The determinantal regions of complex sign pattern matrices and ray pattern matrices

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Abstract

This paper studies the determinantal regions $S_A$ of complex sign pattern matrices and the determinantal regions $R_A$ of ray pattern matrices. We give an affirmative answer to a problem proposed in [C.A. Eschenbach, F.J. Hall, Z. Li, From real to complex sign pattern matrices, Bull. Australian Math. Soc. 57 (1998) 159–172] concerning the boundary points of the regions $S_A$. We also prove that the region $S_A \{ 0 \}$ (and $R_A \{ 0 \}$) is an open set in the complex plane $C$, except the cases that $S_A$ (or $R_A$) is entirely contained in a line through the origin. We determine all the 21 possible regions $S_A$ of complex sign nonsingular matrices $A$ and all the possible regions $R_A$ of ray nonsingular matrices $A$. We also study some further properties of $S_A$ (and $R_A$) when $A$ is not necessarily complex sign nonsingular (or ray nonsingular).

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1. Introduction

The sign of a real number $a$, denoted by $\text{sgn}(a)$, is defined to be 1, $-1$ or 0, according to $a > 0$, $a < 0$ or $a = 0$. The sign pattern of a real matrix $A$ is the
(0, 1, −1)-matrix obtained from A by replacing each entry by its sign. The set of real matrices with the same sign pattern as A is called the qualitative class of A, and is denoted by \( Q(A) \) [1,3].

The concept of the qualitative class \( Q(A) \) was extended from real matrices to complex matrices in [4–6] in the following two ways.

Let \( M_{m \times n}(\mathbb{C}) \) be the set of \( m \times n \) complex matrices and let \( A = (a_{pq}) \in M_{m \times n}(\mathbb{C}) \). The ray pattern class of \( A \), denoted by \( QR(A) \), is the following set of complex matrices [4–6]:

\[
QR(A) = \{ B = (b_{pq}) \in M_{m \times n}(\mathbb{C}) \mid b_{pq} = c_{pq}a_{pq} \text{ for some positive number } c_{pq} \}.
\] (1.1)

If \( A = A_1 + iA_2 \), where \( A_1 \) and \( A_2 \) are real matrices, then the complex sign pattern class of \( A \), denoted by \( QC(A) \), is the following set of complex matrices [4]:

\[
QC(A) = \{ B_1 + iB_2 \mid B_1 \in Q(A_1) \text{ and } B_2 \in Q(A_2) \}.
\] (1.2)

It is obvious from the definitions that \( QR(A) \subseteq QC(A) \) (since \( QR(A) \subseteq QC(A) \)), and it is clear that \( A \) is ray-nonsingular (or complex sign nonsingular), if and only if \( 0 \notin RA \) (or \( 0 \notin SA \)).

Several problems were proposed in [4] concerning the determinantal regions \( SA \).

Among them are the following three problems.

**Problem 1.** If \( A \) is complex sign nonsingular, are the boundaries of the determinantal region \( SA \) always on the axes in the complex plane?

**Problem 2.** How can the determinantal regions \( SA \) be located for complex sign nonsingular matrices \( A \)?
Problem 3. Determine all the possible determinantal regions $S_A$ for arbitrary complex square matrices $A$.

In this paper, we give (in Theorem 3.2) an affirmative answer to Problem 1 (not only for complex sign nonsingular matrices, but also for arbitrary complex square matrices). We also show (in Theorem 3.3) that all the nonzero boundary points of $S_A$ (resp. $R_A$) are not in $S_A$ (resp. $R_A$) except the cases that $S_A$ (resp. $R_A$) is entirely contained in a line through the origin. It follows from this result that if $S_A$ (resp. $R_A$) is not entirely contained in a line through the origin, then $S_A \setminus \{0\}$ (resp. $R_A \setminus \{0\}$) must be an open set in the complex plane $\mathbb{C}$, and thus must be a disjoint union of open sectors (for $S_A$, each such sector should also be a union of several quadrants).

As for Problem 2, it is proved in [4–Theorem 2.1] that the determinantal region $S_A$ is a sector in the complex plane if $A$ is a complex sign nonsingular matrix. We show in Theorem 4.1 of this paper that there are exactly 21 possible determinantal regions $S_A$ for complex sign nonsingular matrices $A$, and all these 21 possible regions are listed in Section 4. We also determine all the possible determinantal regions $R_A$ for ray nonsingular matrices $A$ in Theorem 4.2. For Problem 3 concerning the determinantal regions $S_A$ for general complex square matrices $A$, we show in Section 5 that there are at least 29 (and at most 54) possible determinantal regions. We list all these 29 regions and conjecture that these 29 regions are all the possible determinantal regions $S_A$ in general cases. In Section 6, we study some further properties of the determinantal regions $R_A$, and propose some research problems concerning the regions $R_A$.

2. Notation and terminology

A complex matrix $A$ of order $n$ is said to have identically zero determinant (or to be combinatorially singular), if $R_A = \{0\}$ (i.e., $\det \tilde{A} = 0$ for all $\tilde{A} \in QR(A)$). It turns out that this property depends only on the zero–nonzero pattern of $A$ (since $R_A = \{0\}$ if and only if each term in the determinantal expansion of $A$ is zero). By using the well-known König’s theorem [2], it is easy to see that the following four statements are equivalent:

(1) $R_A = \{0\}$.
(2) $\rho(A) < n$.
(3) $A$ contains a $p \times q$ zero submatrix with $p + q = n + 1$.
(4) $S_A = \{0\}$.

where $\rho(A)$ (the term rank of $A$) is the maximal cardinality of the set of nonzero entries of $A$ no two of which lie on the same row or same column.

It is pointed out in [6–Lemma 4.5] and [4] that both the regions $R_A$ and $S_A$ are closed under the multiplication by a positive number $k$ (simply by taking a matrix $B$
obtained from $A$ by multiplying the first row of $A$ by $k$. So both $R_A \setminus \{0\}$ and $S_A \setminus \{0\}$ are unions of the open rays originating at (but excluding) the origin.

Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real numbers and the set of complex numbers, respectively. For two subsets $S \subseteq \mathbb{C}$, $T \subseteq \mathbb{C}$ and a complex number $z \in \mathbb{C}$, we define
\begin{equation}
ST = \{ab \mid a \in S \text{ and } b \in T\},
\end{equation}
\begin{equation}
zS = \{za \mid a \in S\},
\end{equation}
\begin{equation}
-S = \{-a \mid a \in S\}.
\end{equation}
The boundary of the set $S \subseteq \mathbb{C}$, denoted by $\text{bd}(S)$, is the set difference of its topological closure $\overline{S}$ and its interior $\text{int}(S)$. Thus a point $z$ is in $\text{bd}(S)$ if and only if only if each neighborhood of $z$ contains both a point in $S$ and a point not in $S$. Notice that a point $z$ in $\text{bd}(S)$ may or may not be in $S$. For example, if $S$ is the open unit disk, then all the points on the unit circle are in $\text{bd}(S)$, but not in $S$.

The real part, imaginary part and the argument of a complex number $z$ are denoted by $\text{Re}(z)$, $\text{Im}(z)$ and $\text{arg}(z)$, respectively.

Given two angles $\alpha \leq \beta$, the open sector from $\alpha$ to $\beta$, denoted by $S(\alpha, \beta)$, is defined to be the following set of nonzero complex numbers:
\begin{equation}
S(\alpha, \beta) = \{z \in \mathbb{C} \mid z \neq 0 \text{ and } \alpha < \text{arg}(z) < \beta\}
\end{equation}
and $\beta - \alpha$ is called the angle of the sector $S(\alpha, \beta)$ if $\beta - \alpha \leq 2\pi$. For example, the sector $S(0, \frac{\pi}{2})$ is just the open first quadrant. It is easy to see that each open sector $S(\alpha, \beta)$ is a connected set (in the complex plane) closed under the multiplication by a positive number.

If both $\alpha$ and $\beta$ are integral multiples of $\frac{\pi}{2}$, then the open sector $S(\alpha, \beta)$ is also called an open quadrant sector.

Concerning the complex sign pattern class $Q_C(A)$ of a matrix $A$, it is easy to see that there are essentially 9 different entries of $A$ (thus 8 different nonzero entries of $A$), since for each entry $a_{pq}$ of $A$, there are 3 choices for the sign of $\text{Re}(a_{pq})$ (from $\{0, 1, -1\}$) and 3 choices for the sign of $\text{Im}(a_{pq})$. If both $\text{Re}(a_{pq})$ and $\text{Im}(a_{pq})$ are not zero, then the entry $\bar{a}_{pq}$ of all $\bar{A} \in Q_C(A)$ can take the values of a whole open quadrant (in other words, $Q_C(a_{pq})$ is an open quadrant). In this case, we call the entry $a_{pq}$ a quadrant entry of $A$. If exactly one of $\text{Re}(a_{pq})$ and $\text{Im}(a_{pq})$ is zero, then $Q_C(a_{pq})$ is an open half-axis. In this case we call the entry $a_{pq}$ an axis entry of $A$. Obviously, there are 4 possible quadrant entries and 4 possible axis entries of complex matrices.

3. The solution of Problem 1

In this section, we give an affirmative answer to Problem 1 in general cases (not only in the complex sign nonsingular cases). We also show that if $S_A$ (resp. $R_A$) is not entirely contained in a line through the origin, then $S_A \setminus \{0\}$ (resp. $R_A \setminus \{0\}$) is always an open set in the complex plane, and thus is a disjoint union of open sectors.
In order to answer Problem 1, we first prove a slightly stronger result in the following theorem.

**Theorem 3.1.** Let \( A = (a_{ij}) \) be a complex matrix of order \( n \) and \( S_A \) be the determinantal region of \( A \). Suppose that \( S_A \) contains some complex number \( z \) in an open quadrant \( S(\alpha, \alpha + \frac{\pi}{2}) \), where \( \alpha \) is an integral multiple of \( \frac{\pi}{2} \). Then for any \( \delta > 0 \), there exist matrices \( B_1 \) and \( B_2 \) in \( QC(A) \) satisfying the following two conditions:

1. \( B_1 \) and \( B_2 \) differ in at most one row.
2. \( \det B_1 \in S(\alpha, \alpha + \delta) \) and \( \det B_2 \in S(\alpha + \frac{\pi}{2} - \delta, \alpha + \frac{\pi}{2}) \).

**Proof.** Let \( A(i j) \) be the submatrix of \( A \) obtained by deleting the \( i \)th row and the \( j \)th column of \( A \). Let \( A_{ij} = (-1)^{i+j} \det A(i j) \) be the algebraic co-minor of \( A(i, j = 1, 2, \ldots, n) \). Then we have

\[
\det A = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}.
\]  
(3.1)

Without loss of generality we may assume that \( \det A = z \in S(\alpha, \alpha + \frac{\pi}{2}) \). We now consider the following cases.

**Case 1.** There exist two indices \( j_1 \) and \( j_2 \) in \( \{1, \ldots, n\} \) such that

\[
a_{1j_1}A_{1j_1} \in S(\alpha - \frac{\pi}{4}, \alpha) \quad \text{and} \quad a_{1j_2}A_{1j_2} \in S(\alpha + \frac{\pi}{4}, \alpha + \pi).
\]

Then since \( z \in S(\alpha, \alpha + \frac{\pi}{2}) \), there exist positive numbers \( \lambda_1 \) and \( \lambda_2 \) such that

\[
z + \lambda_1 a_{1j_1}A_{1j_1} \in S(\alpha, \alpha + \delta)
\]  
(3.2)

and

\[
z + \lambda_2 a_{1j_2}A_{1j_2} \in S(\alpha + \frac{\pi}{4} - \delta, \alpha + \frac{\pi}{4}).
\]  
(3.3)

Now take the matrix \( B_k \) which is obtained from \( A \) by replacing the entry \( a_{1j_k} \) by \( (1 + \lambda_k)a_{1j_k} \) \((k = 1, 2)\). Then clearly \( B_1, B_2 \) are in \( QC(A) \), \( B_1 \) and \( B_2 \) differ only in the first row, and

\[
\det(B_k) = \det A + \lambda_k a_{1j_k}A_{1j_k} = z + \lambda_k a_{1j_k}A_{1j_k} \quad (k = 1, 2).
\]

So \( B_1 \) and \( B_2 \) satisfy condition (2) by (3.2) and (3.3).

**Case 2.** Case 1 does not occur.

Then there must exist some index \( j \) in \( \{1, \ldots, n\} \) such that \( a_{1j}A_{1j} \in S(\alpha, \alpha + \frac{\pi}{2}) \). For otherwise, since Case 1 does not occur, either all the \( \text{Re}(e^{-i\alpha}a_{1p}A_{1p}) \leq 0 \) for \( p = 1, \ldots, n \) or all the \( \text{Im}(e^{-i\alpha}a_{1p}A_{1p}) \leq 0 \) for \( p = 1, \ldots, n \), which implies by (3.2) that either \( \text{Re}(e^{-i\alpha} \det A) \leq 0 \) or \( \text{Im}(e^{-i\alpha} \det A) \leq 0 \), contradicting the fact \( \det A = z \in S(\alpha, \alpha + \frac{\pi}{2}) \) (and thus both the real part and the imaginary part of \( e^{-i\alpha}z \)).
are positive. We now assume \( a_{11} A_{11} \in S_{(\alpha, \alpha + \frac{\delta}{2})} \) and then consider the following three subcases.

**Subcase 2.1.** \( a_{11} \) is an axis entry.

Without loss of generality we assume that \( a_{11} \) is a real positive number. Then \( A_{11} = \text{det} A(1|1) \) is in the open quadrant \( S_{(\alpha, \alpha + \frac{\delta}{2})} \) since \( a_{11} A_{11} \) is in \( S_{(\alpha, \alpha + \frac{\delta}{2})} \). By using induction on the matrix \( A(1|1) \), there exist matrices \( C_1 \) and \( C_2 \) in \( QC(A(1|1)) \) satisfying conditions (1) and (2). Now take \( \varepsilon > 0 \) sufficiently small and take

\[
B_k = \begin{pmatrix}
a_{11} & \varepsilon a_{12} & \cdots & \varepsilon a_{1n} \\
a_{21} & & & \\
& \ddots & & \\
a_{n1} & & C_k
\end{pmatrix} \quad (k = 1, 2).
\]

Then \( B_k \in QC(A) \), \( B_1 \) and \( B_2 \) differ in at most one row (since \( C_1 \) and \( C_2 \) do), and

\[
\text{det} B_k = a_{11} \text{det} C_k + \varepsilon d_k \quad (k = 1, 2),
\]

where \( d_k \) is a constant independent of \( \varepsilon \). Take \( \varepsilon \) small enough, then \( B_1 \) and \( B_2 \) can also satisfy condition (2).

**Subcase 2.2.** \( a_{11} \) is a quadrant entry in some open quadrant \( S_{(\beta_1, \beta_1 + \frac{\delta}{2})} \) and \( A_{11} = \det A(1|1) \) is also in some open quadrant \( S_{(\beta_2, \beta_2 + \frac{\delta}{2})} \).

Then since \( a_{11} A_{11} \in S_{(\alpha, \alpha + \frac{\delta}{2})} \), there exist \( a_1^{(k)} \in S_{(\beta_1, \beta_1 + \frac{\delta}{2})} \) for \( k = 1, 2 \) and \( \tilde{a}_2 \in S_{(\beta_2, \beta_2 + \frac{\delta}{2})} \) such that

\[
a_1^{(1)} \tilde{a}_2 \in S_{(\alpha, \alpha + \frac{\delta}{2})} \quad \text{and} \quad a_1^{(2)} \tilde{a}_2 \in S_{(\alpha + \frac{\delta}{2}, \alpha + \frac{\delta}{2})}.
\] (3.4)

Now using induction on the matrix \( A(1|1) \), there exist \( C_1 \) and \( C_2 \) in \( QC(A(1|1)) \) satisfying conditions (1) and (2) of this theorem (with \( \alpha \) replaced by \( \beta_2 \)). Also \( C_1 \) and \( C_2 \) satisfy condition (2) (for \( \delta > 0 \) small enough) imply that there exists some \( \lambda > 0 \) such that \( \arg(\det C_1 + \lambda \det C_2) = \arg(\tilde{a}_2) \). So by (3.4) we have

\[
a_1^{(1)}(\det C_1 + \lambda \det C_2) \in S_{(\alpha, \alpha + \delta)}
\] (3.5)

and

\[
a_1^{(2)}(\det C_1 + \lambda \det C_2) \in S_{(\alpha + \frac{\delta}{2}, \alpha + \frac{\delta}{2})}.
\] (3.6)

Suppose that \( C_1 \) and \( C_2 \) differ only in the first row, and \( \alpha_1 \) and \( \alpha_2 \) are the first row of \( C_1 \) and \( C_2 \), respectively. Take \( C \) to be the matrix obtained from \( C_1 \) by replacing its first row \( \alpha_1 \) by \( \alpha_1 + \lambda \alpha_2 \). Then \( C \in QC(A(1|1)) \) and \( \det C = \det C_1 + \lambda \det C_2 \). Now take \( \varepsilon > 0 \) and take \( B_1, B_2 \) as
Then $B_1$ and $B_2$ are in $QC(A)$ and they only differ in the first row, also
\[
\det B_k = a^{(k)}_1 \det C + \varepsilon d_k = a^{(k)}_1 (\det C_1 + \lambda \det C_2) + \varepsilon d_k \quad (k = 1, 2).
\]
Take $\varepsilon$ small enough, then $B_1$ and $B_2$ can also satisfy condition (2) by (3.5) and (3.6).

**Subcase 2.3.** $a_{11}$ is a quadrant entry and $A_{11} = \det A(1|1)$ is on some axis.

Then since $a_{11} A_{11} \in S_{(\alpha, \alpha + \frac{\delta}{a_{ii}^2})}$, we can take $a^{(1)}_{11}$ and $a^{(2)}_{11}$ in the quadrant $QC(a_{11})$ such that
\[
a^{(1)}_{11} A_{11} \in S_{(\alpha, \alpha + \delta)}
\]
and
\[
a^{(2)}_{11} A_{11} \in S_{(\alpha + \frac{\delta}{a_{ii}^2}, \alpha + \frac{\delta}{a_{ii}^2})}.
\]
Similar to (3.7) we take $\varepsilon > 0$ and take $B_1, B_2$ as
\[
B_k = \begin{pmatrix}
a^{(k)}_{11} & \varepsilon a_{12} & \cdots & \varepsilon a_{1n} \\
a^{(k)}_{21} & C \\
\vdots \\
a^{(k)}_{n1}
\end{pmatrix} \quad (k = 1, 2).
\]
Then similar to Subcase 2.2 we can check that $B_1$ and $B_2$ are in $QC(A)$ and satisfy conditions (1) and (2) when $\varepsilon$ is small enough.

The following corollary, which is an easy consequence of Theorem 3.1, shows that if $S_A$ contains some interior point of some quadrant, then $S_A$ must contain the whole open quadrant.

**Corollary 3.1.** Let $A$ be a complex square matrix and $z = a + bi$ be a complex number with $ab \neq 0$. Let $QC(z)$ be the open quadrant containing $z$. If $z \in S_A$, then $QC(z) \subseteq S_A$.

**Proof.** Suppose that the open quadrant $QC(z)$ is $S_{(\alpha, \alpha + \frac{\delta}{a_{ii}^2})}$.

For any $u \in S_{(\alpha, \alpha + \frac{\delta}{a_{ii}^2})}$, take $\delta > 0$ such that $u \in S_{(\alpha + \frac{\delta}{a_{ii}^2}, \alpha + \frac{\delta}{a_{ii}^2})}$. By Theorem 3.1 there exist matrices $B_1$ and $B_2$ in $QC(A)$ satisfying the conditions (1) and (2) in Theorem 3.1. By condition (2), there exist positive numbers $\lambda_1$ and $\lambda_2$ such that $\lambda_1 \det B_1 + \lambda_2 \det B_2 = u$. Now suppose that $B_1$ and $B_2$ differ only in the first row which are respectively $\alpha_1$ and $\alpha_2$. Take $A$ to be the matrix obtained from $B_1$ by
replacing its first row \( \alpha_1 \) by \( \lambda_1 \alpha_1 + \lambda_2 \alpha_2 \). Then clearly \( \tilde{A} \in QC(A) \) and \( \det \tilde{A} = \lambda_1 \det B_1 + \lambda_2 \det B_2 = u \). Thus \( u \in S_A \) and so \( QC(z) \subseteq S_A \). \( \square \)

Now we are ready to give an affirmative answer to Problem 1

**Theorem 3.2.** Let \( A \) be a complex matrix of order \( n \) and \( z \) be a boundary point of the region \( S_A \). Then \( z \) lies on some axis.

**Proof.** Suppose to the contrary that \( z \) is not on the axes, then \( z \) is inside some open quadrant \( QC(z) \). Since \( z \) is a boundary point of \( S_A \), any neighborhood of \( z \) contains a point in \( S_A \). In particular, \( QC(z) \) contains a point, say \( w \), in \( S_A \). It follows from Corollary 3.1 that \( QC(z) = QC(w) \subseteq S_A \), so \( z \) is not a boundary point of \( S_A \), a contradiction. \( \square \)

From Theorem 3.2 and the fact that the set \( S_A \) is closed under the multiplication by a positive number, we see that the boundary set \( \text{bd}(S_A) \) is always a union of several closed half-axes (including the origin).

Now we turn to study some other properties related to \( \text{bd}(S_A) \).

We know that a boundary point \( z \in \text{bd}(S_A) \) may or may not be in \( S_A \). In the remaining part of this section, we will show that if \( S_A \) is not entirely contained in some axis, then all the nonzero boundary points of \( S_A \) are not in \( S_A \). This result will imply that \( S_A \setminus \{0\} \) is always an open set in the complex plane, and thus is a disjoint union of some open sectors (and each of such open sectors is a union of several adjacent quadrants), except the cases that \( S_A \) is entirely contained in some axis.

**Note.** All the results of the remaining part of this section will be true for both \( S_A \) and \( R_A \) (i.e., for both complex sign pattern matrices and ray pattern matrices).

We first prove the following lemmas.

**Lemma 3.1.** Let \( A \) be a complex matrix of order \( n \) and \( A_1, A_2 \in QC(A) \) (resp. \( QR(A) \)) satisfying:

1. \( \det A_1 \neq \det A_2 \).
2. \( A_1 \) and \( A_2 \) differ only in one row.

Then there exists an open line segment \( L \) in \( S_A \) (resp. \( R_A \)) containing \( \det A_1 \) and \( \det A_2 \).

**Proof.** Suppose that \( A_1 \) and \( A_2 \) differ only in the first row, and \( \alpha_1 \) and \( \alpha_2 \) are the first row of \( A_1 \) and \( A_2 \), respectively. Take \( M_x \) to be the matrix obtained from \( A_1 \) by replacing its first row \( \alpha_1 \) by \( x\alpha_1 + (1-x)\alpha_2 \). Then there exists \( \varepsilon > 0 \) such that \( M_{x} \in QC(A) \) (resp. \( QR(A) \)) for all \( x \in (-\varepsilon, 1 + \varepsilon) \). Take
\[ L = \{ \det M_x \mid x \in (-\varepsilon, 1 + \varepsilon) \}, \]

where \( \det M_x = x \det A_1 + (1 - x) \det A_2 \), then \( L \) is an open line segment in \( S_A \) (resp. \( R_A \)) containing \( \det A_1 \) and \( \det A_2 \). \( \square \)

The following lemma shows that if some positive number is both in \( S_A \) and \( \text{bd}(S_A) \), then \( S_A \) must be entirely contained in the real axis.

**Lemma 3.2.** Let \( A \) be a complex matrix of order \( n \) with \( \det A > 0 \). If \( \det A \in \text{bd}(S_A) \), then \( S_A \subseteq \mathbb{R} \).

(Similarly, if \( \det A \in \text{bd}(R_A) \), then \( R_A \subseteq \mathbb{R} \).)

**Proof.** Firstly, for each matrix \( B \in Q_C(A) \) with at most one row different from \( A \), \( \det B \) must be real. For otherwise using Lemma 3.1 for \( A \) and \( B \) (together with the fact that \( S_A \) is closed under the positive multiplication), we can derive that \( \det A \) is an interior point of \( S_A \), a contradiction.

Secondly, for \( \varepsilon > 0 \) we define

\[ Q^{(\varepsilon)}_C(A) = \{ B \in Q_C(A) \mid |b_{ij} - a_{ij}| < \varepsilon \text{ for } i, j = 1, \ldots, n \}. \]

Take \( \varepsilon \) small enough such that for each matrix \( B \) in \( Q^{(\varepsilon)}_C(A) \), \( \det B \) is not a nonpositive real number (since \( \det B \) can be close to \( \det A \) enough and \( \det A > 0 \)).

Now we show that for such \( \varepsilon \), each matrix \( B \) in \( Q^{(\varepsilon)}_C(A) \) has \( \det B > 0 \). For this purpose, let

\[ A_0 = A, \quad A_1, \ldots, A_n = B \]

be a sequence of matrices such that each \( A_i \) is in \( Q^{(\varepsilon)}_C(A) \), and \( A_i \) and \( A_{i+1} \) differ in at most one row \((i = 0, 1, \ldots, n - 1) \). By induction \( \det A_i > 0 \). Using the above conclusion on \( A_i \), we deduce that \( \det A_{i+1} \) is real. On the other hand, \( \det A_{i+1} \) is not a nonpositive real number (by the choice of \( \varepsilon \)) since \( A_{i+1} \in Q^{(\varepsilon)}_C(A) \), so we conclude that \( \det A_{i+1} > 0 \). Thus \( \det B = \det A_n > 0 \) by induction on \( i \).

Thirdly, we show that each term in the determinantal expansion of \( A \) is real. For example, we show that \( a_{11}a_{22} \cdots a_{nn} \) is real. Let

\[ A_x = A + x \cdot \text{diag}(a_{11}, a_{22}, \ldots, a_{nn}), \]

where \( \text{diag}(a_{11}, \ldots, a_{nn}) \) denotes the diagonal matrix with diagonal entries \( a_{11}, \ldots, a_{nn} \).

Then \( A_x \in Q^{(\varepsilon)}_C(A) \) for sufficiently small \( x > 0 \). Write

\[ \det A_x = f(x) + ig(x), \]

where \( f(x) \) and \( g(x) \) are real polynomials on \( x \). Then for all sufficiently small \( x > 0 \), we have \( \det A_x > 0 \) since \( A_x \in Q^{(\varepsilon)}_C(A) \). Thus \( g(x) = 0 \) for all sufficiently small \( x > 0 \), and so \( g(x) \) is a zero polynomial. Therefore, \( \det A_x = f(x) \) is a real polynomial. In particular, its coefficient of the term \( x^n \), namely \( a_{11}a_{22} \cdots a_{nn} \), is real.
To show $S_A \subseteq \mathbb{R}$, take any nonzero term in the determinantal expansion of $A$, say $a_{11}a_{22}\cdots a_{nn} \neq 0$. We want to show that each $a_{ii}$ is an axis entry. For otherwise suppose $a_{11}$ is a quadrant entry. Take $\tilde{a}_{11} \in QC(a_{11})$ such that $\tilde{a}_{11}/a_{11}$ is not real and $|\tilde{a}_{11} - a_{11}| < \varepsilon$. Let $\tilde{A}$ be the matrix obtained from $A$ by replacing $a_{11}$ by $\tilde{a}_{11}$. Then clearly $\tilde{A} \in QC^{(\varepsilon)}(A)$ and thus $\det \tilde{A} > 0$ by the above arguments. So similarly using the above arguments on $\tilde{A}$ we conclude that each term in the determinantal expansion of $\tilde{A}$ is also real. It follows that $\tilde{a}_{11}a_{22}\cdots a_{nn}$ is also real, contradicting the fact that $\tilde{a}_{11}/a_{11}$ is not real. So each $a_{ii}$ is an axis entry. Therefore, for each $B \in QC(A)$, each nonzero term $b_{1j_1}b_{2j_2}\cdots b_{nj_n}$ in the determinantal expansion of $B$ is a positive multiple of $a_{1j_1}a_{2j_2}\cdots a_{nj_n}$ which is real, so $b_{1j_1}b_{2j_2}\cdots b_{nj_n}$ is also real and thus $\det B$ is real. This proves that $S_A \subseteq \mathbb{R}$.

(The proof of the result for $R_A$ is similar.) □

Lemma 3.2 is actually a special case of a more general result in the following theorem.

**Theorem 3.3.** Let $A$ be a complex square matrix such that $S_A$ (resp. $R_A$) is not contained in any line through the origin. Then for each $z \in bd(S_A)$ with $z \neq 0$, we have $z \notin S_A$.

**Proof.** Suppose to the contrary that $z \in S_A$, and without loss of generality we may assume that $z = \det A$. By Theorem 3.2, $z$ lies on some axis. So $i^k z > 0$ for some integer $k$.

Now let $B$ be the matrix obtained from $A$ by multiplying its first row by $i^k$. Then $\det B = i^k \cdot \det A = i^k \cdot z > 0$. Also $\det A = z \in bd(S_A)$ implies that $\det B \in bd(S_B)$. So using Lemma 3.2 for $B$ we conclude that $S_B \subseteq \mathbb{R}$, and thus $S_A = i^{-k} \cdot S_B$ will be contained in some line through the origin, a contradiction.

(For the proof of the result for $R_A$, just replace $i^k$ by $\bar{z}$.) □

Theorem 3.3 tells us that, if $S_A$ (resp. $R_A$) is not entirely contained in any line through the origin, then $S_A \setminus \{0\}$ (resp. $R_A \setminus \{0\}$) is an open set in $\mathbb{C}$. It follows that in this case, $S_A \setminus \{0\}$ (resp. $R_A \setminus \{0\}$) is always a disjoint union of open sectors in the complex plane (recalling that $S_A$ and $R_A$ are closed under positive multiplications).

4. The determinantal regions of complex sign-nonsingular matrices and ray-nonsingular matrices

In this section, we consider Problem 2 about the possible determinantal regions $S_A$ for complex sign-nonsingular matrices $A$. We will show that there are exactly 21 possible determinantal regions $S_A$ for complex nonsingular matrices $A$. We will also determine all the possible determinantal regions $R_A$ for ray nonsingular matrices $A$. 
Similar to the real case that the determinantal region of a real matrix is always a connected subset of \( \mathbb{R} \), we have the following.

**Lemma 4.1.** Let \( A = (a_{ij}) \) be a complex matrix of order \( n \). Then \( S_A \) (and also \( R_A \)) is a connected subset of \( \mathbb{C} \).

**Proof.** The matrix class \( QC(A) \) can be viewed as a subset of \( \mathbb{C}^{n^2} \), and in fact can be written as the following cartesian product:

\[
QC(A) = \prod_{i,j=1}^{n} QC(aij).
\]

Since each \( QC(aij) \) is a connected subset of \( \mathbb{C} \), and the cartesian product of connected sets is connected, we know that \( QC(A) \) is a connected set in \( \mathbb{C}^{n^2} \). Now the determinant is a continuous function on the entries of the matrices, so \( S_A \) is the continuous image of the connected set \( QC(A) \). Thus \( S_A \) is also connected. \( \square \)

The \( S_A \) case of the following corollary is essentially the Theorem 2.1 of [4], except that “sector” is replaced by “open sector” here.

**Corollary 4.1.** Let \( A \) be a complex sign-nonsingular (resp. ray-nonsingular) matrix of order \( n \). Then \( S_A \) (resp. \( R_A \)) is either a ray originating at (but excluding) the origin or a single open sector.

**Proof.** By hypothesis \( 0 \not\in S_A \), so the result follows from Lemma 4.1 and Theorem 3.3. \( \square \)

Now if we combine the conclusions of Theorem 3.2 and Corollary 4.1, we will see that if \( A \) is a complex sign-nonsingular matrix, then \( S_A \) must be in one of the following 6 types.

(T1) A half-axis (excluding the origin).
(T2) An open quadrant.
(T3) An open sector consisting of two adjacent quadrants.
(T4) An open sector consisting of three quadrants.
(T5) An open sector consisting of four quadrants.
(T6) \( \mathbb{C}\setminus\{0\} \)

Notice that for each of the types (T1)–(T5), there are exactly four (possible) regions. So altogether there are exactly 21 (possible) regions in the types (T1)–(T6). Actually it can be easily seen from the examples given in [4] that all these 21 regions can be attained as the determinantal region \( S_A \) of some complex sign nonsingular matrix \( A \).
Theorem 4.1. Let \( S \) be a subset of \( \mathbb{C} \). Then \( S = S_A \) for some complex sign nonsingular matrix \( A \) if and only if \( S \) is one of the 21 regions in the types (T1)–(T6).

Proof. The necessity part follows from Theorem 3.2 and Corollary 4.1. For the sufficiency part, just look at the following examples (also see [4, Example 2.2]), where \( I_n \) is the identity matrix of order \( n \) and \( B \oplus C \) denotes the direct sum of the matrices \( B \) and \( C \):

1. Let \( A_k = (i^k) \) be a matrix of order 1, then \( S_{A_k} \) for \( k = 0, 1, 2, 3 \) attains all the 4 possible regions of type (T1).
2. Let \( A_k = (i^k) \oplus (I_1 + iI_1) \), then \( S_{A_k} \) for \( k = 0, 1, 2, 3 \) attains all the 4 possible regions of type (T2).
3. Let \( A_k = (i^k) \oplus (I_2 + iI_2) \), then \( S_{A_k} \) for \( k = 0, 1, 2, 3 \) attains all the 4 possible regions of type (T3).
4. Let \( A_k = (i^k) \oplus (I_3 + iI_3) \), then \( S_{A_k} \) for \( k = 0, 1, 2, 3 \) attains all the 4 possible regions of type (T4).
5. Let \( A_k = (i^k) \oplus (I_4 + iI_4) \), then \( S_{A_k} \) for \( k = 0, 1, 2, 3 \) attains all the 4 possible regions of type (T5).
6. Let \( A = I_5 + iI_5 \), then \( S_A = \mathbb{C}\{0\} \).

□

We notice that all the examples given in Theorem 4.1 are diagonal matrices.

Now we turn to consider possible regions \( R_A \) for ray nonsingular matrices \( A \).

Theorem 4.2. Let \( S \) be a subset of \( \mathbb{C} \). Then \( S = R_A \) for some ray nonsingular matrix \( A \) if and only if \( S \) is in one of the following three cases:

1. \( S \) is a ray originating at (but excluding) the origin.
2. \( S \) is an open sector \( S(\beta_1, \beta_2) \) with \( 0 < \beta_2 - \beta_1 \leq 2\pi \).
3. \( S = \mathbb{C}\{0\} \).

Proof. The necessity part follows from Theorem 3.2 and Corollary 4.1. For the sufficiency part, we consider the following examples:

1. Let \( A = (e^{i\theta}) \) be a matrix of order 1, then \( A \) is ray-nonsingular and \( R_A \) is the ray with argument \( \theta \).
2. Take \( \alpha_1, \ldots, \alpha_k \) with \( 0 < \alpha_i < \frac{\pi}{2} \) \( (i = 1, \ldots, k) \) such that \( \alpha_1 + \cdots + \alpha_k = \beta_2 - \beta_1 \). Take

\[
A = (e^{i\beta_1}) \oplus \left( \begin{array}{cc} 1 & -1 \\ e^{i\alpha_1} & 1 \end{array} \right) \oplus \cdots \oplus \left( \begin{array}{cc} 1 & -1 \\ e^{i\alpha_k} & 1 \end{array} \right).
\]

(4.1)

Then it is not difficult to verify that \( A \) is ray-nonsingular and \( R_A = S(\beta_1, \beta_2) \).
3. If \( A \) is defined as in (4.1) where we take \( \alpha_1, \ldots, \alpha_k \) such that \( \alpha_1 + \cdots + \alpha_k > 2\pi \), then we can verify that \( R_A = \mathbb{C}\{0\} \).

□
5. The determinantal regions $S_A$ for general cases

In this section, we consider Problem 3 concerning the possible determinantal regions $S_A$ for arbitrary complex square matrices $A$ (which are not necessarily complex sign-nonsingular).

From Theorems 3.2 and 3.3, we know that if $A$ is a square complex matrix that is not complex sign-nonsingular, then $S_A$ is either entirely contained in some axis or a disjoint union of $\{0\}$ and several (disjoint) open quadrant sectors. Obviously there are only finitely many such regions. Now we list all the 53 possible such regions in the following types (A1)–(A5) and types (B1)–(B11) (those regions which are equivalent up to rotations and reflections are put into one type):

- (A1) $\{0\}$.
- (A2) The whole complex plane $\mathbb{C}$.
- (A3) (2 regions) The real axis $\mathbb{R}$ or imaginary axis $i\mathbb{R}$.
- (A4) (2 regions) The union of $\{0\}$ and two “opposite” (nonadjacent) open quadrants.
- (A5) (2 regions) $(\mathbb{C}\setminus\mathbb{R}) \cup \{0\}$ or $(\mathbb{C}\setminus i\mathbb{R}) \cup \{0\}$ (A disjoint union of $\{0\}$ and two opposite open half planes).
- (B1) (4 regions) A half-axis including the origin.
- (B2) (4 regions) The union of $\{0\}$ and an open quadrant.
- (B3) (4 regions) The union of $\{0\}$ and an open (upper, lower, right or left) half plane.
- (B4) (4 regions) The union of $\{0\}$ and an open quadrant sector of angle $\frac{3}{2}\pi$.
- (B5) (4 regions) The union of $\{0\}$ and an open quadrant sector of angle $2\pi$.
- (B6) (4 regions) A disjoint union of $\{0\}$ and two “adjacent” open quadrants.
- (B7) (4 regions) A disjoint union of $\{0\}$ and three open quadrants.
- (B8) (1 region) A disjoint union of $\{0\}$ and four open quadrants.
- (B9) (8 regions) A disjoint union of $\{0\}$ and an open half plane and an open quadrant.
- (B10) (4 regions) A disjoint union of $\{0\}$ and an open half plane and two open quadrants.
- (B11) (4 regions) A disjoint union of $\{0\}$ and an open quadrant and an open quadrant sector of angle $\frac{3}{2}\pi$.

Altogether there are 8 possible regions in the types (A1)–(A5) and 45 possible regions in the types (B1)–(B11).

**Theorem 5.1.** All the 8 possible regions in the types (A1)–(A5) is the determinantal region $S_A$ of some complex square matrix $A$.

**Proof**
1. Take $A = O$, then $S_A = \{0\}$.
2. Take $A = \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right)$ ⊕ $B$, then $S_A = S_B \cup \{0\} \cup (-S_B)$.
If we take complex sign-nonsingular matrix $B$ such that $SB$ is some (suitable) one of the types (T1), (T2), (T3) or (T4) (as in Section 4), then $SA$ can be (any) one of the types (A3), (A4), (A5) or (A2). □

From Theorem 5.1 and the results in Section 4, we can see that in order to solve Problem 3 (i.e., determine all the possible determinantal regions for arbitrary complex square matrices), it suffices to determine whether or not each of the regions of types (B1)–(B11) can be the determinantal region $SA$ of some complex square matrix $A$. For this question we propose the following conjecture:

**Conjecture.** There is no complex square matrix whose determinantal region $SA$ is one of types (B1)–(B11).

The above conjecture is obviously equivalent to the following statement:

**Statement.** All the possible determinantal regions $SA$ of the arbitrary complex square matrices $A$ are the 29 regions in the types (T1)–(T6) and the types (A1)–(A5).

So if the above conjecture turns out to be true, then Problem 3 in Section 1 is solved.

As partial results concerning the above conjecture, we now exclude types (B1)–(B3) and types (B6)–(B7) as possible determinantal regions $SA$, as a consequence of the following theorem.

**Theorem 5.2.** Let $A$ be a complex square matrix such that $SA = [0] \cup F_1 \cup \cdots \cup F_k$, where $k \geq 1$ and $F_1, \ldots, F_k$ are disjoint open (quadrant) sectors. Then for each $F_j$, there exists some $z \in F_j$ such that $SA$ contains some open line segment $L$ containing $0$ and $z$.

**Proof.** Let $B$ and $C$ be matrices in $QC(A)$ such that $\det B \in F_j$ and $\det C \notin F_j$. Let $A_0, A_1, \ldots, A_n$ be the sequence of matrices in $QC(A)$ such that $A_0 = B$, $A_0 = C$ and $A_r$ and $A_{r+1}$ differ in at most one row for $r = 0, 1, \ldots, n - 1$. Then there exists some index $r$ such that $\det A_r \in F_j$ and $\det A_{r+1} \notin F_j$.

By Lemma 3.1, $SA$ contains some open line segment $L$ containing $\det A_r$ and $\det A_{r+1}$. Now $L$ must contain $0$, since $\det A_r \in F_j$ and $\det A_{r+1} \notin F_j$ (otherwise some nonzero boundary point of $F_j$ will be in $SA$, a contradiction). Let $z = \det A_r$, then $z \in F_j$ and $L$ contains $0$ and $z$. □

(It is easy to see that Theorem 5.2 is also true for the case of $RA$.)

Notice that Theorem 5.2 (and its proof) is also true if $F_j$ is a half axis.

It now follows easily from Theorem 5.2 that the regions of types (B1)–(B3) and (B6)–(B7) can not be the determinantal region $SA$ of any complex square matrix $A$. 
since for each region \( S \) of such types, there exists some \( F_j \) such that \(-z \notin S\) for all \( z \in F_j \).

6. Some further properties of the determinantal region \( R_A \)

In this section we study some further properties of the determinantal region \( R_A \) of a square ray pattern matrix \( A \).

Recalling that in Sections 3–5, we already mentioned that some properties we obtained there for the regions \( S_A \) also hold for the regions \( R_A \). For convenience, we list these results in the following Theorems 6.1 and 6.2.

**Theorem 6.1** (comparing with Theorem 3.3). Let \( A \) be a complex square matrix such that \( R_A \) is not contained in any line through the origin. Then:

(1) Each nonzero boundary point of \( R_A \) is not in \( R_A \) (Thus \( R_A \setminus \{0\} \) is an open set in the complex plane \( \mathbb{C} \)).

(2) \( R_A \setminus \{0\} \) is a disjoint union of open sectors.

**Theorem 6.2** (comparing with Theorem 5.2). Let \( A \) be a complex square matrix such that \( R_A = \{0\} \cup F_1 \cup \cdots \cup F_k \), where \( F_1, \ldots, F_k \) are disjoint open sectors. Then for each \( F_j \), there exists some \( z \in F_j \) such that \( R_A \) contains some open line segment \( L \) containing 0 and \( z \).

Besides Theorems 6.1–6.2 and Theorem 4.2 in Section 4 (which gives all possible regions \( R_A \) for ray nonsingular matrices \( A \)), we give some further properties of \( R_A \) in the following Theorem 6.3 and Corollary 6.1. First we recall some notations from [6].

For a set \( S \subseteq \mathbb{C} \), the notations \( \text{ri}(S) \) (the relative interior of \( S \)), \( \text{cone}(S) \) (the cone generated by \( S \)) and \( \text{conv}(S) \) (the convex hull of \( S \)) are defined as the same as in [6], while \( \overline{S} \) denotes the topological closure of \( S \).

From the results in convex analysis [7], we know that if \( S \subseteq \mathbb{C} \) and \( |S| < \infty \), then \( z \in \text{ri}(\text{cone}(S)) \) if and only if \( z \) is a positive linear combination of the elements of \( S \).

Let \( A = (a_{ij}) \) be a complex matrix of order \( n \) and let \( \sigma \in S_n \), where \( S_n \) is the set of permutations (i.e., bijective maps) on \( \{1, \ldots, n\} \). We write:

\[
A_{\sigma} = \text{sgn}(\sigma)a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)}. \tag{6.1}
\]

Namely, \( A_{\sigma} \) is a typical term in the standard determinantal expansion of \( A \), or \( \det A = \sum_{\sigma \in S_n} A_{\sigma} \).

Let \( T(A) \) be the collection (multiset) of all nonzero \( A_{\sigma} \) with repetitions allowed and let \( T_0(A) \) be the set of distinct elements of \( T(A) \). Namely,

\[
T_0(A) = \{ A_{\sigma} \mid \sigma \in S_n \text{ and } A_{\sigma} \neq 0 \}. \tag{6.2}
\]

Then \( \det A \) is indeed equal to the sum of the elements of \( T(A) \).
There are close relationships between the determinantal region $R_A$ and the (multi)set $T(A)$ (or $T_0(A)$). For examples, we have the following two propositions.

**Proposition 6.1.** Let $A$ be a complex square matrix. Then

$$R_A \subseteq \text{ri}(\text{cone } T(A)). \quad (6.3)$$

**Proof.** For each $\tilde{A} \in QR(A)$ and each $\sigma \in S_n$, $\tilde{A}_\sigma$ is a positive multiple of $A_\sigma$, say $\tilde{A}_\sigma = p_\sigma A_\sigma$. Then $\tilde{A} = \sum_{\sigma \in S_n} \tilde{A}_\sigma = \sum_{\sigma \in S_n} p_\sigma A_\sigma$ is a positive linear combination of the elements of $T(A)$. Thus $\det \tilde{A} \in \text{ri}(\text{cone } T(A))$ and so (6.3) holds. \hfill \Box

A necessary and sufficient condition on $A$ such that the equality in (6.3) holds will be obtained later in Corollary 6.1.

**Proposition 6.2.** Let $A$ be a complex square matrix. Then $T_0(A) \subseteq \overline{R}_A$.

**Proof.** Take any $A_\sigma \in T_0(A)$ (i.e., $\sigma \in S_n$ and $A_\sigma \neq 0$). Take $\varepsilon > 0$ and $\tilde{A}_\varepsilon = (\tilde{a}_{ij}) \in QR(A)$ such that

$$\tilde{a}_{ij} = \begin{cases} a_{ij} & \text{if } j = \sigma(i), \\ \varepsilon a_{ij} & \text{otherwise.} \end{cases}$$

Then $\det \tilde{A}_\varepsilon = A_\sigma + \varepsilon \cdot g(\varepsilon)$ (where $g(\varepsilon)$ is a polynomial of $\varepsilon$), and $\det \tilde{A}_\varepsilon \in R_A$. Let $\varepsilon \to 0$, then $\tilde{A}_\varepsilon \to A_\sigma$. So $A_\sigma \in \overline{R}_A$ and thus the result follows. \hfill \Box

From the above two propositions it is easy to see that, if $H$ is any closed half plane (with zero on its boundary) in $\mathbb{C}$, then $T_0(A) \subseteq H$ if and only if $R_A \subseteq H$. From this we have the following theorem.

**Theorem 6.3.** Let $A$ be a square ray pattern matrix and $H$ be a closed half plane (with zero on its boundary) in $\mathbb{C}$. If $R_A \subseteq H$, then $R_A$ can be completely determined by $T_0(A)$ according to the following cases:

1. If $T_0(A)$ is empty, then $R_A = \{0\}$ (i.e., $A$ has identically zero determinant).
2. If $|T_0(A)| = 1$, then $R_A$ is a ray originating at (but excluding) the origin.
3. If $|T_0(A)| = 2$ and $T_0(A) = \{a, -a\}$ for some nonzero complex number $a$, then $R_A$ is the line through $a$, $-a$ and 0.
4. In all other cases we write $T_0(A) = \{e^{i\alpha_1}, \ldots, e^{i\alpha_k}\}$ with $k \geq 2$ and $\alpha_1 < \cdots < \alpha_k$, where $\alpha_k - \alpha_1 \leq \pi$ if $k = 2$ and $\alpha_k - \alpha_1 \leq \pi$ if $k \geq 3$ (since $R_A \subseteq H$ is equivalent to $T_0(A) \subseteq H$). Then $R_A$ is equal to the open sector $S(\alpha_1, \alpha_k)$.

**Proof.** The cases (1)–(3) are obvious. For (4), first we have

$$\text{cone } T(A) = \text{cone } T_0(A) = S(\alpha_1, \alpha_k).$$
So by (6.3) we have \( R_A \subseteq \text{ri} (\text{cone} \, T(A)) = \text{ri} (S_{(a_1, a_k)}) = S_{(\alpha_1, \alpha_k)} \). It follows that \( 0 \not\in R_A \). So by Theorem 6.1 and Lemma 4.1, \( R_A \) is a single open sector. On the other hand, \( T_0(A) \subseteq \overline{R_A} \) by Proposition 6.2, which implies that \( \overline{R_A} = \text{cone} \, \overline{R_A} \supseteq \text{cone} \, T_0(A) \supseteq S_{(\alpha_1, \alpha_k)} \), so we must have \( R_A = S_{(\alpha_1, \alpha_k)} \). \( \square \)

Notice that for all cases in Theorem 6.3, we have either \( R_A = \{0\} \) or \( R_A = \text{ri}(\text{cone} \, T(A)) \). Using this we can obtain the following characterization of the equality case of (6.3).

**Corollary 6.1.** Let \( A \) be a complex square matrix with \( R_A \neq \{0\} \). Then \( R_A = \text{ri}(\text{cone} \, T(A)) \) if and only if either \( R_A = \mathbb{C} \) or \( R_A \subseteq H \) for some closed half plane \( H \) with zero on its boundary.

**Proof.** Sufficiency follows from (6.3) and Theorem 6.3. For necessity, we have:

Case 1: \( T_0(A) \subseteq H \) for some closed half plane \( H \).

Then \( R_A \subseteq H \) by (6.3).

Case 2: \( T_0(A) \not\subseteq H \) for any closed half plane \( H \).

Then we have:

\[
\text{cone} \, T(A) = \text{cone} \, T_0(A) = \mathbb{C}.
\]

So \( R_A = \text{ri}(\text{cone} \, T(A)) = \text{ri}(\mathbb{C}) = \mathbb{C} \). \( \square \)

Now we would like to point out the relationships between Theorem 6.3 and [6–Theorem 3.1]. Recalling that [6–Theorem 3.1] asserts that if \( T(A) \) is not empty and \( 0 \not\in \text{ri}(\text{conv} \, T(A)) \), then \( A \) is ray-nonsingular. It can be verified that the hypothesis of [6, Theorem 3.1] will imply that \( T_0(A) \) (and thus \( R_A \)) is contained in some closed half plane \( H \) (with zero on its boundary), \( T_0(A) \) is not empty and \( T_0(A) \neq \{a, -a\} \) for any nonzero complex number \( a \). So by using our Theorem 6.3, we can conclude that \( A \) is ray-nonsingular (if \( A \) satisfies the hypothesis of [6–Theorem 3.1]). Thus in this sense, Theorem 6.3 can be viewed as a generalization of [6–Theorem 3.1].

Finally, we conclude by proposing some problems concerning the determinantal regions \( R_A \) of ray pattern matrices.

**(P1)** If \( 0 \in R_A \) and \( R_A \) is not a line through the origin, is it necessary that \( R_A \) is a disjoint union of \( \{0\} \) and finitely many open sectors?

We can also introduce a parameter related to Problem (P1). Let \( n_R(A) \) be the number of connected components of the set \( R_A \setminus \{0\} \). Then Problem (P1) is equivalent to ask whether \( n_R(A) \) is always finite. We also know from Lemma 4.1 that \( n_R(A) = 1 \) if \( 0 \not\in R_A \). We can also find some examples with \( n_R(A) = 2 \) (similar to type (A4) in Section 5), but we have not seen examples with \( n_R(A) \geq 3 \) up to now.

**(P2)** It seems to be interesting to study the parameter \( n_R(A) \) in general. For example, the maximal possible value of \( n_R(A) \), the characterizations of the cases \( n_R(A) = 1 \) and \( n_R(A) = 2 \), and so on.
(P3) If \( \{0, 1\} \subseteq R_A \), is it necessary that \(-1 \in R_A\)?

Problem (P3) is actually equivalent to ask whether \( R_A \) is necessarily symmetric with respect to the origin when \( 0 \in R_A \).

(P4) If \( A \) does not have identically zero determinant (i.e., \( R_A \neq \{0\} \)), then it is well known that \( A \) is permutation equivalent to a lower triangular blocked matrix each of whose diagonal block \( A_i \) \((i = 1, \ldots, k)\) is (square) fully indecomposable. In this case, we have

\[
R_A = \pm R_{A_1} R_{A_2} \cdots R_{A_k}.
\]

So it is interesting to study \( R_A \) in the special cases when \( A \) is fully indecomposable. For example, determine all the possible determinantal regions \( R_A \) of fully indecomposable ray pattern matrices (and ray nonsingular matrices). The similar problems can also be raised for \( S_A \).

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