \mu_{n,n}\, z^n = \frac{1}{\sqrt{1-4z}} \tag{34}$$

$$\sum_{n \geqslant 0} \chi_{n,n}(\lambda)\, z^n = \frac{1}{\sqrt{1 - 2(\lambda-1)(\lambda-2)\, z + \lambda^2(\lambda-1)^2 z^2}}. \tag{35}$$

7. Supersolvability

We conclude by showing that the lattices $\mathscr{W}_{m,n}$ are *supersolvable.* This property was first defined and studied by Richard Stanley in [8], and has many important combinatorial consequences. For example it follows that $\mathscr{W}_{m,n}$ is *shellable* [1]. We will not discuss the implications of these results in detail. The interested reader should consult [2] for a thorough survey, including many additional references.

We begin by recalling some basic definitions. If L is a ranked lattice, and element $X \in L$ is said to be *modular* if

$$r(X \vee Y) = r(X) + r(Y) - r(X \wedge Y)$$

for all $Y \in L$. If L has a $\mathbf{0}$ and $\mathbf{1}$, then L is *supersolvable* if there exists a maximal chain $\mathbf{0} < X_0 < X_1 < \cdots < X_N = \mathbf{1}$ such that each X_i is a modular element of L. Our first step will be to construct some modular elements in $\mathscr{W}_{m,n}$.

Definition 7.24. Let $\mathbf{u}$ and $\mathbf{v}$ be words in $\mathscr{W}_{m,n} = \mathscr{W}_{\mathbf{x},\mathbf{y}}$, and let $x \in \bar{\mathbf{u}} \cap \bar{\mathbf{v}} \cap \bar{\mathbf{x}}$. We say that x is *crossed* in $\mathbf{u}$ and $\mathbf{v}$ if there exist letters $y, y' \in$ in $\bar{\mathbf{u}} \cap \bar{\mathbf{v}} \cap \bar{\mathbf{y}}$ such that $y \leqslant y'$, and y' precedes x in one of the words and y follows x in the other. We say that $y \in \bar{\mathbf{u}} \cap \bar{\mathbf{v}} \cap \bar{\mathbf{y}}$ is crossed if the analogous condition holds with the roles of $\mathbf{x}$ and $\mathbf{y}$ reversed.

We note that x is crossed in $\mathbf{u}$ and $\mathbf{v}$ if x is one of the letters deleted during step (2) of the construction of $\mathbf{u} \vee \mathbf{v}$, described in Section 2. Similarly, y is crossed if it is deleted in step (2) of the construction of $\mathbf{u} \wedge \mathbf{v}$. In Example 2.2, the crossed letters are B and E in $\bar{\mathbf{x}}$ and z in $\bar{\mathbf{y}}$.

Lemma 7.25. *If* $\mathbf{u}$ and $\mathbf{v}$ *are elements of* $\mathscr{W}_{\mathbf{x},\mathbf{y}}$, *then*

$$r(\mathbf{u} \vee \mathbf{v}) = r(\mathbf{u}) + r(\mathbf{v}) - r(\mathbf{u} \wedge \mathbf{v}) + \Delta(\mathbf{u}, \mathbf{v})$$

where $\Delta(\mathbf{u}, \mathbf{v})$ *denotes the number of crossed x's minus the number of crossed y's.*

Proof. This is an immediate consequence of the construction of $\mathbf{u} \wedge \mathbf{v}$ and $\mathbf{u} \vee \mathbf{v}$ given in Section 2. One obtains

$$r(\mathbf{u} \vee \mathbf{v}) = r(\mathbf{u}) + r(\mathbf{v}) - r(\mathbf{u} \wedge \mathbf{v})$$

if no letters $x \in \mathbf{x}$ or $y \in \mathbf{y}$ are deleted by step (2) of that construction, and $\Delta(\mathbf{u}, \mathbf{v})$ adds the appropriate correction otherwise. ∎

COROLLARY 7.26. *Let* $\mathbf{u} \in \mathscr{W}_{\mathbf{x},\mathbf{y}}$ *be such that* $\mathbf{u} \leqslant \mathbf{x}$ *or* $\mathbf{u} \leqslant \mathbf{y}$. *Then* $\mathbf{u}$ *is a modular element of* $\mathscr{W}_{\mathbf{x},\mathbf{y}}$.

Proof. If $\mathbf{u} \leqslant \mathbf{x}$ or $\mathbf{u} \leqslant \mathbf{y}$, then $\Delta(\mathbf{u}, \mathbf{v})$ is always zero. ∎

THEOREM 7.27. *For all* $\mathbf{x}$, $\mathbf{y}$, $\mathscr{W}_{\mathbf{x},\mathbf{y}}$ *is supersolvable.*

Proof. Let $\mathbf{0} < \mathbf{u}_0 < \mathbf{u}_1 < \cdots < \mathbf{u}_{m+n} = \mathbf{1}$ be any maximal chain in $\mathscr{W}_{\mathbf{x},\mathbf{y}}$ obtained by deleting the letters of $\mathbf{x}$ in some order (obtaining the empty word), then adding the letters of $\mathbf{y}$ in some order. By Corollary 7.26 each of the $\mathbf{u}_i$'s is a modular element of $\mathscr{W}_{\mathbf{x},\mathbf{y}}$. Hence $\mathscr{W}_{\mathbf{x},\mathbf{y}}$ is supersolvable. ∎

We conclude by showing how Theorem 7.27 implies a result obtained by other means in Section 5. If L is a supersolvable lattice, then every maximal chain $\mathbf{0} < X_0 < X_1 < \cdots < X_N = \mathbf{1}$ of modular elements induces a labeling of the join-irreducible elements of L by positive integers, defined by

$$\alpha(p) = \min\{i \mid p \leqslant X_i\}$$

where p denotes a join-irreducible element of L. This in turn induces a labeling of covering pairs $U \prec V$ defined by

$$\lambda(U \prec V) = \min\{\alpha(p) \mid p \leqslant V, p \nleqslant U\}$$

A maximal chain $\mathbf{0} < Y_0 < Y_1 < \cdots < Y_N = \mathbf{1}$ in L is said to be *decreasing* if $\lambda(Y_i \prec Y_{i+1}) > \lambda(Y_{i+1} \prec Y_{i+2})$ for $i = 0, 1, ..., N-2$, i.e., the edge lables decrease. It is shown in [8] that if L is supersolvable, then the number of decreasing maximal chains in L is equal to $(-1)^N \mu_L(\mathbf{0}, \mathbf{1})$, a result which is independent of the chain of modular elements used to define the labels. These ideas are greatly extended in [1] to the class of shellable posets, which include supersolvable lattices as special case.

In $\mathscr{W}_{\mathbf{x},\mathbf{y}}$, consider the maximal chain $\mathbf{0} < \mathbf{u}_0 < \mathbf{u}_1 < \cdots < \mathbf{u}_{m+n} = \mathbf{1}$ obtained by deleting the letters $x_1, x_2, ..., x_m$ of $\mathbf{x}$, then adding the letters $y_1, y_2, ..., y_n$ of $\mathbf{y}$, each time in subscript order. It is not difficult to show that the corresponding edge-labels $\lambda(\mathbf{u} \prec \mathbf{v})$ are given by

$$\lambda(\mathbf{u} \prec \mathbf{v}) = \begin{cases} i & \text{if } \mathbf{v} \text{ is obtained from } \mathbf{u} \text{ by deleting } x_i \in \bar{\mathbf{x}} \\ m+j & \text{if } \mathbf{v} \text{ is obtained from } \mathbf{u} \text{ by adding } y_j \in \bar{\mathbf{y}}. \end{cases}$$

In the terminology of [1], this defines an explicit "EL-labeling" of $\mathscr{W}_{x,y}$ and can be used to give a simple direct proof that $\mathscr{W}_{x,y}$ is shellable.[1]

Finally we note that the decreasing maximal chains in $\mathscr{W}_{\mathbf{x},\mathbf{y}}$ are precisely those obtained as follows:

1. Start at the bottom with $\mathbf{x} = x_1 x_2 \cdots x_m$.

2. Add the letters $y_1, y_2, ..., y_n$ in reverse order (each y_i may, in general, be added in several places).

3. Then delete the letters x_1, x_2, ..., x_m in reverse order.

Clearly, such a chain is uniquely determined by the word obtained after step 2, which is a shuffle of **x** and **y**. Hence the decreasing maximal chains are in one-to-one correspondence with the shuffles of **x** and **y**, and we have again proved the following:

COROLLARY 7.28. $\mu_{m,n} = (-1)^{m+n} \binom{m+n}{m}$.

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[1] This remark is due to Paul Edelman.