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The Regular Ring of a Finite Baer *-Ring

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INTRODUCTION

The subject of this paper originates in the "rings of operators," "continuous geometries," and "regular rings" of von Neumann. Continuous geometries were invented by von Neumann to explain (and abstract) the projection lattices of certain rings of operators, and regular rings were invented to explain (and coordinatize) certain continuous geometries; thus, the place to begin is with rings of operators (or von Neumann algebras, as they are now called).

The classical facts that are pertinent here are that (1) the projection lattice of a finite von Neumann algebra is a continuous geometry, (2) nearly every continuous geometry (and every operatorially defined one) may be realized as the lattice of principal right ideals of a suitable regular ring, and (3) a finite von Neumann algebra can be embedded as a subring of the regular ring associated with its projection geometry. (For literature citations, see the introduction of [1].)

These themes were focused in work of Kaplansky, (a) in his theory of AW^*-algebras [6–8], which abstracted the algebraic features of von Neumann algebras having to do with their projection lattices, and (b) in his proof that certain lattices are automatically continuous geometries [9], in the course of which the theory of complete *-regular rings was developed as a basic tool. The same style of algebra—an idempotent- and annihilator-oriented algebra in rings with involution—underlies (a) and a substantial part of (b). The two enterprises were unified by Kaplansky in his theory of Baer *-rings; in this setting, the AW^*-algebras are the Baer *-rings that happen to be C^*-algebras, and the complete *-regular rings are the Baer *-rings that happen to be regular. Baer *-rings (and Baer rings, their noninvolutive generalizations) were developed in mimeographed lecture notes in 1955 [10].

Invented to explain operator algebras, AW^*-algebras have in fact been cultivated mainly for their own sake; the history of their literature suggests that operator-theorists found them too algebraic, and algebraists, too functional-analytic. Smaller yet is the literature of Baer *-rings, which are algebraically appealing enough but may be handicapped by their early association with C^*-algebras. At any rate, the publication of the elegantly refurbished second edition of Kaplansky's notes [11] completed the evolution of the subject into a chapter in pure ring theory. This left behind a few orphans in the literature of AW^*-algebras; in part, the aim of this paper is to find a new home for one of them, the theory of the regular ring [1].

More precisely, we show that a finite Baer *-ring A, satisfying suitable axioms, may be embedded in a regular Baer *-ring $\mathbf{C}$; the class of rings A for which the construction is successful may be described roughly as the finite Baer *-rings without "purely real" part [cf. 11, p. 130, Theorem A]. The regular ring $\mathbf{C}$ has the same projection lattice as A (hence is the regular ring whose existence is guaranteed by the theory of continuous geometries), and it inherits the properties hypothesized for A (with one striking, and necessary, exception—boundedness). A consequence (though not necessarily an advantage) is that the theory of these finite Baer *-rings is freed from the theory of continuous geometries. As an application, we advance (but do not completely solve) the problem of $n \times n$ matrices over A. The construction of $\mathbf{C}$ is exactly the same as for the case of finite AW^*-algebras [1], but the development of its properties has to be rearranged completely and the use of "spectral theory" attenuated and delayed as much as possible. The outline of the paper is as follows:

(1) Preliminaries

(2) Construction of $\mathbf{C}$

(3) $\mathbf{C}$ is a finite Baer *-ring with generalized comparability, satisfying the parallelogram law

(4) $\mathbf{C}$ has no new partial isometries

(5) Positivity in $\mathbf{C}$

(6) Cayley transform

(7) Regularity of $\mathbf{C}$

(8) Spectral theory in $\mathbf{C}$

(9) $\mathbf{C}$ has no new bounded elements

(10) $n \times n$ matrices

(11) Problems

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research leave at Indiana University during which this work was done. This paper is dedicated to the memory of David Topping.

1. Preliminaries

Our principal references are [1] and [11]. We review here some of the basic definitions. In a ring A with involution (briefly, a *-ring), a *projection* is a self-adjoint idempotent ($e^* = e = e^2$). Projections e, f are called *equivalent* (in [11] this is called "*-equivalence"), written $e \sim f$, if there exists $w \in A$ with $w^*w = e$, $ww^* = f$; w is called a *partial isometry*, with initial projection e and final projection f. For projections e, f, the relation $e = ef$ is denoted $e \leqslant f$ and is a partial ordering; $e \lesssim f$ means that $e \sim e'$ for some $e' \leqslant f$. An element $x \in A$ is *normal* if $x^*x = xx^*$. In a *-ring with unity, an element u is an *isometry* if $u^*u = 1$, and *unitary* if it is an invertible isometry ($uu^* = u^*u = 1$); A is called *finite* if every isometry is unitary (equivalently, $e \sim 1$ implies $e = 1$).

A *Baer *-ring* is a *-ring A such that, for every subset S, the right-annihilator of S is the principal right ideal generated by a projection, that is, $R(S) = gA$ for a suitable projection g. A *Rickart *-ring* [13] is a *-ring in which this is assumed only for singletons $S = \{x\}$. A Rickart *-ring A has a unity element (take $S = \{0\}$) hence for each x one has $R(\{x\}) = (1 - e)A$ for a unique projection e; e is characterized by the properties (1) $xe = x$, and (2) $xy = 0$ implies $ey = 0$; it is called the *right projection* of x, written $e = RP(x)$; similarly, $f = LP(x) = RP(x^*)$ is the unique projection with $A(1 - f) = L(\{x\})$ (the left annihilator of x). If w is a partial isometry in a Rickart *-ring, then $w^*w = RP(w)$, $ww^* = LP(w)$. The projections in a Rickart *-ring form a lattice, with

$$e \cup f = f + RP[e(1 - f)], \qquad e \cap f = e - LP[e(1 - f)]$$

[6, Lemma 5.3; 13]. The Baer *-rings are the Rickart *-rings with complete projection lattices [13, Lemma 6.1]. A Rickart *-ring is said to satisfy the *parallelogram law* if

$$e \cup f - f \sim e - e \cap f \tag{P}$$

for every pair of projections e, f.

We write Z for the center of a *-ring A, and call its elements central. Projections e, f are said to be *generalized comparable* (GC) if there exists a central projection h such that

$$he \lesssim hf, \qquad (1 - h)f \lesssim (1 - h)e;$$

A is said to have GC if every pair of projections is GC.

If A is a Baer *-ring, its central projections form a complete Boolean algebra [11, p. 30, Corollary]. We write $\mathscr{X}$ for the Stone representation space of this Boolean algebra, and we identify a central projection h with the characteristic function of the corresponding clopen subset of $\mathscr{X}$. We write $C(\mathscr{X})$ for the continuous complex-valued functions on $\mathscr{X}$ (but actually make use only of continuous functions with real values between 0 and 1).

*For the rest of the paper, A denotes a finite Baer *-ring.*

PROPOSITION 1.1. *If A is a finite Baer *-ring such that $LP(x) \sim RP(x)$ for all $x \in A$, then* (1) *A satisfies the parallelogram law* (P), (2) *A has* GC, (3) *A has a unique finite dimension function D, and D is completely additive,* (4) *$D(e \cup f) + D(e \cap f) = D(e) + D(f)$ for all projections e, f, and* (5) *$yx = 1$ implies $xy = 1$.*

Proof. (1) Apply the hypothesis on A to $x = e(1 - f)$.

(2) Note first that if e, f are projections with $eAf \neq 0$, then there exist nonzero subprojections e_0, f_0 such that $e_0 \sim f_0$ (if $exf \neq 0$, let $e_0 = LP(exf)$ and $f_0 = RP(exf)$). Suppose now that e, f are arbitrary projections and set $e' = LP(ef)$, $f' = RP(ef)$, $e'' = e - e'$, $f'' = f - f'$; then $e = e' + e''$, $f = f' + f''$ with $e' \sim f'$ and $e''f = ef'' = 0$. Since $e''f'' = 0$, it follows from the above remark that e'', f'' are GC [11, Theorem 35], hence so are e, f.

(3) A finite dimension function for A is a function D defined on the projection lattice of A, with values in $C(\mathscr{X})$, such that (i) $e \sim f$ implies $D(e) = D(f)$, (ii) $D(e) \geqslant 0$, (iii) $D(h) = h$ when h is central, and (iv) $D(e + f) = D(e) + D(f)$ when e, f are orthogonal. It can be shown that every finite Baer *-ring with GC possesses a unique finite dimension function D, and that D is completely additive: if $(e_\iota)_{\iota \in I}$ is an orthogonal family of projections with $e = \sup e_\iota$, then $D(e) = \sum_{\iota \in I} D(e_\iota)$ (the supremum of the finite sums, calculated in the boundedly complete lattice of real-valued continuous functions on the Stone space $\mathscr{X}$). {Alternative paths to D: The parallelogram law can be used to prove that the projection lattice of A is a continuous geometry [11, Theorem 69]. See also [12].}

(4) Immediate from the parallelogram law.

(5) If $yx = 1$ then $RP(x) = 1$, therefore $LP(x) \sim RP(x) = 1$; by finiteness, $LP(x) = 1$. Then $(1 - xy)x = x - x(yx) = x - x = 0$, and $1 - xy = 0$ results from $LP(x) = 1$. ∎

The notation $e_n \uparrow$ means that e_n is an increasing sequence of projections; if, in addition, $e = \sup e_n$, we write $e_n \uparrow e$, and it follows from complete additivity that $\sup D(e_n) = D(e)$, briefly $D(e_n) \uparrow D(e)$. A key application: If $e_n \uparrow 1$, $f_n \sim e_n$ for all n, and $f_n \uparrow$, then $f_n \uparrow 1$.

2. Construction of C

We assume in this section that A is any finite Baer *-ring such that $LP(x) \sim RP(x)$ for all $x \in A$.

The construction of **C** is identical with that in [1], so we suppress nearly all of the details; for convenient reference, we repeat here the key definitions. The motivation behind the definitions, and the peculiar terminology, comes from the theory of von Neumann algebras; we refer the reader to [1] for the rationale.

Definition 2.1. A *strongly dense domain* (SDD) in A is a sequence of projections (e_n) such that $e_n \uparrow 1$.

Lemma 2.2. *If* (e_n) *and* (f_n) *are* SDD, *then* $(e_n \cap f_n)$ *is an* SDD.

Lemma 2.3. *Let* (e_n) *be an* SDD *and let* $x \in A$. *If* $e_n x e_n = 0$ *for all* n, *then* $x = 0$. *If* $e_n x e_n$ *is self-adjoint for all* n, *then* x *is self-adjoint.*

For the proofs, see [1, Lemmas 1.1, 1.2 and Corollary 1.1].

Definition 2.4. An *operator with closure* (OWC) is a pair of sequences (x_n, e_n) with $x_n \in A$ and (e_n) an SDD, such that $m < n$ implies $x_n e_m = x_m e_m$ and $x_n^* e_m = x_m^* e_m$. If $x_n = x$ and $e_n = 1$ for all n, we write simply $(x, 1)$.

Lemma 2.5. *If* (x_n, e_n) *and* (y_n, f_n) *are* OWC, *then so are* (x_n^*, e_n) *and* $(x_n + y_n, e_n \cap f_n)$. *If, in addition,* A *is a* *-*algebra over an involutive field* F, *then* $(\lambda x_n, e_n)$ *is an* OWC *for every* $\lambda \in F$.

Definition 2.6. If $x \in A$ and e is a projection in A, we write $x^{-1}(e)$ for the largest projection g such that $(1 - e)xg = 0$; that is,

$$x^{-1}(e) = 1 - RP[(1 - e)x].$$

Lemma 2.7. *If* $x \in A$ *and* e *is any projection, then* $e \lesssim x^{-1}(e)$.

For the proof, see [1, Lemma 1.3].

Lemma 2.8. *Suppose* (x_n) *is a sequence in* A *and* (e_n) *is an* SDD *such that* $x_n e_m = x_m e_m$ *whenever* $m < n$. *If* (f_n) *is any* SDD *and if*

$$g_n = e_n \cap x_n^{-1}(f_n),$$

then (g_n) *is an* SDD.

Proof. Write $h_n = x_n^{-1}(f_n)$; thus $g_n = e_n \cap h_n$ and h_n is the largest projection such that

$$(1 - f_n)x_n h_n = 0. \qquad (*)$$

We first show that $g_n \uparrow$. If $m < n$ then $x_n g_m = x_n e_m g_m = x_m e_m g_m = x_m g_m = x_m h_m g_m$, therefore

$$(1 - f_n)x_n g_m = (1 - f_n)(1 - f_m)x_m h_m g_m = 0$$

by $(*)$; then $g_m \leqslant h_n$ by the maximality of h_n, and this, together with $g_m \leqslant e_m \leqslant e_n$, yields $g_m \leqslant e_n \cap h_n = g_n$.

Since $D(h_n) \geqslant D(f_n)$ by Lemma 2.7, the relation $1 - g_n = (1 - e_n) \cup (1 - h_n)$ yields, by (4) of Proposition 1.1,

$$D(1 - g_n) \leqslant D(1 - e_n) + D(1 - h_n) \leqslant D(1 - e_n) + D(1 - f_n),$$

therefore, $D(1 - g_n) \downarrow 0$, $g_n \uparrow 1$. ∎

LEMMA 2.9. *If (x_n, e_n) and (y_n, f_n) are* OWC, *and if*

$$k_n = [f_n \cap y_n^{-1}(e_n)] \cap [e_n \cap (x_n{}^*)^{-1}(f_n)],$$

then $(x_n y_n, k_n)$ is an OWC.

Proof. This follows routinely from Lemma 2.8 and Definition 2.6. ∎

DEFINITION 2.10. Two OWC (x_n, e_n), (y_n, f_n) are said to be *equivalent*, written $(x_n, e_n) \equiv (y_n, f_n)$, if there exists an SDD (g_n) such that $x_n g_n = y_n g_n$ and $x_n{}^* g_n = y_n{}^* g_n$ for all n. We then say that the equivalence is *implemented* via the SDD (g_n).

The relation $\equiv$ defined above is an equivalence relation in the set of all OWC, by a routine verification (transitivity depends on Lemma 2.2). The following elementary lemma is useful in the manipulation of representatives of equivalence classes:

LEMMA 2.11. (i) *If (x_n, e_n) is an* OWC *and (g_n) is any* SDD, *then $(x_n, e_n \cap g_n)$ is also an* OWC *and $(x_n, e_n) \equiv (x_n, e_n \cap g_n)$.*

(ii) *If $(x_n, e_n) \equiv (y_n, f_n)$ via an* SDD *(g_n), then, setting $h_n = e_n \cap f_n \cap g_n$, it follows that (x_n, h_n), (y_n, h_n) are* OWC *and $(x_n, h_n) \equiv (y_n, h_n)$ via (h_n).*

DEFINITION 2.12. We write $[x_n, e_n]$ for the equivalence class of an OWC with respect to the equivalence relation $\equiv$ defined above. The set of all equivalence classes is denoted **C**, and its elements are called *closed operators*

(CO); $[x_n, e_n]$ is the CO *determined* by the OWC (x_n, e_n). We denote the elements of $\mathbf{C}$ by boldface letters $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$. If $x \in A$ we write $\bar{x} = [x, 1]$ for the CO determined by $(x, 1)$, and we write $\bar{A} = \{\bar{x}: x \in A\}$ for the image of A in $\mathbf{C}$ under the injective (Lemma 2.3) mapping $x \mapsto \bar{x}$.

Lemmas 2.5 and 2.9 suggest algebraic operations for the OWC: $(x_n, e_n) + (y_n, f_n) = (x_n + y_n, e_n \cap f_n)$, $(x_n, e_n)^* = (x_n{}^*, e_n)$,

$$(x_n, e_n)(y_n, f_n) = (x_n y_n, k_n),$$

and, when A is a *-algebra over an involutive field F, $\lambda(x_n, e_n) = (\lambda x_n, e_n)$. Using Lemma 2.11, it is easy to check that these operations induce well-defined operations in $\mathbf{C}$: if $\mathbf{x} = [x_n, e_n]$ and $\mathbf{y} = [y_n, f_n]$, one defines $\mathbf{x} + \mathbf{y} = [x_n + y_n, e_n \cap f_n]$, $\mathbf{x}^* = [x_n{}^*, e_n]$, $\mathbf{xy} = [x_n y_n, k_n]$, and, when relevant, $\lambda\mathbf{x} = [\lambda x_n, e_n]$. Also, repeated use of Lemma 2.11 yields the algebraic properties of $\mathbf{C}$ asserted in the following theorem:

THEOREM 2.13. *Let A be a finite Baer *-ring such that $LP(x) \sim RP(x)$ for all $x \in A$.*

Define $\mathbf{C}$, and the operations on $\mathbf{C}$, as indicated above. Then (1) *$\mathbf{C}$ is a *-ring with unity $\bar{1}$ (and if A is a *-algebra over an involutive field F, then so is $\mathbf{C}$), and* (2) *the mapping $x \mapsto \bar{x}$ $(x \in A)$ is a *-isomorphism of A onto a *-subring $\bar{A}$ of $\mathbf{C}$.*

We write 1 for the unity element of $\mathbf{C}$, that is, we identify 1 with $\bar{1}$. But in general we refrain from identifying x with $\bar{x}$; there are conceptual advantages to maintaining the distinction between A and $\bar{A}$ until the properties of $\mathbf{C}$ have been fully developed.

Additional properties of $\mathbf{C}$ that require no further hypotheses on A are developed in the next section.

3. C Is a Finite Baer *-Ring with Generalized Comparability, Satisfying the Parallelogram Law

As in the preceding section, A is a finite Baer *-ring such that $LP(x) \sim RP(x)$ for all $x \in A$, and $\mathbf{C}$ is the ring constructed there. It is remarkable that the properties promised in the section heading are attainable without further hypotheses on A; this is a vast improvement over the techniques of [1].

LEMMA 3.1. *If $\mathbf{x} \in \mathbf{C}$, $\mathbf{x} = [x_n, e_n]$, and if (g_n), (h_n) are* SDD *such that $h_n x_n g_n = 0$ for all n, then $\mathbf{x} = 0$.*

Proof. Also $g_n x_n{}^* h_n = 0$. Set

$$k_n = g_n \cap [e_n \cap x_n^{-1}(h_n)] \cap h_n \cap [e_n \cap (x_n{}^*)^{-1}(g_n)];$$

(k_n) is an SDD (Lemmas 2.8 and 2.2), and it is routine to check that it implements $(x_n, e_n) = (0, 1)$. ▮

Proposition 3.2. *If* $\mathbf{x} = [x_n, e_n]$, $\mathbf{y} = [y_n, f_n]$, *and if* (g_n) *is an* SDD *such that*

$$x_n g_n = y_n g_n \qquad \textit{for all} \quad n,$$

then $\mathbf{x} = \mathbf{y}$. *In fact, it suffices to assume that* $h_n x_n g_n = h_n y_n g_n$ *for a pair of* SDD (g_n), (h_n).

Proof. For all n, $h_n(x_n - y_n)g_n = 0$, thus $\mathbf{x} - \mathbf{y} = 0$ by the Lemma.

{The message of the first assertion of the Proposition: in testing for $(x_n, e_n) = (y_n, f_n)$, it is enough to look after the x_n and y_n; their adjoints take care of themselves. However, the adjoint symmetry in the definition of OWC is still needed for setting up the involution in $\mathbf{C}$.} ▮

Lemma 3.3. *If* (x_n) *is a sequence in* A *and* (e_n) *is a* SDD *such that* $x_n e_m = x_m e_m$ *whenever* $m < n$, *and if* $f_n = LP(x_n e_n)$, *then* $f_n \uparrow$.

Proof. If $n > m$ then $f_n(x_m e_m) = f_n x_n e_m = (f_n x_n e_n)e_m = (x_n e_n)e_m = x_m e_m$, therefore $f_n \geqslant f_m$. ▮

Theorem 3.4. $\mathbf{C}$ *is finite.*

Proof. Suppose $\mathbf{x}^*\mathbf{x} = 1$. Say $\mathbf{x} = [x_n, e_n]$; by Lemma 2.11, we can suppose that $\mathbf{x}^*\mathbf{x} = [x_n^* x_n, e_n]$ and that $(x_n^* x_n, e_n) = (1, e_n)$ via (e_n), thus

$$x_n^* x_n e_n = 1 e_n \qquad \text{for all} \quad n. \tag{$*$}$$

Setting $w_n = x_n e_n$, it follows from $(*)$ that $w_n^* w_n = e_n$. Define $f_n = w_n w_n^* = LP(w_n) = LP(x_n e_n)$. By the Lemma, $f_n \uparrow$; since $f_n \sim e_n$ and $e_n \uparrow 1$, it follows that $f_n \uparrow 1$. For all n, we have

$$f_n = w_n w_n^* = x_n e_n x_n^*; \tag{$**$}$$

obviously $[f_n, f_n] = [e_n, e_n] = 1$, and $\mathbf{x}\mathbf{x}^* = \mathbf{x}1\mathbf{x}^* = [x_n e_n x_n^*, g_n]$ for suitable (g_n), therefore $(**)$ yields $1 = \mathbf{x}\mathbf{x}^*$. ▮

Proposition 3.5. *If* $\mathbf{x} \in \mathbf{C}$, $\mathbf{x} = [x_n, e_n]$, *then*

$$\mathbf{x}\bar{e}_m = \overline{x_m e_m}, \qquad \bar{e}_m \mathbf{x} = \overline{e_m x_m}$$

for all m.

Proof. Fix m. Then $\mathbf{x}\bar{e}_m = [x_n e_m, g_n]$ for a suitable SDD (g_n). Define $f_n = 0$ for $n < m$ and $f_n = 1$ for $n \geqslant m$; then (f_n) implements $(x_n e_m, g_n) =$

$(x_m e_m, 1)$, thus $\mathbf{x}\bar{e}_m = \overline{x_m e_m}$. Since $\mathbf{x}^* = [x_n{}^*, e_n]$, it follows that $\mathbf{x}^*\bar{e}_m = \overline{x_m{}^* e_m}$. ∎

At first glance, Proposition 3.5 looks like a stuffy technicality; surprisingly, it turns out to be the key to lifting properties from A to $\mathbf{C}$.

LEMMA 3.6. *If* $\mathbf{x} \in \mathbf{C}$, *then there exists a projection* $f \in A$ *such that* (1) $\bar{f}\mathbf{x} = \mathbf{x}$, *and* (2) $\mathbf{yx} = 0$ *if and only if* $\mathbf{y}\bar{f} = 0$.

Proof. Say $\mathbf{x} = [x_n, e_n]$. Define $f_n = LP(x_n e_n)$ and let $f = \sup f_n$; thus, f is the smallest projection in A such that

$$f x_n e_n = x_n e_n \qquad \text{for all} \quad n. \tag{$*$}$$

Since $\bar{f}\mathbf{x} = [f x_n, g_n]$ for suitable (g_n), $(*)$ shows that $\bar{f}\mathbf{x} = \mathbf{x}$ (Proposition 3.2).

Suppose $\mathbf{yx} = 0$. Say $\mathbf{y} = [y_n, h_n]$. For all m, n, we have, by Proposition 3.5,

$$0 = \bar{h}_n \mathbf{y}\mathbf{x}\bar{e}_m = \overline{h_n y_n x_m e_m},$$

thus $(h_n y_n)(x_m e_m) = 0$ and therefore $(h_n y_n) f_m = 0$; since m is arbitrary, $h_n y_n f = 0$, and since n is arbitrary, $\mathbf{y}\bar{f} = 0$ by Proposition 3.2. Conversely, $\mathbf{y}\bar{f} = 0$ implies $\mathbf{yx} = \mathbf{y}(\bar{f}\mathbf{x}) = 0$. ∎

THEOREM 3.7. $\mathbf{C}$ *has no new projections*; *that is, if* $\mathbf{e} \in \mathbf{C}$ *is a projection, then* $\mathbf{e} = \bar{e}$ *for some projection* e *in* A.

Proof. By the Lemma, there exists a projection e in A such that $\mathbf{C}(1 - \bar{e}) = L(\{\mathbf{e}\})$, the left annihilator of $\mathbf{e}$; but $L(\{\mathbf{e}\}) = \mathbf{C}(1 - \mathbf{e})$, therefore $1 - \bar{e} = 1 - \mathbf{e}$. ∎

COROLLARY 3.8. $\mathbf{C}$ *is a Baer *-ring.*

Proof. Since $\mathbf{C}$ has no new projections, its projections form a complete lattice; but $\mathbf{C}$ is a Rickart *-ring by Lemma 3.6; therefore $\mathbf{C}$ is a Baer *-ring. ∎

If $x \in A$ and $e = RP(x)$, then $RP(\bar{x})$ exists (Corollary 3.8), and it follows from Theorem 3.7 that $RP(\bar{x}) = \bar{e}$.

COROLLARY 3.9. $\mathbf{C}$ *has* GC *and satisfies the parallelogram law* (P).

Proof. Since A has these properties (Proposition 1.1) and $\mathbf{C}$ has no new projections, $\mathbf{C}$ inherits the properties via the embedding $x \mapsto \bar{x}$. ∎

This recaptures many of the properties of A, but we are surprisingly far from proving that $LP(\mathbf{x}) \sim RP(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{C}$; this is proved in Section 8, under heavy additional hypotheses on A (see "Added in Proof").

The following proposition needs no extra hypotheses; this is a convenient place to record it [cf. 1, Lemma 3.1]:

PROPOSITION 3.10. *If* $\mathbf{x} = [x_n, e_n]$ *and the* x_n *are all invertible in* A, *then* $\mathbf{x}$ *is invertible in* $\mathbf{C}$ *and* $\mathbf{x}^{-1} = [x_n^{-1}, h_n]$ *for a suitable* SDD (h_n).

4. C Has No New Partial Isometries

The promise of the section heading is fulfilled under the hypotheses (1°), (2°) described below.

We recall two axioms, concerning the existence of projections and square roots, that play a prominent role in [11]. A Baer *-ring is said to satisfy the (*EP*)-*axiom* if, for each element x, there exists $y^* = y \in \{x^*x\}''$ (the bicommutant of x^*x) such that $(x^*x)y^2 = e$, e a nonzero projection. A Baer *-ring is said to satisfy the (*SR*)-*axiom* if, for each element x, there exists $y^* = y \in \{x^*x\}''$ with $x^*x = y^2$.

The above weak form of square roots would suffice for the present section, but we need a stronger form later on. Since the stronger form simplifies some of the underlying proofs, we assume it in the present section also. First, it is necessary to discuss a notion of positivity available in any *-ring:

DEFINITION 4.1. If B is any *-ring, we call $x \in B$ *positive*, written $x \geqslant 0$, if there exist $y_1, \ldots, y_m \in B$ with $x = y_1^*y_1 + \cdots + y_m^*y_m$. If $x, y \in B$ are self-adjoint, we write $x \leqslant y$ (or $y \geqslant x$) in case $y - x \geqslant 0$.

The following properties are immediate from the definitions: (1) $x \geqslant 0$ implies $x^* = x$; (2) $x \leqslant y$ implies $z^*xz \leqslant z^*yz$ for all z; (3) $x \geqslant 0$ and $y \geqslant 0$ imply $x + y \geqslant 0$. (However, it can happen that $x \geqslant 0$ and $-x \geqslant 0$ for nonzero x; this possibility will be excluded by later axioms.)

DEFINITION 4.2. A *-ring is said to satisfy the *unique positive square-root axiom* (UPSR) if, whenever $x \geqslant 0$, there exists a unique element y such that (1) $y \geqslant 0$, and (2) $x = y^2$. We assume, in addition, that (3) $y \in \{x\}''$.

This is much stronger than the (SR)-axiom, not only because square roots are expected to be positive and unique, but because all elements of the form $x_1^*x_1 + \cdots + x_m^*x_m$ possess square roots (not just elements of the form x^*x).

For the rest of the paper, A is assumed to satisfy the following conditions:

(1°) *A is a finite Baer *-ring satisfying the* (EP)-*axiom and the* (UPSR)-*axiom*;

(2°) *partial isometries in A are addable* (as is the case when A has no abelian summand [11, Theorem 64] or when A is an AW^*-algebra [7, Lemma 20]).

We recall the definition of (2°): If $(w_\iota)_{\iota \in I}$ is a family of partial isometries with orthogonal initial projections e_ι and orthogonal final projections f_ι, then there exists a partial isometry w such that $w^*w = \sup e_\iota$, $ww^* = \sup f_\iota$, and $we_\iota = w_\iota = f_\iota w$ for all ι.

It follows that A admits a strong form of "polar decomposition": If $x \in A$ and r is the unique positive square root of x, then there exists a unique partial isometry w such that $x = wr$, $w^*w = RP(x)$, $ww^* = LP(x)$ [11, Theorem 65]. In particular, $LP(x) \sim RP(x)$ for all $x \in A$, so the results of the preceding sections apply. We take up again the ring $\mathbf{C}$:

THEOREM 4.3. *$\mathbf{C}$ has no new unitaries.*

Proof. Let $\mathbf{u} \in \mathbf{C}$ be unitary. We can suppose (by Lemma 2.11) that $\mathbf{u} = [x_n, e_n]$, $\mathbf{u}^*\mathbf{u} = [x_n{}^*x_n, e_n]$, and $(x_n{}^*x_n, e_n) \equiv (1, e_n)$ via (e_n), thus

$$x_n{}^*x_ne_n = e_n \qquad \text{for all} \quad n. \tag{$*$}$$

Defining $w_n = x_ne_n$, we have $w_n{}^*w_n = e_n$ by $(*)$. Setting $f_n = w_nw_n{}^* = LP(w_n)$, it follows (as in the proof of Theorem 3.4) that $f_n \uparrow 1$. Switching to the partial isometries $w_n - w_{n-1}$, and repeating the argument in [1, Lemma 3.3], we find $u \in A$ with $u^*u = 1$, $uu^* = 1$ and

$$ue_n = w_n = x_ne_n \qquad \text{for all} \quad n; \tag{$**$}$$

it results from $(**)$ that $\bar{u} = \mathbf{u}$ (Proposition 3.2). ∎

COROLLARY 4.4. *$\mathbf{C}$ has no new partial isometries. In particular, if $\mathbf{e}$, $\mathbf{f}$ are projections in $\mathbf{C}$, say $\mathbf{e} = \bar{e}$, $\mathbf{f} = \bar{f}$ (Theorem 3.7), then $\mathbf{e} \sim \mathbf{f}$ in $\mathbf{C}$ if and only if $e \sim f$ in A.*

Proof. Let $\mathbf{w}$ be a partial isometry in $\mathbf{C}$, say $\mathbf{w}^*\mathbf{w} = \mathbf{e}$, $\mathbf{w}\mathbf{w}^* = \mathbf{f}$. Since $\mathbf{C}$ is finite (Theorem 3.4) and has GC (Corollary 3.9), it follows from $\mathbf{e} \sim \mathbf{f}$ that $1 - \mathbf{e} \sim 1 - \mathbf{f}$ [cf. 5, Lemme 4.12], say $\mathbf{v}^*\mathbf{v} = 1 - \mathbf{e}$, $\mathbf{v}\mathbf{v}^* = 1 - \mathbf{f}$. Then $\mathbf{u} = \mathbf{w} + \mathbf{v}$ is unitary and $\mathbf{ue} = \mathbf{w}$. Say $\mathbf{u} = \bar{u}$ (Theorem 4.3) and $\mathbf{e} = \bar{e}$; then $\mathbf{w} = \bar{u}\bar{e} = \overline{ue}$, where ue is a partial isometry in A. ∎

COROLLARY 4.5. *Partial isometries in $\mathbf{C}$ are addable.*

Proof. Immediate from Corollary 4.4 and the assumed addability of partial isometries in A. ∎

In Section 8 it will be proved, under additional hypotheses on A, that $\mathbf{C}$ also satisfies the (EP)-axiom and the (UPSR)-axiom; combined with Corollary 4.5, this will yield polar decomposition in $\mathbf{C}$. Under the present hypotheses, we can prove a fragment of (EP) that will be used in Section 8:

PROPOSITION 4.6. *If* $\mathbf{x} \in \mathbf{C}$, $\mathbf{x} \neq 0$, *then there exists* $a \in A$ *such that* $\mathbf{x}\bar{a} = f$, *f a nonzero projection.*

Proof. Say $\mathbf{x} = [x_n, e_n]$. Since $\mathbf{x} \neq 0$, there exists an index m such that $x_m e_m \neq 0$ (Lemma 3.1). By the (EP)-axiom in A, there exists $b^* = b \in \{e_m x_m{}^* x_m e_m\}''$ such that $(e_m x_m{}^* x_m e_m)b^2 = e$, e a nonzero projection. Thus,

$$be_m x_m{}^* x_m e_m b = e;$$

setting $w = x_m e_m b$, we have $w^*w = e$. Define $f = ww^*$. Citing Proposition 3.5, we have

$$\bar{w} = \overline{x_m e_m b} = \mathbf{x}\bar{e}_m\bar{b},$$

hence $f = \bar{w}\bar{w}^* = \mathbf{x}\overline{e_m b w^*}$; take $a = e_m b w^*$. ▮

5. POSITIVITY IN C

In addition to the hypotheses (1°), (2°) of the preceding section, we now assume:

(3°) *2 is invertible in A;*

(4°) *if $x, y \in A$ and $x^*x + y^*y = 0$, then $x = y = 0$.*

These two conditions will be superseded by two new hypotheses in the next section (see Remarks 1 and 2 at the beginning of Section 6).

The (UPSR)-axiom simplifies the notion of positivity in A (see Definition 4.1): $x \geq 0$ if and only if $x = y^*y$ for some y (indeed, for some $y \geq 0$). It follows that (4°) extends to finitely many terms: if $x_1, \ldots, x_m \in A$ and $x_1{}^*x_1 + \cdots + x_m{}^*x_m = 0$, then $x_1 = \cdots = x_m = 0$. This means that the ordering described in Definition 4.1 is "antisymmetric" ($x \geq 0$ and $-x \geq 0$ imply $x = 0$). Ordering the self-adjoints of $\mathbf{C}$ in the same way, we have antisymmetry in $\mathbf{C}$ as well [cf. 1, Lemma 3.4]:

PROPOSITION 5.1. *If* $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbf{C}$ *and* $\mathbf{x}_1{}^*\mathbf{x}_1 + \cdots + \mathbf{x}_m{}^*\mathbf{x}_m = 0$, *then* $\mathbf{x}_1 = \cdots = \mathbf{x}_m = 0$.

Another application of (4°) [cf. 2, Lemma 3.1]:

LEMMA 5.2. *If* $x, y \in A$ *then* $RP(x^*x + y^*y) = RP(x) \cup RP(y)$.

We remark that the lemma extends to finitely many terms, and to $\mathbf{C}$.

That the fragments of "spectral theory" implicit in the (EP)- and (UPSR)-axioms are enough for the following theorem is a pleasant surprise (cf. the proof of [1, Theorem 1.1]):

THEOREM 5.3. *If* $x \in A$ *and* (e_n) *is a* SDD *such that* $e_n x e_n \geqslant 0$ *for all* n, *then* $x \geqslant 0$.

Proof. Write $e_n x e_n = t_n{}^* t_n$ (e.g., with $t_n \geqslant 0$). We know from Lemma 2.3 that $x^* = x$. Let r be the unique positive square root of $x^*x = x^2$; in particular, $r \in \{x^2\}''$ (see Def. 4.2), therefore $rx = xr$. Let $e = RP(x) = RP(r)$; since x is self-adjoint, $LP(x) = e$. Let $x = wr$ be the factorization of x with $w^*w = ww^* = e$ (see the remarks preceding Theorem 4.3). We note that $w^* = w$; for,

$$r(w^* - w)r = (wr)^*r - r(wr) = x^*r - rx = xr - rx = 0,$$

therefore $e(w^* - w)e = 0$, thus $w^* - w = 0$.

Define $u = w + (1 - e)$; u is a symmetry (a self-adjoint unitary) and $x = ur$. It would suffice to show that $u = 1$; we do this in case (i) below. By (3^0) we can define $g = \frac{1}{2}(1 + u)$; g is a projection, $u = 2g - 1$.

Write $r = s^2$, $s \geqslant 0$, $s \in \{r\}''$. Since r commutes with w (because $x = wr$ with x, w and r self-adjoint) it also commutes with u, hence $su = us$. Then

$$x = ur = us^2 = sus = 2sgs - s^2,$$

therefore, $t_n{}^* t_n = e_n x e_n = 2e_n sgse_n - e_n s^2 e_n$, thus

$$2(gse_n)^*(gse_n) = (se_n)^*(se_n) + t_n{}^* t_n .$$

Citing the Lemma, we have

$$RP(gse_n) = RP(se_n) \cup RP(t_n) \geqslant RP(se_n);$$

but $RP(gse_n) \leqslant RP(se_n)$ trivially, thus,

$$RP(gse_n) = RP(se_n). \qquad (*)$$

We consider two cases:

(i) If $RP(x) = 1$, then $RP(s) = RP(r) = RP(x) = 1$, and it follows that $RP(se_n) = e_n$. Then $(*)$ yields

$$e_n = RP(gse_n) \sim LP(gse_n) \leqslant g,$$

thus, $D(e_n) \leqslant D(g)$; since $D(e_n) \uparrow 1$, we conclude that $g = 1$, thus $u = 2g - 1 = 1$, $x = ur = r \geqslant 0$.

(ii) In the general case, where $RP(x) = e$, set $y = x + (1 - e)$; since $y^*y = y^2 = x^2 + (1 - e)$, we have $RP(y) = e \cup (1 - e) = 1$ by the Lemma. For all n,

$$e_n y e_n = e_n x e_n + e_n(1 - e)e_n ,$$

where $e_n x e_n \geqslant 0$ and $e_n(1 - e)e_n = [(1 - e)e_n]^*[(1 - e)e_n] \geqslant 0$, therefore, $e_n y e_n \geqslant 0$. Then $y \geqslant 0$ by case (i), therefore $eye \geqslant 0$; but $eye = exe = x$.

{It is in this proof that positive square roots are vital; the (SR)-axiom would not suffice.} ▮

An important consequence of Theorem 5.3 is that the notions of positivity in A and $\mathbf{C}$ are consistent:

COROLLARY 5.4. *If* $\mathbf{x} = \bar{x}$, $x \in A$, *then* $\mathbf{x} \geqslant 0$ *in* $\mathbf{C}$ *if and only if* $x \geqslant 0$ *in* A.

Proof. If $x \geqslant 0$ in A, say $x = y_1{}^*y_1 + \cdots + y_m{}^*y_m$, then $\mathbf{x} = \bar{y}_1{}^*\bar{y}_1 + \cdots + \bar{y}_m{}^*\bar{y}_m \geqslant 0$ in $\mathbf{C}$.

Conversely, suppose $\mathbf{x} \geqslant 0$ in $\mathbf{C}$, say $\mathbf{x} = \mathbf{y}_1{}^*\mathbf{y}_1 + \cdots \mathbf{y}_m{}^*\mathbf{y}_m$. From the definition of the operations in $\mathbf{C}$, it is clear that we can write $\mathbf{x} = [x_n, e_n]$ with $x_n \geqslant 0$ for all n. Citing Proposition 3.5, we have

$$\bar{e}_n \bar{x} \bar{e}_n = \bar{e}_n \mathbf{x} \bar{e}_n = \overline{e_n x_n e_n},$$

thus $e_n x e_n = e_n x_n e_n \geqslant 0$ for all n, therefore $x \geqslant 0$ in A by Theorem 5.3. ▮

6. Cayley Transform

To the hypotheses (1°)–(4°) of the preceding section, we add two more:

(5°) *A contains a central element i such that $i^2 = -1$ and $i^* = -i$* (we also write i for the corresponding element of $\mathbf{C}$, that is, we identify i with $\bar{i}$);

(6°) *A is symmetric, that is, $1 + x^*x$ is invertible in A, for all $x \in A$.*

Remarks. (1) The hypothesis (6°) makes (3°) redundant.

(2) Hypotheses (5°) and (6°), combined with (1°), make (4°) redundant. For, suppose $x^*x + y^*y = 0$. Write $x^*x = r^2$, $y^*y = s^2$, with r and s self-adjoint, $r \in \{r^2\}''$, $s \in \{s^2\}''$. Since $s^2 = -r^2$, $r \in \{s^2\}'$, therefore, $rs = sr$. Then $(r + is)^*(r + is) = r^2 + s^2 = 0$ yields $r + is = 0$; taking adjoint, $r - is = 0$, therefore, $2r = 0$, $r = 0$, $s = 0$. Thus $x = y = 0$.

(3) In the presence of (5°), and the availability of square roots, (6°) is clearly equivalent to the invertibility of $x + i$ in A for all self-adjoints $x \in A$.

(4) By (3°) and (5°), every $\mathbf{x} \in \mathbf{C}$ has a unique Cartesian decomposition $\mathbf{x} = \mathbf{y} + i\mathbf{z}$, $\mathbf{y}$ and $\mathbf{z}$ self-adjoint.

(5) It follows from (1°) and (2°) that A may be written as a direct sum, $A = B \oplus C$, where C satisfies (5°) and every element in the center of B is self-adjoint [11, p. 130, Theorem A]; thus, in assuming that A satisfies (5°), we are abandoning the "purely real" part B.

In view of Proposition 3.10, condition (6°) lifts to **C** [cf. 1, Corollary 3.1]:

PROPOSITION 6.1. *For all* $\mathbf{x} \in \mathbf{C}$, $1 + \mathbf{x}^*\mathbf{x}$ *is invertible in* **C**.

COROLLARY 6.2. *If* $\mathbf{x} \in \mathbf{C}$, $\mathbf{x}^* = \mathbf{x}$, *then* $\mathbf{x} + i$ *is invertible in* **C**.

The following proposition, an elementary consequence of Corollary 6.2, can be formulated in any *-ring B with unity, possessing a central element i with $i^2 = -1$ and $i^* = -i$, such that $x + i$ is invertible for every self-adjoint x in B [cf. 1, Lemma 4.1]:

PROPOSITION 6.3. *The formulas*

$$\mathbf{u} = (\mathbf{x} - i)(\mathbf{x} + i)^{-1}$$
$$\mathbf{x} = i(1 + \mathbf{u})(1 - \mathbf{u})^{-1}$$

define mutually inverse bijections between the set of all self-adjoint elements **x** *and the set of all unitary elements* **u** *such that* $1 - \mathbf{u}$ *is invertible in* **C**. *If* **x** *and* **u** *are so paired, then* $\{\mathbf{x}\}' = \{\mathbf{u}\}'$, $\{\mathbf{x}\}'' = \{\mathbf{u}\}''$ (*the commutants are computed in* **C**). *We call* **u** *the Cayley transform of* **x**.

Remarks. (1) With notations as in Proposition 6.3, write $\mathbf{u} = \bar{u}$ with $u \in A$ unitary (Theorem 4.3). Since $1 - \mathbf{u}$ is invertible, $LP(1 - \mathbf{u}) = RP(1 - \mathbf{u}) = 1$, therefore $LP(1 - u) = RP(1 - u) = 1$ (see the remark following Corollary 3.8). It is shown in the next section (under an extra hypothesis) that, conversely, if $u \in A$ is any unitary with $RP(1 - u) = 1$, then $\bar{u}$ is the Cayley transform of some self-adjoint **x** in **C**.

(2) If $x \in A$ then $\{x\}'$ denotes the commutant of x in A, and $\{\bar{x}\}'$ the commutant of $\bar{x}$ in **C**; there can be no confusion as long as we refrain from identifying x with $\bar{x}$.

To show an application of the unaided Cayley transform, we now take up the notion of regularity (although the regularity of **C** will not be proved until the next section, under an extra hypothesis). A ring B is said to be *regular* if, for each $x \in B$, there exists an idempotent e such that $Bx = Be$. A *-ring B is called **-regular* if, for each $x \in B$, there exists a projection e such that $Bx = Be$. The following variant of [14, Theorem 4.5] is convenient for our purposes:

LEMMA 6.4. *If B is a *-ring with unity, the following conditions on B are equivalent*: (a) *B is *-regular*; (b) *B is regular and the involution of B is proper* (*$x^*x = 0$ implies $x = 0$*); (c) *B is regular and is a Rickart *-ring. In such a ring, if $LP(x) = RP(x) = e$, then x is invertible in eBe.*

When A is regular, the extension **C** collapses back to A:

PROPOSITION 6.5. *If A is regular then $\bar{A} = \mathbf{C}$.*

Proof. In view of the Cartesian decomposition, it suffices to show that if $\mathbf{x} \in \mathbf{C}$, $\mathbf{x}^* = \mathbf{x}$, then $\mathbf{x} \in \bar{A}$. Let $\mathbf{u} = \bar{u}$ be the Cayley transform of $\mathbf{x}$. As remarked following Proposition 6.3, $LP(1 - u) = RP(1 - u) = 1$; by the Lemma, $1 - u$ has an inverse b in A, hence $\mathbf{x} = i(1 + \mathbf{u})(1 - \mathbf{u})^{-1} = \bar{x}$, where $x = i(1 + u)b$. ▮

This has some surprising consequences:

COROLLARY 6.6. *If A is regular and (x_n, e_n) is an (OWC), then there exists $x \in A$ with $xe_n = x_n e_n$ and $e_n x = e_n x_n$ for all n.*

Proof. Let $\mathbf{x} = [x_n, e_n]$, write $\mathbf{x} = \bar{x}$ by Proposition 6.5, and quote Proposition 3.5. ▮

COROLLARY 6.7. *If A is regular, (f_n) is a sequence of orthogonal projections, and $a_n \in f_n A f_n$ for all n, then there exists $x \in A$ with $f_n x = x f_n = a_n$ for all n.*

Proof. If the sequence (f_n) is finite, let x be the sum of the a_n. Otherwise, let $f = \sup f_n$ and define $x_n = a_1 + \cdots + a_n$, $e_n = f_1 + \cdots + f_n + (1 - f)$; then (x_n, e_n) is an OWC, and Corollary 6.6 provides a suitable element x. ▮

COROLLARY 6.8. *Suppose that* (1) *for each $x \in A$ there exists a positive integer k such that $x^*x \leq k1$, and* (2) *A contains an infinite sequence (f_n) of nonzero orthogonal projections. Then A is not regular.*

Proof. Assume to the contrary that A is regular. Setting $a_n = nf_n$, Corollary 6.7 provides an $x \in A$ such that $f_n x = x f_n = nf_n$ for all n. Choose k as in (1); then $f_n x^* x f_n \leq kf_n$, thus (*) $n^2 f_n \leq kf_n$. Fix n with $n^2 > k$; thus $n^2 1 - k1 = 1 + \cdots + 1$ ($n^2 - k$ terms) is ≥ 0 and invertible. Then $n^2 1 \geq k1$, $n^2 f_n \geq kf_n$; combined with (*), this yields $(n^2 1 - k1) f_n = 0$ and, in view of the invertibility of $n^2 1 - k1$, the contradiction $f_n = 0$. ▮

To put it another way, if A is regular and satisfies (2), then it can't satisfy (1). We return to these considerations in the discussion of boundedness in Section 9.

This is a convenient place to record the following:

PROPOSITION 6.9. *For any $\mathbf{x} \in \mathbf{C}$, the element $\mathbf{y} = (1 + \mathbf{x}^*\mathbf{x})^{-1}$ satisfies $\mathbf{y}^2 \leq 1$.*

Proof [cf. 18, Lemma 6]. Aside from the invertibility of $1 + \mathbf{x}^*\mathbf{x}$ (Proposition 6.1), the proof is trivial algebra:

$$1 = \mathbf{y}(1 + \mathbf{x}^*\mathbf{x})^2\mathbf{y} = \mathbf{y}^2 + 2\mathbf{y}(\mathbf{x}^*\mathbf{x})\mathbf{y} + \mathbf{y}(\mathbf{x}^*\mathbf{x})^2\mathbf{y},$$

thus $1 - \mathbf{y}^2 = 2\mathbf{y}^*(\mathbf{x}^*\mathbf{x})\mathbf{y} + \mathbf{y}^*(\mathbf{x}^*\mathbf{x})^2\mathbf{y} \geq 0$. ▮

7. Regularity of C

To proceed further, we need some more spectral theory. The appropriate axiom is as follows:

Definition 7.1. We say that A satisfies the (US)-*axiom* (unitary spectral axiom) if, for each unitary $u \in A$ with $RP(1 - u) = 1$, there exists a sequence of projections $e_n \in \{u\}''$ such that $e_n \uparrow 1$ and $(1 - u)e_n$ is invertible in e_nAe_n for all n.

The following proposition assesses the strength of the (US)-axiom:

Proposition 7.2. (a) *The* (US)-*axiom is implied by the hypothesis* (1°) *when A is orthoseparable* (*equivalently, the center of A is orthoseparable*). (b) *Every AW^*-algebra satisfies the* (US)-*axiom.*

Proof. (a) We say that a *-ring is orthoseparable if every orthogonal family of nonzero projections is countable. It can be shown that a finite Baer *-ring with GC is orthoseparable if and only if its center is orthoseparable. (For the case of von Neumann algebras, see [17, Lemma 1.1].)

Let $u \in A$ be unitary with $RP(1 - u) = 1$ (since $1 - u$ is normal, $LP(1 - u) = 1$ too). Set $a = 1 - u$. Obviously $\{a\}' = \{u\}'$; since u is unitary, $xu = ux$ iff $u^*x = xu^*$ iff $x^*u = ux^*$, thus $\{u\}'$ is a *-subring; therefore $\{a\}'' = \{u\}''$ is a commutative *-subring of A.

Let (f_ι) be a maximal orthogonal family of nonzero projections such that, for each ι, there exists $b_\iota \in \{a\}''$ with $ab_\iota = b_\iota a = f_\iota$ (hence af_ι is invertible in $f_\iota Af_\iota$, with inverse $b_\iota f_\iota$). We assert that $\sup f_\iota = 1$. Let $g = 1 - \sup f_\iota$, and note that $g \in \{a\}''$ [11, Theorem 20]. Assume to the contrary that $g \neq 0$. Since $RP(a) = 1$, it follows that $ag \neq 0$. By the (EP)-axiom, there exists $c \in \{ga^*ag\}'' = \{ga^*a\}'' \subset \{a\}''$ with $(ga^*a)c^*c = f$, f a nonzero projection. Thus,

$$f = (ga^*c^*c)a = a(ga^*c^*c). \qquad (*)$$

Clearly $f \leqslant g$, therefore $ff_\iota = 0$ for all ι; setting $b = ga^*c^*c \in \{a\}''$, $(*)$ shows that maximality is contradicted.

By orthoseparability, the family (f_ι) is countable; write it as a (possibly finite) sequence (f_n). Define $e_n = f_1 + \cdots + f_n$ (if there are only finitely many f_n, then $e_n = 1$ for sufficiently large n). Then $e_n \uparrow 1$ and $e_na = ae_n = \sum_1^n af_k$ is invertible in e_nAe_n.

{The argument shows that the (US)-axiom holds in any orthoseparable Baer *-ring satisfying an axiom somewhat weaker than the (EP)-axiom.}

(b) For AW^*-algebras (indeed, for Rickart C^*-algebras), the (US) property is routine spectral theory. ∎

We assume for the rest of the paper that, in addition to the hypotheses (1°)–(6°) *of the preceding section,* (7°) *A satisfies the* (US)-*axiom.*

We can now characterize the unitaries that occur as Cayley transforms:

PROPOSITION 7.3. *If $\mathbf{u} \in \mathbf{C}$ is unitary, then $1 - \mathbf{u}$ is invertible in $\mathbf{C}$ if and only if $RP(1 - \mathbf{u}) = 1$ (equivalently, writing $\mathbf{u} = \bar{u}$ with $u \in A$ unitary, $RP(1 - u) = 1$). Thus, the Cayley transform pairs the self-adjoints $\mathbf{x}$ of $\mathbf{C}$ with the unitaries $u \in A$ such that $RP(1 - u) = 1$.*

Proof. The "only if" part is trivial. Conversely, suppose $RP(1 - \mathbf{u}) = 1$. Writing $\mathbf{u} = \bar{u}$ with $u \in A$ unitary, we have $RP(1 - u) = 1$. By the (US)-axiom, there exists an SDD (e_n) such that $e_n \in \{u\}''$ and $(1 - u)e_n$ has an inverse y_n in $e_n A e_n$, thus

$$(1 - u)y_n = y_n(1 - u) = e_n . \qquad (*)$$

If $m < n$, it follows from the uniqueness of inverses that $y_n e_m = e_m y_n = y_m$, therefore (y_n, e_n) is an OWC; setting $\mathbf{y} = [y_n, e_n]$, $(*)$ yields $(1 - \mathbf{u})\mathbf{y} = \mathbf{y}(1 - \mathbf{u}) = 1$, thus $1 - \mathbf{u}$ is invertible in $\mathbf{C}$. ∎

At this point one could develop the spectral theory in $\mathbf{C}$; we defer this until the next section, preferring instead to drive on to regularity.

The following formulation of the (US)-axiom will be more convenient:

LEMMA 7.4. *If $u \in A$ is unitary, and $e = RP(1 - u)$, then there exists a sequence of projections $e_n \in \{u\}''$ such that $e_n \uparrow e$ and $(1 - u)e_n$ is invertible in $e_n A e_n$.*

Proof. Note that $e \in \{1 - u\}'' = \{u\}''$; $1 - e$ is the largest projection such that $(1 - u)(1 - e) = 0$, that is, $u(1 - e) = 1 - e$ (hence also $u^*(1 - e) = 1 - e$). So to speak, "$u = 1$ on $1 - e$"; we correct this by defining

$$v = ue - (1 - e);$$

clearly v is unitary, $v \in \{u\}''$, and it is routine to show that $RP(1 - v) = 1$. Note that $ve = ue$, thus $(1 - u)e = (1 - v)e$.

By the (US)-axiom, choose a sequence of projections $g_n \in \{v\}'' \subset \{u\}''$ with $g_n \uparrow 1$ and $(1 - v)g_n$ invertible in $g_n A g_n$. Then $eg_n \in \{u\}''$, $eg_n \uparrow e$, and $(1 - u)eg_n = (1 - v)eg_n$ is invertible in $eg_n A eg_n$, thus the sequence $e_n = eg_n$ meets all requirements. ∎

This is the key to constructing the "relative inverses" needed for regularity:

PROPOSITION 7.5. *If $\mathbf{x} \in \mathbf{C}$, $\mathbf{x}^* = \mathbf{x}$, and if $\mathbf{e} = RP(\mathbf{x})$, then there exists a unique $\mathbf{y}$ such that $\mathbf{y} \in \mathbf{eCe}$ and $\mathbf{xy} = \mathbf{yx} = \mathbf{e}$ (thus $\mathbf{x}$ is invertible in $\mathbf{eCe}$); moreover, $\mathbf{y}^* = \mathbf{y}$.*

Proof. Let $\mathbf{u} = \bar{u}$ be the Cayley transform of $\mathbf{x}$; thus, $\mathbf{x} = i(1 - \mathbf{u})^{-1}(1 + \mathbf{u})$. Clearly $RP(\mathbf{x}) = RP(1 + \mathbf{u})$; writing $\mathbf{e} = \bar{e}$, e a projection in A, we thus have $e = RP(1 + u)$ (see the remark following Corollary 3.8). Set $v = -u$; thus v is unitary and $e = RP(1 - v)$. By the Lemma, there exists a sequence of projections $e_n \in \{v\}'' = \{u\}''$ such that $e_n \uparrow e$ and $(1 - v)e_n$ has an inverse z_n in e_nAe_n. Thus

$$(1 + u)z_n = e_n \qquad \text{for all } n. \tag{$*$}$$

As argued in the proof of Proposition 7.3, $z_ne_m = z_m$ when $m < n$. Setting $f_n = e_n + (1 - e)$, it is routine to verify that (z_n, f_n) is an OWC, with $z_nf_m = z_m$ when $m < n$; define $\mathbf{z} = [z_n, f_n]$. From $(*)$ we see that

$$(1 + u)z_nf_n = e_nf_n = e_n = ef_n;$$

it follows that $(1 + \mathbf{u})\mathbf{z} = \bar{e}$ and, since the z_n commute with u, also $\mathbf{z}(1 + \mathbf{u}) = \bar{e}$. Define $\mathbf{y} = -i(1 - \mathbf{u})\mathbf{z}$; then

$$\mathbf{xy} = [i(1 + \mathbf{u})(1 - \mathbf{u})^{-1}][-i(1 - \mathbf{u})\mathbf{z}] = (1 + \mathbf{u})\mathbf{z} = \bar{e}$$

and similarly $\mathbf{yx} = \bar{e}$. Note that $\mathbf{z} \in \mathbf{eCe}$; indeed, $z_ne = (z_ne_n)e = z_ne_n = z_n$ for all n, hence $\mathbf{z}\bar{e} = \mathbf{z}$ and similarly $\bar{e}\mathbf{z} = \mathbf{z}$. It then follows from the defining formula for $\mathbf{y}$ that $\mathbf{y} \in \mathbf{eCe}$. Thus $\mathbf{x}$ is invertible in $\mathbf{eCe}$, with inverse $\mathbf{y}$, and the self-adjointness of $\mathbf{y}$ follows from the uniqueness of inverses. ▮

THEOREM 7.6. **C** *is *-regular.*

Proof. Let $\mathbf{x} \in \mathbf{C}$, $\mathbf{e} = RP(\mathbf{x}) = RP(\mathbf{x}^*\mathbf{x})$; we show that $\mathbf{Cx} = \mathbf{Ce}$. The inclusion $\mathbf{Cx} \subset \mathbf{Ce}$ results from $\mathbf{x} = \mathbf{xe}$. Applying Proposition 7.5 to the self-adjoint element $\mathbf{x}^*\mathbf{x}$, there exists an element $\mathbf{y}$ such that $\mathbf{e} = \mathbf{y}(\mathbf{x}^*\mathbf{x})$, thus $\mathbf{Ce} \subset \mathbf{Cx}$. ▮

DEFINITION 7.7. We call **C** the regular ring of A.

The regular ring is characterized as follows:

PROPOSITION 7.8. *Suppose D is a *-ring with unity, such that* (1) *D is regular,* (2) *A is a *-subring of D,* (3) *D contains no new unitaries,* (4) *the relations $x, y \in D$, $x^*x + y^*y = 0$ imply $x = y = 0$, and* (5) *the element i of A is also central in D. Then D is *-isomorphic with* **C** *(via an extension of the embedding $a \mapsto \bar{a}$ of A in* **C***).*

Proof. By (1) and (4), D is *-regular, hence a Rickart *-ring (Lemma 6.4). By (3), D and A have the same unity element. If e is any projection in D, then the symmetry $u = 2e - 1$ is in A by (3), therefore $e = \frac{1}{2}(1 + u) \in A$; thus D contains no new projections. Since D is a Rickart *-ring with complete projection lattice, D is a Baer *-ring. For any $x \in D$, $LP(1 + x^*x) = RP(1 + x^*x) = 1$ by (4) (see Lemma 5.2), therefore $1 + x^*x$ is invertible in D by regularity (Lemma 6.4). In particular, the Cayley transform is operative in D (remarks preceding Proposition 6.3).

Let $x \in D$. We assert that there exists an SDD (e_n) such that $xe_n \in A$ and $x^*e_n \in A$ for all n. By the Cartesian decomposition (and Lemma 2.2) we can suppose $x^* = x$. Let u be the Cayley transform of x; thus $u \in A$ by (3). Since $1 - u$ is invertible in D, $RP(1 - u) = 1$ (in D or in A—it's the same). Adopt the notations in the proof of Proposition 7.3, in particular, $(1 - u)y_n = e_n$; writing $(1 - u)^{-1}$ for the inverse of $1 - u$ in D, we have $y_n = (1 - u)^{-1}e_n$, therefore $xe_n = i(1 + u)(1 - u)^{-1}e_n = i(1 + u)y_n \in A$ for all n.

Each $x \in D$ determines an element $\phi(x)$ of $\mathbf{C}$ as follows. We can choose, by the preceding, an SDD (e_n) such that $xe_n \in A$ and $x^*e_n \in A$ for all n. Applying Lemma 2.8 to the sequence (xe_n) in A, we have that $g_n = e_n \cap (xe_n)^{-1}(e_n)$ is an SDD; since $(1 - e_n)xe_ng_n = 0$, we have

$$xg_n = xe_ng_n = (e_nxe_n)g_n \qquad \text{for all } n. \qquad (*)$$

Similarly, defining $h_n = e_n \cap (x^*e_n)^{-1}(e_n)$, (h_n) is an SDD and

$$x^*h_n = x^*e_nh_n = (e_nx^*e_n)h_n \qquad \text{for all } n. \qquad (**)$$

Setting $x_n = e_nxe_n$ and $k_n = g_n \cap h_n$, it is clear from $(*)$ and $(**)$ that (x_n, k_n) is an OWC. We propose to define $\phi(x) = [x_n, k_n]$ hence must show that this is well-defined. Suppose also (e_n') is an SDD with $xe_n' \in A$ and $x^*e_n' \in A$ for all n; applying the foregoing construction, we arrive at an OWC (x_n', k_n'), where $x_n' = e_n'xe_n'$. It is elementary that $(e_n \cap e_n')x_n(e_n \cap e_n') = (e_n \cap e_n')x(e_n \cap e_n') = (e_n \cap e_n')x_n'(e_n \cap e_n')$, thus $(x_n, k_n) = (x_n', k_n')$ by Proposition 3.2.

It is routine to check that $\phi: D \to \mathbf{C}$ is a *-monomorphism. Finally, ϕ is onto. For, suppose $\mathbf{x} \in \mathbf{C}$, $\mathbf{x}^* = \mathbf{x}$. If $\mathbf{u} = \bar{u}$ is the Cayley transform of $\mathbf{x}$, then $RP(1 - u) = 1$; hence $1 - u$ is invertible in D, the formula $x = i(1 + u)(1 - u)^{-1}$ defines a self-adjoint element of D whose Cayley transform is also u, and a straightforward argument shows that $\phi(x) = \mathbf{x}$. ∎

These results of course hold for any finite AW^*-algebra, but this is known from [1]. Part (a) of Proposition 7.2 yields a new result, which is worth noting explicitly (recall that (3°) and (4°) are redundant by Remarks 1 and 2 at the beginning of Section 6):

PROPOSITION 7.9. *Suppose that A is a finite Baer *-ring satisfying the* (EP)- *and* (UPSR)-*axioms, partial isometries in A are addable* (*as is the case* *when A has no abelian summand), A has a central element i such that $i^2 = -1$ and $i^* = -i$, and $1 + x^*x$ is invertible in A for every $x \in A$; suppose, in addition, that A is orthoseparable (equivalently, the center of A is orthoseparable).*

Then all of the foregoing results (as well as all results in the next section) apply to A. In particular, A has a regular ring in the sense of Definition 7.7.

Perhaps the most interesting special case: A any Baer *-factor of Type II_1, satisfying the (EP)- and (UPSR)-axioms, in which every element of the form $1 + x^*x$ is invertible, and possessing a central element z such that $z^* \neq z$ [cf. 11, p. 130, Theorem A].

8. Spectral Theory in C

As in the preceding section, we assume that A satisfies the hypotheses (1°)–(7°). In this section we exploit (7°) to show that all properties hypothesized for A lift to **C**.

An important dividend of the (US)-axiom (7°) is that self-adjoint elements $\mathbf{x}$ can be represented in a form suitable for "spectral theory":

Proposition 8.1. *If $\mathbf{x} \in \mathbf{C}$, $\mathbf{x}^* = \mathbf{x}$, and if $\mathbf{u} = \bar{u}$ is the Cayley transform of $\mathbf{x}$, then one can write $\mathbf{x} = [x_n, e_n]$ with $x_n, e_n \in \{u\}''$, $x_n^* = x_n$, $x_n e_n = x_n$.*

Proof. Adopt the notations in the proof of Proposition 7.3 (we know that $RP(1 - u) = 1$ by the trivial half of the proposition). By elementary algebra,

$$y_n \in \{(1 - u)e_n, ((1 - u)e_n)^*\}'' \subset \{u\}''.$$

In e_nAe_n, ue_n is unitary and y_n is the inverse of $e_n - ue_n = (1 - u)e_n$; defining $x_n = i(e_n + ue_n)y_n$, we have $x_n^* = x_n$ by elementary algebra (x_n is the "inverse Cayley transform" of ue_n in e_nAe_n). From

$$x_n = i(1 + u)y_n, \qquad (*)$$

it is clear that $x_n \in \{u\}''$; also, $m < n$ implies $y_n e_m = y_m$, therefore, $x_n e_m = x_m$. Since $x_n^* = x_n$, it follows that (x_n, e_n) is an OWC, and $(*)$ yields

$$[x_n, e_n] = i(1 + \mathbf{u})\mathbf{y} = i(1 + \mathbf{u})(1 - \mathbf{u})^{-1} = \mathbf{x}. \quad \blacksquare$$

The next proposition is a substitute for the assertion that "functions" of a self-adjoint element $\mathbf{x}$ lie in $\{\mathbf{x}\}''$; the proof is the same as in [1, Corollary 4.2]:

Proposition 8.2. *If $\mathbf{x} \in \mathbf{C}$, $\mathbf{x}^* = \mathbf{x}$, if $\mathbf{u} = \bar{u}$ is the Cayley transform of $\mathbf{x}$, and if $\mathbf{y} = [y_n, f_n]$ with $y_n \in \{u\}''$ for all n, then $\mathbf{y} \in \{\mathbf{x}\}''$.*

PROPOSITION 8.3. *If* $\mathbf{x} \in \mathbf{C}$, *then* $\mathbf{x} \geqslant 0$ *if and only if* $\mathbf{x} = \mathbf{y}^*\mathbf{y}$ *for some* $\mathbf{y} \in \mathbf{C}$. *In fact,* $\mathbf{C}$ *satisfies the* (UPSR)-*axiom.*

Proof. Suppose $\mathbf{x} \geqslant 0$ (that is, $\mathbf{x} = \mathbf{y}_1^*\mathbf{y}_1 + \cdots + \mathbf{y}_m^*\mathbf{y}_m$ for suitable $\mathbf{y}_1 ,..., \mathbf{y}_m \in \mathbf{C}$). Write $\mathbf{x} = [x_n , e_n]$ as in the statement of Proposition 8.1. Citing Proposition 3.5, we have $\bar{x}_n = \overline{e_n x_n e_n} = \bar{e}_n \mathbf{x} \bar{e}_n \geqslant 0$ in $\mathbf{C}$, therefore $x_n \geqslant 0$ in A by Corollary 5.4.

Let r_n be the unique positive square root of x_n; in particular, $r_n \in \{x_n\}'' \subset \{u\}''$ (see Def. 4.2). Thus, the x_n , e_n , r_n all lie in the commutative *-subring $\{u\}''$. It follows from the uniqueness of positive square roots that

$$r_n e_m = r_m \qquad \text{when} \quad m < n; \tag{1}$$

for, $r_n e_m = e_m r_n e_m \geqslant 0$ and $(r_n e_m)^2 = r_n^2 e_m = x_n e_m = x_m$. From (1) we see that (r_n , e_n) is an OWC; defining $\mathbf{r} = [r_n , e_n]$, it results from $r_n^2 = x_n$ that

$$\mathbf{x} = \mathbf{r}^2. \tag{2}$$

Since $r_n \in \{u\}''$, we have

$$\mathbf{r} \in \{\mathbf{x}\}'' \tag{3}$$

by Proposition 8.2. Next, we note that

$$\mathbf{r} \geqslant 0; \tag{4}$$

for, if s_n is the unique positive square root of r_n , the above argument shows that $s_n e_m = s_m$ when $m < n$, thus $\mathbf{s} = [s_n , e_n]$ is a self-adjoint element with $\mathbf{r} = \mathbf{s}^2$.

In view of (2), (3), (4), it remains only to show that if $\mathbf{t} \geqslant 0$ and $\mathbf{x} = \mathbf{t}^2$, then $\mathbf{t} = \mathbf{r}$. The proof is the same as in [1, Corollary 6.2]. ∎

DEFINITION 8.4. If $\mathbf{x} \in \mathbf{C}$, $\mathbf{x} \geqslant 0$, we write $\mathbf{x}^{1/2}$ for the unique positive square root of $\mathbf{x}$ given by Proposition 8.3. We know, in addition, that $\mathbf{x}^{1/2} \in \{\mathbf{x}\}''$.

PROPOSITION 8.5. $\mathbf{C}$ *satisfies the* (EP)-*axiom. In fact, given any* $\mathbf{z} \in C$, $\mathbf{z} \neq 0$, *there exists* $y \in A$ *such that* $\bar{y} \in \{\mathbf{z}^*\mathbf{z}\}''$, $y \geqslant 0$, *and* $\mathbf{z}^*\mathbf{z}\bar{y}^2 = \bar{f}$, f *a nonzero projection.*

Proof. Set $\mathbf{x} = (\mathbf{z}^*\mathbf{z})^{1/2}$, let $\mathbf{u} = \bar{u}$ be the Cayley transform of $\mathbf{x}$, and write $\mathbf{x} = [x_n , e_n]$ as in Proposition 8.1. Adopt the notation of the proof of Proposition 4.6; thus $\mathbf{x}\bar{a} = \bar{f}$, where

$$b \in \{e_m x_m^2 e_m\}'' \subset \{u\}'',$$

$w = x_m e_m b \in \{u\}''$, $f = ww^*$, and $a = e_m b w^* \in \{u\}''$. Since $a \in \{u\}''$ we have $\bar{a} \in \{\mathbf{x}\}''$ by Proposition 8.2; then $f = f^*f$ yields $\bar{f} = \overline{a^*\mathbf{x}^2 a} = \mathbf{z}^*\mathbf{z}\bar{a}^*\bar{a}$. Setting $y = (a^*a)^{1/2} \in \{a^*a\}'' \subset \{u\}''$, we have

$$\bar{y} \in \{\mathbf{x}\}'' = \{\mathbf{x}^2\}'' = \{\mathbf{z}^*\mathbf{z}\}'',$$

and $\mathbf{z}^*\mathbf{z}\bar{y}^2 = \mathbf{z}^*\mathbf{z}\overline{a^*a} = \bar{f}$. ▮

As we note in the following proof, Proposition 8.5 completes the proof that **C** inherits all of the properties (1°)–(7°) hypothesized for A:

PROPOSITION 8.6. *$\bar{A} = \mathbf{C}$ if and only if A is regular. In particular,* **C** *is its own regular ring, thus the operation $A \mapsto \mathbf{C}$ is idempotent.*

Proof. If $\bar{A} = \mathbf{C}$ then A is regular by Theorem 7.6; conversely, if A is regular then $\bar{A} = \mathbf{C}$ was shown in Proposition 6.5 [assuming only (1°)–(6°)].

Note that **C** has all the properties hypothesized for A: it is a finite (Theorem 3.4) Baer *-ring (Corollary 3.8) satisfying the (EP)-axiom (Proposition 8.5) and the (UPSR)-axiom (Proposition 8.3), thus it satisfies (1°); it also satisfies (2°)–(6°) by Corollary 4.5, Proposition 5.1, and Proposition 6.1; finally, **C** inherits (7°) from A since it has no new unitaries (Theorem 4.3) or projections (Theorem 3.7).

It follows that **C** has a regular ring **D**; but $\bar{\mathbf{C}} = \mathbf{D}$ by the first part of the proof, thus **C** is its own regular ring. ▮

It follows that the properties that accrue to A in virtue of (1°)–(7°) also accrue to **C**; for example, **C** admits a strong form of polar decomposition:

PROPOSITION 8.7. *If $\mathbf{x} \in \mathbf{C}$ one can write $\mathbf{x} = \bar{w}\mathbf{r}$ with $\mathbf{r} = (\mathbf{x}^*\mathbf{x})^{1/2}$ and w a partial isometry in A such that $\bar{w}^*\bar{w} = RP(\mathbf{x})$, $\bar{w}\bar{w}^* = LP(\mathbf{x})$. In particular, $LP(\mathbf{x}) \sim RP(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{C}$.*

Proof. The remarks preceding Theorem 4.3 are now applicable to **C**. ▮

COROLLARY 8.8. *If $\mathbf{x}, \mathbf{y} \in \mathbf{C}$ and $\mathbf{yx} = 1$, then $\mathbf{xy} = 1$.*

Proof. Since **C** is a finite Baer *-ring such that $LP(\mathbf{x}) \sim RP(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{C}$ (Proposition 8.7), the corollary is covered by Proposition 1.1. ▮

PROPOSITION 8.9. *If $\mathbf{x} \in \mathbf{C}$, then $\mathbf{x} \geqslant 0$ if and only if one can write $\mathbf{x} = [x_n, e_n]$ with $x_n \geqslant 0$ for all n.*

Proof. The "only if" part is trivial. Conversely, suppose $\mathbf{x} = [x_n, e_n]$ with $x_n \geqslant 0$ (or merely $e_n x_n e_n \geqslant 0$) for all n. Citing Proposition 3.5, $\bar{e}_n \mathbf{x} \bar{e}_n = \overline{e_n x_n e_n} \geqslant 0$ for all n; therefore, $\mathbf{x} \geqslant 0$ by Theorem 5.3 (which is applicable to **C**, since **C** also satisfies (1°)–(4°)). ▮

The following results on positivity [cf. 4] pertain to the discussion of boundedness in the next section.

LEMMA 8.10. *If* $\mathbf{x} \geqslant 0$, $RP(\mathbf{x}) = \mathbf{e}$, *and if* $\mathbf{y}$ *is the inverse of* $\mathbf{x}$ *in* $\mathbf{eCe}$, *then* $\mathbf{y} \geqslant 0$.

Proof. Write $\mathbf{x} = \mathbf{r}^2$ with $\mathbf{r}$ self-adjoint. Then $RP(\mathbf{r}) = RP(\mathbf{x}) = \mathbf{e}$. Let $\mathbf{s}$ be the (self-adjoint) inverse of $\mathbf{r}$ in $\mathbf{eCe}$ (Proposition 7.5). Since $\mathbf{xs}^2 = \mathbf{r}^2\mathbf{s}^2 = \mathbf{e}$, it follows from uniqueness of inverses that $\mathbf{y} = \mathbf{s}^2 = \mathbf{s}^*\mathbf{s} \geqslant 0$. ∎

LEMMA 8.11. *If* $\mathbf{z} \in \mathbf{C}$, $\mathbf{z} \geqslant 1$, *then* $\mathbf{z}$ *is invertible in* $\mathbf{C}$ *and* $0 \leqslant \mathbf{z}^{-1} \leqslant 1$.

Proof. Since $\mathbf{z} - 1 \geqslant 0$, we have $\mathbf{z} - 1 = \mathbf{x}^*\mathbf{x}$ for suitable $\mathbf{x}$ (Proposition 8.3). Then $\mathbf{z} = 1 + \mathbf{x}^*\mathbf{x}$ is invertible in $\mathbf{C}$ (Proposition 6.1) and, writing $\mathbf{y} = \mathbf{z}^{-1}$, we have $\mathbf{y}^2 \leqslant 1$ by Proposition 6.9. Thus,

$$(1 - \mathbf{y})(1 + \mathbf{y}) \geqslant 0. \tag{$*$}$$

Since $\mathbf{y} \geqslant 0$ (Lemma 8.10), $1 + \mathbf{y}$ is invertible and $\geqslant 0$; write $1 + \mathbf{y} = \mathbf{r}^2$ with $\mathbf{r} \geqslant 0$, $\mathbf{r} \in \{1 + \mathbf{y}\}'' = \{\mathbf{y}\}''$. Since $\mathbf{r}$ is also invertible, $(*)$ yields

$$0 \leqslant (\mathbf{r}^{-1})^*(1 - \mathbf{y})(1 + \mathbf{y})\mathbf{r}^{-1} = (1 - \mathbf{y})\mathbf{r}^{-1}\mathbf{r}^2\mathbf{r}^{-1} = 1 - \mathbf{y},$$

thus $\mathbf{y} \leqslant 1$. ∎

PROPOSITION 8.12. *If* $\mathbf{x}, \mathbf{y} \in \mathbf{C}$, $0 \leqslant \mathbf{x} \leqslant \mathbf{y}$, *and if* $\mathbf{x}$ *is invertible, then* $\mathbf{y}$ *is invertible and* $0 \leqslant \mathbf{y}^{-1} \leqslant \mathbf{x}^{-1}$.

Proof. Write $\mathbf{x} = \mathbf{r}^2$, $\mathbf{r}$ self-adjoint; then

$$1 = \mathbf{r}^{-1}\mathbf{x}\mathbf{r}^{-1} \leqslant \mathbf{r}^{-1}\mathbf{y}\mathbf{r}^{-1}.$$

By Lemma 8.11, $\mathbf{r}^{-1}\mathbf{y}\mathbf{r}^{-1}$ is invertible (hence so is $\mathbf{y}$) and $\mathbf{r}\mathbf{y}^{-1}\mathbf{r} \leqslant 1$, therefore

$$\mathbf{y}^{-1} = \mathbf{r}^{-1}(\mathbf{r}\mathbf{y}^{-1}\mathbf{r})\mathbf{r}^{-1} \leqslant \mathbf{r}^{-1}1\mathbf{r}^{-1} = \mathbf{x}^{-1}.$$

Finally, $\mathbf{y}^{-1} \geqslant 0$ by Lemma 8.10. ∎

Implicit in the proof of Lemma 8.11: If $\mathbf{y} \geqslant 0$ and $\mathbf{y}^2 \leqslant 1$, then $\mathbf{y} \leqslant 1$. Conversely, $0 \leqslant \mathbf{y} \leqslant 1$ implies $\mathbf{y}^{1/2}\mathbf{y}\mathbf{y}^{1/2} \leqslant \mathbf{y}^{1/2}1\mathbf{y}^{1/2}$, thus $\mathbf{y}^2 \leqslant \mathbf{y} \leqslant 1$. Citing Proposition 6.9, we have:

PROPOSITION 8.13. *If* $\mathbf{y} \in \mathbf{C}$, $\mathbf{y} \geqslant 0$, *then* $\mathbf{y} \leqslant 1$ *if and only if* $\mathbf{y}^2 \leqslant 1$. *In particular*, $0 \leqslant (1 + \mathbf{x}^*\mathbf{x})^{-1} \leqslant 1$ *for all* $\mathbf{x} \in \mathbf{C}$.

9. C Has No New Bounded Elements

We assume, as in Sections 7 and 8, that A satisfies the hypotheses (1^o)–(7^o), and will shortly add another.

Motivated by the operatorial example, one can define a notion of boundedness in abstract *-rings:

DEFINITION 9.1. An element x of a *-ring with unity is said to be *bounded* if there exists a positive integer k such that $x^*x \leqslant k1$ (in the sense of the ordering in Def. 4.1).

The following two propositions, implicit in Sections 6–8, pertain to this concept:

PROPOSITION 9.2. *Suppose that* (1) *every element of A is bounded, and* (2) *A contains an infinite sequence of nonzero orthogonal projections; then A is not regular, and $\bar{A}$ is contained in* $\mathbf{C}$ *properly.*

Proof. The nonregularity of A is proved in Corollary 6.8 [assuming only (1^o)–(6^o)]; then $\bar{A} \neq \mathbf{C}$ results from the regularity of $\mathbf{C}$ (Theorem 7.6). ▮

PROPOSITION 9.3. *For any* $\mathbf{x} \in \mathbf{C}$, $(1 + \mathbf{x}^*\mathbf{x})^{-1}$ *is bounded. If* $\mathbf{x}^* = \mathbf{x}$, *then* $(\mathbf{x} + i)^{-1}$ *is bounded.*

Proof. Setting $\mathbf{y} = (1 + \mathbf{x}^*\mathbf{x})^{-1}$, the first assertion follows from $\mathbf{y}^*\mathbf{y} = \mathbf{y}^2 \leqslant 1$ (Proposition 6.9).

If $\mathbf{x}^* = \mathbf{x}$ and $\mathbf{z} = (\mathbf{x} + i)^{-1}$, then $\mathbf{z}^*\mathbf{z} = (\mathbf{x} - i)^{-1}(\mathbf{x} + i)^{-1} = (1 + \mathbf{x}^2)^{-1} \leqslant 1$ by Proposition 8.13, thus $\mathbf{z}$ is bounded. ▮

To validate the claim in the section heading, we now assume:

(8^o) *A satisfies the* (PS)-*axiom* (positive sums axiom): If (f_n) is an orthogonal sequence of projections in A with $\sup f_n = 1$, and if, for each n, we are given $a_n \in f_nAf_n$ with $0 \leqslant a_n \leqslant 1$, then there exists $a \in A$ such that $af_n = a_n$ for all n.

Remarks. Assume the notations of (8^o).

(1) The conditions "$a_n \in f_nAf_n$ and $0 \leqslant a_n \leqslant 1$" are equivalent to "$0 \leqslant a_n \leqslant f_n$".

(2) The elements a_n, f_m all commute: $a_nf_n = f_na_n = a_n$, and $m \neq n$ implies $a_nf_m = 0$, $a_na_m = 0$. In our applications of the (PS)-axiom, it is sufficient to assume that $a_n, f_n \in \{u\}''$, where $u \in A$ is unitary and $RP(1 - u) = 1$.

(3) In an AW^*-algebra (or a Rickart C^*-algebra), the construction of a is easy spectral theory; in fact, writing the commutant of the set of all f_n

as a C^*-sum [6, Lemma 2.5], one need only assume that the $a_n \in f_n A f_n$ are bounded in norm.

(4) The element a is unique since $\sup f_n = 1$. Moreover, $0 \leqslant a \leqslant 1$; for, setting $e_n = f_1 + \cdots + f_n$, we have $e_n \uparrow 1$ and $e_n a e_n = a_1 + \cdots + a_n \geqslant 0$ for all n, therefore $a \geqslant 0$ by Theorem 5.3; also,

$$e_n(1 - a)e_n = (f_1 - a_1) + \cdots + (f_n - a_n) \geqslant 0$$

for all n, thus $a \leqslant 1$.

(5) The condition $\sup f_n = 1$ can be dropped, by adjoining $1 - \sup f_n$ to the sequence (but then a need not be unique).

(6) The point of the (PS)-axiom is that $a \in A$; one can always construct an $\mathbf{a} \in \mathbf{C}$ in the obvious way (cf. the proof of Corollary 6.7).

THEOREM 9.4. *Let $\mathbf{x} \in \mathbf{C}$. In order that $0 \leqslant \mathbf{x} \leqslant 1$, it is necessary and sufficient that $\mathbf{x} = \bar{a}$ for some $a \in A$ with $0 \leqslant a \leqslant 1$.*

Proof. Suppose $0 \leqslant \mathbf{x} \leqslant 1$. Let $\mathbf{u} = \bar{u}$ be the Cayley transform of $\mathbf{x}$, and write $\mathbf{x} = [x_n, e_n]$ as in Proposition 8.1. Then $0 \leqslant \bar{e}_n \mathbf{x} \bar{e}_n \leqslant \bar{e}_n$; since $\bar{e}_n \mathbf{x} \bar{e}_n = \overline{e_n x_n e_n} = \bar{x}_n$, we have $0 \leqslant x_n \leqslant e_n$ by Corollary 5.4, hence

$$0 \leqslant x_n \leqslant 1 \qquad \text{for all } n. \tag{1}$$

Note that

$$x_1 \leqslant x_2 \leqslant x_3 \leqslant \ldots; \tag{2}$$

for, if $m < n$ then, since $x_n^{1/2} \in \{x_n\}'' \subset \{u\}''$, we have

$$x_m = x_n e_m = x_n^{1/2} e_m x_n^{1/2} \leqslant x_n^{1/2} 1 x_n^{1/2} = x_n .$$

Set $e_0 = x_0 = 0$ and define $f_n = e_n - e_{n-1}$, $a_n = x_n - x_{n-1}$. From (1) and (2) we have $0 \leqslant a_n \leqslant 1$, and it is easy to see that $a_n \in f_n A f_n$. The (PS)-axiom yields $a \in A$ with

$$a f_n = a_n = x_n - x_{n-1}, \tag{3}$$

hence

$$a e_n = \sum_1^n a f_j = \sum_1^n (x_j - x_{j-1}) = x_n ,$$

thus $\bar{a} = \mathbf{x}$. Moreover, $0 \leqslant a \leqslant 1$ by Remark (4) above.

Conversely, if $a \in A$ and $0 \leqslant a \leqslant 1$, then $0 \leqslant \bar{a} \leqslant 1$ by the trivial half of Corollary 5.4. ∎

COROLLARY 9.5. *If* $\mathbf{x} \in \mathbf{C}$, *then* $\mathbf{x}$ *is bounded in* $\mathbf{C}$ *if and only if* $\mathbf{x} = \bar{a}$ *with* a *bounded in* A. *More precisely,* $\mathbf{x}^*\mathbf{x} \leqslant k1$ (k *a positive integer*) *if and only if* $\mathbf{x} = \bar{a}$ *with* $a^*a \leqslant k1$.

Proof. Suppose $\mathbf{x}^*\mathbf{x} \leqslant k1$. Set $\mathbf{y} = k^{-1/2}\mathbf{x}$ (note that $k^{1/2}1$ exists in A by the (UPSR)-axiom). Then $\mathbf{y}^*\mathbf{y} \leqslant 1$, hence by Theorem 9.4 there exists $b \in A$, $0 \leqslant b \leqslant 1$, such that

$$\mathbf{y}^*\mathbf{y} = \bar{b}. \tag{1}$$

Write $b = c^2$ with $c \in A$, $c \geqslant 0$. Then (1) yields

$$\mathbf{x}^*\mathbf{x} = \overline{kb} = \overline{(k^{1/2}c)^2}. \tag{2}$$

Write $\mathbf{x} = \bar{w}\mathbf{r}$ with $\mathbf{r} = (\mathbf{x}^*\mathbf{x})^{1/2}$ and $w \in A$ a partial isometry (Proposition 8.7). By the uniqueness of positive square roots (Proposition 8.3), it follows from (2) that $\mathbf{r} = \overline{k^{1/2}c}$, thus $\mathbf{x} = \bar{w}\mathbf{r} = \overline{wk^{1/2}c}$; setting $a = k^{1/2}wc$, we have $\mathbf{x} = \bar{a}$ and $\overline{a^*a} = \mathbf{x}^*\mathbf{x} \leqslant k1$, therefore $a^*a \leqslant k1$ by Corollary 5.4.

Conversely, if $a \in A$ and $a^*a \leqslant k1$, then $\bar{a}^*\bar{a} \leqslant k1$ by the trivial half of Corollary 5.4. ∎

COROLLARY 9.6. *If* $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbf{C}$ *and*

$$\mathbf{x}_1{}^*\mathbf{x}_1 + \cdots + \mathbf{x}_m{}^*\mathbf{x}_m = 1,$$

then $\mathbf{x}_i = \bar{a}_i$ *for suitable* $a_i \in A$ *with* $a_i{}^*a_i \leqslant 1$.

Proof. Since $1 - \mathbf{x}_i{}^*\mathbf{x}_i = \sum_{j \neq i} \mathbf{x}_j{}^*\mathbf{x}_j \geqslant 0$, we have $0 \leqslant \mathbf{x}_i{}^*\mathbf{x}_i \leqslant 1$; quote Corollary 9.5. ∎

COROLLARY 9.7. *If every* $a \in A$ *is bounded, then* $\bar{A}$ *coincides with the set of all bounded elements of* $\mathbf{C}$.

Proof. Immediate from Corollary 9.5. ∎

The corollary applies, in particular, to the AW^* case [cf. 1, Lemma 5.1].

10. $n \times n$ MATRICES

We assume, as in the preceding section, that A satisfies the hypotheses (1°)–(8°).

Fix a positive integer n. We consider the *-rings A_n and $\mathbf{C}_n$ of $n \times n$ matrices over A and $\mathbf{C}$, with the usual operations. It is convenient to identify A with $\bar{A}$; then A_n is a *-subring of $\mathbf{C}_n$. The problem of $n \times n$ matrices:

Is A_n a Baer *-ring? In Theorem 10.4 we show that A_n is at least a Rickart *-ring, but in general the problem remains open. (For AW^*-algebras, the problem is solved affirmatively in [3].) We write $X = (\mathbf{x}_{ij})$, $Y = (\mathbf{y}_{ij})$,... for elements of $\mathbf{C}_n$.

The first proposition requires only (1°)–(4°):

PROPOSITION 10.1. *If $X_1, \ldots, X_m \in \mathbf{C}_n$ and $X_1{}^*X_1 + \cdots + X_m{}^*X_m = 0$, then $X_1 = \cdots = X_m = 0$. In particular, the involution of $\mathbf{C}_n$ is proper.*

Proof. Say $X_k = (\mathbf{x}_{ij}^k)$, $k = 1, \ldots, m$. The (j, j) coordinate of the given equation reads

$$\sum_{k=1}^{m} \left(\sum_{i=1}^{n} \mathbf{x}_{ij}^{k*} \mathbf{x}_{ij}^{k} \right) = 0,$$

hence $\mathbf{x}_{ij}^k = 0$ (Proposition 5.1). ∎

The next proposition requires only (1°)–(7°):

PROPOSITION 10.2. *$\mathbf{C}_n$ is *-regular, hence is a Rickart *-ring.*

Proof. Since $\mathbf{C}$ is regular (Theorem 7.6), $\mathbf{C}_n$ is regular by a general theorem of von Neumann [14, Theorem 2.13]. Moreover, the involution of $\mathbf{C}_n$ is proper (Proposition 10.1). Quote Lemma 6.4. ∎

In the following proposition, we make use of the full force of the hypotheses (1°)–(8°):

PROPOSITION 10.3. *If $X_1, \ldots, X_m \in \mathbf{C}_n$ and $X_1{}^*X_1 + \cdots + X_m{}^*X_m = I$ (the identity matrix), then $X_1, \ldots, X_m \in A_n$. In particular, $\mathbf{C}_n$ has no new projections (unitaries, isometries, partial isometries).*

Proof. Say $X_k = (\mathbf{x}_{ij}^k)$. The (j, j) coordinate of the given equation reads

$$\sum_{k=1}^{m} \left(\sum_{i=1}^{n} \mathbf{x}_{ij}^{k*} \mathbf{x}_{ij}^{k} \right) = 1,$$

hence $\mathbf{x}_{ij}^k \in A$ (Corollary 9.6).

If W is a partial isometry in $\mathbf{C}_n$, say $W^*W = E$, E a projection, then $W \in A_n$ results from the equation $W^*W + (I - E)^*(I - E) = I$. Thus, $\mathbf{C}_n$ has no new partial isometries; in particular, it has no new projections, isometries or unitaries. (We do not know whether $\mathbf{C}_n$ is finite, hence we refrain from identifying isometries with unitaries.) ∎

THEOREM 10.4. *A_n is a Rickart *-ring.*

Proof. This is immediate from Proposition 10.2 and the fact that all projections of $\mathbf{C}_n$ are in A_n (Proposition 10.3). ▌

It appears to be hard to come by any additional properties of A_n. For example: does A_n (equivalently, $\mathbf{C}_n$) satisfy the parallelogram law (P)? The following remarks are pertinent, but inconclusive. Suppose E, F are projections in $\mathbf{C}_n$. Setting $X = E(I - F)$, we know that

$$E \cup F - F = RP(X), \qquad E - E \cap F = LP(X)$$

(see Section 1). By regularity, there exists $Y \in \mathbf{C}_n$ with $YX = RP(X)$, $XY = LP(X)$, thus $E \cup F - F$ and $E - E \cap F$ are "algebraically equivalent", that is, equivalent in the sense of Baer rings [11, Section 2]. The problem is to find a partial isometry that implements the equivalence.

11. PROBLEMS

(1) Can the regularity of $\mathbf{C}$ be reached with fewer axioms?

(2) Can orthoseparability be dropped in Proposition 7.9?

(3) Conditions (1°)–(4°) are relatively mild, and condition (6°) is to be expected if one is to arrive at a *-regular ring [cf. 18, Lemma 6]. Condition (7°) comes free of charge in an orthoseparable ring, hence is acceptable. Condition (8°) is restrictive, but one must clearly have something like it to get the results on boundedness—and Theorem 10.4. Condition (5°) seems the most severe; in assuming it, we are leaving aside the "purely real" case [cf. 11, p. 130, Theorem A], in which $z^* = z$ for all central elements z. What can be said for the purely real case? (To treat it, a substitute will have to be found for Cayley transform methods.)

(4) One could study systematically the "bounded subring" of A or of $\mathbf{C}$ [cf. 2, Lemma 3.11]; Vidav did so for a *-regular ring satisfying the condition in Proposition 5.1, and showed the bounded subring to be remarkably like a C^*-algebra [18]. The hypotheses (1°)–(8°) are quite restrictive; it is conceivable that a Baer *-ring satisfying them could be given a more concrete representation.

(5) The study of $\mathbf{C}$ in [1] was extended by Saitô [16] to arbitrary AW^*-algebras (the essential case is the semifinite one). Presumably, something similar can be done for Baer *-rings. We remark that this would in no way advance the $n \times n$ matrix ring problem (it being trivial that A_n is a Baer *-ring when A is a properly infinite Baer *-ring with GC [cf. 3, Introduction]).

The following problems refer to Theorem 10.4.

(6) Does A_n satisfy the parallelogram law (P)? If A_n has anything like square roots, it is easy to see that $LP(X) \sim RP(X)$ for all $X \in A_n$, and therefore A_n satisfies (P) [cf. 3, Section 2]. [For example, it suffices to assume that for each $X \in A_n$, there exists $Y \in A_n$ such that $X^*X = Y^*Y$ and $LP(Y) = RP(Y)$.]

(7) Does A_n have GC? The answer is easily seen to be yes if A_n satisfies (P) [cf. 3, Lemma 3.3].

(8) Is A_n (equivalently, $\mathbf{C}_n$) finite? The answer is easily seen to be yes if (a) every sequence of orthogonal projections in A_n has a supremum, and (b) A_n satisfies the "square root" condition mentioned in Problem (6) (cf. [3, Lemma 4.1]; the word "exactly" in the proof of the cited lemma is not quite correct). More generally, one can show that if B is a finite Rickart *-ring with GC, and if n is a positive integer such that (i) B_n is a Rickart *-ring, (ii) every sequence of orthogonal projections in B_n has a supremum, and (iii) B_n satisfies (P), then B_n is finite [cf. 11, Theorem 56].

(9) Is A_n (equivalently, $\mathbf{C}_n$) a Baer *-ring? The affirmative answer in the case of AW^*-algebras is known [3].

(10) J. E. Roos [15] has observed that any Baer *-ring A may be enlarged to a regular ring $\tilde{A}$ ($\tilde{A}$ is the maximal ring of right fractions constructed by Utumi). One could extend the construction of $\mathbf{C}$ to any Baer *-ring A satisfying $LP(x) \sim RP(x)$ for all $x \in A$ [cf. 16] (but if A has no finite projections, then the construction collapses: $\mathbf{C} = \bar{A}$). What is the precise relation between the two constructions? Can properties of A be lifted to $\tilde{A}$ as they are to $\mathbf{C}$? The relation may be complicated; the regular ring $\tilde{A}$ is available for an arbitrary Baer *-ring A, but a regular Baer *-ring can only be finite [9, Theorem 1].

(11) One can generalize from sequences to well-ordered families, as follows. Let A be a finite Baer *-ring satisfying the (EP)- and (SR)-axioms, such that partial isometries in A are addable. Fix a limit ordinal Λ (heretofore, $\Lambda = \omega$). Instead of sequences, consider families in A indexed by ordinals n, $1 \leqslant n < \Lambda$, and define SDD, OWC, CO, and the ring $\mathbf{C}$, formally the same as for sequences. With "sequence" replaced by "family", all of Sections 2 and 3 extend routinely. So does Theorem 4.3 (one applies addability to the partial isometries $w_n(e_n - \sup_{m<n} e_m)$, and a straightforward transfinite induction establishes (**)), as well as its corollaries. Assuming, in addition, that (5°) and (6°) hold, the proofs of Proposition 6.5 and Corollary 6.6 are unchanged. We suppress the routine alterations needed to adapt the rest of the paper to well-ordered families. (Note, however, that if Λ is "too large", then the ring $\mathbf{C}$ may collapse back to A.) Problem: How does the generalization to well-ordered families bear on Problem 10?

Added in proof (June 20, 1972). It can be shown that **C** satisfies LP $\sim$ RP under the hypotheses of Section 3; details appear in a forthcoming book ["Baer *-rings," Springer-Verlag, New York, in press]. The answer is affirmative for Problem (8) [op. cit., § 58, Exer. 3]. The answers are affirmative for Problems (6), (7) and (9) in the Type II factorial case [op. cit., § 62, Exer. 9].

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