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A HEURISTIC ALGORITHM FOR THE MULTI-CRITERIA SET-COVERING PROBLEMS

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Abstract—A simple greedy heuristic algorithm for the multi-criteria set-covering problem is presented. This result is a multi-criteria generalization of the results established previously by Chvatal.

1. INTRODUCTION

In the Multi-Criteria Set-Covering (MCSC) problem, we are given a finite collection $\mathcal{P} = \{P_j : 1 \le j \le n\}$ of subsets of $\mathcal{I} = \{1, 2, \dots, m\}$. We associate each $P_j \in \mathcal{P}$ with a nonnegative and nonzero cost k-vector $c_j^t = (c_{1j}, c_{2j}, \dots, c_{kj})$. A subcollection $\mathcal{P}^*$ of $\mathcal{P}$ is called a cover if $\bigcup_{P_j \in \mathcal{P}^*} P_j = \mathcal{I}$; the associated cost is a k-vector $\sum_{P_j \in \mathcal{P}^*} c_j$. The problem is to find an "efficient" cover. That is, to find a cover such that if any improvement of one objective function can be achieved it is only at the expense of another. Since it is NP-hard to find an efficient cover, a heuristic solution for (MCSC) is important. This note presents a heuristic algorithm to find a near optimal solution and a tight bound on the worst case of this relatively simple heuristic.

An (MCSC) has the following formulation:

$$\begin{aligned}
&\min \begin{cases} \sum_{j=1}^{n} c_{1j} x_j \\ \sum_{j=1}^{n} c_{2j} x_j \\ \quad \vdots \\ \sum_{j=1}^{n} c_{kj} x_j \end{cases} \\
&\text{s.t.} \sum_{j=1}^{n} a_{1j} x_j \ge 1 \\
&\qquad \vdots \\
&\qquad \sum_{j=1}^{n} a_{mj} x_j \ge 1, \qquad x_j \in \{0,1\}, \quad \forall j = 1, 2, \dots, n.
\end{aligned} \tag{MCSC}$$

Where

$$a_{ij} = \begin{cases} 1 & \text{if } i \in P_j, \\ 0 & \text{otherwise,} \end{cases}$$

and a solution

$$x_j = \begin{cases} 1 & \text{if } P_j \in \mathcal{P}^* \\ 0 & \text{otherwise.} \end{cases}$$

We associate a Boolean vector $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ with a subcollection $\bar{\mathcal{P}}$ of $\mathcal{P}$.

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A feasible solution $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ of (MCSC) is called an efficient solution [1] (Pareto optimal) if, and only if, there is no other feasible solution $x = (x_1, x_2, \ldots x_n)$ such that

$$\sum_{j=1}^{n} c_{ij} x_j \leq \sum_{j=1}^{n} c_{ij} \bar{x}_j, \qquad \text{for all } i = 1, 2, \ldots, k,$$

and

$$\sum_{j=1}^{n} c_{ij} x_j < \sum_{j=1}^{n} c_{ij} \bar{x}_j, \qquad \text{for at least one } i.$$

A cover $\bar{\mathcal{P}}$ is called an efficient cover if its associated vector $\bar{x}$ is efficient.

2. MAIN RESULTS

To develop a heuristic algorithm for finding a near optimal solution intuitively in the single criterion case, one would select P_j to cover j provided $|P_j|/c_j$, the number of points covered by P_j per unit cost, is maximized. However, in a multi-criteria case, c_j is a vector, and this intuition is not useful. In order to do this, we consider an alternative for a multi-criteria case by choosing $c_j^* = \max\{c_{ij} : i = 1, 2, \ldots, k\} > 0$. Due to the different scales used in each objective, this choice cannot be meaningful unless the coefficients of the objectives are normalized. To do this, without loss of generality, we may assume

$$\sum_{j=1}^{n} c_{ij} = 1, \qquad \text{for all } i = 1, 2, \ldots, k.$$

Then the ratio $|P_j|/c_j^*$ means the worst case of unit cost for covering j using P_j among all criteria. This leads to the following heuristic:

THE (MCSC) GREEDY HEURISTIC ALGORITHM.

Step 0

$\mathcal{P}^* \leftarrow \Phi$

Step 1

If $P_j = \Phi$, $\forall j$, stop.

$\mathcal{P}^*$ is a cover, otherwise, find P_k such that $|P_k|/c_k^* = \max\{|P_j|/c_j^* : P_j \in \mathcal{P} - \mathcal{P}^* \text{ and } c_j^* = \max_{1 \leq j \leq k} c_{ij}\}$.

Step 2

$\mathcal{P}^* \leftarrow \mathcal{P}^* \cup \{P_k\}$

$P_j \leftarrow P_j - P_k$, $\forall j \neq k$.

Go to Step 1.

Let $\mathcal{P}^*$ be a cover obtained from the above heuristic, $\bar{\mathcal{P}}$ be any efficient cover, and let x^* and $\bar{x}$ be the associated incident vectors of $\mathcal{P}^*$ and $\bar{\mathcal{P}}$, repectively. Also, let $M = \max_{1 \leq j \leq n} |P_j|$. Then in the single objective case (i.e., $k = 1$) the algorithm is Chvatal's greedy heuristic, and he has shown that "the cost of cover returned by the greedy heuristic is at most $\sum_{l=1}^{M} \frac{1}{l}$ times the cost of an optimal cover." [2] The following general theorem is now presented ($k \geq 1$).

THEOREM. *Let x^* be a solution obtained by the (MCSC) greedy heuristic and $\bar{x}$ be any efficient solution of (MCSC). Then*

$$\sum_{j=1}^{n} c_{ij} x_j^* \leq \left(\sum_{l=1}^{M} \frac{1}{l} \right) \left(\sum_{j=1}^{n} c_j^* \bar{x}_j \right), \qquad \forall i = 1, 2, \ldots, k.$$

PROOF. For each feasible $x = (x_1, x_2, \ldots, x_n)$, since $c_j^* = \max_{1 \le i \le k} c_{ij}$, we have

$$\sum_{j=1}^{n} c_{ij} x_j \le \sum_{j=1}^{n} c_j^* x_j.$$

Therefore, by Chvatal's theorem, for each $i = 1, 2, \ldots, k$,

$$\sum_{j=1}^{n} c_{ij} x_j^* \le \sum_{j=1}^{n} c_j^* x_j^* \le \left(\sum_{l=1}^{M} \frac{1}{l} \right) \left(\sum_{j=1}^{n} c_j^* \bar{x}_j \right).$$

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